mathematics

MDPI

Article

# Existence of Fuzzy Fixed Points and Common Fuzzy Fixed Points for $\mathcal{FG}$-Contractions with Applications

**Dur-e-Shehwar Sagheer [1], Zainab Rahman [1], Samina Batul [1], Ahmad Aloqaily [2,3] and Nabil Mlaiki [2,*]**

[1] Department of Mathematics, Capital University of Science and Technology, Islamabad 44000, Pakistan; d.e.shehwar@cust.edu.pk (D.-e.-S.S.); zainabsaif3040@gmail.com (Z.R.); samina.batul@cust.edu.pk (S.B.)
[2] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; maloqaily@psu.edu.sa
[3] School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney 2150, Australia
[*] Correspondence: nmlaiki2012@gmail.com or nmlaiki@psu.edu.sa

**Abstract:** This article contains results of the existence of fuzzy fixed points of fuzzy mappings that satisfy certain contraction conditions using the platform of partial $b$-metric spaces. Some non-trivial examples are provided to authenticate the main results. The constructed results in this work will likely stimulate new research directions in fuzzy fixed-point theory and related hybrid models. Eventually, some fixed-point results on multivalued mappings are established. These theorems provide an excellent application of main theorems on fuzzy mappings. The results of this article are extensions of many already existing results in the literature.

**Keywords:** partial $b$-metric space (pbMS); closed ball; multi-valued contraction; fuzzy sets

**MSC:** 54H25; 47H10

check for updates

**Citation:** Sagheer, D.-e.-S.; Rahman, Z.; Batul, S.; Aloqaily, A.; Mlaiki, N. Existence of Fuzzy Fixed Points and Common Fuzzy Fixed Points for $\mathcal{FG}$-Contractions with Applications. *Mathematics* **2023**, *11*, 3981. https://doi.org/10.3390/math11183981

Academic Editors: Salvador Romaguera and Manuel Sanchis

Received: 11 July 2023
Revised: 14 September 2023
Accepted: 15 September 2023
Published: 19 September 2023



## 1. Introduction and Preliminaries

Although it is not as old as some classical subjects, fixed-point theory has become an important branch of mathematics. One hundred years ago, in 1922, Banach [1] gave their famous Banach Contraction Principle (BCP). Innumerable extensions of this result have been given over the years. One such extension was established by Nadler [2] for a multi-valued contraction mapping, in which the Hausdorff function $H_d$, endowed with the metric function $d$, plays an important part. Fixed-point theory provides a technique to assure the existence of solutions of many differential and integral equations. Many recent research works can be seen from this perspective. Zhane et al. [3] obtained the non-negative stable approximate solutions to ill-posed linear operator equations in a Hilbert space setting which are based on fixed-point iterations in combination with preconditioning ideas. In [4], Shcheglov et al. used the method of successive approximations to develop a novel iterative algorithm to estimate sorption isotherms.

Matthews [5] defined the partial metric space (pMS) as a generalization of the metric space. Such spaces are important structures in computer science and logic programming semantics. Matthews proved a fixed-point theorem for contractions in partial metric spaces which are analogous to the BCP. A lot of literature has investigated partial metric spaces. See, for example, [6–9]. The $b$-metric space (bMS) was first propounded in the works of Bourbaki [10] and Bakhtin [11]. Czerwik [12] gave a formal definition for $b$-metric spaces, giving a weaker triangular inequality. Furthermore, we refer the reader to see [13,14]. He also generalized the Banach contraction principle.

Fuzzy set theory was initiated by Zadeh [15] in 1965. Weiss [16] and Butnariu [17] introduced fuzzy mappings as a subclass of multi-valued mappings and demonstrated certain fixed-point theorems. The result proved by Heilpern [18] in 1981 on fuzzy mappings is also a noticeable milestone. This theorem is a generalization of the theorem for multi-valued mappings. With many applications in the modern world, it is easily warranted that fuzzy logic and fuzzy set theory are subjects of immense importance and applications. In 2022, Batul et al. [19] introduced the notion of $(\alpha*, F)$ fuzzy contractive mappings and established few results for the existence of $\alpha$ fuzzy fixed points of an $(\alpha*, F)$ contraction and a pair of $(\alpha*, F)$ contractions. One can find some very good results on fuzzy mappings in [19,20]. Shukla [21] combined the concepts of pMSs and bMSs, giving the notion of partial $b$-metric spaces as a generalization of both. He then went on to establish an analogous result to the Banach and Kannan-type fixed-point theorems. This new platform opened doors for many researchers to establish the existence of fixed points for different mappings.

Shoaib et al. [22] provided results for the existence of fixed points of fuzzy mappings in a dislocated bMS, confining the space to a closed ball. In this paper, we extend their results and consider fuzzy mappings defined on a partial $b$-metric space (pbMS) and we provide two fixed-point theorems. The first is a result proving the existence of a fixed point for a single fuzzy mapping and the second is a result in which we provide a common fixed point for two fuzzy mappings defined on the same space. Theorem 2.1 of [22] becomes the special case of our result. As an application of our results, we established two results to prove the existence of fixed points of multi-valued mappings. These theorems are also the special cases of results established in this research.

The following are some definitions and results which are useful for the proof of our main theorems.

**Definition 1** ([21])**.** *Consider a non-empty set $\Xi$. A mapping $b : \Xi \times \Xi \to \mathbb{R}$ is called a b-metric on $\Xi$ if there is a constant $\beta \geq 1$ such that for any $\phi, \chi, \xi \in \Xi$, the following axioms are satisfied:*

*B1:* $b(\phi, \chi) \geq 0$;
*B2:* $b(\chi, \phi) = b(\phi, \chi)$;
*B3:* $b(\phi, \chi) = 0 \Leftrightarrow \phi = \chi$;
*B4:* $b(\phi, \chi) \leq \beta\big[b(\phi, \xi) + b(\xi, \chi)\big]$.

**Definition 2** ([23])**.** *A dislocated (metric-like) function on a non-empty set $\Xi$ is a function $d_l : \Xi \times \Xi \to [0, \infty)$ such that for all $\phi, \chi, \xi \in \Xi$:*

*DL1: if* $d_l(\phi, \chi) = 0$ *then* $\phi = \chi$;
*DL2:* $d_l(\phi, \chi) = d_l(\chi, \phi)$;
*DL3:* $d_l(\phi, \chi) \leq d_l(\phi, \xi) + d_l(\xi, \chi)$,

*and the pair $(\Xi, d_l)$ is called a dislocated (metric-like) space.*

**Definition 3** ([7])**.** *A mapping $p : \Xi \times \Xi \to \mathbb{R}^+$, where $\Xi$ is a non-empty set, is said to be a partial metric on $\Xi$ if for any $\phi, \chi, \xi \in \Xi$:*

*P1:* $\phi = \chi \iff p(\phi, \phi) = p(\phi, \chi) = p(\chi, \chi)$;
*P2:* $p(\phi, \phi) \leq p(\phi, \chi)$;
*P3:* $p(\phi, \chi) = p(\chi, \phi)$;
*P4:* $p(\phi, \chi) \leq p(\phi, \xi) + p(\xi, \chi) - p(\xi, \xi)$.

*The pair $(\Xi, p)$ is then called a partial metric space.*

**Definition 4** ([21])**.** *Consider a non-empty set $\Xi$ and a mapping $p_b : \Xi \times \Xi \to \mathbb{R}^+$. We call $p_b$ a partial b-metric on $\Xi$ if for a constant $\mathfrak{b} \geq 1$ and for all $\phi, \chi, \xi \in \Xi$, the following axioms are satisfied:*

*($\mathcal{P}1$):* $\phi = \chi$ *iff* $p(\phi, \phi) = p(\phi, \chi) = p(\chi, \chi)$;
*($\mathcal{P}2$):* $p(\phi, \phi) \leq p(\phi, \chi)$;
*($\mathcal{P}3$):* $p(\phi, \chi) = p(\chi, \phi)$;
*($\mathcal{P}4$):* $p(\phi, \chi) \leq \mathfrak{b}\big[p(\phi, \xi) + p(\xi, \chi)\big] - p(\xi, \xi)$.

**Example 1.** *Define a function $p_b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $p_b(\phi, \chi) = \big[\max\{\phi, \chi\}\big]^b + |\phi - \chi|^b$. It can be easily verified that $(\Xi, p_b)$ is a pbMS with $\mathfrak{b} = 2^{b-1}$.*

**Definition 5** ([8])**.** *Consider a pbMS $(\Xi, p_b)$. Let $\varkappa^*$ be an element in $\Xi$ and let $\rho > 0$ be a real number. Open and closed balls with centre $\varkappa^*$ and radius $\rho$ are defined below:*

*(a)* *Open Ball: $B_{p_b}(\varkappa^*;\rho) = \{\varkappa \in \Xi : p_b(\varkappa, \varkappa^*) - p_b(\varkappa^*, \varkappa^*) < \rho\}$;*

*(b)* *Closed Ball: $\overline{B_{p_b}(\varkappa^*;\rho)} = \{\varkappa \in \Xi : p_b(\varkappa, \varkappa^*) - p_b(\varkappa^*, \varkappa^*) \le \rho\}$.*

**Definition 6.** *Consider a pbMS $(\Xi, p_b)$ and a non-empty subset $\Phi$ of $\Xi$. Suppose each $\varkappa \in \Xi$ has a minimum one best approximation in $\Phi$. Such a set $\Phi$ is said to be a proximinal set and $\mathcal{P}(\Xi)$ is the family of all proximinal sets of $\Xi$.*

**Definition 7.** *Consider a pbMS $(\Xi, p_b)$ and $\mathcal{P}(\Xi)$. The partial Hausdorff b-metric on $\mathcal{P}(\Xi)$ is defined by*

$$H_{p_b}(\Phi, \mathcal{X}) = \max\{\sup_{\phi\in\Phi} p_b(\phi, \mathcal{X}), \sup_{\chi\in\mathcal{X}} p_b(\chi, \Phi)\}.$$

**Definition 8** ([22])**.** *Let $\Xi$ be a nonempty set. A function whose domain is $\Xi$ and has values in $[0,1]$ is called a fuzzy set in $\Xi$. We denote the family of all fuzzy sets in $\Xi$ by $\mathcal{F}(\Xi)$.*

*For a fuzzy set $\mathfrak{F}$ in $\Xi$, the function $\mathfrak{F}(\varkappa)$ gives the degree (or grade) of membership of $\varkappa$ in $\mathfrak{F}$. For a number $\alpha \in (0,1]$, $[\mathfrak{F}]_\alpha$ denotes the $\alpha$-level set of a fuzzy set $\mathfrak{F}$, which is defined as $[\mathfrak{F}]_\alpha = \{\varkappa \in \Xi : \mathfrak{F}(\varkappa) \ge \alpha\}$, and $[\mathfrak{F}]_0 = \overline{\{\varkappa \in \Xi : \mathfrak{F}(\varkappa) > 0\}}$.*

*Let $\Xi$ be a non-empty set and $Y \subseteq \Xi$. A mapping from $\Xi$ to $\mathcal{F}(Y)$ is called a fuzzy mapping. A fuzzy set $\mathfrak{T}$ is a subset of $\Phi \times \Psi$ having a membership function $\mathfrak{T}(\phi)(\psi)$ which represents the degree of membership of $\psi$ in $\mathfrak{T}(\phi)$. The $\alpha$-level set of $\mathfrak{T}(\phi)$ is denoted by $[\mathfrak{T}\phi]_\alpha$.*

**Example 2.** *Let $\Xi = \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5\}$ be a set of students in a class. Let $\mathfrak{F}(\varkappa_i) \in [0,1]$ be the intelligence level of each student. The fuzzy set $\mathfrak{F}$ will look something like*

$$\mathfrak{F} = \{(\varkappa_1, 0.75), (\varkappa_2, 0.32), (\varkappa_3, 0.51), (\varkappa_4, 1), (\varkappa_5, 0.84)\}.$$

*We say the degree of membership of $\varkappa_1$ in $\mathfrak{F}$ is 0.75.*

**Definition 9** ([22])**.** *Consider a fuzzy mapping $\mathfrak{T} : \Xi \to \mathcal{F}(\Xi)$ and an element $\varkappa^* \in \Xi$. If there exist $\alpha \in (0,1]$ such that $\varkappa^* \in [\mathfrak{T}\varkappa^*]_\alpha$, then $\varkappa^*$ is a fuzzy fixed point of $\mathfrak{T}$.*

The following results are useful in obtaining our main results. The proposition below is modified from [24].

**Proposition 1.** *If $(\Xi, p_b)$ is a pbMS, then the following conditions are equivalent:*

*(i)* *For all $\varkappa \in \Xi$, $p_b(\varkappa, \varkappa) = 0$.*

*(ii)* *$p_b$ is a b-metric.*

*(iii)* *For all $\varkappa \in \Xi$ and all $\rho \in \mathbb{R}^+$, we have $B_{p_b}(\varkappa;\rho) \neq \emptyset$.*

**Proof.** It is easy to see that

$$(i) \Rightarrow (ii) \text{ and } (iii),$$
$$(ii) \Rightarrow (i) \text{ and consequently } (iii).$$

We will show that $(iii) \Rightarrow (i)$ and consequently $(ii)$.

For any $\rho > 0, B_{p_b}(\varkappa;\rho) \neq \emptyset$. This means there exist $\phi \in \Xi$ such that $p_b(\varkappa, \phi) < \rho$. From $(\mathcal{P}2)$, we have

$$p_b(\varkappa, \varkappa) \le p_b(\varkappa, \phi) < \rho,$$

i.e., $p_b(\varkappa, \varkappa) < \rho$ for all $\rho > 0$ and so $p_b(\varkappa, \varkappa) = 0$. □

The lemmas for partial $b$-metric spaces are taken from [25].

**Lemma 1.** *For a pbMS $(\Xi, p_b)$, if $\Phi$ is a non-empty subset of $\Xi$, then for $\varkappa \in \Xi$,*

$$\varkappa \in \overline{\Phi} \Leftrightarrow p_b(\varkappa, \Phi) = p_b(\varkappa, \varkappa).$$

**Corollary 1.** *For a pbMS $(\Xi, p_b)$, if $\Phi$ is a non-empty subset of $\Xi$ and for some $\varkappa \in \Xi$ we have $p_b(\varkappa, \Phi) = 0$, then $\varkappa \in \overline{\Phi}$.*

**Lemma 2.** *For a pbMS $(\Xi, p_b)$ and subsets $\Phi, \mathcal{X}, \Psi \in CB(\Xi)$, we have for any $\phi \in \Phi$ and $\chi \in \mathcal{X}$,*

$$\begin{aligned} p_b(\chi, \Psi) &\le \mathfrak{b}\big[p_b(\chi, \phi) + p_b(\phi, \Psi)\big] - p_b(\phi, \phi) \\ &\le \mathfrak{b}\big[p_b(\chi, \phi) + p_b(\phi, \Psi)\big]. \end{aligned}$$

**Lemma 3.** *Consider a pbMS $(\Xi, p_b)$. Let $\Phi, \mathcal{X} \in CB(\Xi)$ and $h > 1$ be a constant. For any $\phi \in \Phi$ there exist $\chi_\phi \in \mathcal{X}$ such that*

$$p_b(\phi, \chi_\phi) \le h H_{p_b}(\Phi, \mathcal{X}).$$

**Corollary 2.** *Consider a pbMS $(\Xi, p_b)$ and let $\Phi, \mathcal{X} \in CB(\Xi)$ and $h > 1$ be a constant. From the definition of $p_b(\phi, \mathcal{X})$, we must have for all $\phi \in \Phi$*

$$p_b(\phi, \mathcal{X}) \le h H_{p_b}(\Phi, \mathcal{X}).$$

## 2. Main Results

In this section, we will discuss our main results. For our first result, we will consider a single fuzzy mapping $\mathfrak{T}$ defined on a pbMS.

**Definition 10.** *Let $(\Xi, p_b)$ be a pbMS and $\mathfrak{T} : \Xi \to \mathcal{F}(\Xi)$ be a fuzzy mapping. Then $\mathfrak{T}$ is said to be a multi-valued fuzzy generalized contraction ($\mathcal{FG}$-contraction) if*

$$\begin{aligned} H_{p_b}(\mathfrak{T}\phi, \mathfrak{T}\chi) \le{}& \zeta_1 p_b(\phi, [\mathfrak{T}\phi)]_{\alpha(\phi)}) + \zeta_2 p_b(\chi, [\mathfrak{T}\chi]_{\alpha(\chi)}) + \zeta_3 p_b(\phi, [\mathfrak{T}\chi]_{\alpha(\chi)}) \\ &+ \zeta_4 p_b(\chi, [\mathfrak{T}\phi)]_{\alpha(\phi)}) + \zeta_5 p_b(\phi, \chi) \\ &+ \zeta_6 \frac{p_b(\phi, [\mathfrak{T}\phi]_{\alpha(\phi)})\big(1 + p_b(\phi, [\mathfrak{T}\phi]_{\alpha(\phi)})\big)}{1 + p_b(\phi, \chi)} \end{aligned}$$

*for all $\phi, \chi \in \Xi$ and $\zeta_i \ge 0$, $i = 1, 2, \ldots, 6$ with*

$$\zeta_1 + \zeta_2 + 2\mathfrak{b}\zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 < 1.$$

**Theorem 1.** *Consider a complete pbMS $(\Xi, p_b)$ with $\mathfrak{b} \ge 1$ and a fuzzy mapping $\mathfrak{T} : \Xi \to \mathcal{F}(\Xi)$. Let $h > 1$ be a constant. Further, let $\varkappa_0$ be an arbitrary point of $\Xi$. Suppose there exists an $\alpha(\phi) \in (0, 1]$ for all $\phi \in \Xi$ such that $\mathfrak{T}$ can be classified as a multi-valued $\mathcal{FG}$-contraction and*

$$p_b\big(\varkappa_0, [\mathfrak{T}\varkappa_0]_{\alpha(\varkappa_0)}\big) \le \eta(1 - \mathfrak{b}h\eta)\rho \tag{1}$$

*for all $\phi, \chi \in \overline{B_{p_b}(\varkappa_0; \rho)}$, $\rho > 0$ and $\mathfrak{b}h\eta < 1$ where*

$$\eta = \frac{\zeta_1 + \mathfrak{b}\zeta_3 + \zeta_5 + \zeta_6}{1 - h(\zeta_2 + \mathfrak{b}\zeta_3)}$$

*with $h(\zeta_2 + \mathfrak{b}\zeta_3) \ne 1$ and $\mathfrak{b}h(\zeta_2 + \zeta_3) \ne 1$. Further, $\zeta_i \ge 0$ $(i = 1, 2, \ldots, 6)$,*

$$\sum_{i=1}^{6} \zeta_i < 1$$

*and*

$$h\big(\mathfrak{b}\zeta_1 + \zeta_2 + \mathfrak{b}(\mathfrak{b}+1)\zeta_3 + \mathfrak{b}(\zeta_5 + \zeta_6)\big) < 1,$$

*then there is* $\varkappa^* \in \overline{B_{p_b}(\varkappa_0;\rho)}$ *such that* $\varkappa^* \in [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}$.

**Proof.** Let $\varkappa_0$ be an arbitrary point of $\Xi$ such that $\varkappa_1 \in [\mathfrak{T}\varkappa_0]_{\alpha(\varkappa_0)}$. Form a sequence $\{\varkappa_n\}$ in $\Xi$ such that

$$\varkappa_n \in [\mathfrak{T}\varkappa_{n-1}]_{\alpha(\varkappa_{n-1})}.$$

We must first show that $\{\varkappa_n\} \in \overline{B_{p_b}(\varkappa_0;\rho)}$. Using (1), we have

$$\begin{aligned} &p_b(\varkappa_0,\varkappa_1) = p_b\big(\varkappa_0,[\mathfrak{T}\varkappa_0]_{\alpha(\varkappa_0)}\big) \le \eta(1-\mathfrak{b}h\eta)\rho < \rho \\ &\Rightarrow \varkappa_1 \in \overline{B_{p_b}(\varkappa_0;\rho)}. \end{aligned}$$

Now, let $\varkappa_2,\varkappa_3,\ldots,\varkappa_j \in \overline{B_{p_b}(\varkappa_0;\rho)}$, $j \in \mathbb{N}$. Using Lemmas 2 and 3 we have

$$\begin{aligned} p_b(\varkappa_j,\varkappa_{j+1}) &\le hH_{p_b}\big([\mathfrak{T}\varkappa_{j-1}]_{\alpha(\varkappa_{j-1})},[\mathfrak{T}\varkappa_j]_{\alpha(\varkappa_j)}\big) \\ &\le h\bigg[\zeta_1 p_b\big(\varkappa_{j-1},[\mathfrak{T}\varkappa_{j-1}]_{\alpha(\varkappa_{j-1})}\big) + \zeta_2 p_b\big(\varkappa_j,[\mathfrak{T}\varkappa_j]_{\alpha(\varkappa_j)}\big) \\ &\quad + \zeta_3 p_b\big(\varkappa_{j-1},[\mathfrak{T}\varkappa_j]_{\alpha(\varkappa_j)}\big) + \zeta_4 p_b\big(\varkappa_j,[\mathfrak{T}\varkappa_{j-1}]_{\alpha(\varkappa_{j-1})}\big) + \zeta_5 p_b(\varkappa_{j-1},\varkappa_j) \\ &\quad + \zeta_6 \frac{p_b\big(\varkappa_{j-1},[\mathfrak{T}\varkappa_{j-1}]_{\alpha(\varkappa_{j-1})}\big)\Big(1+p_b\big(\varkappa_{j-1},[\mathfrak{T}\varkappa_{j-1}]_{\alpha(\varkappa_{j-1})}\big)\Big)}{1+p_b(\varkappa_{j-1},\varkappa_j)}\bigg] \\ &\le h\bigg[\zeta_1 p_b(\varkappa_{j-1},\varkappa_j) + \zeta_2 p_b(\varkappa_j,\varkappa_{j+1}) \\ &\quad + \zeta_3 p_b(\varkappa_{j-1},\varkappa_{j+1}) + \zeta_4 p_b(\varkappa_j,\varkappa_j) + \zeta_5 p_b(\varkappa_{j-1},\varkappa_j) \\ &\quad + \zeta_6 \frac{p_b(\varkappa_{j-1},\varkappa_j)(1+p_b(\varkappa_{j-1},\varkappa_j))}{1+p_b(\varkappa_{j-1},\varkappa_j)}\bigg] \\ &\le h\bigg[\zeta_1 p_b(\varkappa_{j-1},\varkappa_j) + \zeta_2 p_b(\varkappa_j,\varkappa_{j+1}) \\ &\quad + \mathfrak{b}\zeta_3\big[p_b(\varkappa_{j-1},\varkappa_j) + p_b(\varkappa_j,\varkappa_{j+1})\big] + \zeta_5 p_b(\varkappa_{j-1},\varkappa_j) \\ &\quad + \zeta_6 p_b(\varkappa_{j-1},\varkappa_j)\bigg]. \end{aligned}$$

Thus,

$$\big[1 - h(\zeta_2 + \mathfrak{b}\zeta_3)\big]p_b(\varkappa_j,\varkappa_{j+1}) \le h(\zeta_1 + \mathfrak{b}\zeta_3 + \zeta_5 + \zeta_6)p_b(\varkappa_{j-1},\varkappa_j).$$

Hence,

$$p_b(\varkappa_j,\varkappa_{j+1}) \le h\frac{\zeta_1 + \mathfrak{b}\zeta_3 + \zeta_5 + \zeta_6}{1 - h(\zeta_2 + \mathfrak{b}\zeta_3)}p_b(\varkappa_{j-1},\varkappa_j).$$

That is,

$$\begin{aligned} p_b(\varkappa_j,\varkappa_{j+1}) &\le h\eta p_b(\varkappa_{j-1},\varkappa_j) \\ &\le (h\eta)^2 p_b(\varkappa_{j-2},\varkappa_{j-1}) \\ &\;\;\vdots \\ &\le (h\eta)^j p_b(\varkappa_0,\varkappa_1), \quad j \in \mathbb{N}. \end{aligned}$$

Now, consider

$$\begin{aligned}
p_b(\varkappa_0,\varkappa_{j+1}) &\le \mathfrak{b}p_b(\varkappa_0,\varkappa_1)+\mathfrak{b}^2p_b(\varkappa_1,\varkappa_2)+\cdots+\mathfrak{b}^{j+1}p_b(\varkappa_j,\varkappa_{j+1})\\
&\le \mathfrak{b}p_b(\varkappa_0,\varkappa_1)+\mathfrak{b}^2h\eta p_b(\varkappa_0,\varkappa_1)+\cdots+\mathfrak{b}^{j+1}(h\eta)^j p_b(\varkappa_0,\varkappa_1)\\
&= \mathfrak{b}\big[1+\mathfrak{b}h\eta+\cdots+(\mathfrak{b}h\eta)^j\big]p_b(\varkappa_0,\varkappa_1)\\
&= \mathfrak{b}\frac{1-(\mathfrak{b}h\eta)^{j+1}}{1-\mathfrak{b}h\eta}p_b(\varkappa_0,\varkappa_1)\\
&\le \mathfrak{b}\frac{1-(\mathfrak{b}h\eta)^{j+1}}{1-\mathfrak{b}h\eta}\eta(1-\mathfrak{b}h\eta)\rho\\
&= \mathfrak{b}\eta\big(1-(\mathfrak{b}h\eta)^{j+1}\big)\rho\\
&< \rho.
\end{aligned}$$

That is,

$$\varkappa_{j+1}\in\overline{B_{p_b}(\varkappa_0;\rho)}.$$

By a mathematical induction, we have $\varkappa_n\in\overline{B_{p_b}(\varkappa_0;\rho)}$ for all $n\in\mathbb{N}$, hence we obtain

$$p_b(\varkappa_n,\varkappa_{n+1})\le(h\eta)^n p_b(\varkappa_0,\varkappa_1),\ \text{ for all } n\in\mathbb{N}. \tag{2}$$

The next step is to show that $\{\varkappa_n\}$ is a Cauchy sequence. For this, choose two integers $m$ and $n$ with $m<n$ and consider

$$\begin{aligned}
p_b(\varkappa_m,\varkappa_n) &\le \mathfrak{b}p_b(\varkappa_m,\varkappa_{m+1})+\mathfrak{b}^2p_b(\varkappa_{m+1},\varkappa_{m+2})+\cdots+\mathfrak{b}^{n-m}p_b(\varkappa_{n-1},\varkappa_n)\\
&\le \mathfrak{b}(h\eta)^m p_b(\varkappa_0,\varkappa_1)+\mathfrak{b}^2(h\eta)^{m+1}p_b(\varkappa_0,\varkappa_1)\\
&\quad+\cdots+\mathfrak{b}^{n-m}(h\eta)^{n-1}p_b(\varkappa_0,\varkappa_1)\\
&= \mathfrak{b}(h\eta)^m\big(1+\mathfrak{b}h\eta+\cdots+(\mathfrak{b}h\eta)^{n-m-1}\big)p_b(\varkappa_0,\varkappa_1)\\
&= \mathfrak{b}(h\eta)^m\frac{1-(\mathfrak{b}h\eta)^{n-m}}{1-\mathfrak{b}h\eta}p_b(\varkappa_0,\varkappa_1)\\
&< \frac{\mathfrak{b}(h\eta)^m}{1-\mathfrak{b}h\eta}p_b(\varkappa_0,\varkappa_1)\to 0 \text{ as } m\to\infty.
\end{aligned}$$

This means $\{\varkappa_n\}$ is a Cauchy sequence and so we have $\{\varkappa_n\}$, which converges to $\varkappa^*\in\overline{B_{p_b}(\varkappa_0;\rho)}$.

The final step is to show that $\varkappa^*$ is the desired fixed point. For this, we use Lemma 2 and Corollary 2 and consider

$$\begin{aligned}
p_b\big(\varkappa^*,[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big) &\le \mathfrak{b}\Big[p_b(\varkappa^*,\varkappa_{n+1})+p_b\big(\varkappa_{n+1},[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big)\Big]\\
&\le \mathfrak{b}\Big[p_b(\varkappa^*,\varkappa_{n+1})+hH_{p_b}\big([\mathfrak{T}\varkappa_n]_{\alpha(\varkappa_n)},[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big)\Big]\\
&\le \mathfrak{b}\Bigg[p_b(\varkappa^*,\varkappa_{n+1})+h\Bigg[\zeta_1p_b\big(\varkappa_n,[\mathfrak{T}\varkappa_n]_{\alpha(\varkappa_n)}\big)+\zeta_2p_b\big(\varkappa^*,[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big)\\
&\quad+\zeta_3p_b\big(\varkappa_n,[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big)+\zeta_4p_b\big(\varkappa^*,[\mathfrak{T}\varkappa_n]_{\alpha(\varkappa_n)}\big)+\zeta_5p_b(\varkappa_n,\varkappa^*)\\
&\quad+\zeta_6\frac{p_b\big(\varkappa_n,[\mathfrak{T}\varkappa_n]_{\alpha(\varkappa_n)}\big)\Big(1+p_b\big(\varkappa_n,[\mathfrak{T}\varkappa_n]_{\alpha(\varkappa_n)}\big)\Big)}{1+p_b(\varkappa_n,\varkappa^*)}\Bigg]\Bigg]\\
&\le \mathfrak{b}\Bigg[p_b(\varkappa^*,\varkappa_{n+1})+h\Bigg[\zeta_1p_b(\varkappa_n,\varkappa_{n+1})+\zeta_2p_b\big(\varkappa^*,[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big)\\
&\quad+\zeta_3p_b\big(\varkappa_n,[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big)+\zeta_4p_b(\varkappa^*,\varkappa_{n+1})+\zeta_5p_b(\varkappa_n,\varkappa^*)
\end{aligned}$$

$$+\zeta_6 \frac{p_b(\varkappa_n, \varkappa_{n+1})\big(1+p_b(\varkappa_n, \varkappa_{n+1})\big)}{1+p_b(\varkappa_n, \varkappa^*)}\Bigg]\Bigg].$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}
p_b\big(\varkappa^*, [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big) \le \mathfrak{b}\Bigg[p_b(\varkappa^*, \varkappa^*) + h\Bigg[&\zeta_1 p_b(\varkappa^*, \varkappa^*) + \zeta_2 p_b\big(\varkappa^*, [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big) \\
&+ \zeta_3 p_b\big(\varkappa^*, [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big) + \zeta_4 p_b(\varkappa^*, \varkappa^*) + \zeta_5 p_b(\varkappa^*, \varkappa^*) \\
&+ \zeta_6 \frac{p_b(\varkappa^*, \varkappa^*)\big(1+p_b(\varkappa^*, \varkappa^*)\big)}{1+p_b(\varkappa^*, \varkappa^*)}\Bigg]\Bigg] \\
= \mathfrak{b}h\Big[&\zeta_2 p_b\big(\varkappa^*, [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big) + \zeta_3 p_b\big(\varkappa^*, [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big)\Big].
\end{aligned}$$

One writes

$$\big[1 - \mathfrak{b}h(\zeta_2 + \zeta_3)\big] p_b\big(\varkappa^*, [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big) \le 0.$$

That is,

$$p_b\big(\varkappa^*, [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}\big) = 0.$$

We have

$$\varkappa^* \in \overline{[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}} = [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}.$$

Hence, $\xi^*$ is a fixed point of $\mathfrak{T}$. □

**Lemma 4.** *Every partial metric space is a dislocated metric space, but the converse is not true. The following counterexample is given to illustrate this fact.*

**Example 3.** *Consider the space $\Xi = \{0,1\}$ with a dislocated metric on $\Xi$ defined by*

$$d_l(\phi, \chi) = \begin{cases} 2, & \text{if } \phi = \chi = 0, \\ 1, & \text{otherwise.} \end{cases}$$

*We see that $(\Xi, d_l)$ cannot be a partial metric space, since the property of least self-distance is not satisfied:*

$$d_l(0,0) \nleq d_l(0,1).$$

**Remark 1.** *Theorem 2.1 of [22] is a special case of Theorem 1 by Lemma 4.*

**Definition 11.** *Consider a pbMS $(\Xi, p_b)$ and assume that $\mathfrak{S}, \mathfrak{T}$ be two fuzzy mappings defined on $\Xi$. The pair $(\mathfrak{S}, \mathfrak{T})$ satisfies a $\mathcal{FG}$-contraction condition if*

$$\begin{aligned}
H_{p_b}([\mathfrak{T}\phi]_{\alpha(\phi)}, [\mathfrak{S}\chi]_{\alpha(\chi)}) \le \zeta_1 p_b(\phi, [\mathfrak{T}\phi]_{\alpha(\phi)}) &+ \zeta_2 p_b(\chi, [\mathfrak{S}\chi]_{\alpha(\chi)}) + \zeta_3 p_b(\phi, [\mathfrak{S}\chi]_{\alpha(\chi)}) \\
&+ \zeta_4 p_b(\chi, [\mathfrak{T}\phi]_{\alpha(\phi)}) + \zeta_5 p_b(\phi, \chi)
\end{aligned}$$

*for all $\phi, \chi \in \Xi$ with $\zeta_i \ge 0$ $(i = 1, 2, \ldots, 5)$ and*

$$\sum_{i=0}^{5} \zeta_i < 1.$$

**Definition 12.** *Let $(\Xi, d)$ be a metric space and let $S, T : \Xi \to \mathcal{F}(\Xi)$ be two fuzzy mappings of $\Xi$. A point $\xi \in \Xi$ satisfying $\xi \in [S\xi]_{\alpha_S(\xi)}$ and $\xi \in [T\xi]_{\alpha_T(\xi)}$ for some $\alpha_S(\xi), \alpha_T(\xi) \in (0,1]$ is called a common fuzzy fixed point of $S$ and $T$.*

The next theorem guarantees a common fixed point for two fuzzy mappings $\mathfrak{T}$ and $\mathfrak{S}$ of a complete pbMS.

**Theorem 2.** *Consider a complete pbMS $(\Xi, p_b)$ with $\mathfrak{b} \geq 1$. Let $\mathfrak{S}$ and $\mathfrak{T}$ be two fuzzy mappings defined on $\Xi$. Let $h > 1$ be a constant. Further, let $\varkappa_0$ be an arbitrary point of $\Xi$. Suppose there exist $\alpha_{\mathfrak{S}}(\phi), \alpha_{\mathfrak{T}}(\phi) \in (0,1]$ for all $\phi \in \Xi$ such that the pair $(\mathfrak{S}, \mathfrak{T})$ satisfies a $\mathcal{FG}$-contraction condition and*

$$p_b\big(\varkappa_0, [\mathfrak{T}\varkappa_0]_{\alpha_{\mathfrak{T}}(\varkappa_0)}\big) \leq \eta(1 - \mathfrak{b}h\eta)\rho \tag{3}$$

*for all $\phi, \chi \in \overline{B_{p_b}(\varkappa_0; \rho)}$, $\rho > 0$ and $\mathfrak{b}h\eta < 1$, where*

$$\eta = \frac{\zeta_1 + \zeta_2 + \mathfrak{b}(\zeta_3 + \zeta_4) + 2\zeta_5}{1 - h\big[\zeta_1 + \zeta_2 + \mathfrak{b}(\zeta_3 + \zeta_4)\big]}$$

*with $h\big[\zeta_1 + \zeta_2 + \mathfrak{b}(\zeta_3 + \zeta_4)\big] \neq 1$, $h(\zeta_2 + \mathfrak{b}\zeta_3) \neq 1$, $h(\zeta_1 + \mathfrak{b}\zeta_4) \neq 1$,and$\mathfrak{b}h(\zeta_2 + \gamma_3) \neq 1$. Further, $\zeta_i \geq 0$ $(i = 1, 2, \ldots, 5)$,*

$$\sum_{i=1}^{5} \zeta_i < 1$$

*and*

$$h\big[(\mathfrak{b}+1)(\zeta_1 + \zeta_2) + \mathfrak{b}(\mathfrak{b}+1)(\zeta_3 + \zeta_4) + 2\mathfrak{b}\zeta_5\big] < 1,$$

*then there exists an element $\varkappa^* \in \overline{B_{p_b}(\varkappa_0; \rho)}$, that is, a fixed point for both $\mathfrak{T}$ and $\mathfrak{S}$.*

**Proof.** Consider an arbitrary $\varkappa_0$ in $\Xi$ such that $\varkappa_1 \in [\mathfrak{T}\varkappa_0]_{\alpha_{\mathfrak{T}}(\varkappa_0)}$. A sequence is constructed $\{\varkappa_n\}$ in $\Xi$ such that for $i = 0, 1, 2, \ldots$

$$\varkappa_{2i+1} \in [\mathfrak{T}\varkappa_{2i}]_{\alpha_{\mathfrak{T}}(\varkappa_{2i})} \text{ and } \varkappa_{2i+2} \in [\mathfrak{S}\varkappa_{2i+1}]_{\alpha_{\mathfrak{S}}(\varkappa_{2i+1})}.$$

We will first show that $\{\varkappa_n\} \in \overline{B_{p_b}(\varkappa_0; \rho)}$. Using (3), we have

$$\begin{aligned}
&p_b(\varkappa_0, \varkappa_1) = p_b\big(\varkappa_0, [\mathfrak{T}\varkappa_0]_{\alpha_{\mathfrak{T}}(\varkappa_0)}\big) \leq \eta(1 - \mathfrak{b}h\eta)\rho < \rho \\
&\Rightarrow \varkappa_1 \in \overline{B_{p_b}(\varkappa_0; \rho)}.
\end{aligned}$$

Now, suppose $\varkappa_2, \varkappa_3, \ldots, \varkappa_j \in \overline{B_{p_b}(\varkappa_0; \rho)}$, $j \in \mathbb{N}$.

**Case 1.** Let $j = 2i + 1$, $i = 0, 1, 2, \ldots, \frac{j-1}{2}$. By Lemma 2, we have

$$\begin{aligned}
p_b(\varkappa_{2i+1}, \varkappa_{2i+2}) &\leq hH_{p_b}\big([\mathfrak{T}\varkappa_{2i}]_{\alpha_{\mathfrak{T}}(\varkappa_{2i})}, [\mathfrak{S}\varkappa_{2i+1}]_{\alpha_{\mathfrak{S}}(\varkappa_{2i+1})}\big) \\
&\leq h\Big[\zeta_1 p_b\big(\varkappa_{2i}, [\mathfrak{T}\varkappa_{2i}]_{\alpha_{\mathfrak{T}}(\varkappa_{2i})}\big) + \zeta_2 p_b\big(\varkappa_{2i+1}, [\mathfrak{S}\varkappa_{2i+1}]_{\alpha_{\mathfrak{S}}(\varkappa_{2i+1})}\big) \\
&\quad + \zeta_3 p_b\big(\varkappa_{2i}, [\mathfrak{S}\varkappa_{2i+1}]_{\alpha_{\mathfrak{S}}(\varkappa_{2i+1})}\big) + \zeta_4 p_b\big(\varkappa_{2i+1}, [\mathfrak{T}\varkappa_{2i}]_{\alpha_{\mathfrak{T}}(\varkappa_{2i})}\big) \\
&\quad + \zeta_5 p_b(\varkappa_{2i}, \varkappa_{2i+1})\Big] \\
&\leq h\big[\zeta_1 p_b(\varkappa_{2i}, \varkappa_{2i+1}) + \zeta_2 p_b(\varkappa_{2i+1}, \varkappa_{2i+2}) \\
&\quad + \zeta_3 p_b(\varkappa_{2i}, \varkappa_{2i+2}) + \zeta_4 p_b(\varkappa_{2i+1}, \varkappa_{2i+1}) \\
&\quad + \zeta_5 p_b(\varkappa_{2i}, \varkappa_{2i+1})\big] \\
&\leq h\Big[\zeta_1 p_b(\varkappa_{2i}, \varkappa_{2i+1}) + \zeta_2 p_b(\varkappa_{2i+1}, \varkappa_{2i+2}) \\
&\quad + \mathfrak{b}\zeta_3\big[p_b(\varkappa_{2i}, \varkappa_{2i+1}) + p_b(\varkappa_{2i+1}, \varkappa_{2i+2})\big] \\
&\quad + \zeta_5 p_b(\varkappa_{2i}, \varkappa_{2i+1})\Big].
\end{aligned}$$

We obtain

$$\big[1 - h(\zeta_2 + \mathfrak{b}\zeta_3)\big] p_b(\varkappa_{2i+1}, \varkappa_{2i+2}) \leq h(\zeta_1 + \mathfrak{b}\zeta_3 + \zeta_5) p_b(\varkappa_{2i}, \varkappa_{2i+1}).$$

That is,

$$p_b(\varkappa_{2i+1},\varkappa_{2i+2}) \leq h\frac{\zeta_1+\mathfrak{b}\zeta_3+\zeta_5}{1-h(\zeta_2+\mathfrak{b}\zeta_3)}p_b(\varkappa_{2i},\varkappa_{2i+1}). \tag{4}$$

Now, consider $p_b(\varkappa_{2i+2},\varkappa_{2i+1})$:

$$\begin{aligned}
p_b(\varkappa_{2i+2},\varkappa_{2i+1}) &\leq hH_{p_b}\big([\mathfrak{S}\varkappa_{2i+1}]_{\alpha_{\mathfrak{S}}(\varkappa_{2i+1})},[\mathfrak{T}\varkappa_{2i}]_{\alpha_{\mathfrak{T}}(\varkappa_{2i})}\big)\\
&\leq h\Big[\zeta_1 p_b\big(\varkappa_{2i+1},[\mathfrak{S}\varkappa_{2i+1}]_{\alpha_{\mathfrak{S}}(\varkappa_{2i+1})}\big)+\zeta_2 p_b\big(\varkappa_{2i},[\mathfrak{T}\varkappa_{2i}]_{\alpha_{\mathfrak{T}}(\varkappa_{2i})}\big)\\
&\quad+\zeta_3 p_b\big(\varkappa_{2i+1},[\mathfrak{T}\varkappa_{2i}]_{\alpha_{\mathfrak{T}}(\varkappa_{2i})}\big)+\zeta_4 p_b\big(\varkappa_{2i},[\mathfrak{S}\varkappa_{2i+1}]_{\alpha_{\mathfrak{S}}(\varkappa_{2i+1})}\big)\\
&\quad+\zeta_5 p_b(\varkappa_{2i+1},\varkappa_{2i})\Big]\\
&\leq h\big[\zeta_1 p_b(\varkappa_{2i+1},\varkappa_{2i+2})+\zeta_2 p_b(\varkappa_{2i},\varkappa_{2i+1})\\
&\quad+\zeta_3 p_b(\varkappa_{2i+1},\varkappa_{2i+1})+\zeta_4 p_b(\varkappa_{2i},\varkappa_{2i+2})\\
&\quad+\zeta_5 p_b(\varkappa_{2i+1},\varkappa_{2i})\big]\\
&\leq h\Big[\zeta_1 p_b(\varkappa_{2i+1},\varkappa_{2i+2})+\zeta_2 p_b(\varkappa_{2i},\varkappa_{2i+1})\\
&\quad+\mathfrak{b}\zeta_4\big[p_b(\varkappa_{2i},\varkappa_{2i+1})+p_b(\varkappa_{2i+1},\varkappa_{2i+2})\big]\\
&\quad+\zeta_5 p_b(\varkappa_{2i},\varkappa_{2i+1})\Big].
\end{aligned}$$

One writes

$$\big[1-h(\zeta_1+\mathfrak{b}\zeta_4)\big]p_b(\varkappa_{2i+1},\varkappa_{2i+2}) \leq h(\zeta_2+\mathfrak{b}\zeta_4+\zeta_5)p_b(\varkappa_{2i},\varkappa_{2i+1}).$$

That is,

$$p_b(\varkappa_{2i+1},\varkappa_{2i+2}) \leq h\frac{\zeta_2+\mathfrak{b}\zeta_4+\zeta_5}{1-h(\zeta_1+\mathfrak{b}\zeta_4)}p_b(\varkappa_{2i},\varkappa_{2i+1}). \tag{5}$$

Adding (4) and (5) and discarding the negative terms of the numerator and the positive terms of the denominator, we obtain

$$\begin{aligned}
p_b(\varkappa_{2i+1},\varkappa_{2i+2}) &\leq 2p_b(\varkappa_{2i+1},\varkappa_{2i+2})\\
&\leq h\frac{\zeta_1+\zeta_2+\mathfrak{b}(\zeta_3+\zeta_4)+2\zeta_5}{1-h\big[\zeta_1+\zeta_2+\mathfrak{b}(\zeta_3+\zeta_4)\big]}p_b(\varkappa_{2i},\varkappa_{2i+1}),
\end{aligned}$$

i.e.,

$$p_b(\varkappa_{2i+1},\varkappa_{2i+2}) \leq h\eta\, p_b(\varkappa_{2i},\varkappa_{2i+1}). \tag{6}$$

**Case 2.** Applying the same method for $j=2i+2,\ \ i=0,1,2,\dots,\frac{j-2}{2}$, we obtain

$$p_b(\varkappa_{2i+2},\varkappa_{2i+3}) \leq h\eta\, p_b(\varkappa_{2i+1},\varkappa_{2i+2}). \tag{7}$$

Applying (6) and (7) repeatedly gives

$$p_b(\varkappa_{2i+1},\varkappa_{2i+2}) \leq (h\eta)^{2i+1}p_b(\varkappa_0,\varkappa_1)$$

and

$$p_b(\varkappa_{2i+2},\varkappa_{2i+3}) \leq (h\eta)^{2i+2}p_b(\varkappa_0,\varkappa_1).$$

Combining the above two inequalities gives the general inequality

$$p_b(\varkappa_j,\varkappa_{j+1}) \leq (h\eta)^j p_b(\varkappa_0,\varkappa_1),\ \ j\in\mathbb{N}. \tag{8}$$

Now, consider

$$\begin{aligned}
p_b(\varkappa_0, \varkappa_{j+1}) &\leq \mathfrak{b} p_b(\varkappa_0, \varkappa_1) + \mathfrak{b}^2 p_b(\varkappa_1, \varkappa_2) + \cdots + \mathfrak{b}^{j+1} p_b(\varkappa_j, \varkappa_{j+1}) \\
&\leq \mathfrak{b} p_b(\varkappa_0, \varkappa_1) + \mathfrak{b}^2 h\eta p_b(\varkappa_0, \varkappa_1) + \cdots + \mathfrak{b}^{j+1} (h\eta)^j p_b(\varkappa_0, \varkappa_1) \\
&= \mathfrak{b}\left[1 + \mathfrak{b}h\eta + \cdots + (\mathfrak{b}h\eta)^j\right] p_b(\varkappa_0, \varkappa_1) \\
&= \mathfrak{b} \frac{1 - (\mathfrak{b}h\eta)^{j+1}}{1 - \mathfrak{b}h\eta} p_b(\varkappa_0, \varkappa_1) \\
&\leq \mathfrak{b} \frac{1 - (\mathfrak{b}h\eta)^{j+1}}{1 - \mathfrak{b}h\eta} \eta(1 - \mathfrak{b}h\eta)\rho \\
&< \mathfrak{b}\eta\rho \\
&< \rho.
\end{aligned}$$

That is,

$$\varkappa_{j+1} \in \overline{B_{p_b}(\varkappa_0; \rho)}.$$

By a mathematical induction, we have $\{\varkappa_n\} \in \overline{B_{p_b}(\varkappa_0; \rho)}$ and (8) can be rewritten as

$$p_b(\varkappa_n, \varkappa_{n+1}) \leq (h\eta)^n p_b(\varkappa_0, \varkappa_1), \quad \text{for all } n \in \mathbb{N}.$$

To show that $\{\varkappa_n\}$ is a Cauchy sequence, we take two integers $m$ and $n$ with $m < n$ and consider

$$\begin{aligned}
p_b(\varkappa_m, \varkappa_n) &\leq \mathfrak{b} p_b(\varkappa_m, \varkappa_{m+1}) + \mathfrak{b}^2 p_b(\varkappa_{m+1}, \varkappa_{m+2}) + \cdots + \mathfrak{b}^{n-m} p_b(\varkappa_{n-1}, \varkappa_n) \\
&\leq \mathfrak{b}(h\eta)^m p_b(\varkappa_0, \varkappa_1) + \mathfrak{b}^2 (h\eta)^{m+1} p_b(\varkappa_0, \varkappa_1) \\
&\quad + \cdots + \mathfrak{b}^{n-m} (h\eta)^{n-1} p_b(\varkappa_0, \varkappa_1) \\
&= \mathfrak{b}(h\eta)^m \left[1 + \mathfrak{b}h\eta + \cdots + (\mathfrak{b}h\eta)^{n-m-1}\right] p_b(\varkappa_0, \varkappa_1) \\
&= \mathfrak{b}(h\eta)^m \frac{1 - (\mathfrak{b}h\eta)^{n-m}}{1 - \mathfrak{b}h\eta} p_b(\varkappa_0, \varkappa_1) \\
&\leq \frac{\mathfrak{b}(h\eta)^m}{1 - \mathfrak{b}h\eta} p_b(\varkappa_0, \varkappa_1) \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Hence, $\{\varkappa_n\}$ is a Cauchy sequence, converging to $\varkappa^* \in \overline{B_{p_b}(\varkappa_0; \rho)}$.

Finally, it is only left to show that $\varkappa^*$ is the common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$. For this, we will once again use Lemma 2 and Corollary 2 and consider

$$\begin{aligned}
p_b\left(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\right) &\leq \mathfrak{b}\left[p_b(\varkappa^*, \varkappa_{2n+1}) + p_b\left(\varkappa_{2n+1}, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\right)\right] \\
&\leq \mathfrak{b}\left[p_b(\varkappa^*, \varkappa_{2n+1}) + hH_{p_b}\left([\mathfrak{T}\varkappa_{2n}]_{\alpha_{\mathfrak{T}}(\varkappa_{2n})}, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\right)\right] \\
&\leq \mathfrak{b}\Big[p_b(\varkappa^*, \varkappa_{2n+1}) + h\Big[\zeta_1 p_b\left(\varkappa_{2n}, [\mathfrak{T}\varkappa_{2n}]_{\alpha_{\mathfrak{T}}(\varkappa_{2n})}\right) + \zeta_2 p_b\left(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\right) \\
&\quad + \zeta_3 p_b\left(\varkappa_{2n}, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\right) + \zeta_4 p_b\left(\varkappa^*, [\mathfrak{T}\varkappa_{2n}]_{\alpha_{\mathfrak{T}}(\varkappa_{2n})}\right) \\
&\quad + \zeta_5 p_b(\varkappa_{2n}, \varkappa^*)\Big]\Big] \\
&\leq \mathfrak{b}\Big[p_b(\varkappa^*, \varkappa_{2n+1}) + h\Big[\zeta_1 p_b(\varkappa_{2n}, \varkappa_{2n+1}) + \zeta_2 p_b\left(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\right) \\
&\quad + \zeta_3 p_b\left(\varkappa_{2n}, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\right) + \zeta_4 p_b(\varkappa^*, \varkappa_{2n+1}) + \zeta_5 p_b(\varkappa_{2n}, \varkappa^*)\Big]\Big].
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} p_b\big(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\big) &\leq \mathfrak{b}\Big[p_b(\varkappa^*, \varkappa^*) + h\Big[\zeta_1 p_b(\varkappa^*, \varkappa^*) + \zeta_2 p_b\big(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\big) \\ &\qquad + \zeta_3 p_b\big(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\big) + \zeta_4 p_b(\varkappa^*, \varkappa^*) + \zeta_5 p_b(\varkappa^*, \varkappa^*)\Big]\Big] \\ &= \mathfrak{b}h(\zeta_2 + \zeta_3) p_b\big(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\big) \\ &\Rightarrow \big[1 - \mathfrak{b}h(\zeta_2 + \zeta_3)\big] p_b\big(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\big) \leq 0 \\ &\Rightarrow p_b\big(\varkappa^*, [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}\big) = 0. \end{aligned}$$

That is,

$$\varkappa^* \in \overline{[\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}} = [\mathfrak{S}\varkappa^*]_{\alpha_{\mathfrak{S}}(\varkappa^*)}.$$

Hence, $\varkappa^*$ is the desired fixed point of $\mathfrak{S}$. To show that $\varkappa^*$ is also a fixed point of $\mathfrak{T}$, we can adopt a similar method to that shown above. □

**Remark 2.**

1. *Theorem 2.2 of [22] is a special case of the above result.*
2. *In the case that $\mathfrak{b} = 1$, the space $(\Xi, p_b)$ becomes a partial metric space. We see the result is valid for such spaces and hence also holds in dislocated metric spaces.*

## 3. Examples

The following examples are illustrations of the above theorems. First, we will find the fixed point of fuzzy mapping on a complete partial bMS.

**Example 4.** *Let $\Xi = \{0, 1, 2\}$. We define a function $p_b : \Xi \times \Xi \to \mathbb{R}$ as*

$$p_b(\phi, \chi) = \begin{cases} 0, & \phi = \chi \in \{0, 1\} \\ 1, & \phi = \chi = 2 \\ \max\{\phi, \chi\} \times \mid \phi - \chi \mid, & \phi \neq \chi. \end{cases}$$

*It can be verified that $(\Xi, p_b)$ forms a complete pbMS with $\mathfrak{b} = \frac{4}{3}$. Let $\varkappa_0 = 0$ and $\rho = 4$. This gives $\overline{B_{p_b}(\varkappa_0; \rho)} = \Xi$. Next, define a fuzzy mapping $\mathfrak{T}$ on $\Xi$ by*

$$(\mathfrak{T}0)(\phi) = (\mathfrak{T}1)(\phi) = \begin{cases} \frac{3}{4}, & \phi = 0 \\ 0, & \phi \in \{1, 2\}, \end{cases}$$

$$(\mathfrak{T}2)(\phi) = \begin{cases} \frac{3}{4}, & \phi = 1 \\ 0, & \phi \in \{0, 2\}. \end{cases}$$

*Let $\alpha_{\mathfrak{T}}(\varkappa) = \alpha$ where $\alpha \in (0, \frac{3}{4}]$. This gives us*

$$[\mathfrak{T}\varkappa]_\alpha = \begin{cases} \{0\}, & \varkappa = 0 \\ \{0\}, & \varkappa = 1 \\ \{1\}, & \varkappa = 2. \end{cases}$$

*The partial Hausdorff b-metric will have the following values*

$$\begin{aligned} &H_{p_b}\big([\mathfrak{T}\varkappa]_\alpha, [\mathfrak{T}\varkappa]_\alpha\big) = 0, \quad \varkappa = 0, 1, 2; \\ &H_{p_b}\big([\mathfrak{T}0]_\alpha, [\mathfrak{T}1]_\alpha\big) = 0; \\ &H_{p_b}\big([\mathfrak{T}0]_\alpha, [\mathfrak{T}2]_\alpha\big) = 1; \\ &H_{p_b}\big([\mathfrak{T}1]_\alpha, [\mathfrak{T}2]_\alpha\big) = 1. \end{aligned}$$

*For* $h = \frac{11}{10} > 1,\ \mathfrak{b} = \frac{4}{3} \geq 1,\ \zeta_1 = \zeta_3 = 0,\ \zeta_2 = \frac{1}{9},\ \zeta_4 = \zeta_5 = \frac{1}{4}$, *and* $\zeta_6 = \frac{1}{3}$, *we have*

$$\eta = \frac{105}{158}.$$

*It is necessary to check that all the conditions of the theorem are satisfied, some of which are verified below:*

$$\frac{11}{10}\left(\frac{1}{9} + \frac{4}{3}\left(\frac{1}{4} + \frac{1}{3}\right)\right) < 1,$$

*i.e.,*

$$\frac{44}{45} < 1.$$

*We give two cases for the condition of the partial Hausdorff b-metric:*
*For* $\phi = 0$ *and* $\chi = 1$,

$$H_{p_b}\big([\mathfrak{T}0]_\alpha, [\mathfrak{T}1]_\alpha\big) \leq \frac{1}{9}\, p_b\big(1, [\mathfrak{T}1]_\alpha\big) + \frac{1}{4}\, p_b\big(1, [\mathfrak{T}0]_\alpha\big) + \frac{1}{4}\, p_b(0,1) + \frac{1}{3}\, \frac{p_b\big(0, [\mathfrak{T}0]_\alpha\big)\Big(1 + p_b\big(0, [\mathfrak{T}0]_\alpha\big)\Big)}{1 + p_b(0,1)}.$$

*That is,*

$$0 \leq \frac{1}{9} + \frac{1}{4} + \frac{1}{4} = \frac{22}{36}.$$

*For* $\phi = 1$ *and* $\chi = 2$,

$$H_{p_b}\big([\mathfrak{T}1]_\alpha, [\mathfrak{T}2]_\alpha\big) \leq \frac{1}{9}\, p_b\big(2, [\mathfrak{T}2]_\alpha\big) + \frac{1}{4}\, p_b\big(2, [\mathfrak{T}1]_\alpha\big) + \frac{1}{4}\, p_b(1,2) + \frac{1}{3}\, \frac{p_b\big(1, [\mathfrak{T}1]_\alpha\big)\Big(1 + p_b\big(1, [\mathfrak{T}1]_\alpha\big)\Big)}{1 + p_b(1,2)}.$$

*That is,*

$$1 \leq \frac{2}{9} + \frac{4}{4} + \frac{2}{4} + \frac{2}{9} = \frac{70}{36}.$$

*Similarly, we can check all the various cases of the values of* $\phi$ *and* $\chi$ *and we see that all conditions of Theorem 1 are met. The fixed point of the fuzzy mapping T is* $0 \in \overline{B_{p_b}(0;4)}$.

In the next example, we will find a common fixed point of two fuzzy mappings on a complete partial bMS.

**Example 5.** *Let* $\Xi = \{0,1,2\}$ *and define a function* $p_b : \Xi \times \Xi \to \mathbb{R}$ *as*

$$p_b(\phi,\chi) = \begin{cases} 0, & \phi = \chi \in \{0,1\} \\ 1, & \phi = \chi = 2 \\ \max\{\phi,\chi\} \times \mid \phi - \chi \mid, & \phi \neq \chi. \end{cases}$$

$(\Xi, p_b)$ *is a complete partial b-metric space with* $\mathfrak{b} = \frac{4}{3}$. *Let* $\varkappa_0 = 0$ *and* $\rho = 4$. *This gives* $\overline{B_{p_b}(\varkappa_0;\rho)} = \Xi$. *Next, we define two fuzzy mappings* $\mathfrak{T}$ *and* $\mathfrak{S}$ *on* $\Xi$ *by*

$$(\mathfrak{T}0)(\phi) = (\mathfrak{T}1)(\phi) = \begin{cases} \frac{3}{4}, & \phi = 0 \\ 0, & \phi \in \{1,2\}, \end{cases}$$

$$(\mathfrak{T}2)(\phi) = \begin{cases} \frac{3}{4}, & \phi = 1 \\ 0, & \phi \in \{0,2\} \end{cases}$$

*and*

$$(\mathfrak{S}0)(\phi) = (\mathfrak{S}1)(\phi) = (\mathfrak{S}2)(\phi) = \begin{cases} \frac{3}{4}, & \phi = 0 \\ 0, & \phi \in \{1,2\}. \end{cases}$$

*Further, let* $\alpha_{\mathfrak{T}}(\varkappa) = \alpha_{\mathfrak{S}}(\varkappa) = \alpha$ *where* $\alpha \in (0, \frac{3}{4}]$. *This gives us*

$$[\mathfrak{T}\varkappa]_\alpha = \begin{cases} \{0\}, & \varkappa = 0 \\ \{0\}, & \varkappa = 1 \\ \{1\}, & \varkappa = 2, \end{cases}$$

*and*

$$[\mathfrak{S}0]_\alpha = [\mathfrak{S}1]_\alpha = [\mathfrak{S}2]_\alpha = \{0\}.$$

*The partial Hausdorff b-metric will have the following values for all* $\chi \in \Xi$:

$$H_{p_b}\left([\mathfrak{T}\phi]_\alpha, [\mathfrak{S}\chi]_\alpha\right) = \begin{cases} H_{p_b}\left(\{0\}, \{0\}\right) = 0, & \phi \in \{0,1\} \\ H_{p_b}\left(\{1\}, \{0\}\right) = 1, & \phi = 2. \end{cases}$$

*For* $h = \frac{11}{10} > 1,\ \mathfrak{b} = \frac{4}{3} \geq 1,\ \zeta_1 = \zeta_2 = \zeta_4 = 0,\ \zeta_3 = \frac{19}{77}$ *and* $\zeta_5 = \frac{1}{77}$, *we have*

$$\eta = \frac{410}{737}.$$

*Again, it is necessary to check all the conditions of the theorem, some of which are shown below:*

$$\frac{11}{10}\left(\frac{4}{3} \times \frac{7}{3} \times \frac{19}{77} \ + \ \frac{8}{3} \times \frac{1}{77}\right) < 1.$$

*That is,*

$$\frac{6116}{6930} < 1.$$

*We give two cases for the condition of the partial Hausdorff b-metric:*
*For* $\phi = 1$ *and* $\chi = 2$,

$$H_{p_b}\left([\mathfrak{T}1]_\alpha, [\mathfrak{S}2]_\alpha\right) \leq \frac{19}{77} p_b\left(1, [\mathfrak{S}2]_\alpha\right) + \frac{1}{77} p_b(1,2).$$

*Hence,*

$$0 \leq \frac{19}{77} + \frac{2}{77} = \frac{21}{77}.$$

*For* $\phi = 2$ *and* $\chi = 1$,

$$H_{p_b}\left([\mathfrak{T}2]_\alpha, [\mathfrak{S}1]_\alpha\right) \leq \frac{19}{77} p_b\left(2, [\mathfrak{S}1]_\alpha\right) + \frac{1}{77} p_b(2,1).$$

*That is,*

$$1 \leq \frac{76}{77} + \frac{2}{77} = \frac{78}{77}.$$

*Similarly, other cases of the values of* $\phi$ *and* $\chi$ *can be verified and we see that all conditions of Theorem 2 are met.*

*The desired fixed point of* $\mathfrak{S}$ *and* $\mathfrak{T}$ *is* $0 \in \overline{B_{p_b}(0;4)}$.

## 4. Applications

The results given in Section 2 can be modified for multi-valued mappings defined on pbMS that are complete. Two theorems are given below for the fixed points of multi-valued mappings in a complete PMS. The first is about a generalized multi-valued contraction mapping.

**Theorem 3.** *Consider a complete pbMS* $(\Xi, p_b)$ *with* $\mathfrak{b} \geq 1$ *and let* $\varkappa_0$ *be an arbitrary point in* $\Xi$. *Let* $h > 1$ *be a constant. Let* $\mathfrak{R} : \Xi \to \mathcal{P}(\Xi)$ *be a multi-valued generalized contraction [26] for all* $\phi, \chi \in \overline{B_{p_b}(\varkappa_0; \rho)}, \rho > 0$. *Furthermore, let*

$$p_b(\varkappa_0, \mathfrak{R}\varkappa_0) \leq \eta(1 - \mathfrak{b}h\eta)\rho$$

*where* $\mathfrak{b}h\eta < 1$ *and*

$$\eta = \frac{\zeta_1 + \mathfrak{b}\zeta_3 + \zeta_5 + \zeta_6}{1 - h(\zeta_2 + \mathfrak{b}\zeta_3)}$$

*with* $h(\zeta_2 + \mathfrak{b}\zeta_3) \neq 1$ *and* $\mathfrak{b}h(\zeta_2 + \zeta_3) \neq 1$. *Further,* $\zeta_i \geq 0, \ \ i = 1, 2, \ldots, 6$,

$$\sum_{i=1}^{6} \zeta_i < 1$$

*and*

$$h\big(\mathfrak{b}\zeta_1 + \zeta_2 + \mathfrak{b}(\mathfrak{b}+1)\zeta_3 + \beta(\zeta_5 + \zeta_6)\big) < 1,$$

*then there is* $\varkappa^* \in \overline{B_{p_b}(\varkappa_0; \rho)}$ *such that* $\varkappa^* \in \mathfrak{R}\varkappa^*$,, *i.e.,* $\varkappa^*$ *is a fixed point of* $\mathfrak{R}$.

**Proof.** We define an arbitrary mapping $\alpha : \Xi \to (0, 1]$ and a fuzzy mapping $\mathfrak{T} : \Xi \to \mathcal{F}(\Xi)$ by

$$(\mathfrak{T}\varkappa)(\varpi) = \begin{cases} \alpha(\varkappa), & \varpi \in \mathfrak{R}\varkappa \\ 0, & \varpi \notin \mathfrak{R}\varkappa. \end{cases}$$

By definition, the $\alpha(\varkappa)$ level set of $\mathfrak{T}$ is

$$\begin{aligned}[\mathfrak{T}\varkappa]_{\alpha(\varkappa)} &= \{\varpi \in \Xi : \mathfrak{T}\varkappa(\varpi) \geq \alpha(\varkappa)\} \\ &= \{\varpi \in \Xi : \varpi \in \mathfrak{R}\varkappa\} \\ &= \mathfrak{R}\varkappa.\end{aligned}$$

We have satisfied all the conditions of Theorem 1 and so there must be a point $\varkappa^* \in \overline{B_{d_{lb}}(\varkappa_0; \rho)}$ such that

$$\varkappa^* \in [\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)} = \mathfrak{R}\varkappa^*, \text{ i.e., } \varkappa^* \in \mathfrak{R}\varkappa^*.$$

□

The next result is about two multi-valued mappings in a complete partial $b$-metric space.

**Theorem 4.** *Consider a pbMS* $(\Xi, p_b)$ *with* $\mathfrak{b} \geq 1$. *Let* $(\Xi, p_b)$ *be complete and* $h > 1$ *be a constant. Let* $\varkappa_0$ *be an arbitrary point in* $\Xi$ *and* $\mathfrak{R}, \mathfrak{Q} : \Xi \to \mathcal{P}(\Xi)$ *be non-self multi-valued mappings satisfying the generalized contraction condition for a pair of multi-valued mappings for all* $\phi, \chi \in \overline{B_{p_b}(\varkappa_0; \rho)}, \rho > 0$. *Furthermore, let*

$$p_b(\varkappa_0, \mathfrak{R}\varkappa_0) \leq \eta(1 - \mathfrak{b}h\eta)\rho$$

*where* $\mathfrak{b}h\eta < 1$ *and*

$$\eta = \frac{\zeta_1 + \zeta_2 + \mathfrak{b}\zeta_3 + \mathfrak{b}\zeta_4 + 2\mathfrak{b}\zeta_5}{1 - h(\zeta_1 + \zeta_2 + \mathfrak{b}\zeta_3 + \mathfrak{b}\zeta_4)}$$

*with* $h(\zeta_1 + \zeta_2 + \mathfrak{b}\zeta_3 + \mathfrak{b}\zeta_4) \neq 1$, $h(\zeta_2 + \mathfrak{b}\zeta_3) \neq 1$, $h(\zeta_1 + \mathfrak{b}\zeta_4) \neq 1$ *and* $\mathfrak{b}h(\zeta_2 + \zeta_3) \neq 1$. *Further,* $\zeta_i \geq 0, \ \ i = 1, 2, \ldots, 5$,

$$\sum_{i=1}^{5} \zeta_i < 1$$

*and*

$$h\big((\mathfrak{b}+1)(\zeta_1+\zeta_2)+\mathfrak{b}(\mathfrak{b}+1)(\zeta_3+\zeta_4)+2\mathfrak{b}\zeta_5\big)<1,$$

*then there exists* $\varkappa^* \in \overline{B_{p_b}(\varkappa_0;\rho)}$, *which is the desired common fixed point of* $\mathfrak{R}$ *and* $\mathfrak{Q}$.

**Proof.** We define an arbitrary mapping $\alpha : \Xi \to (0,1]$ and two fuzzy mappings $\mathfrak{S}, \mathfrak{T} : \Xi \to \mathcal{F}(\Xi)$ by

$$(\mathfrak{S}\varkappa)(\varpi)=\begin{cases}\alpha(\varkappa), & \varpi\in\mathfrak{R}\varkappa\\ 0, & \varpi\notin\mathfrak{R}\varkappa\end{cases}$$

and

$$(\mathfrak{T}\varkappa)(\varpi)=\begin{cases}\alpha(\varkappa), & \varpi\in\mathfrak{Q}\varkappa\\ 0, & \varpi\notin\mathfrak{Q}\varkappa.\end{cases}$$

By definition, the $\alpha(\varkappa)$ level sets of $\mathfrak{S}$ and $\mathfrak{T}$ is

$$\begin{aligned}[\mathfrak{S}\varkappa]_{\alpha(\varkappa)} &= \{\varpi\in\Xi : \mathfrak{S}\varkappa(\varpi)\geq\alpha(\varkappa)\}\\ &= \{\varpi\in\Xi : \varpi\in\mathfrak{R}\varkappa\}\\ &= \mathfrak{R}\varkappa\end{aligned}$$

and

$$\begin{aligned}[\mathfrak{T}\varkappa]_{\alpha(\varkappa)} &= \{\varpi\in\Xi : \mathfrak{T}\varkappa(\varpi)\geq\alpha(\varkappa)\}\\ &= \{\varpi\in\Xi : \varpi\in\mathfrak{Q}\varkappa\}\\ &= \mathfrak{Q}\varkappa.\end{aligned}$$

Hence, all conditions of Theorem 2 are met and so there must be a point $\varkappa^* \in \overline{B_{d_{lb}}(\varkappa_0;\rho)}$ such that

$$\varkappa^* \in [\mathfrak{S}\varkappa^*]_{\alpha(\varkappa^*)}\cap[\mathfrak{T}\varkappa^*]_{\alpha(\varkappa^*)}=\mathfrak{R}\varkappa^*\cap\mathfrak{Q}\varkappa^*.$$

□

## 5. Conclusions

Using some well-established results for a partial bMS, we extended the results of Shoaib et al. [22] with significant changes to establishing the existence of fixed points of fuzzy mappings in a complete pbMS. We illustrated how our results can be applied to actual problems with the help of two examples. We also presented applications showing how our results can be modified for the problem of multi-valued mappings in a partial pbMS and the existence of their fixed points. Finally, it is worth mentioning that for $\mathfrak{b}=1$, the space becomes a partial metric space. The results presented are also valid for metric spaces and partial metric spaces. In the future, one can check the uniqueness of fixed points for these results. Considering the technique of Hausdroff distance used in the above results, one can further extend these results on extended partial $b$-metric spaces.

**Author Contributions:** Conceptualization, D.-e.-S.S. and A.A.; methodology, N.M.; investigation, Z.R. and S.B.; writing—original draft preparation, D.-e.-S.S., A.A. and N.M.; writing—review and editing, Z.R. and S.B.; supervision, D.-e.-S.S. and N.M. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Not applicable

**Acknowledgments:** The authors A. Aloqaily and N. Mlaiki would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References


1. Banach, S. Sur les operations dans les ensembles abstracts et leur applications aux equations integrals. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Nadler, S.B. Multivalued contraction mappings. *Pac. J. Math.* **1969** *30*, 475–488. [CrossRef]
3. Zhang, Y.; Hofmann, B. Two new non-negativity preserving iterative regularization methods for ill-posed inverse problems. *arXiv* **2020**, arXiv:2002.04474.
4. Shcheglov, A.; Ali, J.; Wang, C.; Ilin, A.; Zhang, Y. Reconstructing the Absorption Function in a Quasi-Linear Sorption Dynamic Model via an Iterative Regularizing Algorithm. *Adv. Appl. Math. Mech.* **2023**. [CrossRef]
5. Matthews, S.G. Partial metric topology. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
6. Mustafa, Z.; Roshan, J.R.; Parvaneh, V. Some common fixed point results in ordered partial $b$-metric spaces. *J. Inequal. Appl.* **2013**, *2013*, 562. [CrossRef]
7. Harandi, A.A. Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.* **2012**, *2012*, 204. [CrossRef]
8. Aydi,H.; Abbas, M.; Vetro, C. Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces. *Topol. Appl.* **2012**, *159*, 3234–3242. [CrossRef]
9. Gunasekar, T.; Karpagam, S.; Zlatanov, B. On $p$-Cyclic Orbital MK Contractions in a Partial Metric Space. *Mathematics* **2018**, *6*, 116. [CrossRef]
10. Bourbaki, N. *Topologie Generale*; Herman: Paris, France, 1974.
11. Bakhtin, I.A. The contraction mapping principle in almost metric spaces. *J. Funct. Anal.* **1989**, *30*, 26–37.
12. Czerwik, S. Contraction mappings in $b$-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
13. Shatanawi, W.; Shatnawi, T.A.M. A new fixed point results in controlled metric-type spaces based on new contractive conditions. *AIMS Math.* **2023**, *8*, 9314–9330. [CrossRef]
14. Rezazgui, A.Z.; Tallafha, A.A.; Shatanawi, W. Common fixed point results via $A_v$-$\alpha$-contractions with a pair and two pairs of self-mappings in the frame of an extended quasi $b$-metric space. *AIMS Math.* **2023**, *8*, 7225–7241. [CrossRef]
15. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965** *8*, 338–353. [CrossRef]
16. Weiss, M.D. Fixed points and induced fuzzy topologies for fuzzy sets. *J. Math. Anal. Appl.* **1975**, *50*, 142–150. [CrossRef]
17. Butnariu, D. Fixed point for fuzzy mapping. *Fuzzy Sets Syst.* **1982**, *7*, 191–207. [CrossRef]
18. Heilpern, S. Fuzzy mappings and fixed point theorem. *J. Math. Anal. Appl.* **1981**, *83*, 566–569. [CrossRef]
19. Batul, S.; Sagheer, D.E.S.; Anwar, M.; Aydi, H.; Parvaneh, V. Fuzzy Fixed Point Results of Fuzzy Mappings on $b$-Metric Spaces via $(\alpha * -F)$-Contractions. *Adv. Math. Phys.* **2022**, *2022*, 4511632. [CrossRef]
20. Kumam, W.; Sukprasert, P.; Kumam, P.; Shoaib, A.; Shahzad, A.; Mahmood, Q. Some fuzzy fixed point results for fuzzy mappings in complete $b$-metric spaces. *Cogent Math. Stat.* **2018**, *5*, 1458933. [CrossRef]
21. Shukla, S. Partial $b$-metric spaces and fixed point theorems. *Mediterr. J. Med.* **2014**, *11*, 703–711. [CrossRef]
22. Shoaib, A.; Kumam, P.; Shahzad, A.; Phiangsungnoen, S.; Mahmood, Q. Fixed point results for fuzzy mappings in a $b$-metric space. *Fixed Point Theory Appl.* **2018**, *2018*, 2. [CrossRef]
23. Karapinar, E.; Salimi, P. Dislocated metric space to metric spaces with some fixed point theorems. *Fixed Point Theory Appl.* **2013**, *2013*, 222. [CrossRef]
24. Hussain, N.; Roshan, J.R.; Parvaneh, V.; Abbas, M. Common fixed point results for weakly contractive mappings in ordered $b$-dislocated metric spaces with applications. *J. Inequal. Appl.* **2013**, *2013*, 486. [CrossRef]
25. Felhi, A. Some fixed point results for multi-valued contractive mappings in partial $b$-metric spaces. *J. Adv. Math. Stud.* **2016**, *9*, 208–225.
26. Joseph, J.M.; Roselin, D.D.; Marudai, M. Fixed point theorem on multi-valued mappings in $b$-metric spaces. *SpringerPlus* **2016**, *5*, 217. [CrossRef]