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Asymptotic Behavior of a Nonparametric Estimator of the Renewal Function for Random Fields

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Abstract: In this paper, we study the asymptotic normality of a nonparametric estimator of the renewal function associated with a sequence of absolutely continuous nonnegative two-dimensional random fields. We prove that this estimator is asymptotically unbiased. The asymptotic normality of this estimator is established.

Keywords: random fields; renewal function; renewal process; asymptotic normality

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1. Introduction

An arrival process where the interarrival intervals $X_1, X_2, \ldots$ are independent and identically distributed $(i.i.d.)$, nonnegative random variables is called a renewal process because it probabilistically starts over at each arrival time; that is, the nth arrival occurs at $S_n = \sum_{i=1}^{n} X_i$, and the jth subsequent arrival occurs at $S_{n+j} - S_n = \sum_{i=1}^{j} X_{n+i}$.

Denote for all $t > 0$, $N(t) = \sup\{n \geq 0, S_n \leq t\}$ the renewal counting process with interarrival time $\{X_n{:}n \geq 1\}$ and define its renewal function by $H(t) = E\big(N(t)\big)$.

Set $F(t) = P(X \leq t)$ and $F^{*k}(t) = P(S_k \leq t)$ for $k \geq 1$. It is well known that the renewal function is given for all $t > 0$ by

$$H(t) = \sum_{k=1}^{\infty} F^{*k}(t).$$

The estimation of the function $H(t)$ is studied in the literature (see, e.g., [1–4]).

Ref. [1] introduced a nonparametric estimator of $H(t)$ given by

$$H_n(t) = \sum_{k=1}^{m(n)} F_n^{(k)}(t), \tag{1}$$

where

$$F_n^{(k)}(t) = \begin{pmatrix} n \\ k \end{pmatrix}^{-1} \sum_{(n,k)} \mathbb{I}\big(X_{i_1} + \ldots + X_{i_k} \leq t\big),$$

and $\sum_{(n,k)}$ denotes the sum over all $\begin{pmatrix} n \\ k \end{pmatrix}$ distinct combinations of $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, and $m(n) \leq n$ is an integer sequence fulfilling $m(n) \uparrow \infty$ as $n \uparrow \infty$. He studied the almost sure convergence and the asymptotic normality of this estimator by using a method of reversed martingales for $i.i.d.$ random variables, Considering a slightly different estimator, ref. [5] showed some results on consistency, asymptotic normality, and asymptotic validity of bootstrap confidence regions. Then, using a linearization of the process, ref. [2] studied the weak convergence of this estimator on the Skorohod topology.

Generalization to multivariate and/or multidimensional renewal processes may be of a practical interest. For example, the case of customers arriving at a single-server service station. A customer is immediately served if the server is idle, and he or she waits if the server is busy. The process is bivariate and bidimensional because the distribution function of the interarrival times and that of the service times of successive customers are generally different.

Since ref. [6], the studies of higher dimensional renewal processes, more particulary in dimension two, have known some developments. For example, ref. [7] constructed a bivariate renewal model and explained its inclusion in basic and preventive maintenance. Ref. [8] investigated several decision models estimating the expected total cost incurred under various types of two-attribute warranty policies. Ref. [9] studied two-dimensional failure modeling for a system where degradation is due to age and usage.

Ref. [3] generalized the results of ref. [2] by introducing the multidimensional–multivariate renewal function and its estimator with an appropriate Skorohod topology.

The expansion of applications using random fields instead of random variables can be performed in spaces of dimensions greater than two. This will be the case, for example, in the number of people affected by a contagious disease before a time t, for which the geographical distance of two people affected simultaneously, will be considered. The first index is that of a region and the second one is the order of contamination in the region. The greater the difference between the numbers of the first index, the further the regions are. The random fields are considered in many papers and the major literature is studied (see, e.g., [10–12]).

The renewal function based on random fields was never investigated in the literature, and that is why we study the asymptotic normality of the estimator of the renewal function based on the two-dimensional nonnegative random fields. In this paper, we use a sequence of *i.i.d*. absolutely continuous positive random fields, and we also use an estimator of the renewal function as a sum of empirical function. We study the asymptotic behavior of this estimator. A renewal process is a set of arrival processes in which the associated arrival intervals are nonnegative random fields $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{N}^{*2}}$. General assumptions, including some notations and definitions, are given in Section 2. A concrete application is detailed in Section 3. In Section 4, we examine the asymptotic normality of the empirical distribution function, and in Section 5, we investigate the asymptotic normality of the estimator of the renewal function. We give in Section 6 the motivation for future research.

2. General Assumptions

Let $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{N}^{*2}}$ be a sequence of *i.i.d*. absolutely continuous positive random fields. Let n be a positive integer, and we describe the summation path of $\mathbf{X_i}$ illustrated by Figure 1, according to an order relation which will be defined as in the following:

$$(i_1, i_2) < (i'_1, i'_2) \iff i_1 + i_2 < i'_1 + i'_2 \text{ or } i_1 + i_2 = i'_1 + i'_2 \text{ and } \left\{ \begin{array}{c} i_1 < i'_1 \text{ and } i_1 + i_2 \equiv 1[2] \\ \text{or else} \\ i_2 < i'_2 \text{ and } i_1 + i_2 \equiv 0[2] \end{array} \right\}.$$

Knowing that the random fields are independent, our order was chosen only to facilitate the proofs and the obtained results of convergences will remain true whatever the order defined.

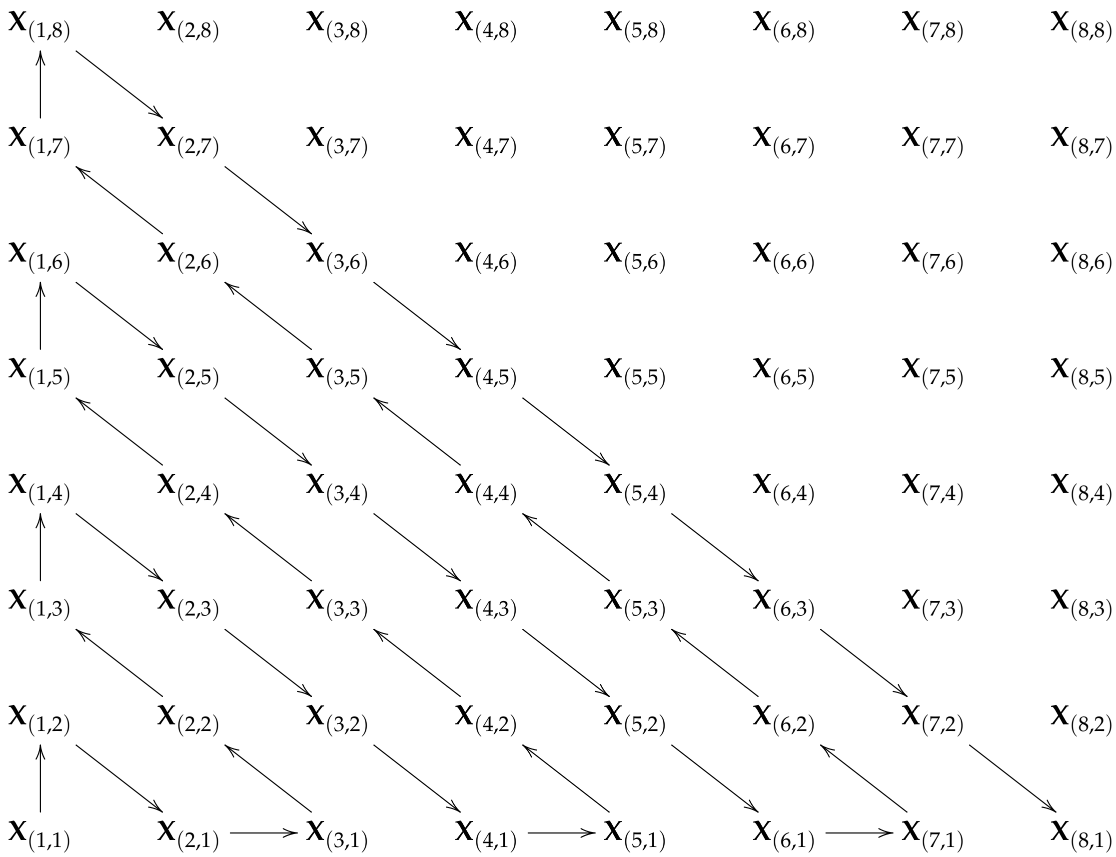

Figure 1. Represents the order of summation of the random fields $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{N}^{*2}}$ on the two-dimensional Euclidian space $\mathbb{N}^2$.

According to this order, we define the triangular domain $\widehat{\mathbf{I}}_n$ of the summation path as follows:

$$\widehat{\mathbf{I}}_n = \{(i_1, i_2) \text{ such that } i_1 + i_2 \leq n+1, \;\; i_1 = 1, \ldots, n, \;\; i_2 = 1, \ldots, n\}.$$

Denote by S_k the sum of k first random fields according to the order established above. Define

$$N(t) = \sup\{n, S_n \leq t\}.$$

$N(t)$ is the number of events by time t and called the counting process.

Let

$$H(t) = E\Big(N(t)\Big),$$

H is called renewal function.

Given a sequence of random fields $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{N}^{*2}}$, for all $\mathbf{i} \in \mathbb{N}^{*2}$, $\mathbf{X_i}$ has a cumulative distribution function $F_{\mathbf{i}}$ with density function $f_{\mathbf{i}}$.

Put $F^{(k)}$ as the distribution function defined as follows:

$$F^{(k)}(t) \;\; = \;\; P(S_k \leq t).$$

Since $N(t)$ is a process with integer value, then the renewal function can be also defined by

$$H(t) \;\; = \;\; \sum_{k=1}^{\infty} F^{(k)}(t).$$

Denote by $\mathbf{I}_k^*$ the set of the k first indices $\{\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_k\}$ according to the order. A sequence $\mathbf{I}_k$ of k indices is called equivalent to $\mathbf{I}_k^*$ if for each subset consisting of all indices which have the same first component; the second components starting from 1 are consecutive, which we note $\mathbf{I}_k \sim \mathbf{I}_k^*$. Denote by $\mathcal{I}_k = \{\mathbf{I}_k; \mathbf{I}_k \sim \mathbf{I}_k^*\}$ the set of k indices equivalent to $\mathbf{I}_k^*$. The random fields being *i.i.d.*, we deduce that

$$P\left(\sum_{\mathbf{i}\in\mathbf{I}_k} \mathbf{X}_\mathbf{i} \le t\right) = P\left(\sum_{\mathbf{i}\in\mathbf{I}_k^*} \mathbf{X}_\mathbf{i} \le t\right), \text{ for any } \mathbf{I}_k \in \mathcal{I}_k.$$

Denote by ω_n the number of random fields inclued in the triangular domain $\widehat{\mathbf{I}}_n$ such that

$$\omega_n = \frac{n(n+1)}{2}$$

and $\widehat{\mathbf{w}}_n(k)$ the number of independent block of k random fields such that

$$\widehat{\mathbf{w}}_n(k) = \left\lfloor \frac{\omega_n}{k} \right\rfloor,$$

where $\lfloor x \rfloor$ stands for the integer part of x.

Denote by $\left\{\mathbf{I}_{k,n}^{(l)}\right\}_{1\le l\le \widehat{\mathbf{w}}_n(k)}$ the set of the $\widehat{\mathbf{w}}_n(k)$ sequences of k indices $\{\mathbf{i}_1^l, \ldots, \mathbf{i}_k^l\}$ such that $\mathbf{i}_1^l < \mathbf{i}_2^l < \ldots < \mathbf{i}_k^l$, $\mathbf{i}_k^l < \mathbf{i}_k^{l+1}$ and $\mathbf{i}_1^1 = (1,1)$. $\mathbf{i}_k^{\widehat{\mathbf{w}}_n(k)}$ is the $k\widehat{\mathbf{w}}_n(k)$-th index in the set $\widehat{\mathbf{I}}_n$ with respect to the order defined above.

Now, put

$$Y_{l,n}^{(k)} = \sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}} \mathbf{X}_\mathbf{i}, \text{ for } 1 \le l \le \widehat{\mathbf{w}}_n(k).$$

Note that the sequence $Y_{l,n}^{(k)}$ is an independent and identically distributed random variable.

Let $\{\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_k\}$ and $\{\mathbf{j}_1, \mathbf{j}_2, \ldots, \mathbf{j}_{k'}\}$ be two subsets of $\mathbb{N}^{*2}$ that have $u \le \min(k, k')$ elements in common; we define

$$F_u^{(k,k')}(t) = P\Big((\mathbf{X}_{\mathbf{i}_1} + \mathbf{X}_{\mathbf{i}_2} + \ldots + \mathbf{X}_{\mathbf{i}_k} \le t) \cap (\mathbf{X}_{\mathbf{j}_1} + \mathbf{X}_{\mathbf{j}_2} + \ldots + \mathbf{X}_{\mathbf{j}_{k'}} \le t)\Big).$$

Let $m = m(n)$ be an integer sequence fulfilling $m(n) \uparrow \infty$, as $n \uparrow \infty$ and $m(n) \le n$; it seems natural to estimate H for a finite sum by an estimator $\widehat{H}_n$ defined by

$$\widehat{H}_n(t) = \sum_{k=1}^{m(n)} \widehat{F}_n^{(k)}(t),$$

where $\widehat{F}_n^{(k)}$ is the unbiased estimator of the distribution function $F^{(k)}$ defined as

$$\widehat{F}_n^{(k)}(t) = \frac{1}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} \mathbb{I}\Bigg(\sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}} \mathbf{X}_\mathbf{i} \le t\Bigg), \tag{2}$$

where $\mathbb{I}(.)$ is the indicator function.

3. Concrete Application

Consider a contagious disease propagation in France whose penetration is between the departments. Suppose that the disease is already spreading in another country; the time to infect the first person in France follows a law of random field $\mathbf{X}_{(1,1)}$. The first component 1 of the index is that of the infected person's department and the second component indicates the first infected person. The infection time from the first infected person to the second person in the same department is the law of the random field $\mathbf{X}_{(1,2)}$. Or else, if the person is in another department, the first component will be 2, and the infection time of the second person is the law of the random field $\mathbf{X}_{(2,1)}$. The index of the infection time of the law of random fields will be assigned to the same process as for the first two people. The contamination time of k persons follows a law of the sum of k random fields such that $\sum_{\mathbf{i}\in\mathbf{I}_k} \mathbf{X_i}$ where $\mathbf{I}_k \in \mathcal{I}_k$. $N(t)$, the number of infected people by time t, will be defined by

$$N(t) = \sup\left\{k, \sum_{\mathbf{i}\in\mathbf{I}_k} \mathbf{X_i} \leq t\right\},$$

where the set $\mathbf{I}_k$ of the k indices of the path of the $\mathbf{X_i}$ belongs to $\mathbf{I}_k$. The renewal function $H(t)$, expectation of $N(t)$, can be estimated by our results in Section 5.

Moreover, we can estimate the renewal function restricted to different regions from an adapted counting process $N_{\mathbf{j}_1,\mathbf{j}_2,\ldots,\mathbf{j}_m}$ where $\mathbf{j}_1, \mathbf{j}_2, \ldots, \mathbf{j}_m$ are the indices of the departments of the region and m is the number of departments in the region.

4. Asymptotic Normality of $\widehat{F}_n^{(k)}(t)$

The first main result on the central limit theorem of the empirical estimator $\widehat{F}_n^{(k)}$ is a generalization of that of ref.[13] for random variables. Our proofs are based on the convergence of the characteristic function, and we show that its second-order moments of Taylor expansion converge to a characteristic function of a normal random variable and the fourth-order converges to zero.

Theorem 1. *Suppose that the sequence of random fields $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{N}^{*2}}$ is i.i.d. and absolutely continuous positive and the summation path of $\mathbf{X_i}$ illustrated in Figure 1 holds; then, the process $n\left(\widehat{F}_n^{(k)}(t) - F^{(k)}(t)\right)$, $t > 0$, converges in distribution to $\mathcal{N}(0, \sigma_k^2)$ where $\mathcal{N}$ is the centered normal random variable with variance defined by $\sigma_k^2 = F^{(k)}(t)\left(1 - F^{(k)}(t)\right)$.*

Proof. Our first stage is to show that the empirical function $\widehat{F}_n^{(k)}$ is an unbiased estimator of the distribution function $F^{(k)}$; one has

$$\begin{aligned}
E\left(\widehat{F}_n^{(k)}(t)\right) &= E\left(\frac{1}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} \mathbb{I}\left(\sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}} \mathbf{X_i} \leq t\right)\right) \\
&= \frac{1}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} E\left(\mathbb{I}\left(\sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}} \mathbf{X_i} \leq t\right)\right) \\
&= \frac{1}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} E\left(\mathbb{I}\left(Y_{l,n}^{(k)} \leq t\right)\right) \\
&= \frac{\widehat{\mathbf{w}}_n(k)}{\widehat{\mathbf{w}}_n(k)} E\left(\mathbb{I}\left(Y_{1,n}^{(k)} \leq t\right)\right) \\
&= F^{(k)}(t).
\end{aligned}$$

Now, we prove the asymptotic normality. From (2), one has

$$n\left(\widehat{F}_n^{(k)}(t) - F^{(k)}(t)\right) = n\left(\frac{1}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} \mathbb{I}\left(\sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}} \mathbf{X_i} \leq t\right) - F^{(k)}(t)\right)$$

$$
\begin{aligned}
&= \frac{n}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} \left(\mathbb{I}\Big(\sum_{\mathbf{i} \in \mathbf{I}_{k,n}^{(l)}} \mathbf{X_i} \le t \Big) - F^{(k)}(t) \right) \\
&= \frac{n}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} \left(\mathbb{I}\Big(Y_{l,n}^{(k)} \le t \Big) - F^{(k)}(t) \right). \qquad (3)
\end{aligned}
$$

Put $\{A_{l,n}^{(k)}\}$ as the sequence of *i.i.d*. random variables defined as

$$
A_{l,n}^{(k)} = \mathbb{I}\Big(Y_{l,n}^{(k)} \le t \Big) - F^{(k)}(t), \text{ for all } 1 \le l \le \widehat{\mathbf{w}}_n(k). \qquad (4)
$$

Denote by ϕ_n the characteristic function of the process $n\Big(\widehat{F}_n^{(k)}(t) - F^{(k)}(t)\Big)$ such that

$$
\phi_n(u) = E\left[\exp\left(iun \left(\frac{1}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} \mathbb{I}\Big(\sum_{\mathbf{i} \in \mathbf{I}_{k,n}^{(l)}} \mathbf{X_i} \le t \Big) - F^{(k)}(t) \right) \right) \right], \; u \in \mathbb{R},
$$

where i is the complex number unity.

Using the Taylor expansion neighborhood of zero,

$$
E\left(e^{iuA_{l,n}^{(\mathbf{k})}} \right) = 1 + (iu)E(A_{l,n}^{(\mathbf{k})}) + \frac{(iu)^2 E\Big(A_{l,n}^{(\mathbf{k})}\Big)^2}{2!} + \ldots + \frac{(iu)^n E\Big(A_{l,n}^{(\mathbf{k})}\Big)^n}{n!} + o(|u|^n), \qquad (5)
$$

and

$$
\ln(1-u) = -u - \frac{u^2}{2} + o(|u|^2), \qquad (6)
$$

one has

$$
\begin{aligned}
\exp\Big[\ln\Big(\phi_n(u)\Big) \Big] &= \exp\left[\ln\left(E\left(\exp\left(\frac{iun}{\widehat{\mathbf{w}}_n(\mathbf{k})} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} \left(\mathbb{I}\Big(\sum_{\mathbf{i} \in \mathbf{I}_{k,n}^{(l)}} \mathbf{X_i} \le t \Big) - F^{(k)}(t) \right) \right) \right) \right) \right] \\
&= \exp\left[\ln\left(E\left(\exp\left(\frac{iun}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} A_{l,n}^{(k)} \right) \right) \right) \right] \\
&= \exp\left[\ln\left(E\left(\exp\left(\frac{iun}{\widehat{\mathbf{w}}_n(k)} (A_{l,n}^{(k)}) \right) \right) \right)^{\widehat{\mathbf{w}}_n(k)} \right] \\
&= \exp\left[\widehat{\mathbf{w}}_n(k) \ln\left(E\left(\exp\left(\frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{l,n}^{(k)} \right) \right) \right) \right] \qquad (7) \\
&= \exp\left[\widehat{\mathbf{w}}_n(k) \ln\left(1 - \frac{u^2 n^2}{2\Big(\widehat{\mathbf{w}}_n(k)\Big)^2} E\Big(A_{l,n}^{(k)}\Big)^2 \right) + o\left(\left| \frac{u^2}{\widehat{\mathbf{w}}_n(\mathbf{k})} \right| \right) \right] \\
&= \exp\left[-\frac{u^2 n^2 \widehat{\mathbf{w}}_n(k)}{2\Big(\widehat{\mathbf{w}}_n(k)\Big)^2} E\Big(A_{l,n}^{(\mathbf{k})}\Big)^2 - \frac{u^4 n^4 \Big(\widehat{\mathbf{w}}_n(k)\Big)}{8\Big(\widehat{\mathbf{w}}_n(k)\Big)^4} \left(E\Big(A_{l,n}^{(k)}\Big)^2 \right)^2 \right. \\
&+ \left. o\left(\left| \frac{u}{\widehat{\mathbf{w}}_n(k)} \right|^4 \right) \right].
\end{aligned}
$$

Indeed, the fourth-order terms $\frac{u^4 n^4 \left(\widehat{\mathbf{w}}_n(k)\right)}{8\left(\widehat{\mathbf{w}}_n(k)\right)^4}\left(E\left(A_{l,n}^{(k)}\right)^2\right)^2 + o\left(\left|\frac{u}{\widehat{\mathbf{w}}_n(k)}\right|^4\right)$ converge to zero as $n \to \infty$ because $\mathcal{O}\left(\widehat{\mathbf{w}}_n(k)\right) = n^2$.

From (4), one has

$$\begin{aligned} E\left(A_{l,n}^{(k)}\right)^2 &= E\left(\mathbb{I}\left(Y_{l,n}^{(k)} \leq t\right) - F^{(k)}(t)\right)^2 \\ &= E\left(\left(\mathbb{I}\left(Y_{l,n}^{(k)} \leq t\right)\right)^2 - 2\left(\mathbb{I}\left(Y_{l,n}^{(k)} \leq t\right)\right)F^{(k)}(t) + \left(F^{(k)}(t)\right)^2\right) \\ &= E\left(\mathbb{I}\left(Y_{l,n}^{(k)} \leq t\right)\right) - 2E\left(\mathbb{I}\left(Y_{l,n}^{(k)} \leq t\right)\right)F^{(k)}(t) + \left(F^{(k)}(t)\right)^2 \\ &= F^{(k)}(t) - \left(F^{(k)}(t)\right)^2 \\ &= F^{(k)}(t)\left(1 - F^{(k)}(t)\right) \\ &= \sigma_k^2. \end{aligned}$$

As the characteristic function ϕ_n converges to a Gaussian characteristic function ϕ, we deduce that the process $n\left(\widehat{F}_n^{(k)}(t) - F^{(k)}(t)\right)$, $t > 0$ converges in distribution to a centered normal random variable $\mathcal{N}(0, \sigma_k^2)$. $\square$

5. Asymptotic Normality of $\widehat{H}_n(t)$

The second main result relating to the central limit theorem of the empirical estimator $\widehat{H}_n$ is also a generalization of that of ref. [13] for random variables.

For each $k \geq 1$, define

$$m^{-1}(k) = \inf\{n : m(n) \geq k\}.$$

Define

$$\xi_{kk'}(c) = \mathrm{Cov}\left(F^{(k-c)}(t - (\mathbf{X}_{\mathbf{i}_1} + \ldots + \mathbf{X}_{\mathbf{i}_c})), F^{(k'-c)}(t - (\mathbf{X}_{\mathbf{j}_1} + \ldots + \mathbf{X}_{\mathbf{j}_c}))\right).$$

The central limit theorem is stated in the following theorem.

Theorem 2. *Suppose that the sequence of random fields $(\mathbf{X}_\mathbf{i})_{\mathbf{i} \in \mathbb{N}^{*2}}$ is i.i.d. and absolutely continuous positive and the summation path of $\mathbf{X}_\mathbf{i}$ illustrated in Figure 1 holds; suppose that either for $r > 4$,*

$$E|\mathbf{X}_\mathbf{i}|^r < \infty \text{ and } n = \mathcal{O}\left(m^{\frac{r-4}{2}}\right) \tag{8}$$

then, the process $n\left(\widehat{H}_n(t) - H(t)\right)$, $t > 0$, converges in distribution to $\mathcal{N}(0, \varsigma^2)$ where $\mathcal{N}$ is the centered normal random variable with variance defined as

$$\varsigma^2 = \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \xi_{kk'}(c).$$

Proof. We start by studying the bias between the renewal function and its estimator.

$$
\begin{aligned}
\lim_{n\to\infty} E\Big(\widehat{H}_n(t)\Big) &= \lim_{n\to\infty} E\left(\sum_{k=1}^{m}\left(\frac{1}{\widehat{\mathbf{w}}_n(k)}\sum_{l=1}^{\widehat{\mathbf{w}}_n(k)}\mathbb{I}\Big(\sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}}\mathbf{X_i}\le t\Big)\right)\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^{m}\left(\frac{1}{\widehat{\mathbf{w}}_n(k)}\sum_{l=1}^{\widehat{\mathbf{w}}_n(k)}E\left(\mathbb{I}\Big(\sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}}\mathbf{X_i}\le t\Big)\right)\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^{m}\left(\frac{1}{\widehat{\mathbf{w}}_n(k)}\sum_{l=1}^{\widehat{\mathbf{w}}_n(k)}E\Big(\mathbb{I}\Big(Y_{l,n}^{(k)}\le t\Big)\Big)\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^{m}F^{(k)}(t)\\
&= H(t).
\end{aligned}
$$

From this computation, we can deduce that $\widehat{H}_n$ is an asymptotically unbiased estimator of H.

Proceeding as in the proof of Theorem 1, let Φ_n be the characteristic function of the process $\widehat{n}\Big(\widehat{H}_n(t)-H(t)\Big)$.

From Condition (7), one has

$$
\begin{aligned}
n\Big(\widehat{H}_n(t)-H(t)\Big) &= n\left(\sum_{k=1}^{m}\frac{1}{\widehat{\mathbf{w}}_n(k)}\sum_{l=1}^{\widehat{\mathbf{w}}_n(k)}\mathbb{I}\Big(\sum_{\mathbf{i}\in\mathbf{I}_{k,n}^{(l)}}\mathbf{X_i}\le t\Big)-F^{(k)}(t)\right)-n\sum_{k>m}F^{(k)}(t)\\
&= \sum_{k=1}^{m(n)}\left(\frac{n}{\widehat{\mathbf{w}}_n(k)}\sum_{l=1}^{\widehat{\mathbf{w}}_n(k)}\Big(\mathbb{I}\Big(Y_{l,n}^{(k)}\le t\Big)-F^{(k)}(t)\Big)\right)-n\sum_{k>m}F^{(k)}(t)\\
&= \sum_{k=1}^{m}\frac{n}{\widehat{\mathbf{w}}_n(k)}\sum_{l=1}^{\widehat{\mathbf{w}}_n(k)}A_{l,n}^{(k)}-n\sum_{k>m}F^{(k)}(t),
\end{aligned}
$$

where $\{A_{l,n}^{(k)}\}_{1\le l\le\widehat{\mathbf{w}}_n(k)}$ is the sequence of random variables defined in (4).

For the continuation of our proof, we will proceed in two steps.

Our first step is to show that $n\sum_{k>m}F^{(k)}(t)$ is negligible and to do this we need the following lemmas.

Lemma 1. *Let $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{N}^{*2}}$ be a sequence of random fields such that $E|\mathbf{X_i}|^r<\infty$, $r>4$, $E(\mathbf{X_i})>0$ for each $\mathbf{i}\in\mathbb{N}^{*2}$; denote by S_k the sum of k random fields, and we have*

$$
P(S_k\le t)\le\mathcal{O}\Big(k^{-\frac{r}{2}}\Big),\ \ for\ t>0.
$$

Proof. Put $E(\mathbf{X_i})=\mu$ and choose ε sufficiently small such that $\varepsilon<\mu$ and $t\le(\mu-\varepsilon)k$, by using Markov's inequality; one has

$$
\begin{aligned}
P\{|S_k-k\mu|\ge k\varepsilon\} &= P\{|S_k-k\mu|^r\ge(k\varepsilon)^r\}\\
&\le \frac{1}{(k\varepsilon)^r}E\big(|S_k-k\mu|^r\big).
\end{aligned}
$$

Put $\mathbf{X}_\mathbf{i}^*=\mathbf{X_i}-\mu$; we have

$$
P\{|S_k-k\mu|\ge k\varepsilon\}\ \le\ \frac{1}{(k\varepsilon)^r}\big(E|S_k^*|^r\big).
$$

Since $(\mathbf{X}_{\mathbf{i}}^*)_{\mathbf{i}\in\mathbb{N}^{*2}}$ is a sequence of centered random fields, from the moment inequality of ref. [14], we deduce that there exists a positive constant C such that

$$E|S_k^*|^r < Ck^{\frac{r}{2}}. \tag{9}$$

Then,

$$\begin{aligned} P\{|S_k - k\mu| \geq k\varepsilon\} &\leq \frac{C}{(k\varepsilon)^r}k^{\frac{r}{2}} \\ &= \frac{C}{\varepsilon^r}k^{-\frac{r}{2}}. \end{aligned}$$

As

$$|S_k - k\mu| \geq k\varepsilon \Leftrightarrow \begin{cases} S_k - k\mu \geq k\varepsilon \\ S_k - k\mu \leq -k\varepsilon. \end{cases}$$

By using the particular inequality $S_k - k\mu \leq -k\varepsilon$, one has

$$P(S_k \leq (\mu - \varepsilon)k) \leq \frac{C}{\varepsilon^r}k^{-\frac{r}{2}}.$$

For k sufficiently large,

$$\begin{aligned} F^{(k)}(t) &= P(S_k \leq t) \\ &\leq P(S_k \leq (\mu - \varepsilon)k) \\ &\leq \mathcal{O}\left(k^{-\frac{r}{2}}\right). \end{aligned}$$

It achieves the proof of Lemma 1. □

Lemma 2. *Under Condition* (8)*, one has*

$$\lim_{n\to\infty} n \sum_{k>m} F^{(k)}(t) = 0.$$

Proof. From the definition of $m(n)$, the condition $n = \mathcal{O}\left(m^{\frac{r-4}{2}}\right)$ implies $m^{-1}(n) = \mathcal{O}\left(n^{\frac{r-4}{2}}\right)$; using (8)

$$n \sum_{k>m} F^{(k)}(t) = \mathcal{O}\left(\sum_{k>m} k^{\frac{r-4}{2}} F^{(k)}(t)\right).$$

Using Lemma 1, one has

$$\begin{aligned} \lim_{n\to\infty} n \sum_{k>m} F^{(k)}(t) &= \lim_{n\to\infty} \mathcal{O}\left(\sum_{k>m} k^{\frac{r-4}{2}} F^{(k)}(t)\right) \\ &\leq \lim_{n\to\infty} \mathcal{O}\left(\sum_{k>m} k^{\frac{r-4}{2}-\frac{r}{2}}\right) \\ &\leq \lim_{n\to\infty} \mathcal{O}\left(\sum_{k>m} k^{-2}\right) \\ &\leq \lim_{n\to\infty} \mathcal{O}\left(\sum_{k>m} \frac{1}{k^2}\right) \\ &= 0. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} n \sum_{k>m} F^{(k)}(t) = 0.$$

It achieves the proof of Lemma 2. □

Our second step consists of studying the characteristic function of $\sum_{k=1}^{m} \frac{n}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} A_{l,n}^{(k)}$. From (9) and Lemma 2, one has

$$\begin{aligned}
\Phi_n(u) &= \exp\left[\ln\left(\Phi_n(u)\right)\right] \\
&= \exp\left[\ln\left(E\left(\exp\left(iun\left(\widehat{H}_n(t) - H(t)\right)\right)\right)\right)\right] \\
&= \exp\left[\ln\left(E\left(\exp\left(\sum_{k=1}^{m} \frac{iu}{\widehat{\mathbf{w}}_n(k)} \sum_{l=1}^{\widehat{\mathbf{w}}_n(k)} A_{l,n}^{(k)}\right)\right)\right)\right] \\
&= \exp\left[\ln\left(E\left(\exp\left(\sum_{k=1}^{m} \frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{1,n}^{(k)} + \ldots + \sum_{k=1}^{m} \frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{\widehat{\mathbf{w}}_n(k),n}^{(k)}\right)\right)\right)\right] \\
&= \exp\left[\ln\left(E\left(\exp\left(\sum_{k=1}^{m} \frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{1,n}^{(k)}\right)\right) \times \ldots \times E\left(\exp\left(\sum_{k=1}^{m} \frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{\widehat{\mathbf{w}}_n(k),n}^{(k)}\right)\right)\right)\right] \\
&= \exp\left[\ln\left(E\left(\exp\left(\sum_{k=1}^{m} \frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{1,n}^{(k)}\right)\right)\right)^{\widehat{\mathbf{w}}_n(k)}\right] \\
&= \exp\left[\widehat{\mathbf{w}}_n(k) \ln\left(E\left(\exp\left(\sum_{k=1}^{m} \frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{1,n}^{(k)}\right)\right)\right)\right].
\end{aligned}$$

Using successively the expansion Formula (5) and (6) , one has

$$E\left(\exp\left(\sum_{k=1}^{m} \frac{iun}{\widehat{\mathbf{w}}_n(k)} A_{1,n}^{(k)}\right)\right) = 1 - \frac{1}{2} E\left(\sum_{k=1}^{m} \frac{un}{\widehat{\mathbf{w}}_n(k)} \left(A_{1,n}^{(k)}\right)\right)^2 + o\left(\left|\sum_{k=1}^{m} \frac{u}{\widehat{\mathbf{w}}_n(k)}\right|^2\right).$$

Then,

$$\begin{aligned}
\exp\left[\ln\left(\phi_n(u)\right)\right] &= \exp\left[\widehat{\mathbf{w}}_n(k) \ln\left(1 - \frac{1}{2} E\left(\sum_{k=1}^{m} \frac{un}{\widehat{\mathbf{w}}_n(k)} \left(A_{1,n}^{(k)}\right)\right)^2\right)\right. \\
&+ \left. o\left(\left|\sum_{k=1}^{m} \frac{u}{\widehat{\mathbf{w}}_n(k)}\right|^2\right)\right] \\
&= \exp\left[-\frac{1}{2}\widehat{\mathbf{w}}_n(k) E\left(\sum_{k=1}^{m} \frac{un}{\widehat{\mathbf{w}}_n(k)} \left(A_{1,n}^{(k)}\right)\right)^2\right. \\
&- \frac{1}{8}\widehat{\mathbf{w}}_n(k) E\left(\sum_{k=1}^{m} \frac{un}{\widehat{\mathbf{w}}_n(k)} \left(A_{1,n}^{(k)}\right)\right)^4 \\
&+ \left. \widehat{\mathbf{w}}_n(k) \times o\left(\left|\sum_{k=1}^{m} \frac{un}{\widehat{\mathbf{w}}_n(k)}\right|^4\right)\right] \\
&= \exp\left[-\frac{u^2}{2} Q_n(k) - \frac{u^4}{8} K_n(k) + o\left(\widehat{\mathbf{w}}_n(k) \left|\sum_{k=1}^{m} \frac{un}{\widehat{\mathbf{w}}_n(k)}\right|^4\right)\right].
\end{aligned}$$

where

$$Q_n(k) = \widehat{\mathbf{w}}_n(k) E\left(\sum_{k=1}^{m} \frac{n}{\widehat{\mathbf{w}}_n(k)} \left(A_{l,n}^{(k)} \right) \right)^2,$$

and

$$K_n(k) = \widehat{\mathbf{w}}_n(k) E\left(\sum_{k=1}^{m} \frac{n}{\widehat{\mathbf{w}}_n(k)} \left(A_{l,n}^{(k)} \right) \right)^4.$$

We have to show that $\lim_{n\to\infty} Q_n(k)$ is finite and $K_n(k)$ converges to zero.

For n sufficiently large, using the approximation

$$\mathcal{O}\left(\widehat{\mathbf{w}}_n(k) \right) \simeq n^2, \tag{10}$$

thus,

$$\frac{n}{\widehat{\mathbf{w}}_n(k)} \simeq \frac{1}{n}. \tag{11}$$

One has

$$\begin{aligned} \lim_{n\to\infty} Q_n(k) &= \lim_{n\to\infty} \widehat{\mathbf{w}}_n(k) E\left(\sum_{k=1}^{m} \frac{n}{\widehat{\mathbf{w}}_n(k)} \left(A_{l,n}^{(k)} \right) \right)^2 \\ &\simeq \lim_{n\to\infty} n^2 E\left(\sum_{k=1}^{m} \frac{1}{n} \left(A_{l,n}^{(k)} \right) \right)^2 \\ &= \lim_{n\to\infty} E\left(\sum_{k=1}^{m} \left(A_{l,n}^{(k)} \right) \right)^2. \end{aligned}$$

To calculate $E\left(\sum_{k=1}^{m} \left(A_{l,n}^{(k)} \right) \right)^2$, we first examine the covariance between $A_{l,n}^{(k)}$ and $A_{l,n}^{(k')}$. Let $\{\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_k\}$ and $\{\mathbf{j}_1, \mathbf{j}_2, \dots, \mathbf{j}_{k'}\}$ be two subsets of $\mathbb{N}^{*2}$ that have $c \leq \min(k, k')$ elements in common.

$$\begin{aligned} \mathrm{Cov}\left(A_{l,n}^{(k)}, A_{l,n}^{(k')} \right) &= E\left(\mathbb{I}(\mathbf{X}_{\mathbf{i}_1} + \mathbf{X}_{\mathbf{i}_2} + \ldots + \mathbf{X}_{\mathbf{i}_k} \leq t) \mathbb{I}(\mathbf{X}_{\mathbf{j}_1} + \mathbf{X}_{\mathbf{j}_2} + \ldots + \mathbf{X}_{\mathbf{j}_{k'}} \leq t) \right) \\ &- F^{(k)}(t) F^{(k')}(t) \\ &= F_c^{(k,k')}(t) - F^{(k)}(t) F^{(k')}(t) \\ &= \xi_{kk'}(c). \end{aligned}$$

Then,

$$E\left(\sum_{k=1}^{m} \left(A_{l,n}^{(k)} \right) \right)^2 = \sum_{k=1}^{m} \sum_{k'=1}^{m} \xi_{kk'}(c).$$

Thus,

$$\lim_{n\to\infty} Q_n(k) = \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \xi_{kk'}(c).$$

Since

$$F_c^{(k,k')}(t) \leq \min\left(F^{(k)}(t), F^{(k')}(t) \right),$$

and

$$\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} F^{(k)}(t)F^{(k')}(t) = \sum_{k=1}^{\infty}\sum_{k=1}^{\infty} \mathcal{O}\left(k^{-\frac{r}{2}}k'^{-\frac{r}{2}}\right)$$
$$= \sum_{k_1=1}^{\infty}\sum_{k'=1}^{\infty} \frac{1}{k^{\frac{r}{2}}}\frac{1}{k'^{\frac{r}{2}}}$$
$$< \infty,$$

because $\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} \frac{1}{k^{\frac{r}{2}}}\frac{1}{k'^{\frac{r}{2}}}$ is a convergent Riemann's series for $r > 4$, we deduce that the variance $\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} \xi_{kk'}(c)$ is finite.

To achieve the proof, we show that $K_n(k)$ converges to zero. Using Conditions (10) and (11), one has

$$\lim_{n\to\infty} K_n(k) = \lim_{n\to\infty} \widehat{\mathbf{w}}_n(k) E\left(\sum_{k=1}^{m} \frac{n}{\widehat{\mathbf{w}}_n(k)}\left(A_{l,n}^{(k)}\right)\right)^4$$
$$\simeq \lim_{n\to\infty} n^2 E\left(\sum_{k=1}^{m} \frac{1}{n}\left(A_{l,n}^{(k)}\right)\right)^4$$
$$\simeq \lim_{n\to\infty} \frac{1}{n^2} E\left(\sum_{k=1}^{m}\left(A_{l,n}^{(k)}\right)\right)^4.$$

Using Inequality (9), there exists a positive constant C such that

$$E\left(\sum_{k=1}^{m}\left(A_{l,n}^{(k)}\right)\right)^4 \leq Cm^2.$$

Then,

$$\lim_{n\to\infty} K_n(k) \leq \lim_{n\to\infty} \frac{Cm^2}{n^2}$$
$$= \lim_{n\to\infty} \frac{C}{m^{r-4}}$$
$$= 0.$$

Finally,

$$\lim_{n\to\infty} \exp\left[\ln\left(\phi_n(u)\right)\right] = \exp\left(-\frac{u^2}{2}\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} \xi_{kk'}(c)\right)^2.$$

It achieves the proof of Theorem 2. □

6. Conclusions

The present paper suggests some ideas that can be worth exploring in the future:

- Extend the results for dependent random fields. In the case of contagious disease, the time for one person to be affected will depend of his or her distance from the other person already affected.
- Extend the results for nonstationary random fields. Always in the case of contagious disease, the law of a person's infection time may change depending on the period.
- Extend the results for multi-dimensional random fields with dimensions strictly more than two.

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