Undominated Maximals: General Definition and Characterizations

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Abstract: This paper proposes a general definition of an undominated maximal of a relation on a constraint set. No specific requirement is imposed on either the asymmetry of the objective relation or the constraint set (which might, or might not, coincide with the ground set of the objective relation). Several characterizations are formulated that express undominated maximals of an objective relation as maximals of some trace associated with that objective relation. By means of some of these characterizations, the structure of the entire set of undominated maximals is examined in the particular case of relations induced by open and closed convex cones—among them, the weak and strong Pareto dominance—and, in the case of semiorders, that admit certain types of representability. The results of the last part of the examination allow the construction of many examples of relations whose entire sets of maximals and undominated maximals are completely identifiable in an elementary way.

Keywords: undominated maximal; optimization; characterization; cone; Pareto dominance; representable interval order; semiorder

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1. Introduction

By making use of a version of Duncan Luce's famous *sugar example* in [1], the article [2] by Peris and Subiza makes the point that the maximal elements of a relation may be different from one another. The observation is motivated by the fact that two maximals of a relation can bear different connections with the other feasible alternatives and that this difference might make a maximal "more attractive" than another to a decision maker. More precisely, this difference might induce an ordering on the alternatives that complements the primitive preference–indifference relation and that the decision maker might take into consideration when making a choice. Driven by this motivation, the authors originally proposed an optimality notion, called undominated maximality, which selects from the set of maximal elements of an *asymmetric* objective relation the alternatives that are also maximal elements of the global trace of the objective relation. Undominated maximality is the subject of the present work.

The original definition of an undominated maximal proposed in [2]—as well as that adopted by [3,4]—restricts attention to asymmetric relations and to constraint sets that coincide with the entire ground set of the objective relation. A quite dissatisfying aspect of this definition is its lack of generality and applicability. Many objective relations of economic interest are not asymmetric relations, and hence, one cannot apply the original definition of an undominated maximal when trying to refine the set of maximals of those relations. Furthermore, many optimization problems of economic interest, especially in consumer theory, are properly constrained optimization problems (see, e.g., [5] p. 472) and, perhaps even more importantly, many objective relations of economic interest (e.g., the two usual Pareto dominance relations on spaces of "utility levels") do not possess unconstrained maximals and so one cannot hope to select anything by means of the original definition of an

undominated maximality. There is therefore a clear need to extend the original definition of undominated maximality. But how? How can we extend the original definition of an undominated maximal so as to deal with objective relations that are not asymmetric? How can we extend the definition of an undominated maximal so as to deal with constrained optimization problems?

The previous two questions are not equally problematic. The extension of the definition of an undominated maximal that accommodates to relations that are not necessarily asymmetric can be carried out by applying the original definition of an undominated maximal of an asymmetric relation to the sole asymmetric part of an objective relation. This seems to be, by far, the most natural extension that accommodates to relations that are not necessarily asymmetric; however, admittedly, it is just one of the possible generalizations. The extension of the definition of an undominated maximal that accommodates to constrained optimization problems is much less obvious. By relying on the fact that the maximals of a relation on a constraint set are equal to the maximals of the restriction of the objective relation to that constraint set, one might be tempted to define an undominated maximal on a constraint set as an unconstrained undominated maximal of the restriction of the relation to the constraint set: probably, this is the definition of constrained undominated maximality *tacitly* used in Example 2 in [6]. On the other hand, it is natural to define constrained undominated maximals as those constrained maximals that are also constrained maximals of the global trace of the asymmetric part of the primitive objective relation. The problem is that these two possible definitions of a constrained undominated maximal are different definitions. Asymmetrization and restriction commute with one another, and hence the asymmetric part of the restriction of a relation coincides with the restriction of the asymmetric part of that relation: this is the reason why the maximals of a relation on a constraint set are equal to the maximals of the restriction of that relation to that constraint set. However, the global trace of the asymmetric part of the restriction of a relation (*does coincide* with the global trace of the restriction of the asymmetric part of that relation but) *does not generally coincide* with the restriction of the global trace of the asymmetric part of that relation: this is the reason why the maximals of the global trace of the asymmetric part of a relation on a certain constraint set do not generally coincide with the maximals of the restriction to that constraint set of the global trace of the asymmetric part of that relation. Two simple examples in Section 3.4 illustrate the point.

The first, and main, purpose of this work is to provide a general definition of an undominated maximal that accommodates to possibly not asymmetric objective relations and to constrained optimization problems. Once we have provided this general definition, we characterize the set of undominated maximals of an objective quasi-transitive relation as the set of maximals of the global trace associated with the asymmetric part of that objective relation. This identification allows the direct application of many results of the large amount of literature on maximals (in particular, of many existence results). For classes of quasi-transitive objective relations that satisfy some regularity conditions—called $\mathcal{T}$-regularity and quasi-coherence (precisely defined and discussed in Sections 2.2 and 2.4 as well as in Section 5)—we further clarify the connection with the set of maximals of the right trace of the asymmetric part of the objective relation and with the very set of maximals. Establishing the exact structure of the set of undominated maximals can be a hard exercise even when one is dealing with a seemingly simple objective relation and a seemingly simple constraint set. The last part of this work focuses on two classes of $\mathcal{T}$-regular relations and obtains the exact structure of their sets of undominated maximals.

This paper is organized as follows. Section 2 recalls and introduces basic notions. Section 3 proposes a definition of undominated maximality and discusses its generality. Section 4 provides several characterizations of the set of undominated maximals. Section 5 presents two classes of $\mathcal{T}$-regular and quasi-transitive relations, while Sections 6 and 7 examine the structure of the sets of undominated maximals of those two classes. Section 8 concludes and indicates some open issues. Appendix A contains some auxiliary results.

2. Basic Definitions

Section 2 provides all of the basic definitions and notation used in the rest of the paper. Sections 2.1–2.4 pertain to relations, while Section 2.5 relates to vector spaces. (For an introduction to relational mathematics see, e.g., [7,8]: for its applications to social sciences and, in particular, to mathematical economics see, e.g., [9–11], while see [12] for its connections with the related field of vector optimization. Many examples can be found in the books just mentioned as well as in various authoritative handbooks of mathematical economics not cited here). With the exception of quasi-coherence and $\mathcal{T}$-regularity, no definition provided here is new. In point of fact, not even quasi-coherence is new as it was previously examined in [13]. Furthermore, a regularity notion somehow related to $\mathcal{T}$-regularity was used in [14,15] (and sufficient conditions for the $\mathcal{T}$-regularity of a relation are also implicit in Theorem 3.5 in [16]).

2.1. Basic Relational Definitions

A relation B on a (ground) set X is a subset of $X \times X$. Let X be a set and B be a relation on X. We view the second factor of $X \times X$ as the domain of B and the first as its codomain. The membership $(y, x) \in B$ reads "y is preferred or indifferent to x through B": this reading is, however, immaterial to the goal of the present paper. The **upper section of B at** $x \in X$ is the set $B(x) = \{y : (y, x) \in B\}$. The **restriction of B to a subset** Y of X is the relation $(Y \times Y) \cap B$ on Y. The **converse** of B is the relation $B^{\mathfrak{c}}$ on X defined by $B^{\mathfrak{c}}(x) = \{y : (x, y) \in B\}$ at all $x \in X$. The **lower section of B at** $x \in X$ is the upper section of $B^{\mathfrak{c}}$ at x. The **right trace of** B is the relation $\mathcal{T}_B$ on X defined by

$$\mathcal{T}_B(x) = \{y \in X : B(y) \subseteq B(x)\}$$

at all $x \in X$. The relation $\mathcal{T}_{B^{\mathfrak{c}}}^{\mathfrak{c}}$ is the **left trace of** B. Therefore,

$$\mathcal{T}_{B^{\mathfrak{c}}}^{\mathfrak{c}}(x) = \{y \in X : B^{\mathfrak{c}}(x) \subseteq B^{\mathfrak{c}}(y)\}$$

at all $x \in X$. The relation

$$\mathcal{G}_B = \mathcal{T}_B \cap \mathcal{T}_{B^{\mathfrak{c}}}^{\mathfrak{c}}$$

on X is the **global trace of** B. The **irreflexive part** of B is the relation $B^{\mathfrak{i}}$ on X defined by $B^{\mathfrak{i}}(x) = B(x) \backslash \{x\}$ at all $x \in X$. The **asymmetric part** of B is the relation $B^{\mathfrak{a}}$ on X defined by $B^{\mathfrak{a}}(x) = B(x) \backslash B^{\mathfrak{c}}(x)$ at all $x \in X$. The membership $(y, x) \in B^{\mathfrak{a}}$ reads "y is preferred to x through B": this reading is, however, immaterial to the goal of the present paper. For all $(\mathfrak{p}, \mathfrak{q}) \in \{\mathfrak{a}, \mathfrak{c}\} \times \{\mathfrak{a}, \mathfrak{c}\}$, we write $B^{\mathfrak{p}\mathfrak{q}}$ instead of $(B^{\mathfrak{p}})^{\mathfrak{q}}$.

Remark 1. *Conversion and asymmetrization commute with each other and hence $B^{\mathfrak{a}\mathfrak{c}} = B^{\mathfrak{c}\mathfrak{a}}$. Asymmetrization is idempotent and hence $B^{\mathfrak{a}\mathfrak{a}} = B^{\mathfrak{a}}$.*

A relation B on a set X is: **irreflexive** if $B = B^{\mathfrak{i}}$ (if the negated membership $x \notin B(x)$ is true for all $x \in X$); **asymmetric** if $B = B^{\mathfrak{a}}$ (if B is irreflexive and $B = B^{\mathfrak{a}}$, if the implication $y \in B(x) \Rightarrow x \notin B(y)$ is true for all $(y, x) \in X \times X$); **total** if $B \cup B^{\mathfrak{c}} = X \times X$ (if either $y \in B(x)$ or $x \in B(y)$ for all $(y, x) \in X \times X$); **reflexive** if $\mathcal{T}_B \subseteq B$ (if the membership $x \in B(x)$ is true for all $x \in X$); **transitive** if $B \subseteq \mathcal{T}_B$ (if the implication $[z \in B(y)$ and $y \in B(x)] \Rightarrow z \in B(x)$ is true for all $(z, y, x) \in X \times X \times X$); a **strict partial order** if $B \subseteq \mathcal{T}_B^{\mathfrak{a}}$ (if B is irreflexive and transitive, if B is asymmetric and transitive); a **coherent order** if $B = \mathcal{T}_B^{\mathfrak{a}}$; a **preorder** if $B = \mathcal{T}_B$ (if B is transitive and reflexive); **Ferrers** if $\mathcal{T}_B$ is total (if the implication $[z \in B(y)$ and $x \in B(w)] \Rightarrow [z \in B(w)$ or $x \in B(y)]$ is true for all $(z, y, x, w) \in X \times X \times X \times X$); **interval order** if B is irreflexive and $\mathcal{T}_B$ is total (if B is irreflexive and Ferrers; if B is a Ferrers strict partial order); a **semiorder** if B is irreflexive and $\mathcal{G}_B$ is total (if B is an interval order satisfying the implication $[z \in B(y)$ and $y \in B(x)] \Rightarrow [$either $w \in B(x)$ or $z \in B(w)]$ for all $(z, y, x, w) \in X \times X \times X \times X$).

2.2. Quasi-Transitivity, Quasi-Coherence, $\mathcal{T}$-Regularity

Henceforth, in this Section, let X be a set and B a relation on X. The relation B is **quasi-transitive** if $B^{\mathfrak{a}}$ is transitive (if $B^{\mathfrak{a}} \subseteq \mathcal{T}_{B^{\mathfrak{a}}}$, if $B^{\mathfrak{a}} \subseteq \mathcal{T}_{B^{\mathfrak{a}}}^{\mathfrak{a}}$).

Remark 2. *By the idempotence of asymmetrization, B is quasi-transitive if and only if $B^{\mathfrak{a}}$ is quasi-transitive. It is readily checked that B is quasi-transitive if B is transitive.*

The relation B is $\mathcal{T}$-**regular** if $\mathcal{G}_{B^{\mathfrak{a}}} = \mathcal{T}_{B^{\mathfrak{a}}}$ (if $\mathcal{T}_{B^{\mathfrak{a}}} \subseteq \mathcal{T}_{B^{\mathfrak{ca}}}^{\mathfrak{c}}$).

Remark 3. *By the idempotence of asymmetrization, B is $\mathcal{T}$-regular if and only if $B^{\mathfrak{a}}$ is $\mathcal{T}$-regular. It is readily noted that B is a semiorder if B is a $\mathcal{T}$-regular interval order.*

The relation B is **quasi-coherent** if $B^{\mathfrak{a}}$ is a coherent order (if $B^{\mathfrak{a}} = \mathcal{T}_{B^{\mathfrak{a}}}^{\mathfrak{a}}$).

Remark 4. *By the idempotence of asymmetrization, B is quasi-coherent if and only if $B^{\mathfrak{a}}$ is quasi-coherent. Moreover, B is quasi-coherent if B is a coherent order. (If B is a coherent order, then $B = \mathcal{T}_B^{\mathfrak{a}}$ and hence $B^{\mathfrak{a}} = \mathcal{T}_B^{\mathfrak{aa}}$ with $\mathcal{T}_B^{\mathfrak{aa}} = \mathcal{T}_B^{\mathfrak{a}}$. If B is a coherent order, then B is asymmetric, and hence, $B = B^{\mathfrak{a}}$. Thus, the equality $B^{\mathfrak{a}} = \mathcal{T}_{B^{\mathfrak{a}}}^{\mathfrak{a}}$ is true if B is a coherent order).*

2.3. Nonsatiation and Topological Conditions

Henceforth, in this Section, let X be a set and B a relation on X. We say that B is **quasi-nonsatiated** if $[(y,x) \in X \times X$ and $B^{\mathfrak{a}}(x) \neq \varnothing = B^{\mathfrak{a}}(y)] \Rightarrow y \in B^{\mathfrak{a}}(x)$; **nonsatiated** if $B^{\mathfrak{a}}(x) \neq \varnothing$ for all $x \in X$. Henceforth, in this Section, assume that X is a set endowed with some topology τ and hence that (X, τ) is a topological space. As usual, a subset S of X is said to be τ-compact if S endowed with the relative topology is compact. The topological closure of a subset S of X is denoted by cl S, while the topological interior of S is denoted by int S. We say that B is **conditionally locally nonsatiated for** τ if $[x \in X$ and $B^{\mathfrak{a}}(x) \neq \varnothing] \Rightarrow x \in \text{cl } B^{\mathfrak{a}}(x)$; **locally quasi-nonsatiated for** τ if B is quasi-nonsatiated and conditionally locally nonsatiated for τ; and **locally nonsatiated for** τ if B is nonsatiated and conditionally locally nonsatiated for τ (if $x \in \text{cl } B^{\mathfrak{a}}(x)$ for all $x \in X$). (Figure 1 in [13] conveniently summarizes some implications about the various notions of nonsatiation. Clearly, conditional local quasi-nonsatiation (is implied by local quasi-nonsatiation but) neither implies nor is implied by quasi-nonsatiation. For an economic interpretation of (local) nonsatiation the reader is referred to [17]). Finally, we say that B is **open-valued for** τ if $B(x)$ is a τ-open subset of X for all $x \in X$; **closed-valued for** τ if $B(x)$ is a τ-closed subset of X for all $x \in X$.

2.4. $\mathcal{T}$-Regularity and Quasi-Transitivity of Quasi-Coherence

Every quasi-coherent relation—and hence every coherent order—is $\mathcal{T}$-regular and quasi-transitive.

Proposition 1. *Let X be a set and B a quasi-coherent relation on X. The relation B is $\mathcal{T}$-regular and quasi-transitive.*

Proof. It is clear from Section 2.2 that B is quasi-transitive. Thus, to conclude the proof, we only need to show that B is $\mathcal{T}$-regular and hence that $\mathcal{T}_{B^{\mathfrak{a}}} \subseteq \mathcal{T}_{B^{\mathfrak{ca}}}^{\mathfrak{c}}$. Suppose $(y,x) \in X \times X$ and $y \in \mathcal{T}_{B^{\mathfrak{a}}}(x)$. The last membership is equivalent to

$$B^{\mathfrak{a}}(y) \subseteq B^{\mathfrak{a}}(x). \tag{1}$$

Now, by way of contradiction, suppose $y \notin \mathcal{T}_{B^{\mathfrak{ca}}}^{\mathfrak{c}}(x)$. The last negated membership is equivalent to $B^{\mathfrak{ca}}(x) \not\subseteq B^{\mathfrak{ca}}(y)$ and hence to the nonemptiness of $B^{\mathfrak{ca}}(x) \backslash B^{\mathfrak{ca}}(y)$. Pick an arbitrary $z \in B^{\mathfrak{ca}}(x) \backslash B^{\mathfrak{ca}}(y)$. Then $x \in B^{\mathfrak{a}}(z)$ and $y \notin B^{\mathfrak{a}}(z)$. As B is quasi-coherent, the last membership and the last negated membership are equivalent to $B^{\mathfrak{a}}(x) \subset B^{\mathfrak{a}}(z)$ and $B^{\mathfrak{a}}(y) \not\subset B^{\mathfrak{a}}(z)$, respectively; a contradiction with (1). □

Many simple classes of quasi-coherent relations are considered in [13]: see Theorem 6, Theorem 9 (and Corollary 3), Theorem 10 (and Corollary 4) and Theorem 12 of the mentioned article. Corollary 4 in [13] serves as a lemma in the rest of the paper.

Lemma 1 ([13], Corollary 4). *Let (X, τ) be a topological space and B a transitive relation on X. If B is closed-valued for τ and locally quasi-nonsatiated for τ, then B is quasi-coherent.*

2.5. Vector Spaces

The set of the reals (of nonnegative reals, of positive reals) is denoted by $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$). For every positive integer n, the n-ary Cartesian power of $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$) is denoted by $\mathbb{R}^n$ ($\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$). The usual preorder relation on the set $\mathbb{R} \cup \{+\infty, -\infty\}$ is denoted by $\geq$ (its asymmetric part by $>$, its converse by $\leq$, the asymmetric part of its converse by $<$). When S is a subset of $\mathbb{R} \cup \{+\infty, -\infty\}$, the supremum of S is denoted by $\sup S$ ($\sup S = -\infty$ when $S = \varnothing$); if a maximum of S exists in $\mathbb{R}$, it is denoted by $\max S$. A vector space over the reals (a real vector space) is abbreviated by **RVS**. A topological vector space over the reals (a real topological vector space) is abbreviated by **RTVS**. When X is an RVS and the sets S and T are subsets thereof, the set $\{x \in X : -x \in S\}$ is denoted by $-S$, and the **Minkowski sum** of S and T is denoted by $S + T$, whereas the sum of S and $-T$ is denoted by $S - T$; when S is a singleton, say $\{s\}$, we just write $s + T$ and $s - T$ instead of $\{s\} + T$ and $\{s\} - T$, respectively. By a **cone** in X, we mean any subset C of X satisfying the implication $x \in C \Rightarrow \lambda x \in C$ for every $\lambda \in \mathbb{R}_{++}$. A cone might be empty and need not contain the zero vector.

3. Optimality Notions

This Section recalls the usual definitions of a maximal and of a greatest element and provides a general definition of an undominated maximal element.

3.1. Maximals and Greatest Elements: Definition

Let X be a set and B a relation on X. The set of **maximals of B on a subset Y of X** is the set $\mathcal{M}(B, Y)$ defined by

$$\mathcal{M}(B, Y) = \{y \in Y : B^{\mathfrak{a}}(y) \cap Y = \varnothing\}.$$

The set of **unconstrained maximals of B** is the set $\mathcal{M}(B)$ defined by

$$\mathcal{M}(B) = \{y \in X : B^{\mathfrak{a}}(y) = \varnothing\}.$$

Clearly, $\mathcal{M}(B) = \mathcal{M}(B, X)$.

Remark 5. *If B is an asymmetric relation on X, then $\mathcal{M}(B, Y) = \{y \in Y : B(y) \cap Y = \varnothing\}$ for all $Y \subseteq X$ and $\mathcal{M}(B) = \{y \in Y : B(y) = \varnothing\}$.*

The maximals of a relation are the maximals of the asymmetric part of that relation.

Remark 6. *Let X be a set and B a relation on X. By the idempotence of asymmetrization, $\mathcal{M}(B, Y) = \mathcal{M}(B^{\mathfrak{a}}, Y)$ for all $Y \subseteq X$ and $\mathcal{M}(B) = \mathcal{M}(B^{\mathfrak{a}})$.*

The set of **greatest elements of B on a subset Y of X** is the set $\mathfrak{M}(B, Y)$ defined by

$$\mathfrak{M}(B, Y) = \{y \in Y : Y \subseteq B^{\mathfrak{c}}(y)\}.$$

The set of **unconstrained greatest elements of B** is the set $\mathfrak{M}(B)$ defined by

$$\mathfrak{M}(B) = \{y \in X : X \subseteq B^{\mathfrak{c}}(y)\}.$$

Clearly, $\mathfrak{M}(B) = \mathfrak{M}(B, X)$. In addition, $\mathfrak{M}(B, Y) \subseteq \mathcal{M}(B, Y)$ for all $Y \subseteq X$ and $\mathfrak{M}(B) \subseteq \mathcal{M}(B)$.

Remark 7. *Let X be a set and B a total relation on X. Then $\mathfrak{M}(B, Y) = \mathcal{M}(B, Y)$ for all $Y \subseteq X$ and $\mathfrak{M}(B) = \mathcal{M}(B)$.*

3.2. Undominated Maximality: Definition

Let X be a set and B a relation on X. The set of **undominated maximals of B on a subset Y of X** is the set $\mathcal{U}(B, Y)$ defined by

$$\mathcal{U}(B, Y) = \mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{G}_{B^a}, Y).$$

(For clarity, $\mathcal{U}(B, Y) = \mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{T}_{B^a} \cap \mathcal{T}_{B^{ca}}^c, Y) = \mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{T}_{B^a} \cap \mathcal{T}_{B^{ac}}^c, Y)$ for all $Y \subseteq X$ and $\mathcal{U}(B) = \mathcal{M}(B) \cap \mathcal{M}(\mathcal{T}_{B^a} \cap \mathcal{T}_{B^{ca}}^c) = \mathcal{M}(B) \cap \mathcal{M}(\mathcal{T}_{B^a} \cap \mathcal{T}_{B^{ac}}^c)$).

The set of **unconstrained undominated maximals of B** is the set $\mathcal{U}(B)$ defined by

$$\mathcal{U}(B) = \mathcal{M}(B) \cap \mathcal{M}(\mathcal{G}_{B^a}).$$

Clearly, $\mathcal{U}(B) = \mathcal{U}(B, X)$.

Remark 8. *If B is an asymmetric relation on X, then $\mathcal{U}(B, Y) = \mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{G}_B, Y)$ for all $Y \subseteq X$ and $\mathcal{U}(B) = \mathcal{M}(B) \cap \mathcal{M}(\mathcal{G}_B)$.*

The undominated maximals of a relation are the undominated maximals of the asymmetric part of that relation.

Remark 9. *Let X be a set and B a relation on X. By the idempotence of asymmetrization, $\mathcal{U}(B, Y) = \mathcal{U}(B^a, Y)$ for all $Y \subseteq X$ and $\mathcal{U}(B) = \mathcal{U}(B^a)$.*

Remark 10—that follows from Remark 7—points out that every undominated maximal of a total relation is a greatest element.

Remark 10. *Let X be a set and B a total relation on X. Then $\mathcal{U}(B, Y) \subseteq \mathfrak{M}(B, Y)$ for all $Y \subseteq X$ and $\mathcal{U}(B) \subseteq \mathfrak{M}(B)$.*

Example 1 shows that a greatest element of a total and quasi-transitive relation need not be an undominated maximal. Example 2 shows that an undominated maximal of a reflexive and quasi-transitive relation need not be a greatest element even when at least one greatest element exists.

Example 1. *Let $\{x, y, z\} = X$ be a set with three distinct elements and B the total and quasi-transitive relation on X defined by $B(x) = \{x, y\}$ and $B(y) = B(z) = \{x, y, z\}$. It is readily checked that*
$$\mathcal{U}(B) = \{x\} \subset \{x, y\} = \mathcal{M}(B) = \mathfrak{M}(B).$$

Example 2. *Let $\{x, y, z\} = X$ be a set with three distinct elements and B the reflexive and quasi-transitive relation on X defined by $B(x) = \{x, z\}$, $B(y) = \{y, z\}$ and $B(z) = \{x, y, z\}$. It is readily checked that*
$$\mathfrak{M}(B) = \{z\} \subset \{x, y, z\} = \mathcal{M}(B) = \mathcal{U}(B).$$

3.3. Connections with Previous Definitions

The definition of *unconstrained* undominated maximality was first provided in [2] for the case of *at least asymmetric* relations with a finite ground set. The finiteness assumption was relaxed in [3,4]. Remark 11 clarifies that in the case of an asymmetric objective relation, the definition of an *unconstrained* undominated maximal given here coincides with the one

that can be found in the literature: attention will be focused on the definition provided in [4].

Remark 11. *Henceforth, in this Remark, assume that B is an asymmetric relation. We have already pointed out in Remark 8 that $\mathcal{U}(B) = \mathcal{M}(B) \cap \mathcal{M}(\mathcal{G}_B)$. As the idempotence of asymmetrization implies $\mathcal{M}(\mathcal{G}_B^\mathfrak{a}) = \mathcal{M}(\mathcal{G}_B)$, we infer*

$$\mathcal{U}(B) = \mathcal{M}(B) \cap \mathcal{M}(\mathcal{G}_B^\mathfrak{a}). \tag{2}$$

If we identify B with the asymmetric relation denoted by $\succ$ in [4], then the global trace $\mathcal{G}_B$ is the relation denoted in [4] by $\succsim^D$ (the right and left traces $\mathcal{T}_B$ and $\mathcal{T}_{B^\complement}^\complement$ are denoted in [4] by $\succsim^{**}$ and $\succsim^*$, respectively). If we denote the asymmetric part of $\succsim^D$ by $\succ^D$, then we can equivalently rewrite the equality in (2) as

$$\mathcal{U}(\succ) = \mathcal{M}(\succ) \cap \mathcal{M}(\succ^D).$$

Thus, there should be no doubt that the notion of an unconstrained undominated maximal of an asymmetric relation is exactly the same as that in [4], where attention is restricted to unconstrained *undominated maximals of* (at least) asymmetric *relations.*

3.4. Discussion

It is well known and readily verified that a constrained maximal is an unconstrained maximal of the restriction of the objective relation to the constraint set. An analogous statement does not hold true, in general, in the case of undominated maximals. (A similar observation extends to the Fishburn shading as defined in [18]: see Section 5.4 in [19] for a longer and more detailed discussion). The following Remark clarifies the previous point.

Remark 12. *Let X be a set and B be a relation on X. Suppose $Y \subseteq X$ and denote the restriction of B to Y by R. It is readily verified that*

$$\mathcal{M}(B, Y) = \mathcal{M}(R).$$

(If needed, see part 2 of Proposition A.1 in [20] for proof).

However—and this is clear from Examples 3 and 4 below—it is well possible that

$$\mathcal{U}(B, Y) \neq \mathcal{U}(R).$$

Example 3 shows that the set of undominated maximals of a relation on a constraint set might be strictly included in the set of unconstrained undominated maximals of the restriction of that relation to that constraint set. Example 4 shows that the set of undominated maximals of a relation on a constraint set might strictly include the set of unconstrained undominated maximals of the restriction of that relation to that constraint set.

Example 3. *Put $X = \mathbb{R}^2$ and let Y be the subset of X defined by*

$$Y = \{(3,3), (3,2)\}.$$

Consider the relation B on X defined by $B(x) = x + \mathbb{R}_{++}^2$ and denote the restriction of B to Y by R. Then

$$\mathcal{U}(B, Y) = \{(3,3)\} \neq \{(3,3), (3,2)\} = \mathcal{U}(R)$$

and hence

$$\mathcal{U}(B, Y) \subset \mathcal{U}(R).$$

Example 4. *Put $X = \mathbb{R}^2$ and let Y be the subset of X defined by*

$$Y = \{(3,3), (2,2), (4,1)\}.$$

Consider the relation B on X defined by $B(x) = x + \mathbb{R}_+^2$ and denote the restriction of B to Y by R. Then

$$\mathcal{U}(B, Y) = \{(3,3), (4,1)\} \neq \{(3,3)\} = \mathcal{U}(R)$$

and hence

$$\mathcal{U}(R) \subset \mathcal{U}(B, Y).$$

Thus far, the literature has provided a definition only of an unconstrained undominated maximal and only for asymmetric relations. When generalizing that definition to arbitrary relations and when allowing for constraint sets, several alternative definitions are possible, in principle. Needless to say, from a purely mathematical viewpoint, there is no compelling reason to favor one definition over another. In the personal opinion of the author of the present paper, the conceptually most suitable definition of an undominated maximal should make use of the traces generated by the asymmetric part of the primitive objective relation (namely, by the primitive preference relation of a decision maker) rather than by its many restrictions to the various constraint sets that can be admitted by the ground set of the primitive objective relation (and that are compatible with the decision problem faced by the decision maker).

4. Undominated Maximals of Quasi-Transitive Relations

The set of undominated maximals of a quasi-transitive relation B equals the set of maximals of the global trace of the asymmetric part of B. By making use of this fact, we can provide several characterizations of the set of undominated maximals of relations that are at least quasi-transitive. Let X_1 and X_2 be sets, B_1 a relation on X_1 and B_2 a relation on X_2. We say that B_1 **is quasi-extended by** B_2 if $X_1 = X_2$ and $B_1^\mathfrak{a} \subseteq B_2^\mathfrak{a}$. Put $X = X_1$ and suppose B_1 is quasi-extended by B_2: note that $\mathcal{M}(B_2, Y) \subseteq \mathcal{M}(B_1, Y)$ for all $Y \subseteq X$.

Lemma 2. *Let X be a set and B a quasi-transitive relation on X. Then B is quasi-extended by $\mathcal{G}_{B^\mathfrak{a}}$.*

Proof. Suppose $(y, x) \in X \times X$ and $y \in B^\mathfrak{a}(x)$. Then

$$B^\mathfrak{a}(y) \subset B^\mathfrak{a}(x) \tag{3}$$

in that $B^\mathfrak{a}$ is a strict partial order. Clearly, $x \in B^{\mathfrak{a}\mathfrak{c}}(y)$. The converse of any strict partial order is a strict partial order. Thus, $B^{\mathfrak{a}\mathfrak{c}}$ is a strict partial order and the last membership implies $B^{\mathfrak{a}\mathfrak{c}}(x) \subset B^{\mathfrak{a}\mathfrak{c}}(y)$: so

$$B^{\mathfrak{c}\mathfrak{a}}(x) \subset B^{\mathfrak{c}\mathfrak{a}}(y) \tag{4}$$

in that conversion and asymmetrization commute with each other. The inclusions in (3) and (4) immediately imply $y \in \mathcal{G}_{B^\mathfrak{a}}^\mathfrak{a}(x)$: to see this, just note that the previous membership is true if and only if either $[B^\mathfrak{a}(y) \subset B^\mathfrak{a}(x)$ and $B^{\mathfrak{c}\mathfrak{a}}(x) \subseteq B^{\mathfrak{c}\mathfrak{a}}(y)]$ or $[B^\mathfrak{a}(y) \subseteq B^\mathfrak{a}(x)$ and $B^{\mathfrak{c}\mathfrak{a}}(x) \subset B^{\mathfrak{c}\mathfrak{a}}(y)]$. □

Theorem 1. *Let X be a set, Y a subset of X and B a relation on X.*

1. $\mathcal{U}(B, Y) = \mathcal{M}(\mathcal{G}_{B^\mathfrak{a}}, Y)$ *provided B is quasi-transitive.*
2. $\mathcal{U}(B, Y) = \mathcal{M}(\mathcal{T}_{B^\mathfrak{a}}, Y)$ *provided B is $\mathcal{T}$-regular and quasi-transitive.*
3. $\mathcal{U}(B, Y) = \mathcal{M}(B, Y)$ *provided B is quasi-coherent.*

Proof. Suppose B is quasi-transitive. Lemma 2 ensures that B is quasi-extended by $\mathcal{G}_{B^\mathfrak{a}}$. Therefore, $\mathcal{M}(\mathcal{G}_{B^\mathfrak{a}}, Y) \subseteq \mathcal{M}(B, Y)$ and hence $\mathcal{U}(B, Y) = \mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{G}_{B^\mathfrak{a}}, Y) = \mathcal{M}(\mathcal{G}_{B^\mathfrak{a}}, Y)$. This proves part 1 of Theorem 1. Part 2 of Theorem 1 is a consequence of part 1 of Theorem 1 and of the definition of $\mathcal{T}$-regularity. The idempotence of asymmetrization implies $\mathcal{M}(B, Y) = \mathcal{M}(B^\mathfrak{a}, Y)$ and $\mathcal{M}(\mathcal{T}_{B^\mathfrak{a}}, Y) = \mathcal{M}(\mathcal{T}_{B^\mathfrak{a}}^\mathfrak{a}, Y)$; recalling that the quasi-coherence of B is equivalent to the validity of the equality $B^\mathfrak{a} = \mathcal{T}_{B^\mathfrak{a}}^\mathfrak{a}$, part 3 of Theorem 1 is a consequence of Proposition 1 and part 2 of Theorem 1. □

Corollary 1. *Let X be a set, Y a subset of X and B a relation on X.*

1. $\mathcal{U}(B, Y) = \mathcal{M}(\mathcal{G}_B, Y)$ *provided B is a strict partial order.*
2. $\mathcal{U}(B, Y) = \mathcal{M}(\mathcal{T}_B, Y)$ *provided B is a $\mathcal{T}$-regular strict partial order.*
3. $\mathcal{U}(B, Y) = \mathcal{M}(B, Y)$ *provided B is a coherent order.*

Proof. Every strict partial order is asymmetric and so $B = B^{\mathfrak{a}}$ if B is a strict partial order. Furthermore, every strict partial order is quasi-transitive. Finally, Remark 4 has already pointed out that every coherent order is a quasi-coherent relation. Recalling these facts, Corollary 1 is a consequence of Theorem 1. □

Corollary 2. *Let X be a set, Y a subset of X and B a relation on X.*

1. $\mathcal{U}(B, Y) = \mathfrak{M}(\mathcal{G}_B, Y)$ *provided B is a semiorder.*
2. $\mathcal{U}(B, Y) = \mathfrak{M}(\mathcal{T}_B, Y)$ *provided B is a $\mathcal{T}$-regular semiorder.*

Proof. Every semiorder is a strict partial order possessing a total global trace. Furthermore, every interval order is a strict partial order possessing a total right trace. Finally, Remark 3 has already pointed out that every $\mathcal{T}$-regular interval order is a ($\mathcal{T}$-regular) semiorder. Recalling these facts, Corollary 2 is a consequence of Remark 7 and parts 1 and 2 of Corollary 1. □

Remark 13. *It should be clear from Remark 6 that Theorem 1 and Corollary 1 hold invariantly true if one replaces $\mathcal{M}(\mathcal{G}_{B^{\mathfrak{a}}}, Y)$, $\mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}}, Y)$, $\mathcal{M}(B, Y)$, $\mathcal{M}(\mathcal{G}_B, Y)$, $\mathcal{M}(\mathcal{T}_B, Y)$ with $\mathcal{M}(\mathcal{G}_{B^{\mathfrak{a}}}^{\mathfrak{a}}, Y)$, $\mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}}^{\mathfrak{a}}, Y)$, $\mathcal{M}(B^{\mathfrak{a}}, Y)$, $\mathcal{M}(\mathcal{G}_B^{\mathfrak{a}}, Y)$, $\mathcal{M}(\mathcal{T}_B^{\mathfrak{a}}, Y)$, respectively.*

By means of Theorem 1 and Corollaries 1 and 2, one can infer several statements on (and even characterizations of) the existence of undominated maximals of quasi-transitive relations. For instance, keeping in mind Remark 13 and recalling that traces are transitive, one can use Theorem 4 in [6] and part 1 of Theorem 1 above to characterize the existence of an unconstrained undominated maximal of a quasi-transitive relation. Just to provide other examples, one can use Theorem 4 in [20] and part 1 of Theorem 1 above to characterize the existence of an undominated maximal of a quasi-transitive relation on every nonempty compact subset of its ground set, or else (note that the global and right traces of any semiorder are total preorders), one can use Corollaries 3.1 and 3.2 in [21] and part 1 of Corollary 2 above to characterize the existence of an unconstrained undominated maximal of a semiorder as well as the existence of an undominated maximal of a semiorder on every nonempty compact subset of its ground set. The range of possible applications is not limited to the previous examples. In the rest of this paper, we do not go into the details of such applications, and the existence of undominated maximals are considered only marginally; rather, we focus on deriving simple descriptions of the entire set of undominated maximals of some classes of $\mathcal{T}$-regular relations that illustrate when and how undominated maximality actually selects maximality. Before providing such descriptions, this Section shows some special consequences for the unconstrained undominated maximals of (a relation whose asymmetric part is) an interval order and for the undominated maximals of a relation whose asymmetric part is a semiorder.

Corollary 3. *Let X be a set and B an interval order on X. Then $\mathfrak{M}(\mathcal{T}_{B^c}^{\mathfrak{c}}) = \mathcal{M}(\mathcal{T}_{B^c}^{\mathfrak{c}}) = \mathcal{M}(\mathcal{T}_{B^c}^{\mathfrak{ca}}) \subseteq \mathcal{U}(B)$.*

Proof. The left trace of any interval order is total and hence $\mathfrak{M}(\mathcal{T}_{B^c}^{\mathfrak{c}}) = \mathcal{M}(\mathcal{T}_{B^c}^{\mathfrak{c}})$ by Remark 7; clearly, $\mathcal{M}(\mathcal{T}_{B^c}^{\mathfrak{c}}) = \mathcal{M}(\mathcal{T}_{B^c}^{\mathfrak{ca}})$ by the idempotence of asymmetrization. To conclude the proof we only need to show that $\mathfrak{M}(\mathcal{T}_{B^c}^{\mathfrak{c}}) \subseteq \mathcal{U}(B)$. Assume that $y \in \mathfrak{M}(\mathcal{T}_{B^c}^{\mathfrak{c}})$ and, by way of contradiction, suppose $y \notin \mathcal{U}(B)$. As B is a strict partial order, the last

negated membership is equivalent to $y \notin \mathcal{M}(\mathcal{G}_B)$ by part 1 of Corollary 1. The membership $y \in \mathfrak{M}(\mathcal{T}_{B^c}^{\mathfrak{c}})$ implies

$$B^{\mathfrak{c}}(t) \subseteq B^{\mathfrak{c}}(y) \text{ for all } t \in X. \tag{5}$$

The negated membership $y \notin \mathcal{M}(\mathcal{G}_B)$ implies the existence of $x \in X$ such that either $[B(x) \subset B(y)$ and $B^{\mathfrak{c}}(y) \subseteq B^{\mathfrak{c}}(x)]$ or $[B(x) \subseteq B(y)$ and $B^{\mathfrak{c}}(y) \subset B^{\mathfrak{c}}(x)]$; from (5), we then infer that $B(x) \subset B(y)$ and $B^{\mathfrak{c}}(y) = B^{\mathfrak{c}}(x)$. The last inclusion implies $B(y) \neq \varnothing$, and the fact that B is a strict partial order implies that $B^{\mathfrak{c}}$ is a strict partial order. Pick an arbitrary $z \in B(y)$; then $y \in B^{\mathfrak{c}}(z)$ and hence $B^{\mathfrak{c}}(y) \subset B^{\mathfrak{c}}(z)$ in that $B^{\mathfrak{c}}$ is a strict partial order. However, the inclusion $B^{\mathfrak{c}}(y) \subset B^{\mathfrak{c}}(z)$ is in contradiction with (5). $\square$

By means of Corollary 3, one can infer several statements on (yet not characterizations of) the existence of unconstrained undominated maximals of an interval order B. For instance, when B is an interval order (the left trace of any interval order is a total preorder), one can use Corollary 3.2 in [21] and Corollary 3 above to characterize the existence of an unconstrained greatest element of $\mathcal{T}_{B^c}^{\mathfrak{c}}$, thus obtaining sufficient conditions for the existence of an unconstrained undominated maximal of B. Just to provide another example, when B is an interval order, one can use Theorem 4 in [6] and Corollary 3 above to characterize the existence of an unconstrained maximal of $\mathcal{T}_{B^c}^{\mathfrak{ca}}$, thus obtaining sufficient conditions for the existence of an unconstrained undominated maximal of B; interestingly—keeping in mind Lemma 3 in [4]—Theorem 4 in [6] and Corollary 3 above imply Theorem 1 in [4] (if B is an interval order and if we alternatively denote B by $\succ$, then $\mathcal{T}_{B^c}^{\mathfrak{ca}}$ is the relation denoted by $\succ^*$ in [4]). Corollary 4 and a discussion close this Section.

Corollary 4. *Let X be a set, Y a subset of X and B a relation on X.*

1. $\mathcal{U}(B, Y) = \mathfrak{M}(\mathcal{G}_{B^{\mathfrak{a}}}, Y)$ *provided $B^{\mathfrak{a}}$ is a semiorder.*
2. $\mathcal{U}(B, Y) = \mathfrak{M}(\mathcal{T}_{B^{\mathfrak{a}}}, Y)$ *provided $B^{\mathfrak{a}}$ is a $\mathcal{T}$-regular semiorder.*
3. $\mathcal{U}(B, Y) = \mathfrak{M}(\mathcal{T}_{B^{\mathfrak{a}}}, Y)$ *provided B is $\mathcal{T}$-regular and $B^{\mathfrak{a}}$ is a semiorder.*
4. $\mathfrak{M}(\mathcal{T}_{B^{ca}}^{\mathfrak{c}}) = \mathcal{M}(\mathcal{T}_{B^{ca}}^{\mathfrak{c}}) = \mathcal{M}(\mathcal{T}_{B^{ca}}^{\mathfrak{ca}}) \subseteq \mathcal{U}(B)$ *provided $B^{\mathfrak{a}}$ is an interval order.*

Proof. Recalling that asymmetrization and conversion commute with each other, this is just a consequence of Remarks 3 and 9 and Corollaries 2 and 3. $\square$

Some authors adopt a "dual" definition of an interval order and a semiorder; see, for instance, the definition of an interval order and a semiorder in [22]. A feature of those "dual" definitions of an interval order and a semiorder is that their asymmetric parts are, respectively, an interval order and a semiorder in the sense of the present paper. In a way, Corollary 4 reformulates Corollaries 2 and 3 in terms of those "dual" definitions.

5. On $\mathcal{T}$-Regularity and Quasi-Transitivity

Section 2.4 showed that quasi-coherent relations form a class of $\mathcal{T}$-regular and quasi-transitive relations. This Section presents two other classes of this kind.

5.1. Relations Induced by Cones

Let X be an RVS and C a cone in X. The **relation induced by C on X** is the relation B on X defined by $B(x) = x + C$ for all $x \in X$. If B is the relation induced by C on X, then $B^{\mathfrak{a}}$ is that induced by the cone $C \setminus -C$ (in X) on X and $B^{\mathfrak{c}}$ is that induced by the cone $-C$ (in X) on X. Proposition 2 shows that any relation induced by a cone—whether convex or not—is a $\mathcal{T}$-regular relation.

Proposition 2. *Let X be an RVS, C a cone in X and B the relation induced by C on X.*

1. *The relation B is $\mathcal{T}$-regular.*
2. *The relation B is $\mathcal{T}$-regular and quasi-transitive (in particular, B is $\mathcal{T}$-regular and transitive) if C is convex.*

Proof. The fact that the relation induced on an RVS by a convex cone in it is a transitive—and hence a quasi-transitive—relation is well known and its proof is omitted. We show only the $\mathcal{T}$-regularity of B: the proof does not require the convexity of C. Put $C^\bullet = C \backslash -C$. Suppose $(y, x) \in X \times X$ and $B^{\mathfrak{a}}(y) \subseteq B^{\mathfrak{a}}(x)$. As $B^{\mathfrak{a}}(x) = x + C^\bullet$ and $B^{\mathfrak{a}}(y) = y + C^\bullet$, the last inclusion is equivalent to $y + C^\bullet \subseteq x + C^\bullet$ and hence—by virtue of the properties of a real vector space—to the inclusion $x - C^\bullet \subseteq y - C^\bullet$. The observation that $B^{\mathfrak{ca}}(x) = x - C^\bullet$ and $B^{\mathfrak{ca}}(y) = y - C^\bullet$ concludes the proof. $\square$

5.2. Representable Semiorders

Let X be a set and B a relation on X. We say that B is $\langle u, v \rangle$-**representable** if u and v are real-valued functions on X satisfying the double implication $y \in B(x) \Leftrightarrow u(y) > v(x)$ for all $(y, x) \in X \times X$; and **Scott–Suppes** $\langle u, v, K \rangle$-**representable** if B is $\langle u, v \rangle$-representable and there exists a real number $K \geq 0$ such that $v(x) = u(x) + K$ for all $x \in X$. The number K is sometimes referred to as the "discriminating threshold". Proposition 3 provides sufficient conditions for an irreflexive and nonsatiated relation on a connected topological space to be $\mathcal{T}$-regular and quasi-transitive. For clarity, a real-valued function f on a subset of the reals is **increasing** if the implication $y \geq x \Rightarrow f(y) \geq f(x)$ holds true for all y and x in the domain of f.

Proposition 3. *Let (X, τ) be a connected topological space and B an irreflexive and nonsatiated relation on X. The relation B is $\mathcal{T}$-regular and quasi-transitive (in particular, B is a $\mathcal{T}$-regular semiorder) if B is $\langle u, v \rangle$-representable and u is a continuous function such that $u = f \circ v$ for some increasing $f : v[X] \to \mathbb{R}$.*

Proof. Assume that B is $\langle u, v \rangle$-representable. By Remark A1, the $\langle u, v \rangle$-representable relation B is Ferrers. Consequently, the irreflexive relation B is an interval order and hence a quasi-transitive relation. Now, assume also that u is a continuous function such that $u = f \circ v$ for some increasing $f : v[X] \to \mathbb{R}$. Then the implication $v(y) \geq v(x) \Rightarrow u(y) \geq u(x)$ holds true for all $(y, x) \in X \times X$; from Lemma A1, we immediately infer the validity of the implication $y \in \mathcal{T}_B(x) \Rightarrow y \in \mathcal{T}_{B^c}^{\mathfrak{c}}(x)$ for all $(y, x) \in X \times X$. Therefore, $\mathcal{T}_B \subseteq \mathcal{T}_{B^c}^{\mathfrak{c}}$ and hence B is $\mathcal{T}$-regular in that $B = B^{\mathfrak{a}}$ by the asymmetry of B (and because asymmetrization and conversion commute with each other). By Remark 3, the $\mathcal{T}$-regular interval order B is a semiorder. $\square$

Corollary 5. *Let (X, τ) be a connected topological space and B a $\langle u, v \rangle$-representable relation on X. The relation B is $\mathcal{T}$-regular and quasi-transitive (in particular, B is a nonsatiated $\mathcal{T}$-regular semiorder) if each of the following four conditions is satisfied:*

C1. $u(x) \leq v(x)$ *for all $x \in X$;*
C2.   *u is upper unbounded;*
C3.   *u is continuous;*
C4. $u = f \circ v$ *for some increasing function $f : v[X] \to \mathbb{R}$.*

Proof. A consequence of Remark A1, Lemma A2 and Proposition 3. $\square$

Corollary 6. *Let (X, τ) be a connected topological space and B a Scott–Suppes $\langle u, v, K \rangle$-representable relation on X with u continuous and upper unbounded. Then B is $\mathcal{T}$-regular and quasi-transitive (in particular, B is a nonsatiated $\mathcal{T}$-regular semiorder).*

Proof. A consequence of Remark A1, Corollary 5 and the very definition of Scott–Suppes $\langle u, v, K \rangle$-representability. $\square$

Example 5 shows that the right and left trace of (the asymmetric part) of a relation that satisfies all assumptions of Corollary 6 need not coincide. Example 6 shows that in Corollary 6, we cannot simply drop the assumption that u is upper unbounded.

Example 5. *Let $X = [0, +\infty)$ be endowed with the usual topology inherited from $\mathbb{R}$ and B the relation defined by $B(x) = \{y : y > x + 2\}$. We have that*

$$\mathcal{T}_B(1) = [1, +\infty) \subset [0, +\infty) = \mathcal{T}_{B^\mathfrak{c}}^\mathfrak{c}(1)$$

and hence that $\mathcal{T}_B(1) \neq \mathcal{T}_{B^\mathfrak{c}}^\mathfrak{c}(1)$. (Clearly, $\mathcal{T}_B = \mathcal{T}_{B^\mathfrak{a}}$ and $\mathcal{T}_{B^\mathfrak{c}}^\mathfrak{c} = \mathcal{T}_{B^\mathfrak{c a}}^\mathfrak{c} = \mathcal{T}_{B^\mathfrak{a c}}^\mathfrak{c}$ by the asymmetry of B).

Example 6. *Let $X = (-\infty, 0]$ be endowed with the usual topology inherited from $\mathbb{R}$ and B the relation defined by $B(x) = \{y : y > x + 2\}$. We have that*

$$\mathcal{T}_{B^\mathfrak{c}}^\mathfrak{c}(-1) = [-1, 0] \subset [-2, 0] = \mathcal{T}_B(-1)$$

and hence that the asymmetric relation B is not $\mathcal{T}$-regular. (Clearly, $\mathcal{T}_B = \mathcal{T}_{B^\mathfrak{a}}$ and $\mathcal{T}_{B^\mathfrak{c}}^\mathfrak{c} = \mathcal{T}_{B^\mathfrak{c a}}^\mathfrak{c} = \mathcal{T}_{B^\mathfrak{a c}}^\mathfrak{c}$ by the asymmetry of B).

6. On Relations Induced by Cones

The undominated maximals of the relation induced by an open convex cone in an RTVS are, exactly, the maximals of the relation induced by the topological closure of that cone. The undominated maximals of the relation induced by a closed convex cone in an RTVS are, exactly, the maximals of that relation.

Lemma 3. *Let (X, τ) be an RTVS and C a convex cone in X. Denote the relation induced by C on X by B.*

1. *If C is τ-open and $0 \notin C$, then B is a $\mathcal{T}$-regular strict partial order and $B^\mathfrak{c}$ is open-valued for τ.*
2. *If C is τ-closed and $\varnothing \neq C$, then B is a quasi-coherent preorder and B is closed-valued for τ.*

Proof. Recall that B is $\mathcal{T}$-regular and transitive (and hence quasi-transitive) by part 2 of Proposition 2. Recalling this fact, suppose for a moment that C is τ-open and $0 \notin C$; the last negated membership implies that B is irreflexive, while the assumption that C is τ-open implies that $B^\mathfrak{c}$ is open-valued for τ. (Note that $B^\mathfrak{c}(v) = v - K$ for all $v \in V$ and recall that the topology of any RTVS is invariant under translation and under multiplication by nonzero scalars). We have thus proved part 1 of Lemma 3 and we conclude the proof by showing the validity of part 2 of Lemma 3, as follows. Assume that C is τ-closed and $\varnothing \neq C$. The assumption that C is τ-closed implies that C is closed-valued for τ. (Note that $B^\mathfrak{c}(v) = v - K$ for all $v \in V$ and recall that the topology of any RTVS is invariant under translation and under multiplication by nonzero scalars). As C is a cone, the implication

$$(\lambda, c) \in \mathbb{R}_{++} \times C \Rightarrow \lambda c \in C \tag{6}$$

is true. Suppose $k \in C$. Every neighborhood of the zero vector of an RTVS is absorbing; from (6), we then infer that, for every τ-neighborhood of the zero vector of X, there exists a positive scalar λ that makes the vector λk an element of the intersection of that τ-neighborhood with the cone C. Thus, $0 \in \operatorname{cl} C$ and hence $0 \in C$ by the τ-closedness of C. The last membership implies the reflexivity of B. Thus, B is a preorder. By virtue of Lemma 1, to conclude the proof it suffices to show B is locally quasi-nonsatiated. Put $C^\bullet = C \setminus -C$ and note that $B^\mathfrak{a}$ is the relation induced on X by the cone $C^\bullet$. If $C^\bullet = \varnothing$, then $B^\mathfrak{a}(x) = \varnothing$ for all $x \in X$; if $C^\bullet \neq \varnothing$, then $B^\mathfrak{a}(x) \neq \varnothing$ for all $x \in X$. We then infer that B is quasi-nonsatiated, and hence we are finished if we prove that B is conditionally locally nonsatiated. To show this fact, suppose $x \in X$ and $B^\mathfrak{a}(x) \neq \varnothing$. The inequality $B^\mathfrak{a}(x) \neq \varnothing$ is equivalent to the inequality $C^\bullet \neq \varnothing$. Now, suppose $k \in C^\bullet$. As $C^\bullet$ is a cone, the implication

$$(\lambda, c) \in \mathbb{R}_{++} \times C^\bullet \Rightarrow \lambda c \in C^\bullet \tag{7}$$

is true. Every neighborhood of the zero vector of a real topological vector space is absorbing; from (7), we then infer that, for every τ-neighborhood of the zero vector of X, there exists a positive scalar λ that makes the vector λk an element of the intersection of that τ-neighborhood with the cone $C^{\bullet}$. Thus, $0 \in \mathrm{cl}\, C^{\bullet}$ and hence $x \in x + \mathrm{cl}\, C^{\bullet}$. As the topology of any RTVS is translation invariant, the previous membership is equivalent to $x \in \mathrm{cl}(x + C^{\bullet})$. We conclude that $x \in \mathrm{cl}\, B^{\mathfrak{a}}(x)$, and hence that B is conditionally locally nonsatiated. $\square$

Theorem 2. *Let (X, τ) be an RTVS and C a τ-open convex cone in X. Denote the relation induced by the cone C on X by B and denote the relation induced by the cone $\mathrm{cl}\, C$ on X by R. Then*

$$\mathcal{U}(B, Y) = \mathcal{U}(R, Y) = \mathcal{M}(R, Y) \subseteq \mathcal{M}(B, Y)$$

for every subset Y of X.

Proof. Let $Y \subseteq X$. If $C = \varnothing$, then $\mathrm{cl}\, C = \varnothing$ and the proof of Theorem 2 is obvious in that B and R are the empty relation on X. Thus, suppose $C \neq \varnothing$. In any RTVS, the topological closure of a convex cone is a closed convex cone. Consequently, $\mathrm{cl}\, C$ is a τ-closed convex cone in X. Clearly, $\mathrm{cl}\, C \neq \varnothing$. As C is nonempty and τ-open, the membership $0 \in C$ implies $C = X$ in that every neighborhood of the zero vector of an RTVS is absorbing and C is a cone in X; moreover, in this case, the proof of Theorem 2 is obvious in that $B = R = X \times X$. Thus, suppose $0 \notin C$. Part 1 of Lemma 3 ensures that B is a $\mathcal{T}$-regular strict partial order and $B^{\mathfrak{c}}$ is open-valued for τ. Part 2 of Lemma 3 ensures that R is a quasi-coherent preorder relation and that R is closed-valued for τ. From part 3 of Theorem 1, we then infer that

$$\mathcal{U}(R, Y) = \mathcal{M}(R, Y)$$

and hence that $\mathcal{U}(R, Y) = \mathcal{M}(\mathcal{T}_R, Y)$ by the definition of a preorder provided in Section 2. We have

$$y + C \subseteq x + C \Rightarrow \mathrm{cl}(y + C) \subseteq \mathrm{cl}(x + C) \Leftrightarrow y + \mathrm{cl}\, C \subseteq x + \mathrm{cl}\, C \tag{8}$$

and

$$y + \mathrm{cl}\, C \subseteq x + \mathrm{cl}\, C \Rightarrow \mathrm{int}(y + \mathrm{cl}\, C) \subseteq \mathrm{int}(x + \mathrm{cl}\, C) \Leftrightarrow y + \mathrm{int}(\mathrm{cl}\, C) \subseteq x + \mathrm{int}(\mathrm{cl}\, C) \tag{9}$$

for all $(y, x) \in X \times X$. (The *double* implications in (8) and (9) are true because the topology of any RTVS is translation invariant). In any RTVS, a nonempty open convex set coincides with the interior of the closure of that set; consequently, $\mathrm{int}(\mathrm{cl}\, C) = C$. By making use of the last equality, from (8) and (9) we infer that $\mathcal{T}_B = \mathcal{T}_R$ and hence that $\mathcal{M}(\mathcal{T}_B, Y) = \mathcal{M}(\mathcal{T}_R, Y)$ for every subset Y of X. As B is a strict partial order, part 2 of Corollary 1 and part 1 of Proposition 2 imply $\mathcal{U}(B, Y) = \mathcal{M}(\mathcal{T}_B, Y)$ for every subset Y of X. We are in a position to conclude that

$$\mathcal{U}(B, Y) = \mathcal{U}(R, Y)$$

for every subset Y of X. Clearly, the definition of an undominated maximal entails that

$$\mathcal{U}(B, Y) \subseteq \mathcal{M}(B, Y)$$

for every subset Y of X, and this last observation concludes the proof. $\square$

Theorem 3. *Let (X, τ) be an RTVS and C a τ-closed convex cone in X. Denote the relation induced by the cone C on X by B. Then B is a quasi-coherent transitive relation and*

$$\mathcal{U}(B, Y) = \mathcal{M}(B, Y)$$

for every subset Y of X.

Proof. If $C = 0$, then B is quasi-coherent and transitive in that B is the empty relation on X. If $C \neq 0$, then B is a quasi-coherent preorder by part 2 of Lemma 3. Thus, part 3 of Theorem 1 implies the desired result. $\square$

The assumption that C is τ-open cannot be simply dropped in Theorem 2: Examples 7 and 8 prove this claim.

Example 7. *Let $X = \mathbb{R}^2$ be endowed with the usual topology and understood as an RTVS. Consider the convex cone*

$$C = (\{0\} \times \mathbb{R}_+) \cup (\mathbb{R}_{++} \times \mathbb{R})$$

and its closure

$$\text{cl}\, C = \mathbb{R}_+ \times \mathbb{R}.$$

Let B denote the relation induced by C and R the relation induced by $\text{cl}\, C$. Putting

$$Y = \{(0,0), (0,1)\},$$

it can be checked that

$$\mathcal{U}(B, Y) = \{(0,1)\} \subset \{(0,0), (0,1)\} = \mathcal{M}(R, Y).$$

Example 8. *Let $X = \mathbb{R}^3$ be endowed with the usual topology and understood as an RTVS. Consider the convex cone*

$$C = (\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++}) \cup \{(t, t, 0) \in \mathbb{R}^3 : t \geq 0\}$$

and its closure

$$\text{cl}\, C = \mathbb{R}_+^3.$$

Let B denote the relation induced by C and R the relation induced by $\text{cl}\, C$. Putting

$$Y = \{(0,0,0), (1,0,0)\},$$

it can be checked that

$$\mathcal{M}(R, Y) = \{(1,0,0)\} \subset \{(0,0,0), (1,0,0)\} = \mathcal{U}(B, Y).$$

6.1. A Brief Remark on Existence

While indicating simple conditions for the existence of undominated maximals, Corollary 7 shows some implications of the results shown so far in Section 6.

Corollary 7. *Let (X, τ) be an RTVS and C a convex cone in X and Y be a nonempty compact subset of X. Denote the relation induced by the cone C on X by B and the relation induced by the cone $\text{cl}\, C$ on X by R.*

1. *If C is τ-closed, then $\emptyset \neq \mathcal{U}(B, Y) = \mathcal{U}(R, Y) = \mathcal{M}(R, Y) = \mathcal{M}(B, Y)$.*
2. *If C is τ-open, then $\emptyset \neq \mathcal{U}(B, Y) = \mathcal{U}(R, Y) = \mathcal{M}(R, Y) \subseteq \mathcal{M}(B, Y)$.*

Proof. In any RTVS, the topological closure of a (possibly empty) convex cone is a closed convex cone. Therefore, $\text{cl}\, C$ is a τ-closed convex cone in X. The relation induced by any convex cone in an RVS is a transitive relation. Therefore, R is transitive. As the topology of any RTVS is translation invariant, the τ-closedness of $\text{cl}\, C$ implies the closed-valuedness of R for τ. Thus, one of the versions of Wallace theorem on the existence of maximals (if needed, for a proof of this result essentially due to [23] see Corollary 2 in [20]) ensures that $\mathcal{M}(R, Y) \neq \emptyset$. The τ-closedness of C implies the validity of the equality $\text{cl}\, C = C$, and we thus observe that $\mathcal{U}(B, Y) = \mathcal{U}(R, Y)$ and $\mathcal{M}(R, Y) = \mathcal{M}(B, Y)$ when C is closed. As $\mathcal{M}(R, Y) \neq \emptyset$, the previous observation and Theorems 2 and 3 imply the desired conclusion. $\square$

6.2. About Pareto Dominance

For every positive integer n, endow the n-dimensional Euclidean space with the usual topology. Let $\mathcal{D}_n^S$ and $\mathcal{D}_n^W$ be the relations on $\mathbb{R}^n$, respectively, defined by

$$\mathcal{D}_n^W(x) = x + \mathbb{R}_+^n \text{ and } \mathcal{D}_n^S(x) = x + \mathbb{R}_{++}^n$$

at all $x \in \mathbb{R}^n$. Understanding $\mathbb{R}^n$ as the set of all possible "utility levels" of a group of n agents, we call $\mathcal{D}_n^W$ the **weak Pareto dominance** relation and $\mathcal{D}_n^S$ the **strong Pareto dominance** relation.

Corollary 8. *Let n be a positive integer. Then*

$$\mathcal{U}(\mathcal{D}_n^S, Y) = \mathcal{U}(\mathcal{D}_n^W, Y) = \mathcal{M}(\mathcal{D}_n^W, Y) \subseteq \mathcal{M}(\mathcal{D}_n^S, Y)$$

for every subset Y of $\mathbb{R}^n$ and $\mathcal{U}(\mathcal{D}_n^S, Y) \neq \varnothing$ for every nonempty compact subset Y of $\mathbb{R}^n$.

Proof. A consequence of Theorem 2 and Corollary 7. $\square$

Computing the set of all undominated maximals might well be quite a complicated exercise, even when one considers elementary relations and elementary constraint sets. Corollary 8 shows that the set of undominated maximals of the strong and weak Pareto dominance relation coincides with the set of undominated maximals of the weak Pareto dominance relation. This fact greatly simplifies the computation of the entire sets of undominated maximals of the strong and weak Pareto dominance relations, as in the following Example.

Example 9. *Put $n = 2$ and let Y be the subset of $\mathbb{R}^2$ defined by*

$$Y = Y_1 \cup Y_2$$

with

$$Y_1 = [0, 2] \times [0, 1] \subseteq \mathbb{R}^2 \text{ and } Y_2 = [0, 1] \times [0, 2] \subseteq \mathbb{R}^2.$$

It is readily noted that
$$\mathcal{M}(\mathcal{D}_n^W, Y) = \{(1, 2), (2, 1)\}$$

and that

$$\mathcal{M}(\mathcal{D}_n^S, Y) = ([0, 1] \times \{2\}) \cup (\{1\} \times [1, 2]) \cup ([1, 2] \times \{1\}) \cup (\{2\} \times [0, 1]).$$

From Corollary 8, we immediately infer that

$$\mathcal{U}(\mathcal{D}_n^S, Y) = \mathcal{U}(\mathcal{D}_n^W, Y) = \{(1, 2), (2, 1)\}.$$

7. On Representable Semiorders

The undominated maximals of a $\langle u, v \rangle$-representable relation on a connected ground set satisfying conditions C1–4 of Corollary 5 are, exactly, the (constrained) maximizers of u.

Lemma 4. *Let X be a set, B a relation on X and $Y \subseteq X$. Assume that B is irreflexive and $\langle u, v \rangle$-representable (for clarity, $u[Y] = \{u(y) : y \in Y\}$ and $v[Y]$ is defined analogously).*

1. $\mathcal{M}(B, Y) = \{y \in Y : v(y) \geq \sup u[Y]\}.$
2. $\mathcal{M}(\mathcal{T}_B, Y) = \{y \in Y : v(y) = \sup v[Y]\}$ *if $y \in \mathcal{T}_B(x) \Leftrightarrow v(y) \geq v(x)$ for all $(y, x) \in X \times X$.*

Proof. Remark A1 points out that B is an interval order and hence an asymmetric relation. By Remark 5, $\mathcal{M}(B, Y) = \{y \in Y : x \in Y \Rightarrow x \notin B(y)\}$ and so $\mathcal{M}(B, Y) = \{y \in Y : x \in Y \Rightarrow u(x) \leq v(y)\}$ in that B is $\langle u, v \rangle$-representable; this suffices to infer the validity of

part 1 of Lemma 4. The right trace of any interval order is total, and hence $\mathcal{M}(\mathcal{T}_B, Y) = \mathfrak{M}(\mathcal{T}_B, Y) = \{y \in Y : y \in \mathcal{T}_B(x) \text{ for all } x \in Y\}$. Thus, $\mathcal{M}(\mathcal{T}_B, Y) = \{y \in Y : v(y) \geq v(x) \text{ for all } x \in Y\}$ when the double implication in part 2 of Lemma 4 is true; this suffices to infer the validity of part 2 of Lemma 4. $\square$

Theorem 4. *Let (X, τ) be a connected topological space, B a relation on X and $Y \subseteq X$. Assume that B is $\langle u, v \rangle$-representable. Then*

$$\mathcal{M}(B, Y) = \{y \in Y : v(y) \geq \sup u[Y]\}$$

and

$$\mathcal{U}(B, Y) = \{y \in Y : u(y) = \sup u[Y]\}$$

if conditions C1–4 of Corollary 5 are satisfied.

Proof. Assume that conditions C1–4 of Corollary 5 are satisfied. The mentioned Corollary 5 ensures that B is a nonsatiated $\mathcal{T}$-regular semiorder. Thus, part 1 of Lemma A1 and Lemma 4 imply $\mathcal{M}(B, Y) = \{y \in Y : v(y) \geq \sup u[Y]\}$ and $\mathcal{M}(\mathcal{T}_B, Y) = \{y \in Y : u(y) = \sup u[Y]\}$, while part 2 of Corollary 1 implies $\mathcal{U}(B, Y) = \mathcal{M}(\mathcal{T}_B, Y)$. $\square$

By making use of Example 10 presented in Section 7.2, one can check that in Theorem 4, the assumption that condition C2 is true cannot be simply dropped.

7.1. A Brief Remark on Existence

While indicating simple conditions for the existence of undominated maximals, Corollary 9 shows an implication of Theorem 4.

Corollary 9. *Let (X, τ) be a connected topological space, B a $\langle u, v \rangle$-representable relation on X and Y a nonempty τ-compact subset of X. Then*

$$\varnothing \neq \mathcal{U}(B, Y) = \{y \in Y : u(y) = m\} \subseteq \mathcal{M}(B, Y) = \{y \in Y : v(y) \geq m\}$$

with

$$m = \max u[Y]$$

if conditions C1–4 of Corollary 5 are satisfied.

Proof. Clearly, $\mathcal{U}(B, Y) \subseteq \mathcal{M}(B, Y)$ by the very definition of a undominated maximal. By Weierstrass theorem, m is a well-defined real number satisfying the equality $m = \sup u[Y]$ when u is continuous. Said this, Theorem 4 readily implies the desired conclusion. $\square$

7.2. About Scott-Suppes Representability

We conclude restricting attention to relations that admit a Scott-Suppes representability. The simplicity of the conditions involved, allow the production of many simple, yet interesting examples of semiorders where undominated maximality effectively refines maximality. One of these examples is Example 2 in [4], to which the reader is referred.

Corollary 10. *Let (X, τ) be a connected topological space, B a relation on X and $Y \subseteq X$. Assume that B is Scott–Suppes $\langle u, v, K \rangle$-representable with u continuous and upper unbounded. In addition, suppose Y is nonempty and compact. Then*

$$\varnothing \neq \mathcal{U}(B, Y) = \{y \in Y : u(y) = m\} \subseteq \{y \in Y : u(y) \geq m - K\} = \mathcal{M}(B, Y)$$

with

$$m = \max u[Y].$$

Proof. As B is Scott–Suppes $\langle u, v, K \rangle$-representable, $v(x) = u(x) + K$ for all $x \in X$ for some real number $K \geq 0$. Thus, Corollary 9 immediately implies the desired conclusion. □

The assumption that u is upper unbounded cannot be simply dropped in Corollary 10: Example 10 proves this claim.

Example 10. *Let $\mathbb{R}$ be endowed with the usual topology and let B be the relation on $\mathbb{R}$ that is Scott–Suppes $\langle u, v, K \rangle$-represented by*

$$u(x) = \begin{cases} 9 & \text{if } x \in [9, +\infty) \\ x & \text{if } x \in [0, 9) \\ 0 & \text{if } x \in (-\infty, 0) \end{cases}$$

with threshold $K = 8$. Put $Y = [5, 6]$ and $m = 6$ noting that $m = \max u[Y]$. It can be checked that

$$\mathcal{M}(B, Y) = \{y \in Y : u(y) \geq m - K\} = [5, 6] = \mathcal{U}(B, Y)$$

and hence that

$$\mathcal{U}(B, Y) \neq \{y \in Y : u(y) = m\} = \{6\}.$$

8. Conclusions and Open Questions

This paper proposes a general definition of undominated maximality. Restricting attention to classes of relations that are at least quasi-transitive, several characterizations of the definition of an undominated maximal are provided. Many of these characterizations express undominated maximals as maximals of some trace associated with the objective relation. For the case of quasi-coherent relations, an equivalence is shown between maximality and undominated maximality. Moreover, focusing on some specific classes of "regular" relations, a complete description is provided as to the structure of the set of undominated maximals. In the author's view, the merit of this last part of the examination is that of providing simple tools that can be used to create examples of optimization problems whose entire set of undominated maximals can be easily recovered and that show how and when undominated maximality refines maximality.

There are many open questions left to be answered. For instance, it is unclear whether it is possible to characterize an undominated maximal as a maximal of some relation associated to the primitive objective relation when the latter is not quasi-transitive. The existence of *unconstrained* undominated maximals of relations with a *finite* ground set that are not necessarily quasi-transitive was examined in [2]; however, the problem of the existence of undominated maximals when the objective relation is not quasi-transitive and does not possess a finite ground set is uninvestigated. Moreover, this paper points out (at the end of Section 4) that the existence of undominated maximals is not, in a sense, a real issue in the case of a *quasi-transitive* relation provided one is willing to impose conditions directly on some trace (e.g., the global trace) associated with the objective relation. However, traces are not easy to handle, in general, and it is often unclear what these conditions directly imposed on traces (in particular, on the global trace) actually imply in terms of conditions on the objective relation. These implications need further study.

Appendix A

Remark A1. *Let X be a set and B a $\langle u, v \rangle$-representable relation on X. Then B is Ferrers. Furthermore, the relation B is irreflexive if and only if $u(x) \leq v(x)$ for all $x \in X$ (equivalently, the relation B is an interval order if and only if $u(x) \leq v(x)$ for all $x \in X$). Consequently, the Scott–Suppes $\langle u, v, K \rangle$-representability of B implies the irreflexivity of B.*

There exists a large amount of literature on the $\langle u, v \rangle$-representability of interval orders (in addition to some of the already cited literature see, e.g., Chapter 6 in [9] or the more recent [24–28] as well as [29] and the many references contained in the mentioned articles and books). A result of this literature, Theorem 4.1 in [30], provides some sufficient conditions for the function v of a pair (u, v) of functions that $\langle u, v \rangle$-represent an interval order to be a "utility representation" of the right trace associated to the interval order.

Lemma A1 ([30], Theorem 4.1). *Let (X, τ) be a connected set and B a nonsatiated and irreflexive relation on X. Assume that B is $\langle u, v \rangle$-representable and that u is continuous.*

1. $y \in \mathcal{T}_B(x) \Leftrightarrow v(y) \geq v(x)$ for all $(y, x) \in X \times X$.
2. $y \in \mathcal{T}_{B^c}^c(x) \Leftarrow u(y) \geq u(x)$ for all $(y, x) \in X \times X$.

Proof. Remark A1 points out that B is Ferrers. Thus, the irreflexive relation B is an interval order, and a moment's reflection shows that part 1 of Lemma A1 is nothing but Theorem 4.1 in [30]. Furthermore, observe that the membership $y \in \mathcal{T}_{B^c}^c(x)$ is equivalent to the inclusion $B^c(x) \subseteq B^c(y)$, and hence—by virtue of the $\langle u, v \rangle$-representability of B—to the inclusion $\{t \in X : u(x) > v(t)\} \subseteq \{t \in X : u(y) > v(t)\}$; from this simple observation, one immediately infers the validity of part 2 of Lemma A1. □

Lemma A2 provides a sufficient (but not necessary) condition for the nonsatiation of a $\langle u, v \rangle$-representable relation. Corollary A1, of independent interest, clarifies that such a condition is necessary in the case of a Scott–Suppes $\langle u, v, K \rangle$-representable relation with positive threshold K and nonempty ground set.

Lemma A2. *Let X be a set and B a relation on X. Assume that B is irreflexive and $\langle u, v \rangle$-representable. If u is upper unbounded, then B is nonsatiated.*

Proof. We previously pointed out in Remark A1 that the irreflexivity of B is equivalent to the validity of the inequality $u(x) \leq v(x)$ for all $x \in X$ in that B is $\langle u, v \rangle$-representable. Therefore, $\sup u[X] \leq \sup v[X]$. By virtue of the last inequality, the upper unboundedness of u implies the upper unboundedness of v and hence the existence, for every $x \in X$, of at least one $y_x \in X$ such that $v(y_x) > u(x)$. As B is $\langle u, v \rangle$-representable, we have just shown that the upper unboundedness of u implies the nonsatiation of B. □

Corollary A1. *Let X be a nonempty set and B a relation on X. Assume that B is Scott–Suppes $\langle u, v, K \rangle$-representable with $K > 0$. Then B is nonsatiated if and only if u is upper unbounded.*

Proof. Remark A1 previously pointed out that the Scott-iSuppes $\langle u, v, K \rangle$-representability of B implies the irreflexivity of B. The "if part" of Corollary A1 thus follows from Lemma A2. Henceforth, assume that B is nonsatiated. If u were upper bounded, then the nonemptiness of X would imply $\sup u[X] = l$ for some $l \in \mathbb{R}$, and hence the existence of $x \in X$ satisfying the inequality $u(x) + K > l$ in that K is a positive real number. However, the nonsatiation of B and the Scott–Suppes $\langle u, v, K \rangle$-representability of B would in turn imply the existence of $y \in X$ such that $u(y) > u(x) + K$ and this would lead to a contradiction with $\sup u[X] = l$. Thus, u is upper unbounded. □

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