mathematics

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Distribution of Eigenvalues and Upper Bounds of the Spread of Interval Matrices

Wenshi Liao † and **Pujun Long** *,†

College of Mathematics, Physics and Data Science, Chongqing University of Science and Technology, Chongqing 401331, China; 2016011@cqust.edu.cn
* Correspondence: 2017011@cqust.edu.cn
† These authors contributed equally to this work.

Abstract: The distribution of eigenvalues and the upper bounds for the spread of interval matrices are significant in various fields of mathematics and applied sciences, including linear algebra, numerical analysis, control theory, and combinatorial optimization. We present the distribution of eigenvalues within interval matrices and determine upper bounds for their spread using Geršgorin's theorem. Specifically, through an equality for the variance of a discrete random variable, we derive upper bounds for the spread of symmetric interval matrices. Finally, we give three numerical examples to illustrate the effectiveness of our results.

Keywords: interval matrices; distribution; eigenvalues; spread; upper bound

MSC: 15A18; 05B20; 34L15

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Citation: Liao, W.; Long, P. Distribution of Eigenvalues and Upper Bounds of the Spread of Interval Matrices. *Mathematics* **2023**, *11*, 4032. https://doi.org/10.3390/math11194032

Academic Editors: Fei Xue, Qifeng Liao and Yuanzhe Xi

Received: 17 August 2023
Revised: 14 September 2023
Accepted: 20 September 2023
Published: 22 September 2023

1. Introduction

The distribution of eigenvalues of complex matrices is a topic widely researched by some researchers (see Refs. [1,2]). However, in practical problems, matrix elements are often obtained through measurements and computations, which can produce perturbations due to round-off errors and measurement inaccuracies. Consequently, we cannot precisely determine the elements of a matrix. Instead, we can only establish upper and lower bounds for the intervals within which these elements are constrained. As a result, the eigenvalues of real interval matrices become uncertain due to the inherent uncertainties in the matrix elements. This uncertainty complicates the task of locating and estimating the eigenvalues of interval matrices. Recently, a growing number of researchers have begun to study the eigenvalues of interval matrices (see Refs. [3–11]).

Throughout this paper, $tr(A)$ denotes the trace of matrix A; $\lambda_i(A)(i = 1,2,\ldots,n)$ denotes the eigenvalues of the matrix; $Re\lambda_i(A)$ and $Im\lambda_i(A)$ stand for the real part and the imaginary part of $\lambda_i(A)$, respectively; $\|\cdot\|_F$ denotes a Frobenius norm, i.e., for a given $n\times n$ matrix A, $\|A\|_F^2 = \sum\limits_{i,j=1}^{n}|a_{ij}|^2$; and $S_n[a,b]$ denotes the set of $n\times n$ real symmetric matrices whose entries are in the interval $[a,b]$.

Let A_I be a set of real matrices defined by [12]:

$$A_I = \{A = (a_{ij})|a_{ij} \in [p_{ij}, q_{ij}], i,j = 1,2,\ldots,n\}.$$

Let $A \in A_I, P = (p_{ij}), Q = (q_{ij})(i,j = 1,2,\ldots,n)$ be real matrices; then, $P \leqslant A \leqslant Q$.

In 1956, Mirsky conducted the first study on the spread of a matrix in [13], denoted by

$$s(A) = \max_{i,j}|\lambda_i(A) - \lambda_j(A)|, i,j = 1,2,\ldots,n,$$

and then, he obtained some meaningful inequalities of the spread of a matrix for normal and hermitian matrices (see Ref. [14]). The spread of an interval matrix is denoted by

$$s(A_I) = \max_{i,j} |\lambda_i(A_I) - \lambda_j(A_I)|, i, j = 1, 2, \ldots.$$

The real spread and the imaginary spread of an interval matrix are defined by

$$s_R(A_I) = \max_{i,j} |Re\lambda_i(A_I) - Re\lambda_j(A_I)|, i, j = 1, 2, \ldots,$$

$$s_I(A_I) = \max_{i,j} |Im\lambda_i(A_I) - Im\lambda_j(A_I)|, i, j = 1, 2, \ldots.$$

Many researchers have studied the spread of matrices (see Refs. [1,15–17]). However, only a few of them were interested in the spread of interval matrices. In [11], upper bounds of the spread of real symmetric interval matrices were established; it has been shown that if $A \in S_n[-a, a]$ with $n \geqslant 2$ and $a > 0$, then

$$s(A) \leq \begin{cases} \sqrt{2}na & \text{if } n \text{ is even,} \\ \sqrt{2n^2 - 1}a & \text{if } n \text{ is odd.} \end{cases} \quad (1)$$

In this article, we focus on the distribution of the eigenvalues and upper bounds for the spread of interval matrices. In Section 2, we present the distribution for the eigenvalues of interval matrices based on Geršgorin's theorem. In Section 3, we obtain some upper bounds of the spread of interval matrices and establish upper bounds for the spread of real symmetric interval matrices. In Section 4, we provide three numerical examples to illustrate the effectiveness of our results.

2. Distribution of the Eigenvalues of Interval Matrices

In this section, we present a theorem concerning the distribution of eigenvalues of interval matrices. To begin, we introduce the following lemma, which is Geršgorin's theorem.

Lemma 1 ([18], Theorem 6.1.1)**.** *Let $A = (a_{ij})$ be an $n \times n$ complex matrix; every eigenvalue $\lambda_i(A), i = 1, 2, \ldots, n$, must lie in at least one of n closed discs:*

$$G(A) = \left\{ \lambda(A) \in \mathbb{C} \middle| \, |\lambda(A) - a_{ii}| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, i = 1, 2, \ldots, n \right\}.$$

Based on Lemma 1, we obtain the distribution of eigenvalues of interval matrices using Geršgorin's theorem.

Theorem 1. *Let $A_I = \{A = (a_{ij}) | a_{ij} \in [p_{ij}, q_{ij}], i, j = 1, 2, \ldots, n\}$ be an $n \times n$ interval matrix and $\lambda(A_I) = \{\lambda_i(A) | A \in A_I, i = 1, 2, \ldots, n\}$ be the set of eigenvalues of A_I:*

$$G(A_I) = \{\lambda \in \mathbb{C} | \, |\lambda - \min p_{ii}| \leqslant \max R_i\} \cup \{\lambda \in \mathbb{C} | \, |\lambda - \max q_{ii}| \leqslant \max R_i\}$$

$$\cup \{\lambda \in \mathbb{C} | \min p_{ii} \leqslant Re\lambda \leqslant \max q_{ii}, |Im\lambda| \leqslant \max R_i\}$$

where $R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \max\{|p_{ij}|, |q_{ij}|\}, i, j = 1, 2, \ldots, n.$

Then, $\lambda(A_I) \subset G(A_I)$.

Proof. Applying the Geršgorin disc theorem, for any $A = (a_{ij}) \in A_I, i, j = 1, 2, \ldots, n$, every $\lambda_i(A), i = 1, 2, \ldots, n$ must lie in at least one of n closed discs, i.e.,

$$|\lambda - a_{ii}| \leqslant \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leqslant R_i, i = 1, 2, \ldots, n.$$

As $a_{ii}(i = 1, 2, \ldots, n)$ locates on the x axis, so we can obtain n small discs centered at $a_{ii}(p_{ii} \leqslant a_{ii} \leqslant q_{ii})$ and with radius R_i. There is an annulus runway which is symmetric with respect to the x axis containing all the n small discs. We can express the annulus runway as follows:

$$\{\lambda \in \mathbb{C} |\ |\lambda - p_{ii}| \leqslant R_i\} \cup \{\lambda \in \mathbb{C} |\ |\lambda - q_{ii}| \leqslant R_i\}$$
$$\cup \{\lambda \in \mathbb{C} |\ |p_{ii} \leqslant Re\lambda \leqslant q_{ii}, |Im\lambda| \leqslant R_i\}.$$

The significance of the above formula is shown in Figure 1.

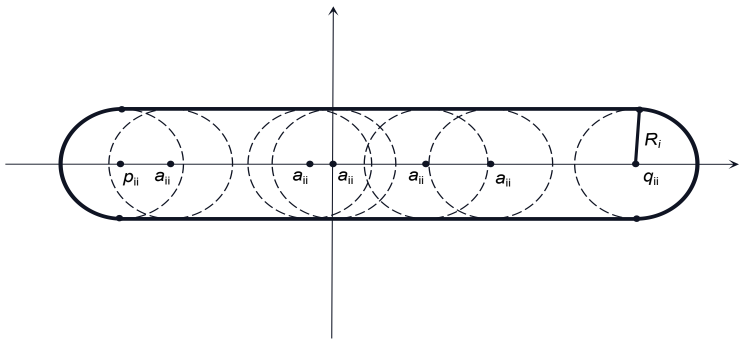

Figure 1. An annulus runway that contains all the small discs.

For an $n \times n$ interval matrix A_I, there are numerous annulus runways like Figure 1, containing all the eigenvalues of interval matrix A_I, so we can use a single big annulus runway containing all the small annulus runways. We can express the big annulus runway as follows:

$$G(A_I) = \{\lambda \in \mathbb{C} |\ |\lambda - \min p_{ii}| \leqslant \max R_i\} \cup \{\lambda \in \mathbb{C} |\ |\lambda - \max q_{ii}| \leqslant \max R_i\}$$
$$\cup \{\lambda \in \mathbb{C} | \min p_{ii} \leqslant Re\lambda \leqslant \max q_{ii}, |Im\lambda| \leqslant \max R_i\}.$$

The significance of the above formula is shown in Figure 2.

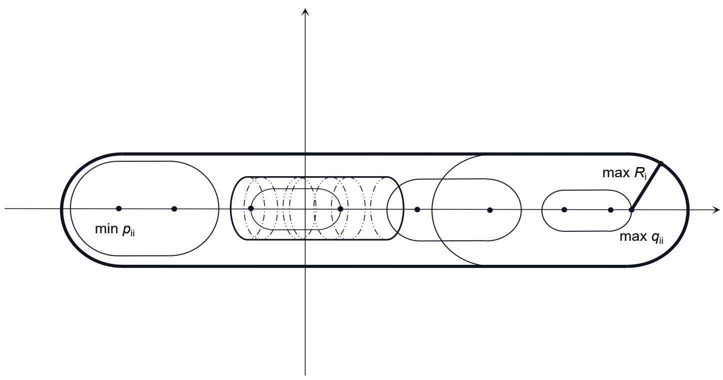

Figure 2. A single big annulus runway that contains the small annulus runways.

Thus, $\lambda(A_I) \subset G(A_I)$, and the proof is completed. □

If all the elements of an interval matrix belong to the same interval, then an application of Theorem 1 can be seen in the following result.

Corollary 1. *Let $A'_I = \{A = (a_{ij}) | a_{ij} \in [a,b], i,j = 1,2,\ldots,n\}$ be an $n \times n$ interval matrix and $\lambda(A'_I) = \{\lambda_i(A) | A \in A'_I, i = 1,2,\ldots,n\}$ be the set of eigenvalues of A'_I:*

$$G(A'_I) = \{\lambda \in \mathbb{C} |\ |\lambda - a| \leqslant \max R'_i\} \cup \{\lambda \in \mathbb{C} |\ |\lambda - b| \leqslant \max R'_i\}$$
$$\cup \{\lambda \in \mathbb{C} |\ |a \leqslant Re\lambda \leqslant b, |Im\lambda| \leqslant \max R'_i\}$$

where $R'_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \max\{|a|,|b|\}$, $\max\{|a|,|b|\}$ is the maximum of $|a_{ij}|(i \neq j), i,j = 1,2,\ldots,n$.

Then, $\lambda(A'_I) \subset G(A'_I)$.

3. The Spread of Interval Matrices

In this section, we give some upper bounds of the spread of general interval matrices and real symmetric interval matrices.

Based on Theorem 1, we have our first result for upper bounds of the spread of general interval matrices.

Theorem 2. *LetA_I be an $n \times n$ interval matrix and the set of eigenvalues of A_I: $\lambda(A_I) = \{\lambda_i(A) | A \in A_I, i = 1,2,\ldots,n\}$. Then,*

$$s(A_I) \leqslant \max q_{ii} - \min p_{ii} + 2\max R_i, i = 1,2,\ldots,n,$$

$$s_R(A_I) \leqslant \max q_{ii} - \min p_{ii} + 2\max R_i, i = 1,2,\ldots,n,$$
$$s_I(A_I) \leqslant 2\max R_i, i = 1,2,\ldots,n.$$

Proof. In Theorem 1, we bind all eigenvalues of a given interval matrix in a single big annulus runway in the complex plane; then, the spread of $s(A_I), s_R(A_I)$ must not exceed the major axis $\max q_{ii} - \min p_{ii} + 2\max R_i$ in the annulus runway, that is

$$s(A_I) \leqslant \max q_{ii} - \min p_{ii} + 2\max R_i, i = 1,2,\ldots,n,$$

$$s_R(A_I) \leqslant \max q_{ii} - \min p_{ii} + 2\max R_i, i = 1,2,\ldots,n,$$

$s_I(A_I)$ must not exceed the minor axis $2\max R_i$ in the annulus runway, that is

$$s_I(A_I) \leqslant 2\max R_i, i = 1,2,\ldots,n.$$

The proof is completed. □

In order to obtain a better upper bound for the spread of interval matrices, we introduce a lemma from reference [13].

Lemma 2. *Let $z_1, z_2, \ldots, z_n$ be any complex numbers, and write*

$$s = \max_{i,j} |z_i - z_j|;$$

then,

$$\frac{1}{2} n s^2 \leqslant \sum_{1 \leqslant i < j \leqslant n} |z_i - z_j|^2, \tag{2}$$

with equality if and only if $z_1, z_2, \ldots, z_n$ satisfy condition φ.

Remark 1. *If n complex numbers $z_1, z_2, \ldots, z_n$ are such that $n-2$ among them are equal to each other and to the arithmetic mean of the remaining two, we shall say that the n numbers satisfy condition φ.*

Based on inequality (2) in Lemma 2, we have another result about the upper bound for the spread of interval matrices.

Theorem 3. *Let A_I be an $n \times n$ interval matrix and $\lambda(A_I) = \{\lambda_i(A) | A \in A_I, i = 1, 2, \ldots, n\}$ be the set of eigenvalues of A_I; then,*

$$s(A_I) \leqslant \left\{ 2 \sum_{i,j=1}^{n} \max\left\{ |p_{ij}|^2, |q_{ij}|^2 \right\} - \frac{2}{n} min \left\{ (\sum_{i=1}^{n} p_{ii})^2, (\sum_{i=1}^{n} q_{ii})^2 \right\} \right\}^{\frac{1}{2}}$$

Proof. According to inequality (2), for any matrix $A \in A_I$, we can obtain

$$\frac{1}{2} n [s(A)]^2 \leqslant \sum_{1 \leqslant i < j \leqslant n} |\lambda_i - \lambda_j|^2$$

for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of matrix A. By Lagrange's identity, it follows that

$$\sum_{1 \leqslant i < j \leqslant n} |\lambda_i - \lambda_j|^2 = n \sum_{i=1}^{n} |\lambda_i|^2 - |\sum_{i=1}^{n} \lambda_i|^2 = n \sum_{i=1}^{n} |\lambda_i|^2 - |trA|^2;$$

thus,

$$\frac{1}{2} n [s(A)]^2 \leqslant n \sum_{i=1}^{n} |\lambda_i|^2 - |trA|^2,$$

and then,

$$s(A) \leqslant (2 \sum_{i=1}^{n} |\lambda_i|^2 - \frac{2}{n} |trA|^2)^{\frac{1}{2}}. \tag{3}$$

By the following inequality in [19],

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \|A\|_F^2;$$

then, we have

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \|A\|_F^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 \leqslant \sum_{i,j=1}^{n} \max\left\{ |p_{ij}|^2, |q_{ij}|^2 \right\}, \tag{4}$$

$$|tr(A)|^2 \geqslant min \left\{ (\sum_{i=1}^{n} p_{ii})^2, (\sum_{i=1}^{n} q_{ii})^2 \right\}, \tag{5}$$

Apply inequalities (4) and (5) to (3), and we can obtain the following conclusion:

$$s(A) \leqslant \left\{ 2 \sum_{i,j=1}^{n} \max\left\{ |p_{ij}|^2, |q_{ij}|^2 \right\} - \frac{2}{n} min \left\{ (\sum_{i=1}^{n} p_{ii})^2, (\sum_{i=1}^{n} q_{ii})^2 \right\} \right\}^{\frac{1}{2}}.$$

The above inequality holds for any matrix $A \in A_I$. The proof is completed. □

Remark 2. *The upper bound of spread of Theorem 3 is more accurate than the result of Theorem 2.*

Next, we will consider a special type of interval matrices whose entries are in the interval $[a, b]$. The following lemma about an equality for the variance of a discrete random variable is necessary.

Lemma 3 ([20], Lemma 1)**.** *Let x_i be discrete random variables, $P(x = x_i) = p_i, i = 1, 2, \ldots, n$, and $b = \max_i\{x_i\}, a = \min_i\{x_i\}, c = \sum_{i=1}^{n} p_i x_i$; then,*

$$Var(x) = (b - c)(c - a) - \sum_{i=1}^{n} p_i(b - x_i)(x_i - a).$$

Based on inequality (3) and Lemma 3, we have the following theorem.

Theorem 4. *Let $A = (a_{ij}) \in S_n[a, b]$ and $n \geqslant 2, a < b$; then,*

$$s(A) \leq \begin{cases} \sqrt{2(n^2 - n)\max\{a^2, b^2\} + \frac{n(b-a)^2}{2}} & \text{if } n \text{ is even}, \\ \sqrt{2(n^2 - n)\max\{a^2, b^2\} + \frac{(n^2-1)(b-a)^2}{2n}} & \text{if } n \text{ is odd}. \end{cases}$$

Proof. By inequality (3) and $\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \|A\|_F^2$, we have

$$s(A) \leqslant \left[2\|A\|_F^2 - \frac{2}{n}(trA)^2\right]^{\frac{1}{2}},$$

Since the elements of A locate in the interval $[a, b]$, $s(A)$ cannot attain the maximum until $\left[2\|A\|_F^2 - \frac{2}{n}(trA)^2\right]^{\frac{1}{2}}$ attains the maximum. Without loss of generality, we have the following inequality:

$$\begin{aligned} s(A) \leqslant \left[2\|A\|_F^2 - \frac{2}{n}(trA)^2\right]^{\frac{1}{2}} &= \sqrt{2}\left[\sum_{i,j=1}^{n}(a_{ij})^2 - \frac{(\sum_{i=1}^{n} a_{ii})^2}{n}\right]^{\frac{1}{2}} \\ &= \sqrt{2}\left[\sum_{\substack{i,j=1 \\ j\neq i}}^{n}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2 - \frac{(\sum_{i=1}^{n} a_{ii})^2}{n}\right]^{\frac{1}{2}}. \end{aligned}$$

So taking $\max\{|a|, |b|\}$ as the maximum of $|a_{ij}|(i \neq j)$, we obtain

$$\max \sum_{\substack{i,j=1 \\ j\neq i}}^{n}(a_{ij})^2 = (n^2 - n)\max\left\{a^2, b^2\right\}. \tag{6}$$

By the variance formula, then we have

$$\frac{1}{n}\left[\sum_{i=1}^{n}(a_{ii})^2 - \frac{(\sum_{i=1}^{n} a_{ii})^2}{n}\right] = \frac{1}{n}\sum_{i=1}^{n}\left[a_{ii} - \frac{\sum_{i=1}^{n} a_{ii}}{n}\right]^2 = Var(a_{ii}), \tag{7}$$

Combining (7) with Lemma 3, then

$$Var(a_{ii}) = (b - \frac{\sum_{i=1}^{n} a_{ii}}{n})(\frac{\sum_{i=1}^{n} a_{ii}}{n} - a) - \sum_{i=1}^{n} \frac{1}{n}(b - a_{ii})(a_{ii} - a),$$

$Var(a_{ii})$ cannot achieve the maximum until $(b - \sum_{i=1}^{n} a_{ii}/n)(\sum_{i=1}^{n} a_{ii}/n - a)$ attains the maximum and $\sum_{i=1}^{n}(b - a_{ii})(a_{ii} - a)/n$ attains the minimum simultaneously.

If n is even, let $(b - a_{ii})(a_{ii} - a) = 0((b - a_{ii})(a_{ii} - a) \geqslant 0)$, that is $a_{ii} = b$ or $a_{ii} = a, i = 1, 2, \ldots, n$. Now, we can consider $(b - \sum_{i=1}^{n} a_{ii}/n)(\sum_{i=1}^{n} a_{ii}/n - a)$ as a function $f(\sum_{i=1}^{n} a_{ii}/n)$, and $Var(a_{ii})$ can achieve the maximum as $\sum_{i=1}^{n} a_{ii}/n = (a+b)/2$ and $a_{ii} = b$ or $a_{ii} = a$. Let $1 < m < n, a_{11} = a, a_{22} = a, \ldots, a_{mm} = a, a_{m+1,m+1} = b, \ldots, a_{nn} = b$; then, $(ma + (n - m)b)/n = (a+b)/2$, solving the equation $m = \frac{n}{2}$, so we conclude that

$$\max Var(a_{ii}) = \frac{(b-a)^2}{4}.$$

Considering (7), we have

$$\sum_{i=1}^{n} a_{ii}^2 - \frac{(\sum_{i=1}^{n} a_{ii})^2}{n} \leqslant \frac{n(b-a)^2}{4}, \tag{8}$$

Combining (6) with (8), we have

$$s(A) \leqslant \sqrt{2(n^2 - n)\max\{a^2, b^2\} + \frac{n(b-a)^2}{2}}.$$

If n is odd, let $(b - a_{ii})(a_{ii} - a) = 0$, that is $a_{ii} = b$ or $a_{ii} = a(i = 1, 2, \ldots, n)$. Similar to n being even, $m = (n-1)/2$, so we conclude that $(b - \sum_{i=1}^{n} a_{ii}/n)(\sum_{i=1}^{n} a_{ii}/n - a) \leqslant \frac{(n^2-1)(b-a)^2}{4n^2}$, and then,

$$Var(a_{ii}) \leqslant \frac{(n^2-1)(b-a)^2}{4n^2},$$

$$\sum_{i=1}^{n} a_{ii}^2 - \frac{(\sum_{i=1}^{n} a_{ii})^2}{n} \leqslant \frac{(n^2-1)(b-a)^2}{4n}, \tag{9}$$

Combining (6) with (9), we have

$$s(A) \leqslant \sqrt{2(n^2 - n)\max\{a^2, b^2\} + \frac{(n^2-1)(b-a)^2}{2n}}.$$

The proof is completed. □

Corollary 2. *Let $A = (a_{ij}) \in S_n[-a, a]$ and $n \geqslant 2, a > 0$; then,*

$$s(A) \leq \begin{cases} \sqrt{2}na & \text{if } n \text{ is even,} \\ \sqrt{\frac{2n^3-2}{n}}a & \text{if } n \text{ is odd.} \end{cases}$$

Remark 3. *If n is even, the conclusion in Corollary 2 is the same as inequality (1), but we have provided a more concise proof.*

4. Numerical Example

In this section, we will give several examples to illustrate the effectiveness of our results.

Example 1.

$$A_I = \begin{bmatrix} [-1,2] & [0.5,1] & [-2,1] \\ [-3,-1] & [-1,1] & [0,1.5] \\ [-2.5,-0.5] & [-0.5,1] & [1,2] \end{bmatrix}.$$

In interval matrix A_I, we have $\min p_{ii} = -1, \max q_{ii} = 2, \max R_i = 4.5$.
Choose a matrix $A \in A_I$,

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -3 & -1 & 1.5 \\ -2.5 & 1 & 2 \end{bmatrix}.$$

The eigenvalues of A are

$$\lambda_1(A) = 4.1910, \lambda_2(A) = -0.5955 + 1.0662i, \lambda_3(A) = -0.5955 - 1.0662i, (i^2 = -1),$$

so we obtain

$$s(A) = 4.9029.$$

From Theorem 1, we can obtain the following region:

$$G(A_I) = \{\lambda \in \mathbb{C}|\ |\lambda - (-1)| \leqslant 4.5\} \cup \{\lambda \in \mathbb{C}|\ |\lambda - 2| \leqslant 4.5\}$$

$$\cup\{\lambda \in \mathbb{C}| -1 \leqslant Re\lambda \leqslant 2, |Im\lambda| \leqslant 4.5\}.$$

Clearly, we can obtain $\lambda_i(A) \in G(A_I), i = 1, 2, 3$.
From Theorem 2, we have

$$s(A_I) \leqslant 9, s_R(A_I) \leqslant 9, s_I(A_I) \leqslant 6.$$

From Theorem 3, we have

$$s(A_I) \leqslant 7.8952,$$

which provides a more precise estimation for the spread of interval matrices than Theorem 2.

Example 2. *Choose a matrix $B \in S_3(1,4)$,*

$$B = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 1 & 4 \\ 1 & 4 & 4 \end{bmatrix}.$$

The eigenvalues of B are $\lambda_1(B) = -3.7163, \lambda_2(B) = 1.4680, \lambda_3(B) = 8.2483$; then, we have

$$s(B) = 11.9646.$$

From Corollary 1, we obtain the following region:

$$G(S_3[1,4]) = \{\lambda \in \mathbb{C}|\ |\lambda - 1| \leqslant 8\} \cup \{\lambda \in \mathbb{C}|\ |\lambda - 4| \leqslant 8\}$$

$$\cup\{\lambda \in \mathbb{C}|1 \leqslant Re\lambda \leqslant 4, |Im\lambda| \leqslant 8\}$$

Clearly, we can obtain $\lambda_i(B) \in G(S_3[1,4]), i = 1, 2, 3$.
From Theorem 2, we have

$$s(S_3[1,4]) \leqslant 19, s_R(S_3[1,4]) \leqslant 19, s_I(S_3[1,4]) \leqslant 16.$$

From Theorem 3, we have

$$s(S_3[1,4]) \leqslant 16.7928.$$

From Theorem 4, we have

$$s(S_3[1,4]) \leqslant 14.2828,$$

and the upper bounds of the spread is more precise than Theorems 2 and 3.

Example 3. *Choose a matrix $C \in S_3[-2,2]$ and $D \in S_4[-2,2]$,*

$$C = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & 2 & 2 & 2 \\ 2 & 2 & -2 & -2 \\ 2 & -2 & -2 & 2 \\ 2 & -2 & 2 & 2 \end{bmatrix}.$$

The eigenvalues of C are $\lambda_1(C) = -4, \lambda_2(C) = 2, \lambda_3(C) = 4$; then, we have

$$s(C) = 8,$$

and the eigenvalues of D are

$$\lambda_1(D) = -5.2263, \lambda_2(D) = -2.1648, \lambda_3(D) = 2.1648, \lambda_4(D) = 5.2263$$

Then,

$$s(D) = 10.4526.$$

From Corollary 2, we have

$$s(S_3[-2,2]) \leqslant 8.3266, s(S_4[-2,2]) \leqslant 11.3137,$$

5. Conclusions

We present the distribution of eigenvalues of interval matrices and establish upper bounds for their spread. Theorem 1 provide the distribution of eigenvalues of interval matrices. Theorems 2 and 3 both offer upper bounds for the spread of general interval matrices. Notably, the upper bound provided by Theorem 4 exhibits higher accuracy compared with that of Theorems 2 and 3. Theorem 4 introduces upper bounds for the spread of symmetric interval matrices, and we obtain the same inequality as (1) when n is even based on a simple proof.

Author Contributions: This work was carried out in collaboration between the authors. P.L. designed the study and guided the research. W.L. performed the analysis and wrote the first draft of the manuscript. P.L. and W.L. managed the analysis of the study. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Science and Technology Research Program of Chongqing Municipal Education Commission (grant No. KJQN201901546 and No. KJQN201901518) and the Science and Technology Research Program of Chongqing Municipal Education Commission (grant No. KJQN202101536).

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Data Availability Statement: Data sharing not applicable.

Acknowledgments: All authors are thankful to the honorable reviewers for their valuable suggestions and comments, which improved the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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