

Article

Optimal Control Strategy for SLBRS with Two Control Inputs

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Abstract: Computer virus attacks result in significant losses each year, drawing considerable attention from enterprises, governments, academic institutions, and various other sectors. Researchers have proposed various approaches to fight against computer viruses, including antivirus software and internet firewalls. In this paper, we focus on investigating computer virus transmission from the perspective of mathematical modeling. Our main contributions in this paper are threefold: (1) we improve the classical SLBRS model by incorporating cure rates, effectively capturing the dynamics of computer network maintenance; (2) we introduce an optimal control system within the SLBRS framework, with the dual objectives of minimizing network detoxification costs and reducing the proportion of broken-out nodes; and (3) by employing Pontryagin's Maximum Principle, we establish the existence and uniqueness of an optimal control strategy for the proposed control system. Furthermore, we perform numerical simulations to demonstrate the effectiveness of our theoretical analyses.

Keywords: computer virus; SLBRS; optimal control; Pontryagin's Maximum Principle; numerical simulations

MSC: 34D12; 49-73; 93E20



Citation: Zhao, X. Optimal Control Strategy for SLBRS with Two Control Inputs. *Mathematics* **2023**, *11*, 4036. <https://doi.org/10.3390/math11194036>

Academic Editors: Rongwei Guo, Cuimei Jiang and Ruimin Xu

Received: 8 August 2023

Revised: 15 September 2023

Accepted: 21 September 2023

Published: 22 September 2023



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1. Introduction

In the era of big data, information exchanges have escalated in our daily lives, thereby amplifying the potential for computer virus propagation within computer networks. Computer virus threats result in substantial losses to enterprises, governments, academic fields, and various sectors worldwide every year (For instance, notable cases like the Stuxnet virus in 2010 and the Ransomware virus in 2017 inflicted tens of billions of dollars within the finance, education, energy, and other sectors). A diverse range of measures must be undertaken to combat computer viruses, including the implementation of firewalls and antivirus software. Nevertheless, delving into the propagation mechanisms of computer viruses is of heightened urgency compared to merely eradicating them from specific networks. Over the past few decades, various ordinary differential equations have been proposed to model the propagation mechanisms of computer viruses. Usually, computers within a network are categorized into distinct compartments: susceptible nodes— $S(t)$, infected nodes— $I(t)$, quarantine nodes— $Q(t)$, recovery nodes— $R(t)$, broken-out nodes— $B(t)$, latent nodes— $L(t)$, etc. In accordance with specific scenarios, certain interdependencies among these compartments are established, leading to the nomenclature of the model based on the compartments involved, i.e., SI, SEIR, SEIRQ, and so on.

In 1991, Kephart, White et al. introduced the SI model, which stands as the pioneering work in the field of applying differential equations to model computer virus transmission [1];

In 2007, Mishra, Saini et al. proposed the SEIR model [2];

In 2010, Mishra, Jha introduced the SEIQRS model [3];

In 2012, Yang, Wen et al. proposed the SLBS model [4].

These early computer virus models are inspired extensively by the results concerning epidemiological virus models, which share a common assumption that an infected

computer in a latent state will not transmit a virus to other computers. However, this assumption does not always hold true for computer virus transmission. In fact, when it comes to computer viruses, the following phenomenon occur:

- (a) Once a computer is infected, it immediately gains the ability to spread the infection;
- (b) Recovered computers can acquire temporary immunity.

In 2012, Yang, Zhang, Li et al. modified the classical SLBS model by incorporating a recovery compartment (R(t)) and used the SLBRS to model computer virus transmission, taking into account the observations in (a) and (b) (see Reference [5]). The detailed relationships among the involved compartments are illustrated in Figure 1.

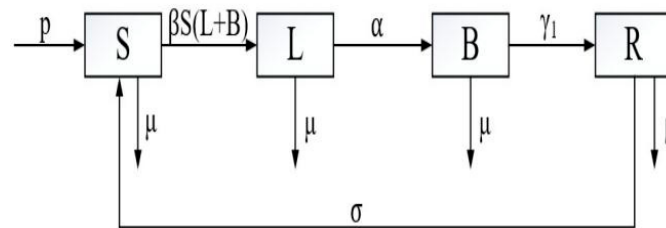


Figure 1. SLBRS model.

When faced with a computer virus attack, computer users often adopt different protective measures to fight against it(see Reference [6]). The following are some examples:

- (c) Some users promptly execute antivirus software to eliminate the virus once they become aware of the threat. If the damage is extensive and recovery is not feasible, users may resort to reinstalling the operating system. This scenario often occurs in broken-out nodes.
- (d) Other users might attempt to clear the virus proactively, even if they are not certain whether the virus is present on their computers. This behavior is typically observed in latent nodes.

Take (c) and (d) into account, people usually make the assumption that breaking-out computers can be cured with specific cure rates γ_1 and γ_3 (either through executing antivirus software or by reinstalling the system), and similarly, latent computers can be cured with a certain cure rate γ_2 (by using antivirus softwares). The graded cure rates $\gamma_1, \gamma_2,$ and γ_3 are integrated into the SLBRS model, resulting in a SLBRS model with cure rates.

Figure 2 can be represented by a system of differential equations, as follows:

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(L + B) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(L + B) - \alpha L - \gamma_2 L - \mu L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - \gamma_1 B, \\ \frac{dR(t)}{dt} = \gamma_2 L - \sigma R - \mu R + \gamma_1 B, \end{cases} \tag{1}$$

where $\gamma_1, \gamma_2,$ and γ_3 represent the cure rates (usually $\gamma_1 > \gamma_2 > \gamma_3$); $\alpha, \beta, \sigma, p,$ and μ represent the conversion rates (non-negative constants); and $S(t), L(t), B(t),$ and $R(t)$ represent the compartments in the network: susceptible nodes, latent nodes, broken-out nodes, recovery nodes.

The intensity of computer virus attacks fluctuates over time, and the countermeasures adopted by computer users also vary (with computer users bolstering protection during hacking threats and vice versa). Hence, a more realistic approach is to hypothesize that the included cure rates are time-varying. We replace the cure rate γ_1 by a Lebesgue measurable function $u_1(t)$.

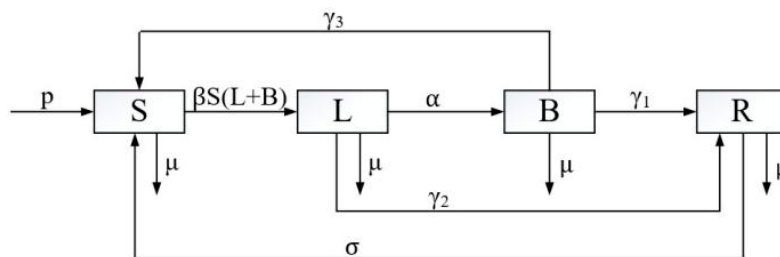


Figure 2. SLBRS with constant cure rates.

Figure 3 can be represented by the following system of differential equations:

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(L + B) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(L + B) - \alpha L - \gamma_2 L - \mu L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - u_1(t)B, \\ \frac{dR(t)}{dt} = \gamma_2 L - \sigma R - \mu R + u_1(t)B. \end{cases} \tag{2}$$

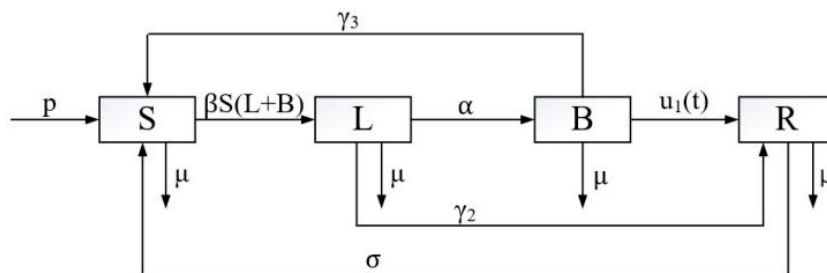


Figure 3. SLBRS with time-varying cure rate $u_1(t)$.

Naturally, the inclusion of more control inputs gains a heightened level of manipulation over the system. We replace the cure rates γ_1 and γ_2 with Lebesgue measurable functions $u_1(t)$ and $u_2(t)$ (the other cases are similar and, thus, we omit the details in this paper).

Figure 4 can be represented by the following system of differential equations:

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(L + B) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(L + B) - \alpha L - \mu L - u_2(t)L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - u_1(t)B, \\ \frac{dR(t)}{dt} = -\sigma R - \mu R + u_1(t)B + u_2(t)L, \end{cases} \tag{3}$$

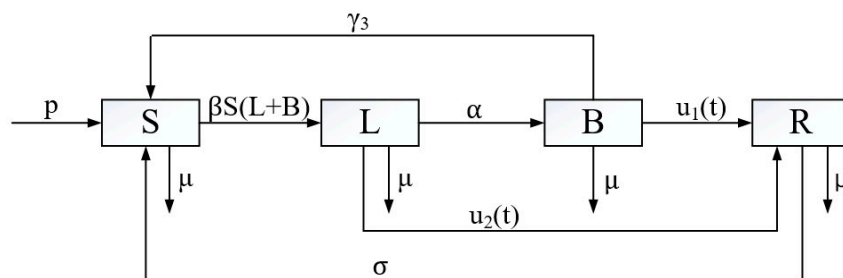


Figure 4. SLBRS with time-varying cure rates $u_1(t)$ and $u_2(t)$.

Research on computer virus models has primarily focused on modeling, stability analysis, and related simulations. As research progresses, various mathematical tools have been applied to the analysis of computer virus models, particularly control theory (see [7–11]). In this paper, we treat the time-varying cure rates $u_1(t)$ and $u_2(t)$ as control input functions and investigate the optimal control problem for SLBRS. The rest of this paper is outlined as below:

In Section 2, we establish the theoretical foundations encompassing the existence and uniqueness of the optimal control strategy, and the necessary conditions for optimal control.

In Section 3, we provide a numerical simulation to demonstrate the effectiveness of the theoretical analyses presented in Section 2.

In Section 4, we draw the conclusion according to the theoretical analyses and the numerical experiment.

2. Optimal Control Strategy for SLBRS

2.1. Optimal Control Problem

Let T be a pre-assigned constant. We define the admissible control set as follows:

$$U = \{u = (u_1, u_2) : 0 \leq u_i(t) \leq 1, t \in [0, T], \text{ for } i = 1, 2\}, \tag{4}$$

To effectively restore a contaminated network, any undertaken measure should aim to minimize the proportion of infected computers ($L(t)$ and $B(t)$) or reduce the cost of system maintenance. With this objective, we present the optimal control problem as follows:

$$\min J(u) = \int_0^T \left[L(t) + B(t) + \frac{\epsilon u_1^2(t)}{2} + \frac{\tau u_2^2(t)}{2} \right] dt, u \in U, \tag{5}$$

which is subject to (3).

Corresponding to (5), we define the Lagrangian as follows:

$$L(L, B, u) = L(t) + B(t) + \frac{\epsilon u_1^2(t)}{2} + \frac{\tau u_2^2(t)}{2}, u = (u_1, u_2)$$

And we define the Hamiltonian as follows:

$$\begin{aligned} H(L, B, u) = & L(L, B, u) + \lambda_1 [p - \beta S(L + B) + \gamma_3 B + \sigma R - \mu S] + \lambda_2 [\beta S(L + B) \\ & - \alpha L - \mu L - u_2(t)L] + \lambda_3 [\alpha L - \gamma_3 B - \mu B - u_1(t)B] \\ & + \lambda_4 [-\sigma R - \mu R + u_1(t)B + u_2(t)L]. \end{aligned}$$

2.2. Main Results and Their Proofs

Firstly, we demonstrate the existence of an optimal solution for the control system (3)–(5) based on the following theorem:

Theorem 1. *There exists an optimal control input $u^* = (u_1^*, u_2^*)$ such that*

$$\min_{u \in U} J(u_1, u_2) = J(u_1^*, u_2^*) \tag{6}$$

subject to the control system (3)–(5) with initial conditions:

$$S(0) = S^0 \geq 0, L(0) = L^0 \geq 0, B(0) = B^0 \geq 0, R(0) = R^0 \geq 0. \tag{7}$$

Proof. To establish the existence of an optimal solution for the control system (3)–(5), it suffices to validate the following general conditions (see [12]):

- (a) The set of control and state variables is nonempty.
- (b) The admissible control set U is both closed and convex.

- (c) The right-hand side of the state system can be bounded by a linear function of the state variables.
- (d) The Lagrangian is concave on the admissible control set U , and there exist constants $k > 1, c_1 > 0$, and c_2 such that

$$L(L, B, u) \geq c_1 + c_2 (|u_1|_2 + |u_2|_2)^{k/2}, \quad u = (u_1, u_2),$$

□

Next, by using Pontryagin’s Maximum Principle, we deduce a necessary condition for the optimal control strategy based on the following theorem:

Theorem 2. *Given an optimal control input $u^*(t) = [u_1^*(t), u_2^*(t)]$ and the corresponding state trajectory S^*, L^*, B^*, R^* of (3), there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that*

$$\frac{d\lambda_1}{dt} = \lambda_1[\beta(L + B) + \mu] - \lambda_2\beta(L + B), \tag{8}$$

$$\frac{d\lambda_2}{dt} = -1 + \lambda_1\beta S - \lambda_2[\beta S - \alpha - \mu - u_2(t)] - \lambda_3\alpha - \lambda_4u_2(t), \tag{9}$$

$$\frac{d\lambda_3}{dt} = -1 + \lambda_1(\beta S - \gamma_3) - \lambda_2\beta S + \lambda_3[\gamma_3 + \mu + u_1(t)] - \lambda_4u_1(t), \tag{10}$$

$$\frac{d\lambda_4}{dt} = -\lambda_1\sigma + \lambda_4(\sigma + \mu), \tag{11}$$

with the transversal conditions

$$\lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 0, \lambda_4(T) = 0. \tag{12}$$

Furthermore, the optimal control inputs were determined by

$$\begin{aligned} u_1^*(t) &= \max\left\{\min\left\{\frac{\lambda_3 - \lambda_4}{\varepsilon} B^*, 1\right\}, 0\right\}, \\ u_2^*(t) &= \max\left\{\min\left\{\frac{\lambda_2 - \lambda_4}{\tau} L^*, 1\right\}, 0\right\}. \end{aligned} \tag{13}$$

Proof. By differentiating the Hamiltonian, we obtain the following:

$$\frac{d\lambda_1}{dt} = -H_{S^*}(t), \quad \frac{d\lambda_2}{dt} = -H_{L^*}(t), \quad \frac{d\lambda_3}{dt} = -H_{B^*}(t), \quad \frac{d\lambda_4}{dt} = -H_{R^*}(t). \tag{14}$$

Consequently, we can reformulate (14) into the co-state Equations (8)–(11). By deducing from the optimal conditions, we derive the control equations as follows:

$$\begin{aligned} \frac{\partial H}{\partial u_1} \Big|_{u_1(t)=u_1^*(t)} &= \varepsilon u_1(t) - \lambda_3 B + \lambda_4 B, \\ \frac{\partial H}{\partial u_2} \Big|_{u_2(t)=u_2^*(t)} &= \tau u_2(t) - \lambda_2 L + \lambda_4 L. \end{aligned} \tag{15}$$

It follows from (15) and the admissible condition

$$U = \{u = (u_1, u_2) : 0 \leq u_i(t) \leq 1, t \in [0, T], \text{ for } i = 1, 2\} \tag{16}$$

that

$$\begin{aligned} u_1^*(t) &= \max\left\{\min\left\{\frac{\lambda_3 - \lambda_4}{\varepsilon} B^*, 1\right\}, 0\right\}, \\ u_2^*(t) &= \max\left\{\min\left\{\frac{\lambda_2 - \lambda_4}{\tau} L^*, 1\right\}, 0\right\}. \end{aligned} \tag{17}$$

Hence, we have successfully derived the state equations:

$$\begin{cases} \frac{dS(t)}{dt} = p - \beta S(L + B) + \gamma_3 B + \sigma R - \mu S, \\ \frac{dL(t)}{dt} = \beta S(L + B) - \alpha L - \mu L - u_2(t)L, \\ \frac{dB(t)}{dt} = \alpha L - \gamma_3 B - \mu B - u_1(t)B, \\ \frac{dR(t)}{dt} = -\sigma R - \mu R + u_1(t)B + u_2(t)L \end{cases} \tag{18}$$

and the co-state equations:

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_1[\beta(L + B) + \mu] - \lambda_2\beta(L + B), \\ \frac{d\lambda_2}{dt} = -1 + \lambda_1\beta S - \lambda_2[\beta S - \alpha - \mu - u_2(t)] - \lambda_3\alpha - \lambda_4u_2(t), \\ \frac{d\lambda_3}{dt} = -1 + \lambda_1(\beta S - \gamma_3) - \lambda_2\beta S + \lambda_3[\gamma_3 + \mu + u_1(t)] - \lambda_4u_1(t), \\ \frac{d\lambda_4}{dt} = -\lambda_1\sigma + \lambda_4(\sigma + \mu), \end{cases} \tag{19}$$

with the initial condition

$$S(0) = S^0 \geq 0, L(0) = L^0 \geq 0, B(0) = B^0 \geq 0, R(0) = R^0 \geq 0, \tag{20}$$

and the transversal condition

$$\lambda_i(T) = 0, i = 1, 2, 3, 4. \tag{21}$$

□

3. Numerical Example

In this section, we conduct numerical simulations to illustrate the influence of optimal control on the SLBRS model.

3.1. Algorithm

We employ an iterative algorithm based on the fourth-order Runge–Kutta method, which involves four distinct steps:

- Step 1. Set the initial states $S(t_0), L(t_0), B(t_0), R(t_0)$ and control $u_0(t_0) = [u_1(t_0), u_2(t_0)]$;
- Step 2. Resolve the state Equation (18) using the fourth-order forward Runge–Kutta method;
- Step 3. Resolve the co-state Equation (19) using the fourth-order backward Runge–Kutta method;
- Step 4. Calculate the optimal control input $u^*(t) = [u_1^*(t), u_2^*(t)]$ by using Equation (17).

3.2. Simulation

We choose the following initial values:

$$S(0) = 0.4, L(0) = 0.3, B(0) = 0.2, R(0) = 0.1. \tag{22}$$

To ensure the stability of the system, we use the parameters presented in Table 1.

Table 1. Values of parameters in the numerical simulation.

Parameters	Value	Parameters	Value
p	0.10	β	0.90
α	0.60	γ ₁	0.15
γ ₂	0.10	γ ₃	0.05
σ	0.05	μ	0.10
ε	2	τ	2

In the visual representation, the red curves denote the states of SLBRS with constant cure rates, while the blue curves denote the states of SLBRS with control inputs. The results of the numerical experiment reveal the following:

- (1) The asymptotic stability of SLBRS is evident as both the red and blue curves converge towards a specific equilibrium.
- (2) When controls are applied, the restoration of the contaminated network becomes evident: susceptible and recovered computers show an increased trend (Figures 5 and 6), whereas latent and breaking-out computers exhibit a decreased trend (Figures 7 and 8).

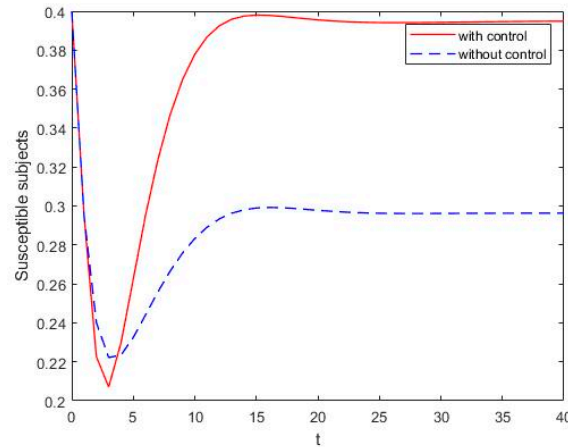


Figure 5. Evolution of susceptible computers.

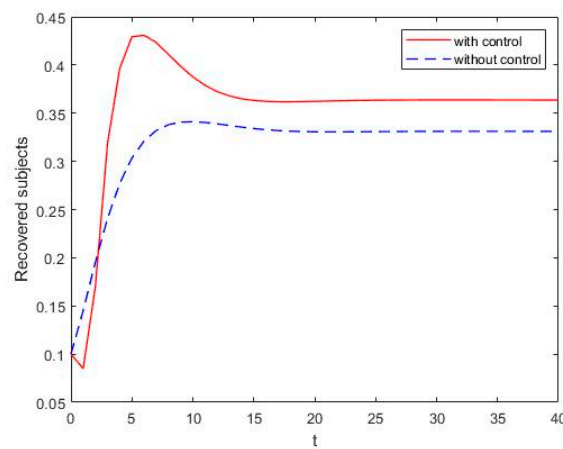


Figure 6. Evolution of recovered computers.

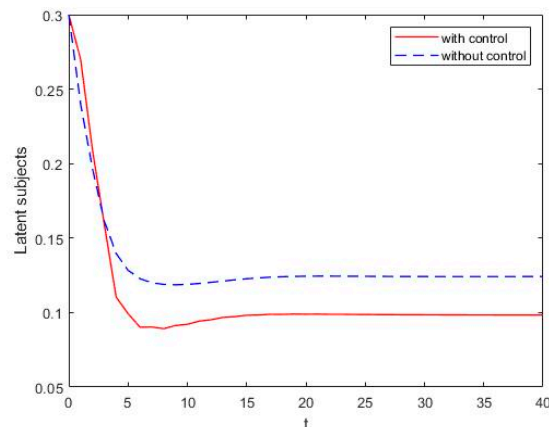


Figure 7. Evolution of latent computers.

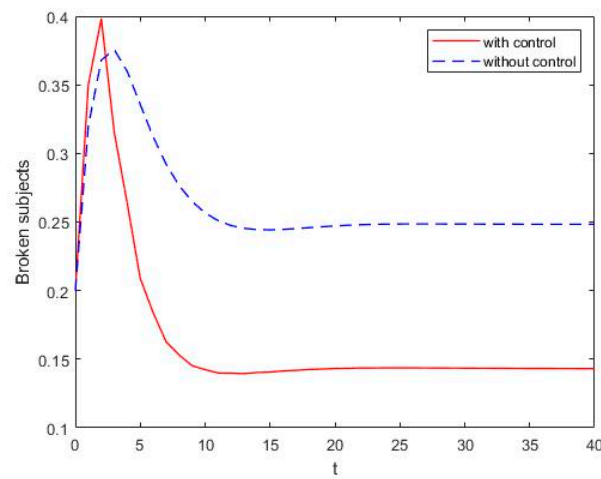


Figure 8. Evolution of breaking-out computers.

The optimal control strategies $u_1^*(t)$ and $u_2^*(t)$ vary with time: initially, when a significant number of computer viruses exist within the network, substantial efforts are dedicated to virus eradication. As the virus count diminishes, the intensity of these efforts gradually decreases (Figures 9 and 10).

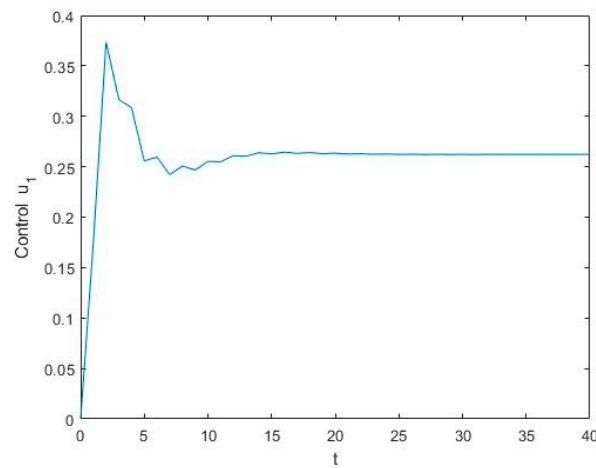


Figure 9. Optimal control variable $u_1^*(t)$.

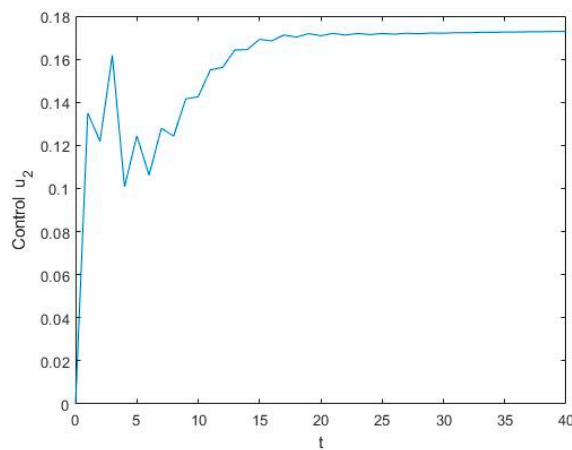


Figure 10. Optimal control variable $u_2^*(t)$.

4. Conclusions

In this paper, we investigate the SLBRS model from the perspectives of mathematical modeling and optimal control theory. The main contributions of this study are as follows:

- (1) We improve the SLBRS model by incorporating time-varying cure rates, thereby effectively capturing the dynamics of computer networks.
- (2) We introduce an optimal control system within the SLBRS framework, with the dual objectives of minimizing network detoxification costs and reducing the number of infected computers.
- (3) By employing Pontryagin's Maximum Principle, we establish the existence and uniqueness of an optimal control strategy for the proposed system.
- (4) We provide numerical demonstrations to highlight the practical effectiveness of our theoretical analyses.

We point out that the control strategies significantly contribute to the restoration of a contaminated network. Taking the same initial condition (22) and using the same parameters (Table 1), we compute the equilibrium points of the aforementioned control systems (1–3). The results are outlined below (Table 2).

Table 2. Comparison of stable values of the state variables in each model.

Model	S*	L*	B	R*
SLBRS with constant cure rates: System (1)	0.30	0.12	0.25	0.33
SLBRS with 1 control input: System (2)	0.36	0.11	0.17	0.36
SLBRS with 2 control inputs: System (3)	0.39	0.10	0.14	0.37

We can conclude, based on the results in Table 2, that the nontoxic compartments S^* and R^* increase while the toxic compartments L^* and B^* decrease. This implies that our proposed measures are indeed effective in counteracting computer virus attacks.

The more control variables there are, the easier it becomes to achieve the desired control effects. Therefore, it is more challenging to study the optimal control problem associated with SLBRS with fewer control inputs. From the technical perspective, the analysis methods for the scenario with three control inputs are nearly identical to those for the scenario with two control inputs. Therefore, our research methods in this paper can be directly applied to solve the optimal control problem for SLBRS with three control inputs.

Funding: This author is supported partially by the Jiangsu Qin-Lan Project of Fostering Excellent Teaching Team, "University Mathematics Teaching Team".

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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