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Similarity Transformations and Nonlocal Reduced Integrable Nonlinear Schrödinger Type Equations

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Abstract: We present three reduced integrable hierarchies of nonlocal integrable nonlinear Schrödinger-type equations, starting from a given vector integrable hierarchy generated from a matrix Lie algebra of B type. The basic tool is the zero curvature formulation. Three similarity transformations are taken to keep the invariance of the involved zero curvature equations. The key is to formulate a matrix solution to a reduced stationary zero curvature equation such that the zero curvature formulation works for a reduced case.

Keywords: zero curvature equation; matrix eigenvalue problem; similarity transformation; integrable hierarchy; nonlinear Schrödinger equations

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1. Introduction

Lax pairs of linear eigenvalue problems are primary objects in the study of nonlinear integrable partial differential equations. It is important to formulate a linear eigenvalue problem from a given matrix Lie algebra [1]. Many integrable hierarchies of nonlinear partial differential equations have been engendered, beginning with the special linear algebras [2–4], and the special orthogonal algebras (see, e.g., [5]). Hamiltonian formulations are normally generated, which is done by applying the trace identity when the underlying matrix Lie algebra is semisimple and the variational identity when the underlying matrix Lie algebra is non-seimsimple. In combination with the recursion structures possessed by the integrable hierarchies, bi-Hamiltonian formulations can often be established, thereby guaranteeing their Liouville integrability.

Motivated by recent studies on nonlocal integrable equations (see, e.g., [6]), we look for similarity transformations and use them to construct integrable reductions from given integrable hierarchies. Such transformations keep the original zero curvature equations invariant. Many local and nonlocal reduced integrable nonlinear Schrödinger (NLS) and modified Korteweg–de Vries (mKdV) equations have been constructed from the Ablowitz–Kaup–Newell–Segur (AKNS) eigenvalue problems (see, e.g., [7–9] and [10–13] for local and nonlocal reduced examples, respectively). Recently, it has been verified that when taking pairs of similarity transformations, it is possible to present novel kinds of both local and nonlocal reduced integrable partial differential equations [14,15]. Such studies have given rise to many new interesting problems in the theory of equations of mathematical physics.

This paper aims to present an application of similarity transformations to generate reduced nonlocal integrable NLS-type equations. In Section 2, we restate a multi-component integrable Hamiltonian hierarchy [16]. In Section 3, we build three types of similarity transformations for the adopted spectral matrix and compute three reduced nonlocal integrable hierarchies. The first two representative examples of reduced nonlocal integrable NLS type equations out of the resultant reduced integrable hierarchies are

$$ir_t = r_{xx} - 2r\gamma\delta r^T(-x,-t)r + r\gamma r^T r(-x,-t)\delta,$$

and

$$ir_t = r_{xx} + 2r\gamma\delta r^T(x,-t)r - r\gamma r^T r(x,-t)\delta,$$

with γ and δ being two constant symmetric and orthogonal matrices which commute and r^T denoting the matrix transpose of the potential matrix r. The third is

$$ir_t = r_{xx} + 2r\gamma\delta r^\dagger(-x,t)r - r\gamma r^T r^*(-x,t)\delta,$$

with γ and δ being two real constant symmetric and orthogonal matrices, which again commute, and $r^\dagger$ and r^* denoting the Hermitian transpose and the complex conjugate of the potential matrix r, respectively. Lastly, in the final section, a conclusion and several remarks are presented.

2. A Vector-Integrable Hamiltonian Hierarchy

In the forthcoming analysis, we recall the multi-component integrable Hamiltonian hierarchy recently computed in [16]. The local integrable hierarchy is constructed from a linear eigenvalue problem, associated with a non-special linear algebra [16]. Let $n \in \mathbb{N}$ be a given number, γ be a given nth-order orthogonal and symmetric matrix, and λ denote the eigenvalue parameter. Supposing that we have the potential vector

$$u = u(x,t) = (r, s^T)^T = (r(x,t), s^T(x,t))^T, \tag{1}$$

where

$$r(x,t) = (r_1(x,t), \cdots, r_n(x,t)),\ s(x,t) = (s_1(x,t), \cdots, s_n(x,t))^T, \tag{2}$$

then, starting from the spatial linear eigenvalue problem below

$$-i\phi_x = M\phi = M(u,\lambda)\phi,\ M = \begin{bmatrix} -\lambda & r & 0 \\ s & 0 & \gamma^T r^T \\ 0 & s^T\gamma^T & \lambda \end{bmatrix}, \tag{3}$$

which is a counterpart of the AKNS eigenvalue problem [2], we can find a matrix solution to the associated stationary zero curvature equation

$$-iZ_x = [M, Z], \tag{4}$$

by assuming the solution to be of Laurent form:

$$Z = \begin{bmatrix} -e & f & 0 \\ g & h & \gamma^T f^T \\ 0 & g^T\gamma^T & e \end{bmatrix} = \sum_{l\geq 0} \lambda^{-l} Z^{[l]}, \tag{5}$$

with the coefficient matrices formulated similarly:

$$Z^{[l]} = \left[\begin{array}{ccc} -e^{[l]} & f^{[l]} & 0 \\ g^{[l]} & h^{[l]} & \gamma^T f^{[l]T} \\ 0 & g^{[l]T}\gamma^T & e^{[l]} \end{array} \right], \; l \geq 0. \tag{6}$$

Obviously, the corresponding stationary zero curvature equation with such a matrix solution determines a recursive way of computing the matrix solution Z:

$$\begin{cases} e_x^{[0]} = 0, \; f^{[0]} = 0, \; g^{[0]} = 0, \; h_x^{[0]} = 0, \\ f^{[l+1]} = i f_x^{[l]} + e^{[l]} r + r h^{[l]}, \; g^{[l+1]} = -i g_x^{[l]} + s e^{[l]} + h^{[l]} s, \\ e_x^{[l+1]} = i(f^{[l+1]} s - r g^{[l+1]}) = -f_x^{[l]} s - r g_x^{[l]}, \\ h_x^{[l+1]} = i(s f^{[l+1]} - g^{[l+1]} r + \gamma^T r^T g^{[l+1]T} \gamma^T - \gamma^T f^{[l+1]T} s^T \gamma^T), \end{cases} \tag{7}$$

in which $l \geq 0$. When choosing

$$e^{[0]} = 1, \; h^{[0]} = 0 \tag{8}$$

and taking the integration constants to be zero,

$$h^{[l]}|_{u=0} = 0, \; e^{[l]}|_{u=0} = 0, \; l \geq 1, \tag{9}$$

a series of differential polynomials $\{e^{[l]}, f^{[l]}, g^{[l]}, h^{[l]} \,|\, l \geq 1\}$ can be computed explicitly.

Upon taking the temporal linear eigenvalue problems

$$-i\phi_t = N^{[k]}\phi = N^{[k]}(u,\lambda)\phi, \; N^{[k]} = (\lambda^k Z)_+ = \sum_{l=0}^{k} \lambda^l Z^{[k-l]}, \; k \geq 0, \tag{10}$$

it can be seen that the compatibility conditions of the two linear eigenvalue problems in (3) and (10), namely, the corresponding zero curvature equations

$$N_t - N_x^{[k]} + i[M, N^{[k]}] = 0, \; k \geq 0, \tag{11}$$

yield the vector integrable hierarchy

$$u_t = \left[\begin{array}{c} r_t^T \\ s_t \end{array} \right] = X^{[k]} = \left[\begin{array}{c} i f^{[k+1]T} \\ -i g^{[k+1]} \end{array} \right], \; k \geq 0. \tag{12}$$

The first example consists of the generalized multi-component integrable nonlinear Schrödinger equations

$$\begin{cases} i r_t = r_{xx} + 2rsr - r\gamma r^T s^T \gamma, \\ i s_t = -s_{xx} - 2srs + \gamma r^T s^T \gamma s, \end{cases} \tag{13}$$

with γ being an arbitrary orthogonal and symmetric matrix.

The Hamiltonian structure for the integrable hierarchy (12), established by the trace identity, is provided by

$$u_t = X_k = J \frac{\delta \mathcal{H}^{[k]}}{\delta u}, \;\; k \geq 1, \tag{14}$$

where the Hamiltonian operator J reads

$$J = i \left[\begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right], \tag{15}$$

with I_n being the nth-order identity matrix, and the Hamiltonian functionals $\mathcal{H}^{[k]}$ are provided by

$$\mathcal{H}^{[l]} = -\int \frac{e^{[l+1]}}{l}\, dx,\ l \geq 1, \tag{16}$$

with $e^{[l]}$ being defined by the corresponding stationary zero curvature equation. The above Hamiltonian formulation presents a relation between symmetries and conserved quantities [17,18]. It is known that infinitely many symmetries commute with each other:

$$[\![X_k, X_l]\!] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \big[(X_k(u+\varepsilon X_l) - X_l(u+\varepsilon X_k)\big] = 0,\ k,l \geq 0, \tag{17}$$

which follows from a Lax operator algebra:

$$[\![V^{[k]}, V^{[l]}]\!] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \big[(V^{[k]}(u+\varepsilon X_l) - V^{[l]}(u+\varepsilon X_k)\big] + [V^{[k]}, V^{[l]}] = 0,\ k,l \geq 0. \tag{18}$$

Furthermore (see, e.g., [19,20]), the Hamiltonian formulation in (14) guarantees that infinitely many conserved functionals commute under the following Poisson bracket:

$$\{\mathcal{H}^{[k]}, \mathcal{H}^{[l]}\}_J = \int \Big(\frac{\delta \mathcal{H}^{[k]}}{\delta u}\Big)^T J \frac{\delta \mathcal{H}^{[l]}}{\delta u}\, dx = 0,\ k,l \geq 0, \tag{19}$$

which is associated with the previous Hamiltonian operator J.

3. Novel Nonlocal Integrable NLS-Type Equations

Let us take a new nth-order orthogonal and symmetric matrix δ which commutes with the previous orthogonal and symmetric matrix γ, and then introduce a higher-order orthogonal matrix Θ by

$$\Theta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \delta & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{20}$$

From this, we can see that the transformed matrix reads

$$\Theta M(\lambda) \Theta^{-1} = \begin{bmatrix} \lambda & s^T \gamma \delta & 0 \\ \gamma \delta r^T & 0 & \delta s \\ 0 & r\delta & -\lambda \end{bmatrix}. \tag{21}$$

3.1. Similarity Transformation 1

We first conduct the similarity transformation

$$\Theta M(x,t,\lambda)\Theta^{-1} = -M(-x,-t,\lambda) \tag{22}$$

for the spectral matrix M. Based on the result in (21), the above transformation equivalently yields

$$s = -\gamma\delta r^T(-x,-t) \text{ or } r = -s^T(-x,-t)\delta\gamma. \tag{23}$$

Further, it can be directly determined that

$$\Theta Z(x,t,\lambda)\Theta^{-1} = -Z(-x,-t,\lambda). \tag{24}$$

This is a consequence of the uniqueness property of the Cauchy problem for the stationary zero-curvature Equation (4). In our case, we have the two Laurent series solutions, $\Theta Z(x,t,\lambda)\Theta^{-1}$ and $Z(-x,-t,\lambda)$ of λ, which solve the corresponding stationary zero-curvature Equation (4), where the spectral matrix $M(x,t,\lambda)$ is replaced with $-M(-x,-t,\lambda)$

and we have the opposite initial values at $\lambda = \infty$. Thus, noting that $N^{[k]} = (\lambda^k Z)_+$, we can have

$$\Theta N^{[k]}(x,t,\lambda)\Theta^{-1} = -N^{[k]}(-x,-t,\lambda),\ k \geq 0. \tag{25}$$

This tells us the relation

$$\begin{aligned} &\Theta\big(M_t(x,t,\lambda) - N_x^{[k]}(x,t,\lambda) + i[M(x,t,\lambda), N^{[k]}(x,t,\lambda)]\big)\Theta^{-1} \\ &= M_t(-x,-t,\lambda) - N_x^{[k]}(-x,-t,\lambda) + i[M(-x,-t,\lambda), N^{[k]}(-x,-t,\lambda)],\ k \geq 0, \end{aligned} \tag{26}$$

between the reduced and unreduced zero curvature equations. A consequence of this is a reduced nonlocal integrable hierarchy

$$r_t = i f^{[k+1]}|_{s=-\gamma\delta r^T(-x,-t)}, \text{ or } s_t = -i g^{[k+1]}|_{r=-s^T(-x,-t)\delta\gamma},\ k \geq 0. \tag{27}$$

Each equation in this reduced integrable hierarchy possesses all characteristic integrable properties that integrable equations exhibit; in particular, the existence of infinitely many symmetries and conserved functionals in the reduced case is guaranteed by the unreduced case. The nonlocal reduced integrable NLS type equation in the resultant hierarchy is provided by

$$i r_t = r_{xx} - 2r\gamma\delta r^T(-x,-t)r + r\gamma r^T r(-x,-t)\delta, \tag{28}$$

with γ and δ being two orthogonal and symmetric matrices which commute with each other.

3.2. Similarity Transformation 2

Second, we conduct the similarity transformation:

$$\Theta M(x,t,\lambda)\Theta^{-1} = M(x,-t,-\lambda), \tag{29}$$

for the spectral matrix M, where Θ is determined by (20).

Upon observing (21), the above similarity transformation generates

$$s(x,t) = \gamma\delta r^T(x,-t) \text{ or } r(x,t) = s^T(x,-t)\gamma\delta. \tag{30}$$

Under either of the two potential reductions, we can obtain

$$\Theta Z(x,t,\lambda)\Theta^{-1} = -Z(x,-t,-\lambda). \tag{31}$$

The reason is similar, that is, it follows from the uniqueness property of the corresponding Cauchy problem. Clearly, the two Laurent series solutions $\Theta Z(x,t,\lambda)\Theta^{-1}$ and $Z(x,-t,-\lambda)$ of λ of the corresponding stationary zero curvature Equation (4), where $M(x,t,\lambda)$ is replaced with $M(x,-t,-\lambda)$, have the opposite initial values at $\lambda = \infty$.

The relation (30) guarantees that

$$\theta N^{[2l]}(x,t,\lambda)\Theta^{-1} = -N^{[2l]}(x,-t,-\lambda),\ l \geq 0, \tag{32}$$

and it then follows that

$$\begin{aligned} &\Theta\big(M_t(x,t,\lambda) - N_x^{[2l]}(x,t,\lambda) + i[M(x,t,\lambda), N^{[2l]}(x,t,\lambda)]\big)\Theta^{-1} \\ &= -\big(M_t(x,-t,-\lambda) - N_x^{[2l]}(x,-t,-\lambda) + i[M(x,-t,-\lambda), N^{[2l]}(x,-t,-\lambda)]\big),\ l \geq 0. \end{aligned} \tag{33}$$

Therefore, the reduced zero curvature equations lead to an integrable hierarchy of nonlocal reduced NLS-type equations:

$$r_t = i f^{[2l+1]}|_{s=\gamma\delta r^T(x,-t)}, \text{ or } s_t = -i g^{[2l+1]}|_{r=s^T(x,-t)\delta\gamma},\ l \geq 0. \tag{34}$$

Under the potential reduction (30), the original infinitely many symmetries and conserved functionals become the reduced ones for this reduced hierarchy. In the reduced hierarchy, the nonlocal reduced integrable NLS-type equation reads

$$ir_t = r_{xx} + 2r\gamma\delta r^T(x,-t)r - r\gamma r^T r(x,-t)\delta, \tag{35}$$

with γ and δ being two constant orthogonal and symmetric matrices which commute.

3.3. Similarity Transformation 3

We now conduct a third similarity transformation

$$\Theta M(x,t,\lambda)\Theta^{-1} = M^*(-x,t,-\lambda^*), \tag{36}$$

for the spectral matrix M, where Θ is again determined by (20) and $*$ denotes the complex conjugate.

Upon recognizing the result in (21), the above similarity transformation precisely engenders

$$s(x,t) = \gamma\delta r^\dagger(-x,t) \text{ or } r(x,t) = s^\dagger(-x,t)\gamma\delta, \tag{37}$$

where we suppose γ and δ to be real and $\dagger$ stands for the Hermitian transpose. Under either of these potential reductions, we can obtain

$$\Theta Z(x,t,\lambda)\Theta^{-1} = -Z^*(-x,t,-\lambda^*), \tag{38}$$

as the two Laurent series solutions $\Theta Z(x,t,\lambda)\Theta^{-1}$ and $Z^*(-x,t,-\lambda^*)$ of the corresponding stationary zero curvature Equation (4), where $M(x,t,\lambda)$ is replaced with $M^*(-x,t,-\lambda^*)$, take the opposite initial values at $\lambda = \infty$. This further ensures that

$$\Theta N^{[2l]}(x,t,\lambda)\Theta^{-1} = -N^{[2l]*}(x,-t,-\lambda^*),\ l \geq 0. \tag{39}$$

Then, it follows that

$$\begin{aligned} &\theta\big(M_t(x,t,\lambda) - N_x^{[2l]}(x,t,\lambda) + i[M(x,t,\lambda), N^{[2l]}(x,t,\lambda)]\big)\Theta^{-1} \\ &= \big(M_t(-x,t,-\lambda^*) - N_x^{[2l]}(-x,t,-\lambda^*) + i[M(-x,t,-\lambda^*), N^{[2l]}(-x,t,-\lambda^*)]\big)^*,\ l \geq 0; \end{aligned} \tag{40}$$

consequently we obtain the following integrable hierarchy of nonlocal reduced NLS-type equations:

$$r_t = if^{[2l+1]}|_{s=\gamma\delta r^\dagger(-x,t)}, \text{ or } s_t = -ig^{[2l+1]}|_{r=s^\dagger(-x,t)\delta\gamma},\ l \geq 0. \tag{41}$$

Their infinitely many symmetries and conserved functionals are similarly guaranteed by reducing the originals through the potential reduction (37). In the resulting reduced hierarchy, the first integrable equation is the nonlocal reduced integrable NLS-type equation

$$ir_t = r_{xx} + 2r\gamma\delta r^\dagger(-x,t)r - r\gamma r^T r^*(-x,t)\delta, \tag{42}$$

with γ and δ being two real constant orthogonal and symmetric matrices, which again commute, and with $r^\dagger$ and r^* standing for the Hermitian transpose and the complex conjugate of the potential matrix r, respectively.

All three nonlocal reduced integrable hierarchies of NLS-type equations presented above are distinct from those that have been presented previously, starting from the multi-component AKNS hierarchy (see, e.g., [6,14]). As counterparts of linear examples of differential equations (see, e.g., [21,22]), those provide new nonlinear extensions (see also, [23,24]) to mathematical theories of integrable differential equations.

4. Concluding Remarks

Under the specific spectral matrix reductions, three reduced integrable hierarchies of nonlocal multi-component NLS type equations have been computed from a new vector integrable Hamiltonian hierarchy. The three similarity transformations presented here are key to the formulation of nonlocal reduced integrable equations.

It would be of interest to look for new nonlocal integrable equations under similarity transformations from other linear eigenvalue problems. A further question is whether soliton solutions to reduced integrable equations could be guaranteed by Darboux transformations [25] or the Riemann–Hilbert technique [6]. Other interesting solutions include lump wave solutions [26,27], complexiton solutions [28], rogue wave solutions [29], Grammian solutions [30], and algebro-geometric solutions [31,32]. Reduced Lax pairs of linear eigenvalue problems and the Hirota bilinear method should be helpful in this regard.

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