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Determination of the Impulsive Dirac Systems from a Set of Eigenvalues

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Abstract: In this work, we consider the inverse spectral problem for the impulsive Dirac systems on $(0,\pi)$ with the jump condition at the point $\frac{\pi}{2}$. We conclude that the matrix potential $Q(x)$ on the whole interval can be uniquely determined by a set of eigenvalues for two cases: (i) the matrix potential $Q(x)$ is given on $\left(0, \frac{(1+\alpha)\pi}{4}\right)$; (ii) the matrix potential $Q(x)$ is given on $\left(\frac{(1+\alpha)\pi}{4}, \pi\right)$, where $0<\alpha<1$.

Keywords: impulsive dirac operator; eigenvalue; inverse spectral problem

MSC: 34A55; 34B24; 47E05

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1. Introduction

Define $\rho(x)=\begin{cases}1, & x<\dfrac{\pi}{2}\\ \alpha, & x>\dfrac{\pi}{2}\end{cases}$ $(0<\alpha<1)$. Consider the following impulsive Dirac systems:

$$ly := By'(x)+Q(x)y(x)=\lambda\rho(x)y(x), \quad x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right), \tag{1}$$

with the boundary conditions

$$y_1(0)=0, \quad y_2(\pi)=0, \tag{2}$$

and the jump conditions

$$y\left(\frac{\pi}{2}+0\right)=Ay\left(\frac{\pi}{2}-0\right), \tag{3}$$

where

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ Q(x)=\begin{pmatrix}p(x) & q(x)\\ q(x) & -p(x)\end{pmatrix},\ y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix},$$

$p(x)$ and $q(x)$ are real-valued functions in $L^2(0,\pi)$, λ is the spectral parameter, and $A=\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}$, $\beta\in\mathbb{R}^+$. The Equations (1)–(3), denoted by $L=L(p(x),q(x),\rho(x),\beta)$, are called a boundary value problem of the Dirac equations with the discontinuity conditions at $\frac{\pi}{2}$.

The discontinuous boundary value problems are related to the discontinuous material characters of an intermediary. This kind of problem has been studied by many authors (see, e.g., references [1–5]).

The inverse problem for the Dirac operator was completely solved by two spectra in references [6,7]. Mochizuki and Trooshin [8] studied the problem $L = L(p(x), q(x), 1, 1)$ with the separable boundary conditions. They gave the uniqueness theorem by a set of values of eigenfunctions in some internal points and spectra. In reference [2], Ozkan and Amirov studied the boundary value problem $L = L(p(x), q(x), 1, \beta)$ and showed that the potential function can be uniquely determined by a set of values of eigenfunctions at some internal points and one spectrum. Amirov [1] gave representations of solutions of the Dirac equation, properties of spectral data and showed that the Dirac operator can be uniquely determined by the Weyl function on a finite interval $(0, \pi)$ for the problem $L = L(p(x), q(x), 1, \beta)$.

There are also related studies on the spectral theory of partial differential operators (see, e.g., references [9–12]). In reference [9], Cao, Diao, Liu and Zou introduced generalized singular lines of the Laplacian eigenfunctions, and studied these singular lines and the nodal lines. The theoretical findings can be applied directly to the inverse scattering problem. Diao, Liu and Wang [12] derived a comprehensive and complete characterisation of the GHP, and they established novel unique identifiability results by, at most, a few scattering measurements.

For the impulsive Dirac operator, Mamedov and Akcay [13] proved that the sequences of eigenvalues and normalizing numbers can uniquely determine the potential and they gave the theorem on the necessary and sufficient conditions for the solvability and a solution algorithm of the inverse problem for the boundary value problem $L = L(p(x), q(x), \rho(x), 1)$. In reference [3], Güldü studied the problem L and proved, using Hochstadt and Lieberman's method [14], that if the potential function $p(x)$ is given on the interval $(\frac{\pi}{2}, \pi)$, then one spectrum can determine $p(x)$ on the whole interval.

In this paper, we consider the problem $L = L(p(x), q(x), \rho(x), \beta)$. It is shown with two cases that (i) if the potential $p(x)$ and $q(x)$ are given on $(0, \frac{(1+\alpha)\pi}{4})$ and (ii) if the potential $p(x)$ and $q(x)$ are given on $(\frac{(1+\alpha)\pi}{4}, \pi)$, respectively, then only a single spectrum is sufficient to determine $p(x)$, $q(x)$ on $(0, \pi)$, $\rho(x)$ and β.

2. Preliminaries

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the solutions of (1), satisfying the initial conditions

$$\varphi(0, \lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \psi(\pi, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the jump condition (3), respectively. Denote $\sigma(x) = \int_0^x \rho(t)dt$, $\tau = \mathrm{Im}\lambda$.

From references [3,15], we can ascertain that $\varphi(x, \lambda)$ has the following representation:

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x K_1(x, t)\varphi_0(t, \lambda)dt,$$

where $\varphi_0(x, \lambda) = (\varphi_{01}(x, \lambda), \varphi_{02}(x, \lambda))^T$ satisfies the following forms:

$$\varphi_{01}(x, \lambda) = \begin{cases} \sin \lambda\sigma(x), & 0 < x < \frac{\pi}{2}, \\ \beta^+ \sin \lambda\sigma(x) + \beta^- \sin \lambda(\pi - \sigma(x)), & \frac{\pi}{2} < x < \pi, \end{cases} \tag{4}$$

$$\varphi_{02}(x, \lambda) = \begin{cases} -\cos \lambda\sigma(x), & 0 < x < \frac{\pi}{2}, \\ -\beta^+ \cos \lambda\sigma(x) + \beta^- \cos \lambda(\pi - \sigma(x)), & \frac{\pi}{2} < x < \pi. \end{cases} \tag{5}$$

Similarly, we can compute that the following representation holds for $\psi(x, \lambda)$:

$$\psi(x, \lambda) = \psi_0(x, \lambda) + \int_x^\pi K_2(x, t)\psi_0(t, \lambda)dt, \tag{6}$$

where $\psi_0(x,\lambda) = (\psi_{01}(x,\lambda), \psi_{02}(x,\lambda))^T$ satisfies the following forms:

$$\psi_{01}(x,\lambda)=\begin{cases} A^+\cos\lambda(\sigma(\pi)-\sigma(x)) \\ \quad -A^-\cos\lambda(\sigma(\pi)+\sigma(x)-\pi), & 0<x<\dfrac{\pi}{2}, \\ \cos\lambda(\sigma(\pi)-\sigma(x)), & \dfrac{\pi}{2}<x<\pi, \end{cases} \tag{7}$$

$$\psi_{02}(x,\lambda)=\begin{cases} A^+\sin\lambda(\sigma(\pi)-\sigma(x)) \\ \quad +A^-\sin\lambda(\sigma(\pi)+\sigma(x)-\pi), & 0<x<\dfrac{\pi}{2}, \\ \sin\lambda(\sigma(\pi)-\sigma(x)), & \dfrac{\pi}{2}<x<\pi, \end{cases} \tag{8}$$

where $\beta^{\pm} = \frac{1}{2}(\beta \pm \frac{1}{\alpha\beta})$, $A^{\pm} = \frac{1}{2}(\frac{1}{\beta} \pm \alpha\beta)$ and $K_n(x,t) = (K_{ijn}(x,t))_{i,j=1,2}(n = 1,2)$ with $K_{ijn}(x,t)$ are real-valued continuous functions for $i,j = 1,2$.

Denote

$$\Delta(\lambda) := W[\varphi(x,\lambda), \psi(x,\lambda)] = \varphi_2(x,\lambda)\psi_1(x,\lambda) - \varphi_1(x,\lambda)\psi_2(x,\lambda). \tag{9}$$

The function $\Delta(\lambda)$ is called the characteristic function of L, which is entire in λ. It follows from (6)–(8) and reference [3] that we have

$$\Delta(\lambda) = \Delta_0(\lambda) + o(\exp|\tau|\sigma(\pi)), \tag{10}$$

where $\Delta_0(\lambda) = -\beta^+\cos\lambda\sigma(\pi) + \beta^-\cos\lambda(\pi - \sigma(\pi))$.

Using the standard method in reference [16], or referring to references [3,17], one can obtain the following lemma.

Lemma 1. (1) *The operator l has an, at most, countable set of eigenvalues such that all of them are real and simple.*
(2) *The eigenvalues denoted by $\{\lambda_n\}_{n\in\mathbb{Z}}$ can be represented by the following asymptotic formula for $|n| \to \infty$:*

$$\lambda_n = \frac{2n}{1+\alpha}\left(1 + O\left(\frac{1}{n}\right)\right), \ \ \lambda \in G_\varepsilon, \tag{11}$$

where $G_\varepsilon := \{\lambda : |\lambda - \lambda_n^0| \geq \varepsilon > 0, n \in \mathbb{Z}\}$.
(3) $|\Delta(\lambda)| \geq C_\varepsilon \exp(|\tau|\sigma(\pi)) = C_\varepsilon \exp\left[\frac{(1+\alpha)\pi|\tau|}{2}\right]$ *for $\lambda \in G_\varepsilon$, where C_ε is a constant.*

3. Main Results

We agree that if a certain symbol v denotes an object related to L, then $\tilde{v}$ denote the analogous object related to $\tilde{L}$. In this paper, the main results are as follows.

Theorem 1. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$, $Q(x) = \tilde{Q}(x)$ on $\left(0, \frac{(1+\alpha)}{4}\pi\right)$, then $Q(x) = \tilde{Q}(x)$ a.e. on $(0,\pi)$, $\beta = \tilde{\beta}$ and $\alpha = \tilde{\alpha}$.*

Theorem 2. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$, $Q(x) = \tilde{Q}(x)$ on $\left(\frac{(1+\alpha)}{4}\pi, \pi\right)$, then $Q(x) = \tilde{Q}(x)$ a.e. on $(0,\pi)$, $\beta = \tilde{\beta}$ and $\alpha = \tilde{\alpha}$.*

Before proving the results, we shall mention the following lemma, which will be needed later.

Lemma 2. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$, then $\rho(x) = \tilde{\rho}(x)$ and $\beta = \tilde{\beta}$.*

Proof. It follows from (11) that $\alpha = \tilde{\alpha}$, that is $\rho(x) = \tilde{\rho}(x)$. We know that $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ are entire functions of λ of order 1. According to the Hadamard's factorization theorem, the characteristic functions can be uniquely determined by the eigenvalues up to multiplicative constants. Similar to reference [17], since $\lambda_n = \tilde{\lambda}_n$ for all $n \in \mathbb{Z}$, we can ascertain that $\Delta(\lambda) = C\tilde{\Delta}(\lambda)$, where $C \neq 0$ is a constant. From (10), we have $\beta^+ = C\tilde{\beta}^+$ and $\beta^- = C\tilde{\beta}^-$. Thus,

$$\frac{1}{2}(\beta \pm \frac{1}{\alpha\beta}) = \frac{C}{2}(\tilde{\beta} \pm \frac{1}{\alpha\tilde{\beta}}).$$

Consequently, $\beta = C\tilde{\beta}$ and $\frac{1}{\beta} = C\frac{1}{\tilde{\beta}}$. In view of $\beta, \tilde{\beta} > 0$, we can obtain that $\beta = \tilde{\beta}$. □

Proof of Theorem 1. By virtue of Lemma 2, we know that $\rho(x) = \tilde{\rho}(x)$ and $\beta = \tilde{\beta}$. For convenience, denote $d = \frac{(1+\alpha)\pi}{4}$. Substituting $\lambda = \lambda_n$ into (9), we can ascertain that for $n \in \mathbb{Z}$,

$$\varphi_2(d, \lambda_n)\psi_1(d, \lambda_n) - \varphi_1(d, \lambda_n)\psi_2(d, \lambda_n) = 0.$$

If $\varphi_2(d, \lambda_n) \neq 0$, then

$$\frac{\varphi_1(d, \lambda_n)}{\varphi_2(d, \lambda_n)} = \frac{\psi_1(d, \lambda_n)}{\psi_2(d, \lambda_n)}, \quad n \in \mathbb{Z}. \tag{12}$$

The same relation holds for $\tilde{L}$:

$$\frac{\tilde{\varphi}_1(d, \lambda_n)}{\tilde{\varphi}_2(d, \lambda_n)} = \frac{\tilde{\psi}_1(d, \lambda_n)}{\tilde{\psi}_2(d, \lambda_n)}, \quad n \in \mathbb{Z}. \tag{13}$$

Since $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $(0, d)$, we can obtain that $\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda)$. That is, $\varphi_1(x, \lambda) = \tilde{\varphi}_1(x, \lambda)$ and $\varphi_2(x, \lambda) = \tilde{\varphi}_2(x, \lambda)$ for $x \in [0, d]$. Together, (12) with (13) yields

$$\psi_2(d, \lambda_n)\tilde{\psi}_1(d, \lambda_n) - \tilde{\psi}_2(d, \lambda_n)\psi_1(d, \lambda_n) = 0. \tag{14}$$

Note that $\varphi_2(d, \lambda_n) = 0$ implies $\psi_2(d, \lambda_n) = \tilde{\psi}_2(d, \lambda_n) = 0$, so this case also leads to (14).

Define

$$A(\lambda) = \psi_2(d, \lambda)\tilde{\psi}_1(d, \lambda) - \tilde{\psi}_2(d, \lambda)\psi_1(d, \lambda).$$

It is obvious that $A(\lambda)$ has zeros $\{\lambda_n\}_{n\in\mathbb{Z}}$. Next, we will show that $A(\lambda) \equiv 0$ in the whole complex plane.

From (6)–(8), and the similar representations for $\tilde{\psi}_1(x, \lambda)$ and $\tilde{\psi}_2(x, \lambda)$, we have

$$A(\lambda) = O(\exp 2|\tau|(\sigma(\pi) - \sigma(d))) = O(\exp |\tau|\sigma(\pi)), \ |\lambda| \to \infty. \tag{15}$$

Define $G(\lambda) := \frac{A(\lambda)}{\Delta(\lambda)}$, which is entire in $\mathbb{C}$. It follows from (15) and (3) in Lemma 1 that

$$|G(\lambda)| \leq B_1, \quad \text{for } \lambda \in G_\varepsilon,$$

where B_1 is a positive constant. Thus, according to Liouville's theorem, we know that $G(\lambda)$ is constant. Furthermore, it follows from (6)–(8), and the Riemann–Lebesque Lemma, that for $\lambda \in \mathbb{R}$,

$$\lim_{\lambda\to\infty} G(\lambda) = 0,$$

which means $G(\lambda) = 0$. Thus, $A(\lambda) = 0$ for all λ in $\mathbb{C}$. Hence

$$\frac{\psi_2(d, \lambda)}{\psi_1(d, \lambda)} = \frac{\tilde{\psi}_2(d, \lambda)}{\tilde{\psi}_1(d, \lambda)}.$$

Note that $\frac{\psi_2(d,\lambda)}{\psi_1(d,\lambda)}$ is the Weyl function, defined in reference [1], of the boundary value problem for (1) on (d,π) with $y_1(d,\lambda)=0$ and the jump condition (3). It has been proved in reference [1] that the Weyl function can uniquely determine the $p(x)$ and $q(x)$ on (d,π). Thus, we can get that $Q(x)=\tilde{Q}(x)$ a.e. on (d,π). This completes the proof. □

Proof of Theorem 2. According to Theorem 1 and Lemma 2, we have $\alpha=\tilde{\alpha}$, $\beta=\tilde{\beta}$, $p(x)=\tilde{p}(x)$ and $q(x)=\tilde{q}(x)$ on (d,π). So, $\psi(x,\lambda)=\tilde{\psi}(x,\lambda)$ on (d,π). From (12) and (13), we show that

$$\varphi_1(d,\lambda_n)\tilde{\varphi}_2(d,\lambda_n)-\tilde{\varphi}_1(d,\lambda_n)\varphi_2(d,\lambda_n)=0.$$

From (4) and (5), and the similar representations for $\tilde{\varphi}_1(x,\lambda)$ and $\tilde{\varphi}_2(x,\lambda)$, we have

$$A_1(\lambda)=O(\exp 2|\tau|\sigma(d))=O(\exp|\tau|\sigma(\pi)),\quad |\lambda|\to\infty. \tag{16}$$

Define $G_1(\lambda):=\frac{A_1(\lambda)}{\Delta(\lambda)}$, which is entire in $\mathbb{C}$. It follows from (16) and (3) in Lemma 1 that

$$|G_1(\lambda)|\le B_2,\quad \text{for } \lambda\in G_\varepsilon,$$

where B_2 is a positive constant. Following the proof of Theorem 1, we have $A_1(\lambda)=0$ for all λ in $\mathbb{C}$, so

$$\frac{\varphi_2(d,\lambda)}{\varphi_1(d,\lambda)}=\frac{\tilde{\varphi}_2(d,\lambda)}{\tilde{\varphi}_1(d,\lambda)}.$$

Note that $\frac{\varphi_2(d,\lambda)}{\varphi_1(d,\lambda)}$ is the Weyl function, defined in reference [1], of the boundary value problem for (1) on $(0,d)$ with $y_1(d,\lambda)=0$ and the jump condition (3). It has been proved in reference [1] that the Weyl function can uniquely determine the $p(x)$ and $q(x)$ on $(0,d)$. Thus, we can get that $Q(x)=\tilde{Q}(x)$ a.e. on $(0,d)$. This completes the proof. □

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