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On Topological and Metric Properties of ⊕-sb-Metric Spaces

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Abstract: In this paper, we study ⊕-sb-metric spaces, which were introduced to generalize the concept of strong b-metric spaces. In particular, we study the properties of the topology induced via an ⊕-sb metric (separation properties, countability axioms, etc.), prove the continuity of the ⊕-sb-metric, establish the metrizability of the ⊕-sb-metric spaces of countable weight, discuss the convergence structure of an ⊕-sb-metric space and prove the Baire category type theorem for such spaces. Most of the results obtained here are new already for strong b-metric spaces, i.e., in the case where an arithmetic sum "+" is taken in the role of ⊕.

Keywords: extended t-conorm; ⊕-metric; strong b-metric; ⊕-sb-metric; topology induced via ⊕-sb-metric; Baire category theorem

MSC: 54E35

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1. Introduction

Metric spaces, introduced by Maurice Frechét in 1906 [1], belong to the most fundamental concepts of modern mathematics. For the convenience of presentation, we recall here this well-known concept.

Definition 1. *A metric on a set X is function $d : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$, satisfying the following axioms:*

(1m) $d(x,x) = 0$ *for all* $x \in X$;
(2m) $d(x,y) = 0 \Longrightarrow x = y$ *for all* $x, y \in X$;
(3m) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
(4m) $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x, y, z \in X$.

The pair (X, d), where d is a metric on the set X, is called a metric space.

Soon after the inception of the notion of a metric, some mathematicians have shown interest in generalizing it, omitting some of the axioms and retaining others. This is how pseudometric spaces [2] (by eliminating axiom (2m)), semi-metric spaces (by eliminating axiom (4m)), and quasimetric spaces (by eliminating axiom (3m)) appeared. Much later, works appeared in which one or more axioms of the metric were revised and replaced by weaker axioms. Among these types of concepts, we include partial metrics [3,4], generalized metrics [5], S-metrics [6–8], S_b metrics [9], b-metrics [10], strong b-metrics [11], etc. An interested reader can learn a lot about this from the monograph by Kirk and Shazad [11]. In turn, as the title shows, in this work, our interests are sb-metric spaces and their generalized analogs, the so-called ⊕-sb-metrics. However, for the sake of completeness, we recall here the more general notion of a b-metric space introduced by S, Czerwik [10] (see also [12,13]).

Definition 2. *Mapping $d : X \times X \to \mathbb{R}^+$ is called a b-metric or, more precisely, a bk-metric if it satisfies axioms (1m)–(3m) of Definition 1 and the following weakened version of axiom (4m):*

(4b) $d(x,z) \leq k \cdot (d(x,y) + d(y,z))$ *for all* $x, y, z \in X$

where $k \geq 1$ is some fixed constant.

Obviously, if $k = 1$, then a b-metric is just an ordinary metric. On the other hand, permission of k to take different values greater than one leads to the fact that the concept of b-metric spaces allows us obtention of different interesting and important examples of metric-type mappings that fail to become metrics. For example, by setting $d(x,y) = | x - y |^2$ for $x, y \in \mathbb{R}$, we receive a b2-metric, which is not a metric. Another example is a b2-metric on the set $C[a,b]$ of continuous real-valued functions on an interval $[a,b] \subset \mathbb{R}$, which is defined using $\int_a^b (g(x) - f(x))^2 dx$ for $f, g \in C[a,b]$.

Unfortunately, there does not exist a "natural" topology induced via a b-metric. The reason for this obstacle is that "open balls" in a b-metric space need not be open (see the detailed comments on this problem in [14,15]). This was the reason for introducing in [11] the notion of a strong b-metric, or an sb-metric for short, which is the intermediate between a b-metric and a metric.

Definition 3. *Mapping $d : X \times X \to \mathbb{R}^+$ is called an sb-metric or, more precisely, an sbk-metric if it satisfies axioms (1m)–(3m) of Definition 1 and the following weakened version of axiom (4m):*

(4sb) $d(x,z) \leq d(x,y) + k \cdot d(y,z)$ *for all* $x, y, z \in X$,

where $k \geq 1$ is some fixed constant.

Obviously, every metric is an sbk-metric for any $k \geq 1$, and every sbk-metric is a bk-metric. On the other hand, the authors of [16] present a series of examples showing that sbk-metrics form a proper class between metrics and bk-metrics.

As far as the already known results about sb-metrics justify, properties of sb-metric spaces have more analogs with properties of metric spaces than in the case of general b-metric spaces. Indeed, in [17], it is shown that an sb-metric space (X,d) has a unique (up to isomorphism) completion (X^*, d^*), which is identical on its subspace (X,d). Some known theorems about fixed points for mapping of metric spaces are extended to the case of mappings of sb-metric spaces, e.g., [18–20]. Mapping $f : (X, d_X) \to (Y, d_Y)$ of sb-metric spaces is continuous if and only if it is continuous as the mapping of the corresponding induced topological spaces [16]. The product of a countable family of sbk-metric spaces (that is, for a fixed k) is the *sbk*-metric [16]. The main purpose of this paper is to further advance the study of topology-related properties of sb-metric spaces. However, following the ideas first presented in [16], in the last, fourth axiom of Definitions 1–3, we replace operation $+$ with a more general operation $\oplus$, which we call a generalized t-conorm. From the theoretical point of view, our observation defining the extended t-conorms is based on the following fact: Ordinary sum and supremum operations on $[0,\infty)$, which are used in the definitions of metric-type and ultrametric-type spaces, have properties similar to the properties of t-conorm defined on the unit interval [0, 1]. Based on this observation, we call such operations defined on $[0,\infty)$ extended t-conorms. Thus, when we define metric-type structures over extended t-conorms, the theories of metric-type and ultrametric-type structures are generalized under a single roof. In addition, such approach allows adjustment of the developed theory to other generalization of metric-type structures based on extended t-conorms, e.g., Example 3. More precisely, the theorems proven in this article are true not only for sb-metric (hence metric) and ultra-sb-metric (ultra-metric) spaces but also for all generalizations obtained using any operation $\oplus$ defined on $[0,\infty)$ that satisfies the properties in Definition 4.

Considering this paper as a definite continuation of our previous article [16], we feel the need to clarify the relationship between the two works. In [16], we focused on the two issues. The first one was replacing, in the third (triangular) axiom, in the definition of a metric-type structure (namely, metrics, pseudometrics, b-metrics, and strong b-metrics) addition $+$ with a more general operation $\oplus$, which we called an extended t-conorm. Proceeding in this direction, we constructed examples of extended t-conorms, which can be used in these definitions, studied some categorical properties of $\oplus$-metric-type spaces, and considered relations between them. The second problem considered in [16] was caused by the question posed by Kirk and Shahzad in [11]. Namely, we constructed a series of examples of strong b-metrics that fail to be metrics. In turn, in this paper, we focus on the study of topological and distance-type properties of $\oplus$-sb-metrics, particularly of strong b-metrics, and conclude that such properties of strong b-metric spaces are much closer to the corresponding properties of metrics than to the properties of b-metrics.

The paper is structured as follows. In Section 2, we collect information about extended t-conorms $\oplus$ necessary for our study. The following Sections 3 and 4 contain the main results of this work: here, we study basic topological and metric properties of $\oplus$-sb-metric spaces. In the last, Conclusion section, we outline some directions that could be of interest for the further research in the context of $\oplus$-sb-metric spaces, in particular in the context of sb-metrics.

2. Preliminaries: Extended *t*-Conorms

Definition 4 ([16])**.** *Let $\mathbb{R}^+ = [0,\infty)$. Binary operation $\oplus : [0,\infty) \times [0,\infty) \to \mathbb{R}^+$ is called an extended t-conorm if for all $a, b, c \in \mathbb{R}^+$ the following properties hold:*

($\oplus_1$) *$\oplus$ is commutative, that is $a \oplus b = b \oplus a$;*
($\oplus_2$) *$\oplus$ is associative, that is $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;*
($\oplus_3$) *$\oplus$ is monotone, that is $a \le b \Longrightarrow a \oplus c \le b \oplus c$;*
($\oplus_4$) *0 is the neutral element for $\oplus$, that is $a \oplus 0 = a$.*

Remark 1. *Note that in case operation $\oplus$ is defined on $[0,1] \times [0,1]$ and takes its values in $[0,1]$, then the definition of an extended t-conorm reduces to the concept of a t-conorm [21]. Just for this observation, we refer to $\oplus$ as an extended t-conorm.*

Sometimes, we also need the following special properties of operation $\oplus$:

Definition 5. *$\oplus$ is called semi-distributive if for all $a, b, k \in \mathbb{R}^+$*

($\oplus_{sd}$) $k \cdot (a \oplus b) \le k \cdot a \oplus k \cdot b$.
$\oplus$ is called distributive if for all $a, b, k \in \mathbb{R}^+$
($\oplus_d$) $k \cdot (a \oplus b) = k \cdot a \oplus k \cdot b$.
$\oplus$ is called compressible if
($\oplus_{\mathrm{cmp}}$) $a \le b \oplus c \iff \frac{a}{a+1} \le \frac{b}{b+1} \oplus \frac{c}{c+1}$.
$\oplus$ is called continuous if
($\oplus_{\mathrm{con}}$) *$\oplus : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous as a two-argument function.*

Remark 2. *Referring to commutativity and monotonicity of $\oplus$, it is easy to prove that $\oplus$ is continuous whenever it is continuous in at least one of the arguments.*

Proposition 1. *If an extended t-conorm is continuous, then for every $r > 0$ there exists $s > 0$ such that $s \oplus s < r$.*

Proof. We take any $0 < p < r$. Then, by continuity of $\oplus$, there exists $q > 0$ such that $p \oplus q < r$. To complete the proof, it is sufficient to take $s = \min\{p, q\}$ and to note that by monotonicity of the operation $\oplus$,

$$s \oplus s \le p \oplus q < r.$$

□

By induction, from this proposition, we can easily prove the next statement:

Corollary 1. *If an extended t-conorm is continuous, then for every $r > 0$ and every $n \in \mathbb{N}$ there exists $s > 0$ such that $\bigoplus_{i=1}^{n} s_n < r$, where $s_n = s$ for all $n \in \mathbb{N}$.*

For a constant $k \geq 1$, we can take $m \in \mathbb{N}$ such that $\frac{k}{m} \leq s$. Therefore, the previous statement can be formulated as follows:

Corollary 2. *If an extended t-conorm $\oplus$ is continuous, then for every $r > 0$, every constant $k \geq 1$, and every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\bigoplus_{i=1}^{n} \frac{k}{m} < r$.*

Moreover, noticing that in case when $\oplus$ is semi-distributive it holds $k \cdot \bigoplus_{i=1}^{n} \frac{1}{m} \leq \bigoplus_{i=1}^{n} \frac{k}{m}$, we have also the following:

Corollary 3. *If an extended t-conorm $\oplus$ is continuous and semi-distributive (at least for constants $k \geq 1$) , then for every $r > 0$), every constant $k \geq 1$ and every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $k \cdot \bigoplus_{i=1}^{n} \frac{1}{m} < r$.*

Below, we offer two basic examples and one additional example of (semi-)distributive continuous extended t-conorms $\oplus$.

Example 1. *We let $a \oplus_L b = a + b$. Thus, $\oplus_L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is an ordinary addition. It is obvious that $+$ satisfies all properties from Definition 4. We can easily see that operation $\oplus_L$ is distributive, compressible and is continuous on the whole space $\mathbb{R}^+ \times \mathbb{R}^+$. When restricted to the triangle $\{x + y \leq 1 \mid x, y \geq 0\} \subset [0,1] \times [0,1]$, operation $\oplus_L$ reduces to the Łukasiewicz t-conorm.*

Example 2. *We let $a \oplus_M b = a \vee b$, where $\vee$ denotes the maximum. It is obvious that $\oplus_M$ satisfies all properties in Definition 4. Thus, $\oplus$ is the extension of the maximum t-conorm $\oplus_M$ from $[0,1] \times [0,1]$ to $\mathbb{R}^+ \times \mathbb{R}^+$. We can easily see that operation $\oplus_M$ is distributive and is continuous. The compressibility of $\vee$ follows from the following obvious inequality:*

$$b \leq c \text{ if and only if } \frac{b}{b+1} \leq \frac{c}{c+1} \text{ for all } a, b, c \in \mathbb{R}^+.$$

Example 3. *We let $a \oplus_T b = a \vee b \vee a \cdot b$. It is obvious that $\oplus_T$ satisfies properties $\oplus_1$, $\oplus_3$ and $\oplus_4$ from Definition 4. We verify associativity of $\oplus_T$ as follows.*

$$\begin{aligned}
a \oplus_T (b \oplus_T c) &= a \oplus_T (b \vee c \vee b \cdot c) \\
&= a \vee (b \vee c \vee b \cdot c) \vee a \cdot (b \vee c \vee b \cdot c) \\
&= a \vee b \vee c \vee b \cdot c \vee a \cdot b \vee a \cdot c \vee a \cdot b \cdot c \\
&= a \vee b \vee a \cdot b \vee c \vee a \cdot c \vee b \cdot c \vee a \cdot b \cdot c \\
&= (a \vee b \vee a \cdot b) \vee c \vee (a \vee b \vee a \cdot b) \cdot c \\
&= (a \vee b \vee a \cdot b) \oplus_T c \\
&= (a \oplus_T b) \oplus_T c.
\end{aligned}$$

The continuity of $\oplus_T$ is obvious. We show that $\oplus_T$ satisfies the property of semi-distributivity for constants $k \geq 1$ as follows:

$$\begin{aligned}
k \cdot (a \oplus_T b) &= k \cdot (a \vee b \vee a \cdot b) = k \cdot a \vee k \cdot b \vee k \cdot a \cdot b \\
&\leq k \cdot a \vee k \cdot b \vee k^2 \cdot a \cdot b \\
&= k \cdot a \oplus_T k \cdot b.
\end{aligned}$$

The extended t-conorm $\oplus_T$ is not compressible. For example, $5 \leq 2 \oplus_T 3$ but $\frac{5}{5+1} \not\leq \frac{2}{2+1} \oplus_T \frac{3}{3+1}$.

3. Topology of an $\oplus$-sb-Metric Space

We let X be a set, $\oplus$ be a continuous extended t-conorm.

Definition 6 ([16]). *Mapping $d : X \times X \to \mathbb{R}^+$ is called an $\oplus$-sb metric, or, more precisely, an $\oplus$-sbk metric if*

(1m) $d(x,x) = 0$ *for all* $x \in X$;
(2m) $d(x,y) = 0 \Longrightarrow x = y$ *for all* $x, y \in X$;
(3m) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
(4$\oplus$sb) $d(x,z) \leq k \cdot d(x,y) \oplus d(y,z)$ *for all* $x, y, z \in X$.

Pair (X,d) is called an $\oplus$-sb-metric space.

Examples of $\oplus$-metric type structures can be found in Section 5 of our previous paper [16] specially devoted to this problem.

Remark 3. *Applying axioms (3m) and (4$\oplus$sb), we have*

$$d(x,z) = d(z,x) \leq k \cdot d(z,y) \oplus d(y,x) = d(x,y) \oplus k \cdot d(y,z),$$

and hence, axiom (4$\oplus$sb) is equivalent to axiom

(4'$\oplus$sb) $d(x,z) \leq d(x,y) \oplus k \cdot d(y,z)$ *for all* $x, y, z \in X$.

3.1. Balls in an $\oplus$-sb-Metric Space

Definition 7. *We let (X,d) be an $\oplus$-sb-metric space and let $a \in X$ and $r > 0$. Then, set*

$$B(a,r) = \{x \mid x \in X, d(a,x) < r\}$$

is called an open ball with center $a \in X$ and radius r.

We let $\mathcal{T}_d$ be the topology on X induced by family $\mathcal{B} = \{B(x,r) \mid x \in X, r > 0\}$ of all open balls as a subbase. As we show further in this subsection, $\mathcal{B}$ is actually a base of $\mathcal{T}_d$.

Proposition 2. *An open ball is open in the topological space (X,d), i.e., given ball $B(a,r)$ and point $x_0 \in B(a,r)$, there exists ball $B(x_0,\varepsilon) \subseteq B(a,r)$.*

Proof. Since $x_0 \in B(a,r)$, it follows that $d(a,x_0) < r$ and hence, by continuity of $\oplus$, we can find $\varepsilon > 0$ such that $d(a,x_0) \oplus \varepsilon < r$. We now let $\delta = \frac{\varepsilon}{k}$. We show that $B(x_0,\delta) \subseteq B(a,r)$. Indeed, we let $z \in B(x_0,\delta)$. Then, taking into account Remark 3, we have

$$d(a,z) \leq d(a,x_0) \oplus k \cdot d(x_0,z) \leq d(a,x_0) \oplus k \cdot \delta \leq d(a,x_0) \oplus \varepsilon < r.$$

□

Proposition 3. *Intersection of a finite family of open balls is an open set in topology $\mathcal{T}_d$.*

Proof. We let $\{B(a,r_1), \ldots, B(a,r_n)\}$ be a family of open balls and let $x_0 \in \bigcap_{i=1}^n B(a,r_i)$. By Proposition 2 for each $i \in \{1,\ldots,n\}$, we can find $\varepsilon_i > 0$ such that $B(x_0,r_i) \subseteq B(a,r_i)$. Let $r_0 = \min\{\varepsilon_1,\ldots,\varepsilon_n\}$. It is clear that $B(x_0,r_0) \subseteq \bigcap_{i=1}^n B(a,r_i)$. □

Corollary 4. *Family $\mathcal{B} = \{B(x,r) \mid x \in X, r > 0\}$ of all open balls is a base for topology $\mathcal{T}_d$.*

Since, obviously, family $\mathcal{B} = \{B(a,\frac{1}{n}) \mid n \in \mathbb{N}\}$ is a local base at point $a \in X$, we also obtain the following corollary:

Corollary 5. *The topological space $(X,\mathcal{T}_d)$ is first-countable.*

We let (X,d) be an $\oplus$-sb-metric space and $\bar{B}(a,r) = \{x \mid x \in X, d(a,x) \leq r\}$. We call set $\bar{B}(a,r)$ closed ball with centre a and radius r.

Theorem 1. *We set $\bar{B}(a,r)$ closed in topology $\mathcal{T}_d$.*

Proof. We let $y \notin \bar{B}(a,r)$, then $d(a,y) > r$. By continuity of $\oplus$, we can find $\varepsilon > 0$ such that $r \oplus k \cdot \varepsilon < d(a,y)$. We claim that $B(y,\varepsilon) \cap \bar{B}(a,r) = \emptyset$.

Indeed, we let $z \in B(y,\varepsilon) \cap \bar{B}(a,r) = \emptyset$. Then,

$$d(a,y) \le d(a,z) \oplus k \cdot d(z,y) \le r \oplus k \cdot \varepsilon < d(a,y),$$

which contradicts our assumption. Hence, $\bar{B}(a,r)$ is a closed set in topology $\mathcal{T}_d$. □

Remark 4. *For completeness, we want to emphasize that the closed ball $\bar{B}(a,r)$ is not necessarily the closure of the open ball $B(a,r)$. In other words, $\oplus$-sb metrics are not necessarily "round" at all. To demonstrate this, consider the following example given in our previous paper [16].*

Example 4. *We let $X_a = \{a\} \times [0,1], X_b = \{b\} \times [0,1], X_c = \{c\} \times [0,1]$ and $X = X_a \cup X_b \cup X_c$. We denote $x = \{i\} \times \{\bar{x}\} \in X$ where $\bar{x} \in [0,1]$ and $i \in \{a,b,c\}$. We define $d : X \times X \to [0,5]$ as follows:*

$$d(x,y) = d(y,x) = \begin{cases} x - y & ,x,y \in X_i \\ 1 & ,x \in X_a, y \in X_b \\ 2 & x \in X_a, y \in X_c \\ 5 & x \in X_b, y \in X_c \end{cases},$$

where d is an sb3-metric (see Example 7 in [16] and notice that $\oplus = +$). For $x = \{a\} \times \{0\}$, we have $B(x,1) = \{a\} \times [0,1)$ and $\bar{B}(a,1) = X_a \cup X_b$ but $\overline{B(x,1)} = X_a$.

3.2. Continuity of an sb-$\oplus$-Metric

As different from the case of a b-metric, an sb-metric is continuous as function $d : X \times X \to R^+$. Within the framework of this paper, we have the following statement:

Theorem 2. *We let $\oplus$ be a continuous t-conorm and let (X,d) be an $\oplus$-sb-metric space. Then, the $\oplus$-sb-metric $d : X \times X \to (R^+, \oplus)$ is continuous.*

Proof. Since the topology induced by an $\oplus$-sb-metric is first countable (see Corollary 5), we can use sequences for the proof of the theorem. Namely, it is sufficient to show that if $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are sequences in the space (X,d) and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$. Explicitly this means that, given $\varepsilon > 0$, we have to find $\delta > 0$ such that

$$d(x,y) \le d(x_n,y_n) \le d(x,y) \oplus \varepsilon \tag{1}$$

whenever $d(x,x_n) < \delta$ and $d(y,y_n) < \delta$.

Instead, we prove inequality

$$d(x,y) \le d(x_n,y_n) \oplus \varepsilon \le (d(x,y) \oplus \varepsilon) \oplus \varepsilon = d(x,y) \oplus (\varepsilon \oplus \varepsilon). \tag{2}$$

Formally, it is weaker than the provable inequality (1); however, taking into account continuity and associativity of the extended t-conorm $\oplus$, both inequalities are equivalent. We proceed as follows.

We let $\varepsilon > 0$ be given. Referring to Corollary 2, we choose $\delta > 0$ such that $k \cdot \delta \oplus k \cdot \delta \le \varepsilon$. From the convergence of sequence $(x_n)_{n\in\mathbb{N}}$ to x and the convergence of sequence $(y_n)_{n\in\mathbb{N}}$ to y, we find $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ it holds that $d(x,x_n) < \delta, d(y,y_n) < \delta$. Now, referring to the triangle inequality of the $\oplus$-sb-metric, Remark 3, the commutativity of $\oplus$, we have the following sequence of inequalities:

$$\begin{aligned} d(x_n,y_n) \le d(y,x_n) \oplus k \cdot d(y_n,y) \le (d(y,x) \oplus k \cdot d(x,x_n)) \oplus k \cdot d(y,y_n) = \\ d(x,y) + k \cdot (d(x,x_n) \oplus d(y,y_n)) \le d(x,y) \oplus (k \cdot \delta \oplus k \cdot \delta) \le d(x,y) \oplus \varepsilon. \end{aligned}$$

In a similar way, we have

$$d(x,y) \le d(x,y_n) \oplus k \cdot d(y_n,y) \le (d(x_n,y_n) \oplus k \cdot d(x_n,x)) \oplus k \cdot d(y_n,y) =$$
$$d(x_n,y_n) \oplus (k \cdot d(x_n,x)) \oplus k \cdot d(y_n,y)) \le d(x_n,y_n) \oplus (k \cdot d(x_n,x)) \oplus (k \cdot d(y_n,y)) \le$$
$$d(x_n,y_n) \oplus (k \cdot \delta \oplus (k \cdot \delta) \le d(x_n,y_n) \oplus \varepsilon.$$

From the above two inequalities, we obtain the required

$$d(x,y) \le d(x_n,y_n) \oplus \varepsilon \le (d(x,y) \oplus \varepsilon) \oplus \varepsilon = \le d(x,y) \oplus (\varepsilon \oplus \varepsilon).$$

□

3.3. Separation and Cardinality Properties of ⊕-sb-Metric Spaces

Proposition 4. *We let ⊕ be a continuous extended t-conorm. Then, topology $\mathcal{T}_d$ of an ⊕-sb-metric space (X,d) is Hausdorff.*

Proof. We let $a,b \in X, a \ne b$ and hence $d(a,b) = r > 0$. By Proposition 1, there exists $s > 0$ such that $s \oplus s < r$. We show that $B(a,\frac{s}{k}) \cap B(b,s) = \emptyset$. Indeed, if $x \in B(a,\frac{s}{k}) \cap B(b,s)$, then it would be

$$d(a,b) \le k \cdot d(x,a) \oplus d(x,b) \le k \cdot \tfrac{s}{k} \oplus s \le s \oplus s < r,$$

contrary to our assumption. □

Theorem 3. *We let ⊕ be a continuous extended t-conorm. Then, topology $\mathcal{T}_d$ of an ⊕-sb metric space (X,d) is normal.*

Proof. We let $A \subset X, C \subset X$ be closed sets in (X,τ_d) and $A \cap C = \emptyset$. Given $x \in A$, we let $d(x,C) = \alpha_x$. By the continuity of the ⊕-sb-metric, it follows that $\alpha_x > 0$. Referring to Proposition 1, there exists $\varepsilon_x > 0$ such that $\varepsilon_x \oplus \varepsilon_x < \alpha_x$. Further, given $y \in C$, we let $d(y,A) = \beta_y$. By continuity of the ⊕-sb-metric, we have $\beta_y > 0$. Again, according to Proposition 1, there exists $\varepsilon_y > 0$ such that $\varepsilon_y \oplus \varepsilon_y < \beta_y$. Without loss of generality, let us assume that $\varepsilon_x \ge \varepsilon_y$. We define open neighbourhoods of closed sets A and C by setting $U_A = \bigcup_{x\in A} B(x,\frac{\varepsilon_x}{k})$ and $U_C = \bigcup_{y\in C} B(y,\varepsilon_y)$. We claim that $U_A \cap U_C = \emptyset$.

Indeed, we let $z \in U_A \cap U_C$. Then, there exist $x \in A$ and $y \in C$ such that $z \in B(x,\frac{\varepsilon_x}{k}) \cap B(y,\varepsilon_y)$. Since $\varepsilon_x \ge \varepsilon_y$, we have $z \in B(x,\frac{\varepsilon_x}{k}) \cap B(y,\varepsilon_x)$. However, this means that

$$d(x,y) \le k \cdot d(x,z) \oplus d(z,y) \le k \cdot \frac{\varepsilon_x}{k} \oplus \varepsilon_x = \varepsilon_x \oplus \varepsilon_x < \alpha_x.$$

But this contradicts our assumption that $d(x,C) = \alpha_x$. Thus, the constructed open neighbourhoods U_A and U_C are disjoint.

Now, we conclude the proof noticing that by Proposition 4, space (X,τ_d) satisfies the T_1 separation axiom. □

From Theorem 3 and referring the Urysohn theorem stating that a second countable regular topological space is metrizable (see, e.g., [22]), we obtain the following interesting fact:

Theorem 4. *A second countable ⊕-sb-metric space, where ⊕ is a continuous extended t-conorm, is metrizable.*

Theorem 5. *We let $\mathcal{T}_d$ be a topology induced by an ⊕-sb-metric d, where ⊕ is a continuous extended t-conorm. Then, the following properties are equivalent for topological space $(X,\mathcal{T}_d)$:*

1. *$(X,\mathcal{T}_d)$ is second countable, i.e., it has a countable base.*
2. *$(X,\mathcal{T}_d)$ is separable.*
3. *$(X,\mathcal{T}_d)$ is Lindelöf.*

Proof.

- Implication $(1) \Longrightarrow (2)$ is true for any topological space.

- To show implication (1) $\Longrightarrow$ (2), we let (X, d) be a separable $\oplus$-sb-metric space and let $A = \{a_i \mid i \in \mathbb{N}\}$ be a dense countable subset of (X, d). For each a_i, we fix a countable local base $\mathcal{B} = \{B(a_i, r_{ij}) \mid j \in \mathbb{N}, r_{ij} \in \mathbb{Q}^+\}$, where $\mathbb{Q}^+$ denotes the set of positive rational numbers. We claim that family $\mathcal{B} = \{B(a_i, r_{ij}) \mid i \in \mathbb{N}, j \in \mathbb{N}\}$ is a base for topology $\mathcal{T}_d$.
  We let $x_0 \in X$ and let $U \in \mathcal{T}_d$ be its open neighbourhood. We take some open ball $B(x_0, r) \subseteq U$. Without loss of generality, we may assume that $r \in \mathbb{Q}^+$. Referring to Proposition 1, we can find $s \in \mathbb{Q}^+$ such that $s \oplus s < r$. Since A is dense in $(X, \mathcal{T}_d)$, there exists $a_i \in A$ such that $d(a_i, x_0) < \frac{s}{k}$. We choose $B(a_i, \frac{s}{k}) \in \mathcal{B}$. Since, obviously, $x_0 \in B(a_i, \frac{s}{k})$, to complete the proof, we have to show that $B(a_i, \frac{s}{k}) \subseteq B(x_0, r)$.
  Indeed, we let $z \in B(a_i, \frac{s}{k})$. Then,
  $$d(z, x_0) \leq k \cdot d(z, a_i) \oplus d(a_i, x_0) \leq k \cdot \tfrac{s}{k} \oplus s = s \oplus s < r.$$
- Implication (1) $\Longrightarrow$ (3) is true for any topological space.
- To prove implication (3) $\Longrightarrow$ (2), we let (X, d) be a Lindelöf $\oplus$-sb-metric space. For every $m \in \mathbb{N}$, we consider cover $\mathcal{U}_m = \{B(x, \frac{1}{m}) \mid x \in X\}$. we let $\mathcal{V}_m = \{B(x_i^m, \frac{1}{m}) \mid i \in \mathbb{N}\}$ be its countable subcover and let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ We claim that countable set
  $$D = \{x_i^m \mid i \in \mathbb{N}, m \in \mathbb{N}\}$$
  is dense in $(X, \mathcal{T}_d)$.
  Indeed, we take any $x \in X$ and let $\varepsilon > 0$. We fix some $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Since $\mathcal{V}_m = \{B(x_i^m, \frac{1}{m}) \mid i \in \mathbb{N}\}$ is a cover of X, there exists $B(x_i^m, \frac{1}{m})$ containing point x and hence $d(x, x_i^m) < \frac{1}{m}$. However, this means that $D = \{x_i^m \mid i \in \mathbb{N}, m \in \mathbb{N}\}$ is a countable dense subset of $(X, \mathcal{T}_d)$, and hence the space of $(X, \mathcal{T}_d)$ is separable.

□

Since a subspace of a second countable space has obviously a countable base, from the previous theorem, we obtain

Corollary 6. *Separability and Lindelöfness are hereditary properties in the class of $\oplus$-sb-metric spaces.*

4. Metric Properties of $\oplus$-sb-Metric Spaces

First, let us clarify in the context of our work the well-known concepts from the theory of metric spaces.

Definition 8. *We let X be a non empty set, $\oplus$ be a continuous extended t-conorm, $d : X \times X \to \mathbb{R}^+$ be an $\oplus$-sb-metric, $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence and $x \in X$.*
(1) $\{x_n\}_{n\in\mathbb{N}}$ is said to converge to x if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. In this case, we denote $\lim\limits_{n\to\infty} x_n = x$.
(2) $\{x_n\}_{n\in\mathbb{N}}$ is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(3) (X, d) is said to be a complete $\oplus$-sb-metric if every Cauchy sequence converges in this space.

Theorem 6. *We let $\oplus$ be a continuous extended t-conorm, $d : X \times X \to \mathbb{R}^+$ be an $\oplus$-sb-metric and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence. If $\{x_n\}_{n\in\mathbb{N}}$ converges, then its limit is unique.*

Proof. We assume that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} x_n = y$. We show that $x = y$. We have

$$\begin{aligned}
d(x, y) &\leq d(x, x_n) \oplus k \cdot d(x_n, y), \\
\lim_{n\to\infty} d(x, y) &\leq \lim_{n\to\infty} (d(x, x_n) \oplus k \cdot d(x_n, y)), \\
\lim_{n\to\infty} d(x, y) &\leq \lim_{n\to\infty} d(x, x_n) \oplus k \cdot \lim_{n\to\infty} d(x_n, y), \\
\lim_{n\to\infty} d(x, y) &\leq 0 \oplus k \cdot 0, \\
\lim_{n\to\infty} d(x, y) &\leq 0.
\end{aligned}$$

This implies that $d(x, y) = 0$, and therefore we have $x = y$. □

Theorem 7. *We let $\oplus$ be a continuous extended t-conorm, $d : X \times X \to \mathbb{R}^+$ be an $\oplus$-sb-metric and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence. If $\{x_n\}_{n\in\mathbb{N}}$ converges, then it is a Cauchy sequence.*

Proof. We assume that $\lim_{n\to\infty} x_n = x$. Then, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) \oplus k \cdot d(x, x_n), \\ \lim_{n,m\to\infty} d(x_n, x_m) &\leq \lim_{n,m\to\infty} (d(x_n, x) \oplus k \cdot d(x, x_m)), \\ \lim_{n,m\to\infty} d(x_n, x_m) &\leq \lim_{n\to\infty} d(x_n, x) \oplus k \cdot \lim_{m\to\infty} d(x, x_m), \\ \lim_{n,m\to\infty} d(x_n, x_m) &\leq 0 \oplus k \cdot 0, \\ \lim_{n,m\to\infty} d(x_n, x_m) &\leq 0. \end{aligned}$$

Therefore, $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $\{x_n\}$ is a Cauchy sequence. □

Theorem 8 (Baire Category Theorem for $\oplus$-sb-metric spaces)**.** *We let $\oplus$ be a continuous extended t-conorm and (X, d) be a complete $\oplus$-sb-metric space. Then, the intersection of a countable family of dense open sets is dense.*

Proof. We let $U_1, U_2, U_3, \ldots$ be a countable family of dense open sets and W be an arbitrary open set. We show that $\bigcap_i U_i \cap W \neq \emptyset$. Since U_1 is dense,

$$W \cap U_1 \neq \emptyset,$$

and we can choose $x_1 \in W \cap U_1$. Since (X, d) is normal (by Theorem 3) and hence is also regular, there exists an open ball $B(x_1, r_1)$ such that

$$x_1 \in \overline{B(x_1, r_1)} \subseteq W \cap U_1,$$

where $r_1 < 1$. Similarly, since U_2 is dense,

$$B(x_1, r_1) \cap U_2 \neq \emptyset,$$

and we can choose $x_2 \in B(x_1, r_1) \cap U_2$. Further, since (X, d) is regular, there exists an open ball $B(x_2, r_2)$ such that

$$x_2 \in \overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap U_2,$$

where $r_2 < \frac{1}{2}$. Continuing in this way, we can choose x_n for all $n \in \mathbb{N}$ in such a way that

$$x_n \in B(x_n, r_n), r_n < \frac{1}{n} \text{ ve } \overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap U_n.$$

We consider sequence $\{x_n\}_{n\in\mathbb{N}}$. Since $x_n, x_m \in B(x_{n_0}, r_{n_0})$, in case $n, m > n_0$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n_0}) \oplus k \cdot d(x_{n_0}, x_m), \\ d(x_n, x_m) &\leq r_0 \oplus k \cdot r_0, \\ d(x_n, x_m) &\leq \frac{1}{n_0} \oplus k \cdot \frac{1}{n_0}, \\ \lim_{n,m\to\infty} d(x_n, x_m) &\leq \lim_{n,m\to\infty} \left(\frac{1}{n_0} \oplus k \cdot \frac{1}{n_0} \right), \\ \lim_{n,m\to\infty} d(x_n, x_m) &\leq \lim_{n_0\to\infty} \frac{1}{n_0} \oplus k \cdot \lim_{n_0\to\infty} \frac{1}{n_0} = 0. \end{aligned}$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete $\oplus$-sb-metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. For every $n \in \mathbb{N}$, subsequence $\{x_m\}_{m\geq n}$ also converges and, in addition, it converges to the same point x. Since all terms of this subsequence are contained in $B(x_n, r_n)$, we conclude that $x \in \overline{B(x_n, r_n)}$. Hence, we have

$$x \in \overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap U_n \subseteq W \cap U_n$$

and

$$x \in W \cap \left(\bigcap_i U_n\right) \neq \emptyset,$$

that is, $\bigcap_i U_i \cap W \neq \emptyset$. □

We specify the standard definition of uniform convergence of a sequence of functions for the case of $\oplus$-sb-metric spaces as follows:

Definition 9. *We let X be a topological space, $\oplus$ be a continuous extended t-conorm, (Y, d) be an $\oplus$-sb-metric space and $f_n, f : X \to Y$ be a family of functions.*

We say that sequence $\{f_n\}$ uniformly converges to f if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow d(f_n(x), f(x)) < \varepsilon \; \forall x \in X.$$

Theorem 9. *We let X be a topological space, $\oplus$ be a continuous extended t-conorm, (Y, d) be an $\oplus$-sb-metric space and let $f_n : X \to Y$ be a family of continuous functions. If sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to function $f : X \to Y$, then function f is also continuous.*

Proof. We let $V \subseteq Y$ be an open set, $x_0 \in f^{-1}(V)$ and $y_0 = f(x_0)$. There exits an open ball $B(y_0, r) \subseteq V$ where $r > 0$. Referring to Corollary 2, we can find $m \in \mathbb{N}$ such that

$$\frac{k}{m} \oplus \frac{k}{m} \oplus \frac{k}{m} < r.$$

Since $\{f_n\}$ converges uniformly to f, for $\frac{1}{m} > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow d(f_n(x), f(x)) < \frac{1}{m}.$$

Since f_n is continuous at x_0, there exists a neighbourhood U of x_0 such that

$$f_n(U) \subseteq B\left(f_n(x_0), \frac{1}{m}\right).$$

For every $x \in U$, we have

$$\begin{aligned} d(f(x), f(x_0)) &\leq k \cdot d(f(x), f_n(x)) \oplus d(f_n(x), f(x_0)) \\ &\leq k \cdot d(f(x), f_n(x)) \oplus (k \cdot d(f_n(x), f_n(x_0)) \oplus d(f_n(x_0), f(x_0))) \\ &\leq k \cdot \frac{1}{m} \oplus \left(k \cdot \frac{1}{m} \oplus \frac{1}{m}\right) \\ &\leq \frac{k}{m} \oplus \frac{k}{m} \oplus \frac{1}{m} \\ &\leq \frac{k}{m} \oplus \frac{k}{m} \oplus \frac{k}{m} \\ &< r. \end{aligned}$$

However, this just means that $f(x) \in B(f(x_0), r) \subseteq V$. Therefore, function f is continuous. □

5. Conclusions

In this article, we continue to study $\oplus$-sb-metric spaces introduced in [16]. In the process of this study, we notice that, in the case of a *continuous* extended t-conorm, the topological properties of $\oplus$-sb-metric spaces are similar to the topological properties of metric spaces. This observation leads us to the question whether every ⊕-sb-metric space is metrizable. Theorem 4 offers a positive answer in the case of spaces with countable weight. In the future, we plan to study the problem of metrizability in a general setting. Another direction of research is to study which of the important properties of metric spaces can be extended to the class of ⊕-sb-metric spaces. In particular, this concerns the properties of the dimension of metric spaces, finite and infinite. In [23], and also in [24], interesting results on transfinite asymptotic dimension [25] are established for metric spaces. It is a tempting task to extend these and other results on asymptotic dimension to ⊕-metric spaces and further to ⊕-sb-metric spaces.

Another question that seems interesting is to view ⊕-sb-spaces as a category. As morphisms in this category, we can take continuous, uniformly continuous or non-expanding mappings. However, we note that when studying ⊕-sb-metric spaces from a categorical point of view, we should distinguish between two significantly different cases: the category of ⊕-sbk metric spaces (that is, when the constant k is the same for all considered spaces) and the category of ⊕-sb-metric spaces (that is, when the constant k can vary between different spaces). The difference between these cases was noticed already in our previous paper [16], see Section 6: the product of a countable family of ⊕-sbk-metric spaces is an ⊕-sbk-metric space, while ⊕-sb-metric spaces are invariant only under finite products. Quite interesting, in our opinion, would be to study relations between the category of ⊕-sb-metric spaces and the category of ⊕-metric spaces: the latter one can be considered as a complete subcategory of the first.

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