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Gilbreath Equation, Gilbreath Polynomials, and Upper and Lower Bounds for Gilbreath Conjecture

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Abstract: Let $S = (s_1, \ldots, s_n)$ be a finite sequence of integers. Then, S is a Gilbreath sequence of length n, $S \in \mathbb{G}_n$, iff s_1 is even or odd and $s_2, \ldots, s_n$ are, respectively, odd or even and $\min \mathbb{K}_{(s_1,\ldots,s_m)} \leq s_{m+1} \leq \max \mathbb{K}_{(s_1,\ldots,s_m)}, \forall m \in [1, n)$. This, applied to the order sequence of prime number P, defines Gilbreath polynomials and two integer sequences, A347924 and A347925, which are used to prove that Gilbreath conjecture GC is implied by $p_n - 2^{n-1} \leqslant \mathcal{P}_{n-1}(1)$, where $\mathcal{P}_{n-1}(1)$ is the $n-1$-th Gilbreath polynomial at 1.

Keywords: Gilbreath conjecture; prime numbers; sequence of integer numbers; Gilbreath polynomials

MSC: 11A41; 11B83; 11C08

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1. Introduction to *GC*

Let the ordered sequence $P = (2, 3, 5, 7, 11, 13, 17, \ldots) = (p_1, \ldots, p_n)$ be formed by prime numbers, and set

$$p_a^b = \begin{cases} p_{a+1} - p_a, & \text{if } b = 1; \\ |p_{a+1}^{b-1} - p_a^{b-1}|, & \text{otherwise} \end{cases} \tag{1}$$

where $b \in [1, n)$. N. L. Gilbreath conjectured that $p_1^b = 1$. It is likely that this conjecture is satisfied by many other sequences of integers, so it is necessary to define the general properties of all sequences that satisfy this conjecture. In particular, H. Croft and others have suggested that any sequence starting with 2 followed by odd numbers which does not increase too fast or too slow (does not have too large gaps) satisfies this conjecture [1]. We equally suggest that the same may be true for any sequence starting with 1 followed by even numbers which does not increase too fast or too slow. Thus, in general, we suggest that any sequence starting with an even or odd number followed by odd or even numbers which does not increase too fast or too slow satisfies this conjecture. In the following sections, we shall call a Gilbreath sequence any sequence that satisfies the Gilbreath conjecture and the upper and lower bounds that satisfy the Gilbreath conjecture for a given sequence.

Let $GC(n)$ denote the GC proved for the ordered sequence of the first n prime numbers $(p_1, \ldots, p_n)$; then, there are some interesting computational proofs of $GC(n)$. R. B. Killgrove and K. E. Ralston showed $GC(63419)$ in 1959 [2] and A. M. Odlyzko showed $GC(10^{13})$ in 1993 [3].

2. Properties of Gilbreath Sequence

Definition 1. *Let $S = (s_1, s_2, s_3, \ldots, s_n)$ be a finite sequence of integers and*

$$s_a^b = \begin{cases} s_{a+1} - s_a, & \textit{if } b = 1; \\ |s_{a+1}^{b-1} - s_a^{b-1}|, & \textit{otherwise} \end{cases} \tag{2}$$

where $b \in [1, n)$; then, S is called a Gilbreath sequence iff $s_1^b = 1, \forall b$.

For example, let $S = (2, 3, 5, 9, 11, 13, 17)$ be a sequence of length $n = 7$ and Gilbreath triangle of S:

$$
\begin{array}{lllllllll}
s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\
s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & \\
\dots & & & & & & & & \\
s_1^{n-2} & s_2^{n-2} & & & & & & & \\
s_1^{n-1} & & & & & & & &
\end{array}
$$

Replacing these values gives

$$
\begin{array}{lllllll}
2 & 3 & 5 & 9 & 11 & 13 & 17 \\
1 & 2 & 4 & 2 & 2 & 4 & \\
1 & 2 & 2 & 0 & 2 & & \\
1 & 0 & 2 & 2 & & & \\
1 & 2 & 0 & & & & \\
1 & 2 & & & & & \\
1 & & & & & &
\end{array}
$$

Let $\mathbb{G}_n$ denote the set of all Gilbreath sequences of length n and $\mathbb{G}$ be the set of all Gilbreath sequences. In the previous example, the first term of every sequence (except for the original sequence S) is equal to 1; then, $S \in \mathbb{G}_7$.

Lemma 1. *Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_{n-1})$ be finite sequences of integers; then, $S' \in \mathbb{G}_{n-1}$.*

Proof. Consider the Gilbreath triangle of S

$$
\begin{array}{lllllllll}
s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\
s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & \\
\dots & & & & & & & & \\
s_1^{n-2} & s_2^{n-2} & & & & & & & \\
s_1^{n-1} & & & & & & & &
\end{array}
$$

where $s_1^1 = \ldots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. Removing the last element of each sequence gives

$$
\begin{array}{llllllll}
s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} \\
s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & \\
\dots & & & & & & & \\
s_1^{n-2} & & & & & & &
\end{array}
$$

which is a Gilbreath triangle of S', $s_1^1 = \ldots = s_1^{n-2} = 1$ as a consequence of $S \in \mathbb{G}_n$; then, $S' \in \mathbb{G}_{n-1}$. $\square$

Definition 2. *Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_n, k)$ be finite sequences of integers. Denote with $\mathbb{K}_S$ the set of integers k such that $S' \in \mathbb{G}_{n+1}$.*

The Gilbreath triangle of S is

$$
\begin{array}{lllllllll}
s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\
s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & \\
\dots & & & & & & & & \\
s_1^{n-2} & s_2^{n-2} & & & & & & & \\
s_1^{n-1} & & & & & & & &
\end{array}
$$

where $s_1^1 = \ldots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. The Gilbreath triangle of S' is

$$
\begin{array}{llllllllll}
s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n & k \\
s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & |s_n - k| & \\
\dots & & & & & & & & & \\
s_1^{n-2} & s_2^{n-2} & \multicolumn{8}{l}{|s_3^{n-3} - |s_4^{n-4} - | \ldots - |s_{n-1}^1 - |s_n - k|| \ldots |||}
\end{array}
$$

$$\begin{array}{ll} s_1^{n-1} & |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \ldots - |s_{n-1}^1 - |s_n - k|| \ldots |||| \\ |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \ldots - |s_{n-1}^1 - |s_n - k|| \ldots ||||| & \end{array}$$

where $s_1^1 = \ldots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. If $s_1^n = |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \ldots - |s_{n-1}^1 - |s_n - k|| \ldots ||||| = 1$, then $S' \in \mathbb{G}_{n+1}$.

Consider the equation

$$|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \ldots - |s_{n-1}^1 - |s_n - k|| \ldots ||||| = 1 \tag{3}$$

We will refer to Equation (3) as a Gilbreath equation of S. There are 2^n values of k that satisfy (3); then, $\mathbb{K}_S = \{k_1, \ldots, k_{2^n}\}$ is the set of all solutions for k:

$$k_{1,\ldots,2^n} = \pm s_1^{n-1} \pm s_2^{n-2} \pm s_3^{n-3} \pm s_4^{n-4} \pm \ldots \pm s_{n-1}^1 + s_n \pm 1 \tag{4}$$

The largest value of k that solves (3) is $\max \mathbb{K}_S = s_1^{n-1} + s_2^{n-2} + s_3^{n-3} + s_4^{n-4} + \ldots + s_{n-1}^1 + s_n + 1$, and the smallest value is $\min \mathbb{K}_S = -s_1^{n-1} - s_2^{n-2} - s_3^{n-3} - s_4^{n-4} - \ldots - s_{n-1}^1 + s_n - 1 = 2s_n - \max \mathbb{K}_S$. A remarkable relation is

$$\max \mathbb{K}_S + \min \mathbb{K}_S = 2s_n \tag{5}$$

Lemma 2. *Let $S = (s_1, \ldots, s_n) \in \mathbb{G}_n$ be a finite sequence of integers where $s_1 \in 2\mathbb{Z}$; then, $s_2, \ldots, s_n \in 2\mathbb{Z} + 1$.*

Proof. Let $S_1 = (s_1)$, where $s_1 \in 2\mathbb{Z}$. From Definition 2, $S_2 = (s_1, k) \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z} + 1$. Let now the sequence $S_3 = (s_1, s_1 \pm 1, k)$, from Definition 2, $S_3 \in \mathbb{G}_3$ if $k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1$. From the previous step, $s_1 \pm 1 \in 2\mathbb{Z} + 1$. Then, $1 + s_1 \pm 1 \in 2\mathbb{Z}$ and $1 + s_1 \pm 1 \pm 1 \in 2\mathbb{Z} + 1$. By induction, this can be proved for every element of S. If $S \in \mathbb{G}_n$ and the first element of S is an even number, then all the other numbers of the sequence will be odd. □

Lemma 3. *Let $S = (s_1, \ldots, s_n) \in \mathbb{G}_n$ be a finite sequence of integers where $s_1 \in 2\mathbb{Z} + 1$; then $s_2, \ldots, s_n \in 2\mathbb{Z}$.*

Proof. This argument is the same argument as Lemma 2. □

Lemma 4. *Let $2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$ denote the sets $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ and let a finite sequence of integers $S = (s_1, \ldots, s_n) \in \mathbb{G}_n$ where $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$. Then, $s_2, \ldots, s_n \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$.*

Proof. See Lemmas 2 and 3. □

Lemma 5. *Let $S = (s_1, \ldots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \ldots, s_n, k) \in \mathbb{G}_{n+1}$ be finite sequences of integers where $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$. Then,*

$$k \in \mathbb{K}_S = \{x \in [\min \mathbb{K}_S, \max \mathbb{K}_S] \wedge x \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\}.$$

Proof. See Definition 2 and Lemma 4, where the symbol $\wedge$ is used to denote the truth-functional operator of logical conjunction AND [4]. □

An important result regarding Equation (4) follows from Lemma 5. Equation (4) generates 2^n solutions for a finite sequence $(s_1, \ldots, s_n, k) \in \mathbb{G}_{n+1}$, where $(s_1, \ldots, s_n) \in \mathbb{G}_n$. From Lemma 5, these solutions are only even or only odd if s_1 is odd or even, respectively. Therefore, the number of distinct solutions generated by (4) is 2^{n-1} since solutions are divided between even and odd.

Theorem 1. *Let $S = (s_1, \ldots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \ldots, s_n, k)$ be finite sequences of integers; then, $k \in \mathbb{K}_S \Leftrightarrow S' \in \mathbb{G}_{n+1}$.*

Proof. Suppose that $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$. Prove the right implication first. From Definition 2, $k \in [\min \mathbb{K}_S, \max \mathbb{K}_S]$ and from Lemma 5, $k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Then, $k \in \mathbb{K}_S \Rightarrow S' \in \mathbb{G}_{n+1}$. Prove the left implication by contradiction. Suppose that $S' \in \mathbb{G}_{n+1}$ but $k \notin \mathbb{K}_S$. Then, $k \in \{x \notin [\min \mathbb{K}_S, \max \mathbb{K}_S] \vee x \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\}$. The symbol $\vee$ is used to denote the truth-functional operator of logical disjunction OR [4]. From Definition 2 and Lemma 5, it is not possible to have $S' \in \mathbb{G}_{n+1}$ if $k > \max \mathbb{K}_S \vee k < \min \mathbb{K}_S \vee k \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Then, it is true that $k \in \mathbb{K}_S \Leftarrow S' \in \mathbb{G}_{n+1}$. □

Corollary 1. *Let $S = (s_1, \ldots, s_n)$ be a finite sequence of integers; then, $S \in \mathbb{G}_n \Leftrightarrow \min \mathbb{K}_{(s_1,\ldots,s_m)} \leq s_{m+1} \leq \max \mathbb{K}_{(s_1,\ldots,s_m)}, \forall m \in [1, n)$.*

Proof. As a consequence of Definition 2 and Equation (4), each m-th element of a sequence S must be within the range between the upper and the lower sequences calculated on all the elements prior to the m-th ones. From Definition 2 and according to the solution of Gilbreath Equation (4), there cannot exist a Gilbreath sequence in which the m-th element is larger than $\max \mathbb{K}_{(s_1,\ldots,s_{m-1})}$, since $\max \mathbb{K}_{(s_1,\ldots,s_{m-1})}$ is the maximum value for the m-th element. The same goes for $\min \mathbb{K}_{(s_1,\ldots,s_{m-1})}$, since it is the smallest value for the m-th element. □

3. Upper and Lower Bound Sequences

Let us now introduce the definition of two notable Gilbreath sequences. Let $S = (s_1, \ldots, s_n) \in \mathbb{G}_n$ be a finite sequence of integers; from (4), any solutions of the Gilbreath equation cannot be greater than $\max \mathbb{K}_S$, so the sequence $(s_1, \ldots, s_n, \max \mathbb{K}_S) \in \mathbb{G}_{n+1}$ is the upper bound sequence for S. Let the new sequence now be $S' = (s_1, \ldots, s_n, \max \mathbb{K}_S)$ and its upper bound sequence be $\left(s_1, \ldots, s_n, \max \mathbb{K}_S, \max \mathbb{K}_{(s_1,\ldots,s_n,\max \mathbb{K}_S)}\right) \in \mathbb{G}_{n+2}$, and so on. Equally, let a finite sequence of integers be $S = (s_1, \ldots, s_n)$; from (4), any value of k cannot be smaller than $\min \mathbb{K}_S$ and the new sequence $S' = (s_1, \ldots, s_n, \min \mathbb{K}_S, k)$ will have a lower limit for $k = \min \mathbb{K}_{(s_1,\ldots,s_n,\min \mathbb{K}_S)}$, and so on. Then, it is possible to introduce the definition of the upper bound Gilbreath sequence and the lower bound Gilbreath sequence.

Definition 3. *Let $S = (s_1, \ldots, s_n) \in \mathbb{G}_n$ be a finite sequence of integers. Let us denote with U_S the upper bound Gilbreath sequence for S and with L_S the lower bound Gilbreath sequence for S:*

$$U_S = (u_1, \ldots) = \left(s_1, \ldots, s_n, \max \mathbb{K}_{(s_1,\ldots,s_n)}, \max \mathbb{K}_{\left(s_1,\ldots,s_n,\max \mathbb{K}_{(s_1,\ldots,s_n)}\right)}, \ldots\right)$$

$$L_S = (l_1, \ldots) = \left(s_1, \ldots, s_n, \min \mathbb{K}_{(s_1,\ldots,s_n)}, \min \mathbb{K}_{\left(s_1,\ldots,s_n,\min \mathbb{K}_{(s_1,\ldots,s_n)}\right)}, \ldots\right)$$

The following recursive definition holds:

$$u_i = \begin{cases} s_i, & \text{if } i \leq n; \\ \max \mathbb{K}_{(u_1,\ldots,u_{i-1})}, & \text{otherwise} \end{cases}$$

and

$$l_i = \begin{cases} s_i, & \text{if } i \leq n; \\ \min \mathbb{K}_{(u_1,\ldots,u_{i-1})}, & \text{otherwise} \end{cases}$$

It is also useful to define a notable sub sequence of U_S and L_S.

Definition 4. *Let $S \in \mathbb{G}_n$ be a finite sequence of integers and its U_S and L_S. Let us define $\tilde{U}_S$ and $\tilde{L}_S$ as follows:*

$$\tilde{U}_S = (\tilde{u}_1, \ldots) = \left(\max \mathbb{K}_S, \max \mathbb{K}_{(S,\max \mathbb{K}_S)}, \cdots\right)$$

$$\tilde{L}_S = (\tilde{l}_1, \ldots) = \left(\min \mathbb{K}_S, \min \mathbb{K}_{(S,\min \mathbb{K}_S)}, \cdots\right)$$

The following recursive definition holds:

$$\tilde{u}_i = \begin{cases} \max \mathbb{K}_S, & \text{if } i = 1; \\ \max \mathbb{K}_{(S,\tilde{u}_1,\ldots,\tilde{u}_{i-1})}, & \text{otherwise} \end{cases}$$

$$\tilde{l}_i = \begin{cases} \min \mathbb{K}_S, & \text{if } i = 1; \\ \min \mathbb{K}_{(S,\tilde{l}_1,\ldots,\tilde{l}_{i-1})}, & \text{otherwise} \end{cases}$$

From Theorem 1, $U_S \in \mathbb{G}$ and $L_S \in \mathbb{G}$, while elements of $\tilde{U}_S$ and $\tilde{L}_S$ are all even or all odd; then, $\tilde{U}_S \notin \mathbb{G}$ and $\tilde{L}_S \notin \mathbb{G}$.

From Definition 3, let $S = (s_1)$; then, $U_S = (s_1, s_1+1, s_1+3, \ldots, s_1+2^n-1)$, $\tilde{U}_S = (s_1+1, s_1+3, \ldots, s_1+2^n-1)$, $L_S = (s_1, s_1-1, s_1-3, \ldots, s_1-2^n+1)$, and $\tilde{L}_S = (s_1-1, s_1-3, \ldots, s_1-2^n+1)$. Table 1 shows some examples of Gilbreath sequences and the closed form for $\tilde{u}_n$.

Table 1. Some examples of Gilbreath sequences and their closed form for $\tilde{u}_n$.

m	$S \in \mathbb{G}_m$	$\tilde{u}_n$
2	$(44, 45)$	$2^{n+1} + 43$
3	$(21, 20, 18)$	$2^{n+2} + 14$
4	$(38, 39, 39, 39)$	$2^{n+3} - n^2 - 5n + 31$
4	$(6, 7, 5, 3)$	$2^{n+3} - n^2 - 3n - 5$
5	$(28, 29, 27, 25, 21)$	$2^{n+4} - n^2 - 5n + 5$
6	$(7, 8, 10, 6, 6, 6)$	$2^{n+5} - 4n^2 - 20n - 26$
6	$(13, 14, 14, 14, 12, 10)$	$2^{n+5} - \frac{n^4}{12} - \frac{5n^3}{6} - \frac{71n^2}{12} - \frac{115n}{6} - 22$
7	$(93, 94, 94, 94, 92, 92, 94)$	$2^{n+6} - \frac{n^4}{6} - \frac{7n^3}{3} - \frac{77n^2}{6} - \frac{122n}{3} + 30$

4. Gilbreath Polynomials

Definition 5. *Let $P = (p_1, \ldots, p_m)$ be the ordered sequence of the first m prime numbers and let $\mathcal{P}_m(n)$ be a function such that $\tilde{u}_n = 2^{m+n-1} + \mathcal{P}_m(n)$, where $\mathcal{P}_m(n) = a_{m,0} + a_{m,1}n + \ldots + a_{m,k}n^k$; then, $\mathcal{P}_m$ is called m-th Gilbreath polynomial.*

Through simple algebra, one can prove that for the ordered sequence of the first m prime numbers, $\mathcal{P}_m(n)$ are represented in Table 2. This provides important information about sequence A347924 [5], which is the triangle read by rows, where row m is the m-th Gilbreath polynomial and column n is the numerator of the coefficient of the k-th degree term. According to Table 2, this sequence contains the integer term of every m-th Gilbreath polynomial. The related sequence A347925 [6] contains the lowest common denominator of m-th Gilbreath polynomial. It is the sequence of denominators of the polynomials in Table 2.

Table 2. Gilbreath polynomials for $m \leq 16$.

m	$\mathcal{P}_m(n)$
1, 2, 3	1
4	$-n^2 - 3n - 1$
5	$-n^2 - 5n - 5$
6	$-\frac{2n^3}{3} - 5n^2 - \frac{55n}{3} - 19$
7	$-\frac{n^4}{6} - \frac{7n^3}{3} - \frac{77n^2}{6} - \frac{116n}{3} - 47$
8	$-\frac{n^5}{30} - \frac{2n^4}{3} - \frac{35n^3}{6} - \frac{85n^2}{3} - \frac{1277n}{15} - 109$
9	$-\frac{n^6}{180} - \frac{3n^5}{20} - \frac{65n^4}{36} - \frac{155n^3}{12} - \frac{5327n^2}{90} - \frac{2579n}{15} - 233$
10	$-\frac{n^7}{1260} - \frac{n^6}{36} - \frac{79n^5}{180} - \frac{151n^4}{36} - \frac{2441n^3}{90} - \frac{1087n^2}{9} - \frac{36481n}{105} - 483$
11	$-\frac{n^9}{181440} - \frac{n^8}{4032} - \frac{169n^7}{30240} - \frac{41n^6}{480} - \frac{8389n^5}{8640} - \frac{1597n^4}{192} - \frac{599441n^3}{11340} - \frac{1202527n^2}{5040} - \frac{177197n}{252} - 993$
12	$-\frac{n^{10}}{1814400} - \frac{13n^9}{362880} - \frac{31n^8}{30240} - \frac{1123n^7}{60480} - \frac{20833n^6}{86400} - \frac{41497n^5}{17280} - \frac{3375899n^4}{181440} - \frac{10094093n^3}{90720} - \frac{12276223n^2}{25200} - \frac{355399n}{252} - 2011$
13	$-\frac{n^{11}}{19958400} - \frac{n^{10}}{259200} - \frac{5n^9}{36288} - \frac{13n^8}{4320} - \frac{27841n^7}{604800} - \frac{46711n^6}{86400} - \frac{46133n^5}{9072} - \frac{991007n^4}{25920} - \frac{50938267n^3}{226800} - \frac{3525203n^2}{3600} - \frac{7851061n}{2772} - 4055$
14	$-\frac{n^{12}}{239500800} - \frac{n^{11}}{2661120} - \frac{49n^{10}}{3110400} - \frac{299n^9}{725760} - \frac{54871n^8}{7257600} - \frac{5093n^7}{48384} - \frac{25465669n^6}{21772800} - \frac{854669n^5}{80640} - \frac{60581657n^4}{777600} - \frac{82179283n^3}{181440} - \frac{102126421n^2}{51975} - \frac{15729347n}{2772} - 8149$
15	$-\frac{n^{13}}{3113510400} - \frac{n^{12}}{29937600} - \frac{389n^{11}}{239500800} - \frac{269n^{10}}{5443200} - \frac{7727n^9}{7257600} - \frac{15781n^8}{907200} - \frac{4919917n^7}{21772800} - \frac{13119557n^6}{5443200} - \frac{58181479n^5}{2721600} - \frac{424785041n^4}{2721600} - \frac{4521951163n^3}{4989600} - \frac{3269429687n^2}{831600} - \frac{292152089n}{25740} - 16337$
16	$-\frac{n^{14}}{43589145600} - \frac{17n^{13}}{6227020800} - \frac{73n^{12}}{479001600} - \frac{2557n^{11}}{479001600} - \frac{5777n^{10}}{43545600} - \frac{4051n^9}{1612800} - \frac{11564263n^8}{304819200} - \frac{20564861n^7}{43545600} - \frac{107393969n^6}{21772800} - \frac{471325651n^5}{10886400} - \frac{18807572041n^4}{59875200} - \frac{2266391933n^3}{1247400} - \frac{595484981809n^2}{75675600} - \frac{818209547n}{36036} - 32715$

Relation between Gilbreath Polynomials and GC

Gilbreath polynomials are closely related to prime numbers and GC. Let a finite sequence of integers $S = (s_1, \ldots, s_n)$, Theorem 1 and the following. The relationship $s_2 = s_1 \pm 1$ must be true; otherwise, it would not be true that $s_1^1 = 1$. As a consequence of Lemma 5, for all elements subsequent to s_1, the absolute difference of two successive elements must be an integer multiple of 2 so as to maintain the absolute difference of two successive elements as an even value. So, if the first element in the sequence is even, the subsequent elements must be odd, and if the first element is odd, the subsequent elements must be even.

Let $P = (p_1, p_2) = (2, 3) \in \mathbb{G}_2$ be a Gilbreath sequence formed by the first two prime numbers. From (5), $\min \mathbb{K}_{(p_1,p_2)} \leq p_2 \leq \max \mathbb{K}_{(p_1,p_2)}$ and from Theorem 1, $(p_1, p_2, p_2) \in \mathbb{G}_3$. By definition of P, $p_n > p_{n-1}$. Since $\min \mathbb{K}_{(p_1,p_2)} \leq p_2$, it is certainly true that $\min \mathbb{K}_{(p_1,p_2)} \leq p_3$. The left inequality is proved for $n = 3$ and it is easy to prove for every n. The proof of $\min \mathbb{K}_{(p_1,\ldots,p_{n-1})} \leq p_n$ is trivial and holds for all prime numbers, hence $p_n \leqslant \max \mathbb{K}_{(p_1,\ldots,p_{n-1})} \Rightarrow GC(n)$. Given Gilbreath polynomials in Definition 5, $\max \mathbb{K}_{(p_1,\ldots,p_{n-1})} = 2^{n-1} + \mathcal{P}_{n-1}(1)$, then

$$p_n - 2^{n-1} \leqslant \mathcal{P}_{n-1}(1) \Rightarrow GC(n) \tag{6}$$

The left side of (6)

$$p_n - 2^{n-1} \leqslant \mathcal{P}_{n-1}(1) \quad (7)$$

consists of a Gilbreath polynomial conjecture whose solution implies GC. Unfortunately, bounds for p_n are not enough good to prove (7); however, this opens the way for a new approach to the GC [7–10].

5. Conclusions and Future Work

Theorem 1, about properties of a Gilbreath sequence, states that if and only if the first element of a finite sequence of integers $(s_1, \ldots, s_n) \in \mathbb{G}_n$ is even or odd, then for any odd or even integer $\max \mathbb{K}_S \leq k \leq \min \mathbb{K}_S$, respectively, the sequence is $(s_1, \ldots, s_n, k) \in \mathbb{G}_{n+1}$.

Theorem 1 proves Equation (6) involving Gilbreath polynomials, and Equation (7) implies GC. The Gilbreath polynomials defined in Definition 5 introduce a new interesting tool for the study of the properties of prime numbers; in particular, we are interested in the matrix of coefficients of Gilbreath polynomials defined as $\mathcal{G} = a_{m,k}$, and a paper on $\mathcal{G}$ will be published in the future.

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