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A Novel Interval-Valued Decision Theoretic Rough Set Model with Intuitionistic Fuzzy Numbers Based on Power Aggregation Operators and Their Application in Medical Diagnosis

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Abstract: Intuitionistic fuzzy information is a potent tool for medical diagnosis applications as it can represent imprecise and uncertain data. However, making decisions based on this information can be challenging due to its inherent ambiguity. To overcome this, power aggregation operators can effectively combine various sources of information, including expert opinions and patient data, to arrive at a more accurate diagnosis. The timely and accurate diagnosis of medical conditions is crucial for determining the appropriate treatment plans and improving patient outcomes. In this paper, we developed a novel approach for the three-way decision model by utilizing decision-theoretic rough sets and power aggregation operators. The decision-theoretic rough set approach is essential in medical diagnosis as it can manage vague and uncertain data. The redesign of the model using interval-valued classes for intuitionistic fuzzy information further improved the accuracy of the diagnoses. The intuitionistic fuzzy power weighted average (IFPWA) and intuitionistic fuzzy power weighted geometric (IFPWG) aggregation operators are used to aggregate the attribute values of the information system. The established operators are used to combine information within the intuitionistic fuzzy information system. The outcomes of various alternatives are then transformed into interval-valued classes through discretization. Bayesian decision rules, incorporating expected loss factors, are subsequently generated based on this foundation. This approach helps in effectively combining various sources of information to arrive at more accurate diagnoses. The proposed approach is validated through a medical case study where the participants are classified into three different regions based on their symptoms. In conclusion, the decision-theoretic rough set approach, along with power aggregation operators, can effectively manage vague and uncertain information in medical diagnosis applications. The proposed approach can lead to timely and accurate diagnoses, thereby improving patient outcomes.

Keywords: intuitionistic fuzzy sets; three-way decision; decision-theoretic rough sets; power aggregation operators; decision making; optimization; efficiency

MSC: 03E72; 94D05

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1. Introduction

1.1. Evaluation of Medical Diagnosis

The accurate diagnosis of medical diseases can be a complex process as they can manifest with a variety of symptoms. Common symptoms may include fever, coughing, fatigue, vomiting, diarrhea, and skin rashes. Physicians rely on a range of diagnostic procedures such as patient history, physical examination, laboratory tests, imaging studies, and other techniques to arrive at a diagnosis. Medical professionals need to consider several factors, including the patient's age, medical history, lifestyle, and genetic predisposition while making decisions during the diagnosis process [1]. In complex medical conditions, physicians may use diagnostic algorithms or decision trees to support the decision-making process. In recent years, medical technology advancements, including new imaging techniques, genetic testing, and AI-based decision-making tools, have significantly improved the accuracy and speed of medical diagnoses [2,3]. These advancements help medical professionals to make more informed decisions and provide patients with better healthcare outcomes [4,5].

1.2. Three-Way Decision in the Medical Field

Rough set (RS) theory is a mathematical structure that deals with incomplete and uncertain data in a systematic way. The theory was established by Zdzislaw Pawlak [6] in the early 1980s as a method for dealing with vague and uncertain information. In medical diagnosis, RS theory can be used to help detect the presence or absence of a particular disease or condition based on incomplete or uncertain information [7]. This can include symptoms, medical history, test results, and other relevant data [8]. The fundamental theory behind RS theory is to divide a set of data into subsets based on their attributes, such as symptoms or test results. This process can help to identify the prominent features or factors that are most closely associated with a particular disease or condition. Once the data have been divided into subsets, RS theory can be used to identify rules or patterns that can be used to make predictions about whether a particular patient has a particular disease or condition. Many researchers worked on this notion to identify novel algorithms for diagnosis of diseases [9]. El-Bably et al. [8,10] introduced the soft and rough approximation and applied it to diagnose the medical problem. Hosny et al. [11] worked on the extension of RS using the maximal right neighborhood system and its application in the medical field. Al-Shami et al. [12] defined maximal rough neighborhoods and applied this approach to medical diseases.

Attansove [13] developed the idea of an intuitionistic fuzzy set (IFS) which is the generalization of a fuzzy set (FS). In IFS, there are two grades of membership and grades of non-membership of an element of universal set, respectively. Intuitionistic fuzzy sets played a very important part in the medical field to identify diseases and problems. The application of IFS in medical diagnosis has been studied in various contexts. One area of application is in the diagnosis of medical conditions where there is significant uncertainty and variability in symptoms and test results. IFS can help to capture this uncertainty and provide more nuanced diagnostic information. For example, in the diagnosis of a complex disease such as cancer, IFS can be used to represent the degree of certainty or uncertainty in the diagnosis based on various diagnostic criteria such as the results of blood tests, imaging studies, and biopsy findings. This can help to provide more accurate and reliable diagnoses, as well as more personalized treatment plans. Jiang et al. [14] used IFS for medical image fusion using entropy measures. Recently, Mehmood et al. [15,16] generalized the intuitionistic fuzzy sets and applied these approaches to medical diagnosis. De et al. [17] also analyzed an application of IFS in medical diagnosis and Davvaz et al. [18] produced a similar technique. Szmidt et al. [19] explored IFS in intelligent data analysis for medical diagnosis. During the decision making for IFS, the aggregation operators help a lot to calculate the values of the attributes. Therefore, experts proposed many aggregation operators; for example, Xu et al. [20] designed power aggregation operators for IFS and applied them in MADM. In 2006, some geometric aggregation operators were produced for IFS by Xu [21]. Wajid et al. [22] presented a novel TWD approach for IHFS. Recently, Senapati and Garg [23,24] also explored some novel operators.

Three-way decision (TWD) is a very important generalization of RS theory introduced by Yao [25,26]. A three-way decision for medical diagnosis involves considering three possible outcomes: positive, negative, or inconclusive. Positive: If the medical diagnosis is positive, it means that the patient has the condition or disease being evaluated for. In this case, the patient would need to receive treatment for the condition and the medical team would need to monitor their progress. Negative: If the medical diagnosis is negative, it means that the patient does not have the condition or disease being evaluated for. In this case, the patient may not require any treatment and the medical team may need to investigate other potential causes of the patient's symptoms. Inconclusive: If the medical diagnosis is inconclusive, it means that the test results are not clear enough to determine whether the patient has the condition or disease being evaluated for. In this case, the patient may need to undergo further testing or evaluation to arrive at a more definitive diagnosis. Recently, Li et al. [27,28] applied TWD techniques for hybrid decision making to diagnose medical problems. Hu et al. [29,30] presented the concept of a lattice model for medical diagnosis using TWD. Jia and Fan [31] composed TWD models for multi-criteria environments. Ye et al. [32] combined the TWD notion with the trending research area fuzzy information system. Similarly, many scholars explored this area and proposed novel approaches in different extensions of fuzzy sets [33–35].

1.3. Motivation for Proposed Work

In the literature, we found that three-way decision TWD models are very useful in diagnosing medical problems. By combining IFS and TWD [36], a very powerful theory is produced to cope with the vagueness and unclear situation. It is noted that for aggregation, the results of many participants based on TWD is a very difficult problem. Researchers used the classical way to calculate the alternatives for TWD [37–40]. In the existing TWD model [25,37], to determine the equivalence classes, an external concept is required. Moreover, the threshold is used to classify the alternatives into three regions.

The main purpose of composing this work is to design a novel algorithm for a TWD model-based decision-theoretic rough set using aggregation operators and an improved TWD decision approach based on interval-valued equivalence classes for IFS. The developed approach fulfils the lackness and resolves the computing problem for TWD. Below is a demonstration of this analysis' major contribution.

i. Construct the concept of intervals for membership grades of IFS using the step size function;
ii. Develop the equivalence classes based on intervals and called interval-valued classes;
iii. To cope with the issues of computing and saving time, IFPWA and IFPWG aggregation operators are developed for the TWD model;
iv. An algorithm is proposed to classify the different patients and to diagnose the disease on the basis of multiple symptoms.

The rest of the article is given as follows: In Section 2, we have overviewed the basic notion of IFS, power aggregation operators, and three-way decision (TWD). In Section 3, we have designed intervals for the membership grade using the step size function. Based on the intervals, equivalence classes are produced and remodel the TWD for IFS. In Section 4, we have designed a proper algorithm with a flow chart and explained the approach step by step. In Section 5, we have discussed a case study and utilized the proposed approach to diagnosis a medical problem to classify the alternatives with power aggregation operators for IFS. Some advantages and benefits of proposed models are discussed in detail. Section 6 includes the conclusion and future plan of the authors.

2. Preliminaries

In this section, we update models for IFSs and several concepts pertaining to power aggregation operators. Table 1 is added to describe the abbreviations of the symbols for easiness of understanding.

Table 1. Symbols with their descriptions.

Symbol	Description	Symbol	Description
FSs	Fuzzy Sets	IHFSs	Intuitionistic Hesitant Fuzzy Sets
IFSs	Intuitionistic Fuzzy Sets	IFN	Intuitionistic Fuzzy Number
TWD	Three-Way Decision	MG	Membership Grade
NMG	Non-membership Grade	DTRS	Decision-Theoretic Rough Set
IFPWA	Intuitionistic Fuzzy Power Weighted Averaging	IFPWG	Intuitionistic Fuzzy Power Weighted Geometric
IFPOWA	Intuitionistic Fuzzy Power Order Weighted averaging	IFPOWG	Intuitionistic Fuzzy Power Order Weighted Geometric
DRs	Decision Rules	IFRS	Intuitionistic Fuzzy Rough Set

2.1. IFSs

As an extension of the FS model, Atanassov [13] proposed the IFS model. IFS simultaneously delivers MG and NMG while FS just delivers the MG of an element in a given set [0, 1].

Definition 1 ([13]). *Let an IFS* W *on E be symbolized by* $m(e)$ *and* $n(e)$*. Mathematically, it is presented as:*

$$\mathrm{W} = \langle e,\ m_{\mathrm{W}}(e), n_{\mathrm{W}}(e)) | e\ \in\ E \rangle \tag{1}$$

where $m_{\mathrm{W}}(e) : E \to [0,\ 1]$ *and* $n_{\mathrm{W}}(e) : E \to [0,\ 1]$ *signify the MG and NMG with condition* $0 \leq\ m(e) + n(e) \leq 1$ *for all* $e \in E$*. Generally, the pair (m,n) represents the IFN.*

Definition 2. *For IFNs,* $\mathrm{W} = (m_{\mathrm{W}},\ n_{\mathrm{W}})$ *and the score function and accuracy functions are denoted and defined as follows:*

$$S(\mathrm{W}) = m_{\mathrm{W}} - n_{\mathrm{W}}, \qquad S(\mathrm{W}) \in [-1,\ 1] \tag{2}$$

$$H(\mathrm{W}) = m_{\mathrm{W}} + n_{\mathrm{W}}, \qquad H(\mathrm{W}) \in [0,\ 1] \tag{3}$$

For comparing two IFNs, W_1 *and* W_2*, the score function and accuracy function provide the assistance as below,*

(i) *If* $S(\mathrm{W}_1) > S(\mathrm{W}_1)$ *then* $\mathrm{W}_1 > \mathrm{W}_2$*;*
(ii) *If* $S(\mathrm{W}_1) < S(\mathrm{W}_1)$ *then* $\mathrm{W}_1 < \mathrm{W}_2$ *;*
(iii) *If* $S(\mathrm{W}_1) = S(\mathrm{W}_1)$ *then;*
  a. *If* $H(\mathrm{W}_1) > H(\mathrm{W}_1)$ *then* $\mathrm{W}_1 > \mathrm{W}_2$*;*
  b. *If* $H(\mathrm{W}_1) < H(\mathrm{W}_1)$ *then* $\mathrm{W}_1 < \mathrm{W}_2$*;*
  c. *If* $H(\mathrm{W}_1) = H(\mathrm{W}_1)$ *then* $\mathrm{W}_1 = \mathrm{W}_2$.

Definition 3. *Suppose* $\mathrm{W}_1 = (m_1, n_1)$ *and* $\mathrm{W}_2 = (m_2, n_2)$ *are IFSs, some basic operations are described as follows:*

(i) $\mathrm{W}_1\ \oplus\ \mathrm{W}_2 = (\{m_1 + m_2 - m_1 m_2\}, \{n_1 n_2\});$
(ii) $\mathrm{W}_1\ \otimes\ \mathrm{W}_2 = (\{m_1 m_2\}, \{n_1 + n_2 - n_1 n_2\});$
(iii) $\gimel \mathrm{W}_1 = \left(1 - (1-m)^{\gimel}, n^{\gimel}\right), \gimel > 0;$
(iv) $\mathrm{W}_1^{\gimel} = \left((m)^{\gimel},\ 1 - (1-n)^{\gimel}\right), \gimel > 0;$
(v) $\mathrm{W}_1^c = (n_1, m_1).$

Definition 4 ([20]). *Assume that* $W_j = (m_j, n_j)$ *is a collection of IFSs; the weights* $\tilde{\omega}_j = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)^T$ *for* W_j *with* $\sum_{j=1}^{n} \tilde{\omega}_j = 1$ *and* $\tilde{\omega}_j \in [0, 1]$. *Thus, the* $IFPWA_\omega$ *operator is a mapping of* $IFPWA_\omega$: $W^n \rightarrow W$ *where*

$$IFPWA_\omega(W_1, W_2, \ldots, W_n) = \frac{\bigoplus_{j=1}^{n} (\tilde{\omega}_j(1+T(W_j)W_j)}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_j))}$$
$$= \left(1 - \prod_{j=1}^{n} (1-(m_j)^{\frac{\tilde{\omega}_j((1+T(W_j))}{\sum_{j=1}^{n} \omega_j(1+T(W_j))}}, \prod_{j=1}^{n} (n_j)^{\frac{\tilde{\omega}_j(1+T(W_j))}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_j))}} \right)$$

where

$$T(W_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \tilde{\omega}_j Sup(W_j, W_i)$$

and

$$Sup(W_j, W_i) = 1 - d(W_j, W_i)$$

$$d(W_j, W_i) = \frac{1}{m} \sum_{\substack{i=1 \\ i \neq j}}^{m} (|m_i - m_j| + |n_i - n_j|)$$

Definition 5 ([20]). *For IFNs,* $W_j = (m_j, n_j)$ *with their weights* $\tilde{\omega}_j = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_j)^T$ *such that* $\tilde{\omega}_j > 0$, $\tilde{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \tilde{\omega}_j = 1$. *A mapping of* $IFPOWA_\omega$: $W^n \rightarrow W$ *is defined as follows:*

$$IFPOWA_{\tilde{\omega}}(W_1, W_2, \ldots, W_n) = \frac{\bigoplus_{j=1}^{n} (\tilde{\omega}_j(1+T(W_{\sigma(j)})W_{\sigma(j)})}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_{\sigma(j)}))}$$
$$= \left(1 - \prod_{j=1}^{n} (1-(m_{\sigma(j)})^{\frac{(\tilde{\omega}_j(1+T(W_{\sigma(j)}))}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_{\sigma(j)}))}}, \prod_{j=1}^{n} (n_{\sigma(j)})^{\frac{(\tilde{\omega}_j(1+T(W_{\sigma(j)}))}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_{\sigma(j)}))}} \right)$$

Definition 6 ([21]). *Assume that* $W_j = (m_j, n_j)$ *is a collection of IFNs and the weights* $\tilde{\omega}_j = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)^T$ *for* W_j *and* $\sum_{j=1}^{n} \tilde{\omega}_j = 1$ *where* $\tilde{\omega}_j \in [0, 1]$. *Then, the* $IFPWG_\omega$ *operator is a mapping of* $IFPWG_\omega$: $W^n \rightarrow W$ *where*

$$IFPWG_\omega(W_1, W_2, \ldots, W_n) = \frac{\bigotimes_{j=1}^{n} (\tilde{\omega}_j(1+T(W_j)W_j)}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_j))}$$
$$= \left(\prod_{j=1}^{n} (m_j)^{\frac{\tilde{\omega}_j(1+T(W_j))}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_j))}}, 1 - \prod_{j=1}^{n} (1-(n_j)^{\frac{\tilde{\omega}_j((1+T(W_j))}{\sum_{j=1}^{n} \omega_j(1+T(W_j))}} \right)$$

where

$$T(W_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \tilde{\omega}_j Sup(W_j, W_i)$$

Definition 7 ([21]). *For IFNs,* $W_j = (m_j,\ n_j)$ *with their weights* $\tilde{\omega}_j = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_j)^T$ *such that* $\tilde{\omega}_j > 0,\ \tilde{\omega}_j \in [0,\ 1]$ *and* $\sum_{j=1}^{n} \tilde{\omega}_j = 1$. *A mapping of* $IFPOWG_{\omega}{:}W^n \rightarrow W$ *is defined as follows:*

$$IFPOWG_{\tilde{\omega}}(W_1,\ W_2,\ \ldots,\ W_n) = \frac{\overset{n}{\underset{j=1}{\otimes}} \left(\tilde{\omega}_j(1+T(W_{\sigma(j)})W_{\sigma(j)}\right)}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_{\sigma(j)}))}$$

$$= \left(\prod_{j=1}^{n} \left(m_{\sigma(j)}\right)^{\frac{(\tilde{\omega}_j(1+T(W_{\sigma(j)}))}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_{\sigma(j)}))}}, 1 - \prod_{j=1}^{n} \left(1 - \left(n_{\sigma(j)}\right)^{\frac{(\tilde{\omega}_j(1+T(W_{\sigma(j)}))}{\sum_{j=1}^{n} \tilde{\omega}_j(1+T(W_{\sigma(j)}))}}\right) \right)$$

2.2. A Review of Decision-Theoretic Rough Set Model

The DTRS theory is a framework that involves a collection of states, X and X', indicating the presence or absence of components in X. The theory employs a series of actions, $\mathscr{A}ct = \{\mathscr{A}_{\mathscr{P}}, \mathscr{A}_{\mathscr{B}}, \mathscr{A}_{\mathscr{N}}\}$, where $\mathscr{A}_{\mathscr{P}}, \mathscr{A}_{\mathscr{B}}$, and $\mathscr{A}_{\mathscr{N}}$ act for the decisions to accept, defer, or reject an object A based on its classification, respectively. The objects are classified into three distinct zones, namely the positive region $\mathscr{P}os(\mathrm{X})$, boundary region $\mathscr{B}nd(\mathrm{X})$, and negative region $\mathscr{N}eg(\mathrm{X})$. Additionally, a matrix Table 2, $\mathscr{M} = \{\daleth_{\sigma\tau}\}_{3\times 2}$ ($\sigma = \mathscr{P}, \mathscr{B}, \mathscr{N}$, *and* $\tau = \mathscr{P}, \mathscr{N}$) delivers the cost parameters. The cost related with actions $\mathscr{A}_{\mathscr{P}}, \mathscr{A}_{\mathscr{B}}$, and $\mathscr{A}_{\mathscr{N}}$, when an object becomes X, is represented by $\daleth_{\mathscr{P}\mathscr{N}}$, $\daleth_{\mathscr{B}\mathscr{N}}$, and $\daleth_{\mathscr{N}\mathscr{N}}$. Conversely, when an item does not belong to X, the corresponding expenses for the three actions are denoted by $\daleth_{\mathscr{P}\mathscr{N}}$, $\daleth_{\mathscr{B}\mathscr{N}}$, and $\daleth_{\mathscr{N}\mathscr{N}}$. The classification losses $\mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A])$ related with the three actions are stated as:

$$\mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A]) = \daleth_{\mathscr{P}\mathscr{P}} P(X|[A]) + \daleth_{\mathscr{P}\mathscr{N}} P\left(X' \middle| [A]\right)$$
$$\mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A]) = \daleth_{\mathscr{B}\mathscr{P}} P(X|[A]) + \daleth_{\mathscr{B}\mathscr{N}} P\left(X' \middle| [A]\right)$$
$$\mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A]) = \daleth_{\mathscr{N}\mathscr{P}} P(X|[A]) + \daleth_{\mathscr{N}\mathscr{N}} P\left(X' \middle| [A]\right)$$

Bayesian decision theory provides the principles for the minimum-loss decision.

1. If $\mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A])$ and $\mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A])$, then $A \in POS(X)$;
2. If $\mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A])$ and $\mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A])$, then $A \in BND(X)$;
3. If $\mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A])$ and $\mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A])$, then $A \in NEG(X)$.

Given the prerequisites of $\daleth_{\mathscr{P}\mathscr{P}} \leq \daleth_{\mathscr{B}\mathscr{P}} \leq \daleth_{\mathscr{N}\mathscr{P}}$, $\daleth_{\mathscr{N}\mathscr{N}} \leq \daleth_{\mathscr{B}\mathscr{N}} \leq \daleth_{\mathscr{P}\mathscr{N}}$ and $P(X|[A]) + P\left(X' \middle| [A]\right) = 1$, the decision rules 1, 2, and 3 can be updated as follows:

4. If $P(X|[A]) \geq \alpha$, *then* $A \in POS(X)$;
5. If $\beta < P(X|[A]) < \alpha$, *then* $A \in BND(X)$;
6. If $P(X|[A]) \leq \beta$, *then* $A \in NEG(X)$.

where

$$\alpha = \frac{\daleth_{\mathscr{P}\mathscr{N}} - \daleth_{\mathscr{B}\mathscr{N}}}{(\daleth_{\mathscr{P}\mathscr{N}} - \daleth_{\mathscr{B}\mathscr{N}}) - (\daleth_{\mathscr{B}\mathscr{P}} - \daleth_{\mathscr{P}\mathscr{P}})} \text{ and } \beta = \frac{\daleth_{\mathscr{B}\mathscr{N}} - \daleth_{\mathscr{N}\mathscr{N}}}{(\daleth_{\mathscr{B}\mathscr{N}} - \daleth_{\mathscr{N}\mathscr{N}}) + (\daleth_{\mathscr{N}\mathscr{P}} - \daleth_{\mathscr{B}\mathscr{P}})}$$

Table 2. Cost matrix.

	X	X'
$\mathscr{A}_{\mathscr{P}}$	$\daleth_{\mathscr{P}\mathscr{P}}$	$\daleth_{\mathscr{P}\mathscr{N}}$
$\mathscr{A}_{\mathscr{B}}$	$\daleth_{\mathscr{B}\mathscr{P}}$	$\daleth_{\mathscr{B}\mathscr{N}}$
$\mathscr{A}_{\mathscr{N}}$	$\daleth_{\mathscr{N}\mathscr{P}}$	$\daleth_{\mathscr{N}\mathscr{N}}$

3. A Novel Decision-Theoretic Rough Set Model Based on Interval-Valued Classes for Intuitionistic Fuzzy Sets

The following section presents a novel approach to modeling DTRS using intuitionistic fuzzy environments and interval design. This novel technique produces interval-valued equivalence classes which can be used to partition the universe into three distinct areas, including $\mathscr{P}os(\mathbf{X})$, $\mathscr{N}eg(\mathbf{X})$, and $\mathscr{B}nd(\mathbf{X})$ regions, for participant classification. To discretize the information system, we have developed interval-valued equivalence classes instead of traditional equivalence classes. This has been achieved with the help of a step-size function that aids in the partitioning of alternatives into intervals. The step-size function is defined as follows:

Definition 8. *We define and denote the intervals $\mathscr{I}_N$ for approximation classes based on MGs for a collection of IFNs $A_i = (m_i\ n_i,)$ where, $i = 1,\ 2, \ldots n$,*

$$\mathscr{I}_N = [Min(m_i),\ Min(m_i) + h] \tag{4}$$

where step size function (h) is defined for the membership grades of IFNs as

$$h = \frac{Max(m_i) - Min(m_i)}{N}$$

where N is the number of intervals $\mathscr{I}_N$ which we require.

As per Yao's parental concept [25], equivalence classes can be used to determine the approximation classes. Additionally, by defining intervals $\mathscr{I}_N$ according to Equation (4), the Nth interval-valued equivalence classes $[A]_I$ can be developed for the alternatives, as follows:

Definition 9. *The interval-valued equivalence classes $[A]_I$ for $I \subseteq At$ for the alternatives A_i are designed as*

$$[A]_I = \{A : A_i \in \mathscr{I}_N\}$$

Definition 10. *The membership function for interval-valued classes $[A_k]$, $k \in \mathbb{N}$ for all IFNs is defined as*

$$P(X|[A_k]_I) = \frac{|X \cap [A_k]|}{|[A_k]|} \tag{5}$$

Interval-Valued Decision-Theoretic Rough Set Model

We have employed Definition [8] to create a new model for three-way decision making under an IF environment. We have established the cost parameter matrix $\mathscr{M}$ based on intuitionistic fuzzy cost values which is shown in Table 3. Using Bayesian theory, we have described the expected losses $\mathscr{E}(\mathscr{A}_\sigma|[A])$, $\sigma = \mathscr{P}, \mathscr{B}, \mathscr{N}$ for taking actions with the given set of states for the interval-valued equivalence class $[A]$ as follows:

$$\begin{aligned}
\mathscr{E}(\mathscr{A}_\mathscr{P}|[A]) &= \beth_{\mathscr{P}\mathscr{P}} P(X|[A]) \oplus \beth_{\mathscr{P}\mathscr{N}} P\left(X' \middle| [A]\right) \\
\mathscr{E}(\mathscr{A}_\mathscr{B}|[A]) &= \beth_{\mathscr{B}\mathscr{P}} P(X|[A]) \oplus \beth_{\mathscr{B}\mathscr{N}} P\left(X' \middle| [A]\right) \\
\mathscr{E}(\mathscr{A}_\mathscr{N}|[A]) &= \beth_{\mathscr{N}\mathscr{P}} P(X|[A]) \oplus \beth_{\mathscr{N}\mathscr{N}} P\left(X' \middle| [A]\right)
\end{aligned}$$

$P(X|[A]) + P\left(X' \middle| [A]\right) + \Delta(A) = 1$ where $\Delta(A)$ is an error function. We have

$$\begin{aligned}
\mathscr{E}(\mathscr{A}_\mathscr{P}|[A]) &= \beth_{\mathscr{P}\mathscr{P}} P(X|[A]) \oplus \beth_{\mathscr{P}\mathscr{N}} [1 - P(X|[A]) - \Delta(A)] \\
\mathscr{E}(\mathscr{A}_\mathscr{B}|[A]) &= \beth_{\mathscr{B}\mathscr{P}} P(X|[A]) \oplus \beth_{\mathscr{B}\mathscr{N}} [1 - P(X|[A]) - \Delta(A)] \\
\mathscr{E}(\mathscr{A}_\mathscr{N}|[A]) &= \beth_{\mathscr{N}\mathscr{P}} P(X|[A]) \oplus \beth_{\mathscr{N}\mathscr{N}} [1 - P(X|[A]) - \Delta(A)]
\end{aligned} \tag{6}$$

Table 3. Intuitionistic fuzzy cost matrix.

	X	X'
$\mathscr{A}_{\mathscr{P}}$	$\gimel_{\mathscr{P}\mathscr{P}} = (m_{\mathscr{P}\mathscr{P}}, n_{\mathscr{P}\mathscr{P}})$	$\gimel_{\mathscr{P}\mathscr{N}} = (m_{\mathscr{P}\mathscr{N}}, n_{\mathscr{P}\mathscr{N}})$
$\mathscr{A}_{\mathscr{B}}$	$\gimel_{\mathscr{B}\mathscr{P}} = (m_{\mathscr{B}\mathscr{P}}, n_{\mathscr{B}\mathscr{P}})$	$\gimel_{\mathscr{B}\mathscr{N}} = (m_{\mathscr{B}\mathscr{N}}, n_{\mathscr{B}\mathscr{N}})$
$\mathscr{A}_{\mathscr{N}}$	$\gimel_{\mathscr{N}\mathscr{P}} = (m_{\mathscr{N}\mathscr{P}}, n_{\mathscr{N}\mathscr{P}})$	$\gimel_{\mathscr{N}\mathscr{N}} = (m_{\mathscr{N}\mathscr{N}}, n_{\mathscr{N}\mathscr{N}})$

Bayesian decision theory offers the decision guidelines for the minimum-loss decision.

7. If $\mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A])$ and $\mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A])$, then $A \in POS(X)$;
8. If $\mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A])$ and $\mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A])$, then $A \in BND(X)$;
9. If $\mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A])$ and $\mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A]) \leq \mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A])$, then $A \in NEG(X)$.

The expected losses are evaluated based on the intuitionistic cost parametric values in Table 3 for interval-valued classes for identifying the thresholds which are designed as follows:

$$\mathscr{E}(\mathscr{A}_{\mathscr{P}}|[A]) = \left[1-(1-m_{\mathscr{P}\mathscr{P}})^{P(X|[A])}(1-m_{\mathscr{P}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}, (n_{\mathscr{P}\mathscr{P}})^{P(X|[A])}, (n_{\mathscr{P}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}\right]$$
$$\mathscr{E}(\mathscr{A}_{\mathscr{B}}|[A]) = \left[1-(1-m_{\mathscr{B}\mathscr{P}})^{P(X|[A])}(1-m_{\mathscr{B}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}, (n_{\mathscr{B}\mathscr{P}})^{P(X|[A])}, (n_{\mathscr{B}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}\right]$$
$$\mathscr{E}(\mathscr{A}_{\mathscr{N}}|[A]) = \left[1-(1-m_{\mathscr{N}\mathscr{P}})^{P(X|[A])}(1-m_{\mathscr{N}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}, (n_{\mathscr{N}\mathscr{P}})^{P(X|[A])}, (n_{\mathscr{N}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}\right]$$

Let $m(\mathscr{E})_\sigma = 1-(1-m_{\sigma\mathscr{P}})^{P(X|[A])}(1-m_{\sigma\mathscr{N}})^{1-P(X|[A])-\Delta(A)}$ and $n(\mathscr{E})_\sigma = (n_{\sigma\mathscr{P}})^{P(X|[A])}(n_{\sigma\mathscr{N}})^{1-P(X|[A])-\Delta(A)}$; here, $\sigma = \mathscr{P}, \mathscr{B}, \mathscr{N}$.

The membership and non-membership grades of the cost parameter are used to calculate the categorization losses, $m(\mathscr{E})_\sigma$ and $n(\mathscr{E})_\sigma$, respectively. Furthermore, according to Bayesian decision theory, the new DRs are clearly examined by $m(\mathscr{E})_\sigma$ for the minimum-loss categorization. The categorization losses $m(\mathscr{E})_\sigma$ and $n(\mathscr{E})_\sigma$ are determined by the MG and NMG of the cost parameter, respectively. Furthermore, the new decision rules are examined based on Bayesian decision theory using $m(\mathscr{E})_\sigma$ for the minimum-loss categorizations.

10. If $m(\mathscr{E})_{\mathscr{P}} \leq m(\mathscr{E})_{\mathscr{B}}$ and $m(\mathscr{E})_{\mathscr{P}} \leq m(\mathscr{E})_{\mathscr{N}}$, then $A \in \mathscr{P}os(X)$;
11. If $m(\mathscr{E})_{\mathscr{B}} \leq m(\mathscr{E})_{\mathscr{P}}$ and $m(\mathscr{E})_{\mathscr{B}} \leq m(\mathscr{E})_{\mathscr{N}}$ then $X \in \mathscr{B}nd(X)$;
12. If $m(\mathscr{E})_{\mathscr{N}} \leq m(\mathscr{E})_{\mathscr{P}}$ and $m(\mathscr{E})_{\mathscr{N}} \leq m(\mathscr{E})_{\mathscr{B}}$ then $X \in \mathscr{N}eg(X)$.

Based on categorization losses, if $m(\mathscr{E})_{\mathscr{P}} \leq m(\mathscr{E})_{\mathscr{B}}$, then

$$\ln\left[1-(1-m_{\mathscr{P}\mathscr{P}})^{P(X|[A])}(1-m_{\mathscr{P}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}\right] \geq \ln\left[1-(1-m_{\mathscr{B}\mathscr{P}})^{P(X|[A])}(1-m_{\mathscr{B}\mathscr{N}})^{1-P(X|[A])-\Delta(A)}\right]$$

By (10)

$$P(X|[A]) \geq (1-\Delta(A))\frac{\ln\left[\frac{1-m_{\mathscr{B}\mathscr{N}}}{1-m_{\mathscr{B}\mathscr{P}}}\right]}{\ln\left(\frac{1-m_{\mathscr{P}\mathscr{P}}}{1-m_{\mathscr{B}\mathscr{P}}} \times \frac{1-m_{\mathscr{B}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right)}$$

Similarly

$$m(\mathscr{E})_{\mathscr{N}} \leq m(\mathscr{E})_{\mathscr{N}} \Rightarrow P(X|[A]) \geq (1-\Delta(A))\frac{\ln\left[\frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right]}{\ln\left(\frac{1-m_{\mathscr{P}\mathscr{P}}}{1-m_{\mathscr{N}\mathscr{P}}} \times \frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right)}$$
$$m(\mathscr{E})_{\mathscr{B}} \leq m(\mathscr{E})_{\mathscr{P}} \Rightarrow P(X|[A]) \leq (1-\Delta(A))\frac{\ln\left[\frac{1-m_{\mathscr{B}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right]}{\ln\left(\frac{1-m_{\mathscr{P}\mathscr{P}}}{1-m_{\mathscr{B}\mathscr{P}}} \times \frac{1-m_{\mathscr{B}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right)}$$
$$m(\mathscr{E})_{\mathscr{B}} \leq m(\mathscr{E})_{\mathscr{N}} \Rightarrow P(X|[A]) \geq (1-\Delta(A))\frac{\ln\left[\frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{B}\mathscr{N}}}\right]}{\ln\left(\frac{1-m_{\mathscr{B}\mathscr{P}}}{1-m_{\mathscr{N}\mathscr{P}}} \times \frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{B}\mathscr{N}}}\right)}$$
$$m(\mathscr{E})_{\mathscr{N}} \leq m(\mathscr{E})_{\mathscr{P}} \Rightarrow P(X|[A]) \leq (1-\Delta(A))\frac{\ln\left[\frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right]}{\ln\left(\frac{1-m_{\mathscr{P}\mathscr{P}}}{1-m_{\mathscr{N}\mathscr{P}}} \times \frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right)}$$
$$m(\mathscr{E})_{\mathscr{N}} \leq m(\mathscr{E})_{\mathscr{B}} \Rightarrow P(X|[A]) \leq (1-\Delta(A))\frac{\ln\left[\frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{B}\mathscr{N}}}\right]}{\ln\left(\frac{1-m_{\mathscr{B}\mathscr{P}}}{1-m_{\mathscr{N}\mathscr{P}}} \times \frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{B}\mathscr{N}}}\right)}$$

The decision rules are constructed as $(P), (B)$, and (N) by using the thresholds are given as,

(P) If $P(X|[A]) \geq \chi_1$ and $P(X|[A]) \geq \psi_1$, then $A \in \mathscr{P}os(\mathrm{X})$
(B) If $P(X|[A]) \leq \chi_1$ and $P(X|[A]) \geq \omega_1$, then $A \in \mathscr{B}nd(\mathrm{X})$
(N) If $P(X|[A]) \leq \omega_1$ and $P(X|[A]) \leq \psi_1$, then $A \in \mathscr{N}eg(\mathrm{X})$,

here

$$\chi_1 = (1 - \Delta(A)) \frac{\ln\left[\frac{1-m_{\mathscr{BN}}}{1-m_{\mathscr{BP}}}\right]}{\ln\left(\frac{1-m_{\mathscr{PP}}}{1-m_{\mathscr{BP}}} \times \frac{1-m_{\mathscr{BN}}}{1-m_{\mathscr{PN}}}\right)} \tag{7}$$

$$\psi_1 = (1 - \Delta(A)) \frac{\ln\left[\frac{1-m_{\mathscr{NN}}}{1-m_{\mathscr{PN}}}\right]}{\ln\left(\frac{1-m_{\mathscr{PP}}}{1-m_{\mathscr{NP}}} \times \frac{1-m_{\mathscr{NN}}}{1-m_{\mathscr{PN}}}\right)} \tag{8}$$

$$\omega_1 = (1 - \Delta(A)) \frac{\ln\left[\frac{1-m_{\mathscr{NN}}}{1-m_{\mathscr{BN}}}\right]}{\ln\left(\frac{1-m_{\mathscr{BP}}}{1-m_{\mathscr{NP}}} \times \frac{1-m_{\mathscr{NN}}}{1-m_{\mathscr{BN}}}\right)} \tag{9}$$

The rules for deciding the elements $(P) - (N)$ have so far been characterized by utilizing three thresholds $\chi_1(e)$, $\psi_1(e)$, and $\omega_1(e)$ from the membership grade perspective. Furthermore, DRs (13)–(15) from the viewpoint of NMG are conferred.

13. If $n(\mathscr{E})_{\mathscr{P}} \leq n(\mathscr{E})_{\mathscr{B}}$ and $n(\mathscr{E})_{\mathscr{P}} \leq n(\mathscr{E})_{\mathscr{N}}$, then $A \in \mathscr{P}os(\mathrm{X})$;
14. If $n(\mathscr{E})_{\mathscr{B}} \leq n(\mathscr{E})_{\mathscr{P}}$ and $n(\mathscr{E})_{\mathscr{B}} \leq n(\mathscr{E})_{\mathscr{N}}$ then $A \in \mathscr{B}nd(\mathrm{X})$;
15. If $n(\mathscr{E})_{\mathscr{N}} \leq n(\mathscr{E})_{\mathscr{P}}$ and $n(\mathscr{E})_{\mathscr{N}} \leq n(\mathscr{E})_{\mathscr{B}}$ then $A \in \mathscr{N}eg(\mathrm{X})$.

Given that $P(X|[A]) = 1 - P\left(X' \middle| [A]\right) - \Delta(A)$, DRs are expressed based on the complement of conditional probability as below.

If $n(\mathscr{E})_{\mathscr{P}} \leq n(\mathscr{E})_{\mathscr{B}}$, then

$$P(X|[A]) \ln\left(\frac{n_{\mathscr{PP}}}{n_{\mathscr{BP}}} \times \frac{n_{\mathscr{BN}}}{n_{\mathscr{PN}}}\right) \geq (1 - \Delta(A)) \ln \frac{\beta_N(n_{\mathscr{BN}})}{\beta_N(n_{\mathscr{BP}})}$$

Thus

$$P(X|[A]) \geq (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{BN}}}{n_{\mathscr{BP}}}}{\ln\left(\frac{n_{\mathscr{PP}}}{n_{\mathscr{BP}}} \times \frac{n_{\mathscr{BN}}}{n_{\mathscr{PN}}}\right)}$$

Similarly

$$\begin{aligned}
n(\mathscr{E})_{\mathscr{P}} \geq n(\mathscr{E})_{\mathscr{N}} &\Longrightarrow P(X|[A]) \\
&\geq (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{NN}}}{n_{\mathscr{PN}}}}{\ln\left(\frac{n_{\mathscr{PP}}}{n_{\mathscr{NP}}} \times \frac{n_{\mathscr{NN}}}{n_{\mathscr{PN}}}\right)} \\
n(\mathscr{E})_{\mathscr{B}} \geq n(\mathscr{E})_{\mathscr{P}} &\Longrightarrow P(X|[A]) \\
&\leq (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{BN}}}{n_{\mathscr{BP}}}}{\ln\left(\frac{n_{\mathscr{PP}}}{n_{\mathscr{BP}}} \times \frac{n_{\mathscr{BN}}}{n_{\mathscr{PN}}}\right)} \\
n(\mathscr{E})_{\mathscr{B}} \geq n(\mathscr{E})_{\mathscr{N}} &\Longrightarrow P(X|[A]) \\
&\geq (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{NN}}}{n_{\mathscr{BN}}}}{\ln\left(\frac{n_{\mathscr{BP}}}{n_{\mathscr{NP}}} \times \frac{n_{\mathscr{NN}}}{n_{\mathscr{BN}}}\right)} \\
n(\mathscr{E})_{\mathscr{N}} \geq n(\mathscr{E})_{\mathscr{P}} &\Longrightarrow P(X|[A]) \\
&\leq (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{NN}}}{n_{\mathscr{PN}}}}{\ln\left(\frac{n_{\mathscr{PP}}}{n_{\mathscr{NP}}} \times \frac{n_{\mathscr{NN}}}{n_{\mathscr{PN}}}\right)} \\
n(\mathscr{E})_{\mathscr{N}} \geq n(\mathscr{E})_{\mathscr{B}} &\Longrightarrow P(X|[A]) \\
&\leq (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{NN}}}{n_{\mathscr{BN}}}}{\ln\left(\frac{n_{\mathscr{BP}}}{n_{\mathscr{NP}}} \times \frac{n_{\mathscr{NN}}}{n_{\mathscr{BN}}}\right)}
\end{aligned}$$

Obviously, the decision rules are easily revised as $(P2) - (N2)$.

$(P2)$ If $P(X|[A]) \geq \chi_2$ and $P(X|[A]) \geq \psi_2$, then $A \in \mathscr{P}os(U)$
$(B2)$ If $P(X|[A]) \leq \chi_2$ and $P(X|[A]) \geq \omega_2$, then $A \in \mathscr{B}nd(U)$
$(N2)$ If $P(X|[A]) \leq \omega_2$ and $P(X|[A]) \leq \psi_2$, then $A \in \mathscr{N}eg(U)$,

where

$$\chi_2 = (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{B}\mathscr{N}}}{n_{\mathscr{B}\mathscr{P}}}}{\ln\left(\frac{n_{\mathscr{P}\mathscr{P}}}{n_{\mathscr{B}\mathscr{P}}} \times \frac{n_{\mathscr{B}\mathscr{N}}}{n_{\mathscr{P}\mathscr{N}}}\right)} \tag{10}$$

$$\psi_2 = (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{N}\mathscr{N}}}{n_{\mathscr{P}\mathscr{N}}}}{\ln\left(\frac{n_{\mathscr{P}\mathscr{P}}}{n_{\mathscr{N}\mathscr{P}}} \times \frac{n_{\mathscr{N}\mathscr{N}}}{n_{\mathscr{P}\mathscr{N}}}\right)} \tag{11}$$

$$\omega_2 = (1 - \Delta(A)) \frac{\ln \frac{n_{\mathscr{N}\mathscr{N}}}{n_{\mathscr{B}\mathscr{N}}}}{\ln\left(\frac{n_{\mathscr{B}\mathscr{P}}}{n_{\mathscr{N}\mathscr{P}}} \times \frac{n_{\mathscr{N}\mathscr{N}}}{n_{\mathscr{B}\mathscr{N}}}\right)} \tag{12}$$

Noticeably, providing the IFN cost parameters are also fulfilled,

$$\frac{\ln\left[\frac{1-m_{\mathscr{P}\mathscr{P}}}{1-m_{\mathscr{B}\mathscr{P}}}\right]}{\ln\left[\frac{1-m_{\mathscr{B}\mathscr{N}}}{1-m_{\mathscr{P}\mathscr{N}}}\right]} < \frac{\ln\left[\frac{1-m_{\mathscr{B}\mathscr{P}}}{1-m_{\mathscr{N}\mathscr{P}}}\right]}{\ln\left[\frac{1-m_{\mathscr{N}\mathscr{N}}}{1-m_{\mathscr{B}\mathscr{N}}}\right]}$$

and

$$\frac{\ln \frac{n_{\mathscr{P}\mathscr{P}}}{n_{\mathscr{B}\mathscr{P}}}}{\ln \frac{n_{\mathscr{B}\mathscr{N}}}{n_{\mathscr{P}\mathscr{N}}}} < \frac{\ln \frac{n_{\mathscr{B}\mathscr{P}}}{n_{\mathscr{N}\mathscr{P}}}}{\ln \frac{n_{\mathscr{N}\mathscr{N}}}{n_{\mathscr{B}\mathscr{N}}}}$$

$\omega_j(A) < \psi_j(A) < \chi_j(A)$ is obtained where $\chi_j(A) \in (0,1]$, $\omega_j(A) \in (0,1]$, and $\psi_j(A) \in (0,1]$, $(j = 1,2)$. Therefore, the general DRs $(P3) - (N3)$ can be described as below:

$(P3)$ If $P(X|[A]) \geq \chi_j$, then $A \in \mathscr{P}os(U)$
$(B3)$ If $\omega_j < P(X|[A]) < \chi_j$, then $A \in \mathscr{B}nd(U)$
$(N3)$ If $P(X|[A]) \leq \omega_j$, then $A \in \mathscr{N}eg(U)$,

From the above obtained results, the TWDM-IFNs are described according to the Bayesian DRs, as below.

16. If $P(X|[A]) \geq \chi_j$, then take $\mathscr{A}_{\mathscr{P}}$;
17. If $\omega_j < P(X|[A]) < \chi_j$, then take $\mathscr{A}_{\mathscr{B}}$;
18. If $P(X|[A]) \leq \omega_j$, then take $\mathscr{A}_{\mathscr{N}}$.

In TWD, two threshold pairs $(\chi_1(A),\ \omega_1(A))$ and $(\chi_2(A),\ \omega_2(A))$ are obtained from various viewpoints in (16)–(18). Thus, actions are taken while $P(X|[A])$ is at the corresponding positive, boundary, and negative region thresholds.

4. Proposing an Algorithm to Apply the Interval-Valued Decision-Theoretic Rough Set Model to an Intuitionistic Fuzzy Environment

This section discusses the detailed application of $IFPWA_\omega$ and $IFPWG_\omega$ aggregation operators under IF information for decision-theoretic rough set models. We have outlined five steps for selecting the three-way DRs for various participants.

Let $E = \{A_1, A_2, \ldots, A_n\}$ be the collection of alternatives and $X = \{Yes, No\}$ be a set that indicates the decision for alternatives, where X is a subset of E. The flow chart of the three-way decision model is displayed in Figure 1.

Step 1. Evaluate the intuitionistic fuzzy information system with conditional and decision attributes.

Step 2. For alternatives, $A_i(i = 1, 2, \ldots, n)$ aggregate all the IF attributes $A_{ij}(j = 1, 2, \ldots, m)$ into a general solution A_i applying $IFPWA_{\tilde{\omega}}$ and $IFPWG_{\omega}$ operators as:

$$IHFPWA_{\omega}(W_1, W_2, \ldots, W_n) = \frac{\bigoplus_{j=1}^{n} (\tilde{\omega}_j(1+T(W_j)W_j)}{\sum_{j=1}^{n}\tilde{\omega}_j(1+T(W_j))}$$
$$= \left(1 - \prod_{j=1}^{n}(1-(m_j)^{\frac{\tilde{\omega}_j((1+T(W_j))}{\sum_{j=1}^{n}\omega_j(1+T(W_j))}}, \prod_{j=1}^{n}(n_j)^{\frac{\tilde{\omega}_j(1+T(W_j))}{\sum_{j=1}^{n}\tilde{\omega}_j(1+T(W_j))}}\right)$$

and

$$IHFPWG_{\omega}(W_1, W_2, \ldots, W_n) = \frac{\bigotimes_{j=1}^{n} (\tilde{\omega}_j(1+T(W_j)W_j)}{\sum_{j=1}^{n}\tilde{\omega}_j(1+T(W_j))}$$
$$= \left(\prod_{j=1}^{n}(m_j)^{\frac{\tilde{\omega}_j(1+T(W_j))}{\sum_{j=1}^{n}\tilde{\omega}_j(1+T(W_j))}}, 1 - \prod_{j=1}^{n}(1-(n_j)^{\frac{\tilde{\omega}_j((1+T(W_j))}{\sum_{j=1}^{n}\omega_j(1+T(W_j))}}\right)$$

Step 3. Compute the interval-valued equivalence classes using the suggested intervals as per Definition 9.

Step 4. Fix the set of states (X, X') and compute the membership function, non-membership function, and error functions for the participants.

Step 5. Calculation of expected losses and thresholds based on the cost parameter matrix referenced in Table 8 using Equation (6).

Step 6. Classification of the elements depending upon their membership values using thresholds given in decision rules 16–18.

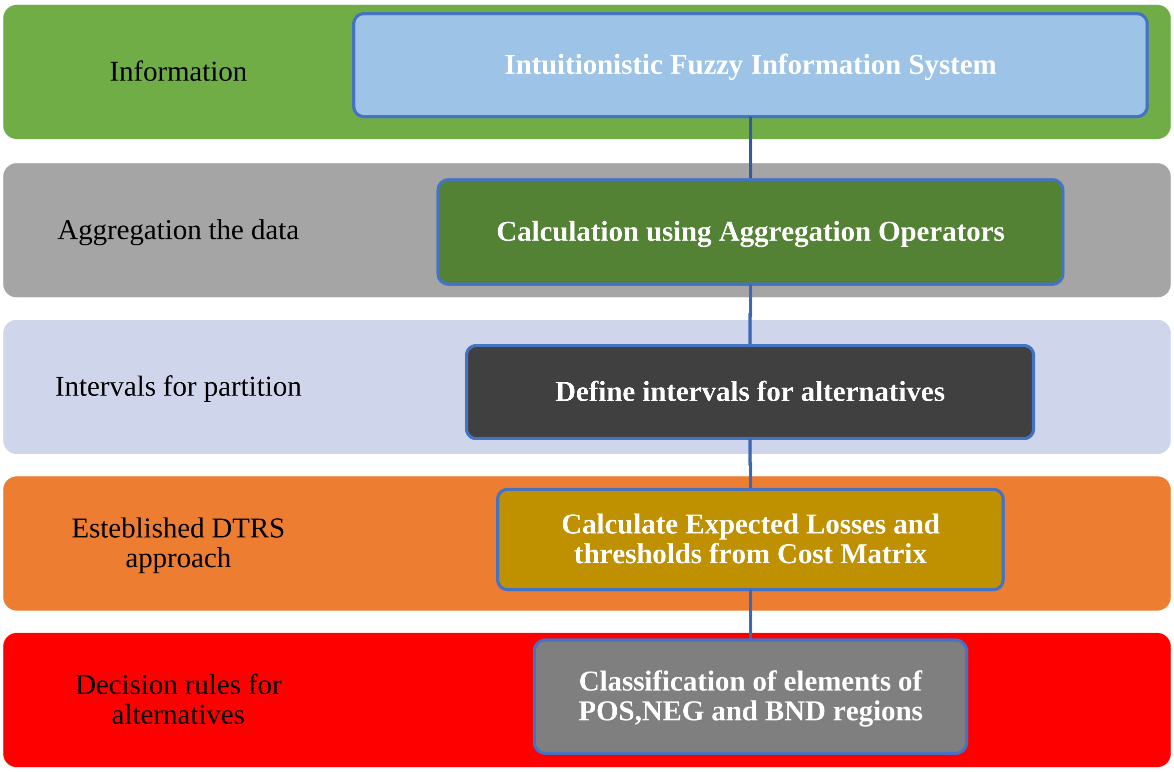

Figure 1. Flow chart of the interval-valued decision-theoretic rough set model.

5. A Case Study

This section includes an illustrative example aimed at determining whether or not a patient has a medical condition through a diagnostic investigation process. The objective is to approve or rule out the existence of the disease.

5.1. Explanation of the Problem

Medical diagnosis is an incredibly crucial task that involves determining which disease or condition a person is suffering from based on their symptoms. Achieving a correct diagnosis is crucial and medical professionals rely on their expertise and experience to make the right decision. With the aid of Intuitionistic Fuzzy Rough Sets (IFRS), healthcare practitioners can enhance their diagnostic accuracy while managing complex linguistic concepts. The use of IFRS has been incredibly successful in medical diagnoses, as demonstrated in numerous studies, including references [17,28]. Figure 2 offers a graphical depiction of the medical diagnosis process that highlights the utility of IFRS in this context and Figure 3 shows the graphical representation of decided elements based on the $\boldsymbol{IFPWA_{\omega}}$ and $\boldsymbol{IFPWG_{\omega}}$ operators.

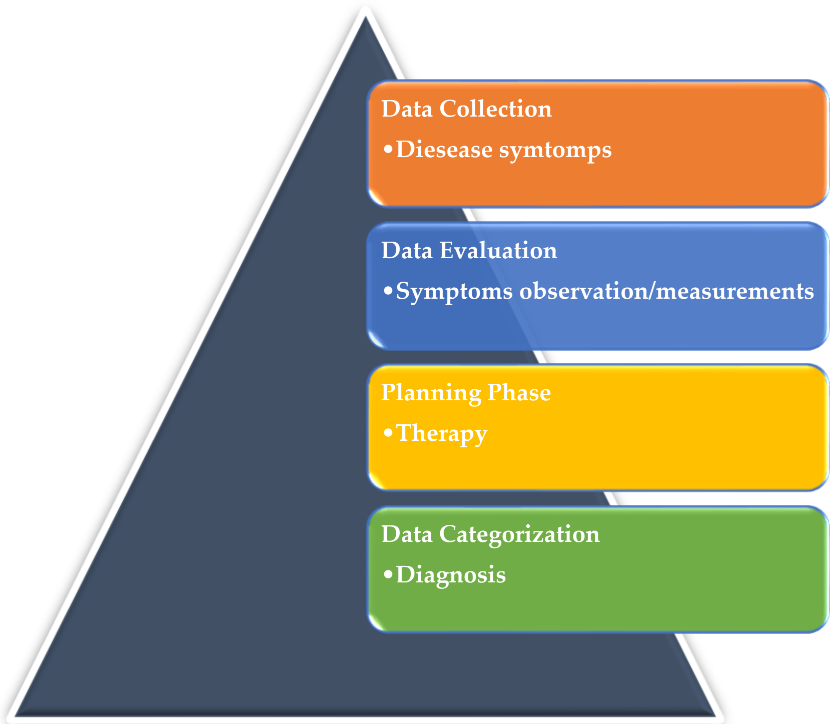

Figure 2. Medical diagnosis diagram.

Assuming there are 15 alternatives (A_i) participating in the diagnosis of the "Coronavirus" disease and a set of conditional attributes $(\boldsymbol{I})$,

$$\boldsymbol{I} = \{I_1(Chestpain), I_2(Fever), I_3(Fatigue), I_4(Cough)\}$$

is considered. Moreover, the decision attribute is represented by the sets as follows, $X = \{A_1, A_2, A_4, A_{15}, A_{11}\}$ and $X' = \{A_3, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{12}, A_{13}, A_{14}\}$ which indicate a positive decision for the existence of the disease. The diagnosis of the disease is made by experts based on the participants' input and the resulting decisions are weighted using a weight vector $\omega = \{\mathbf{0.2}, \mathbf{0.3}, \mathbf{0.4}, \mathbf{0.1}\}$. Next, we present a stepwise algorithm to elaborate on the diagnosis of this disease.

Step 1: The given Table 4 shows the IF information of all the alternatives which participated.

Table 4. An IF information table of alternatives.

Alternatives	I_1	I_2	I_3	I_4	D
A_1	(0.1, 0.3)	(0.4, 0.5)	(0.1, 0.5)	(0.1, 0.5)	Yes
A_2	(0.4, 0.5)	(0.5, 0.4)	(0.5, 0.3)	(0.2, 0.6)	Yes
A_3	(0.2, 0.3)	(0.2, 0.4)	(0.6, 0.2)	(0.4, 0.5)	No
A_4	(0.4, 0.2)	(0.1, 0.2)	(0.7, 0.4)	(0.3, 0.1)	Yes
A_5	(0.5, 0.3)	(0.5, 0.2)	(0.3, 0.2)	(0.4, 0.2)	No
A_6	(0.6, 0.2)	(0.7, 0.1)	(0.4, 0.1)	(0.4, 0.4)	No
A_7	(0.7, 0.1)	(0.2, 0.2)	(0.5, 0.2)	(0.5, 0.2)	No
A_8	(0.3, 0.4)	(0.3, 0.3)	(0.6, 0.2)	(0.2, 0.3)	No
A_9	(0.4, 0.2)	(0.5, 0.2)	(0.7, 0.2)	(0.3, 0.5)	No
A_{10}	(0.5, 0.2)	(0.8, 0.1)	(0.2, 0.3)	(0.4, 0.3)	No
A_{11}	(0.6, 0.2)	(0.9, 0.1)	(0.5, 0.3)	(0.5, 0.4)	Yes
A_{12}	(0.8, 0.1)	(0.0, 0.9)	(0.6, 0.4)	(0.2, 0.2)	No
A_{13}	(0.9, 0.1)	(0.3, 0.2)	(0.4, 0.3)	(0.4, 0.3)	No
A_{14}	(0.1, 0.2)	(0.2, 0.2)	(0.6, 0.3)	(0.3, 0.4)	No
A_{15}	(0.8, 0.1)	(0.1, 0.3)	(0.3, 0.4)	(0.4, 0.2)	Yes

Step 2. For alternatives $A_i(i = 1,\ 2, \ldots,\ 15)$, determine all the conditional attributes numbers utilizing $IFPWA_{\tilde{\omega}}$ or $IFPWG_{\tilde{\omega}}$ operators in the following:

$$IFPWA_{\omega}(\mathrm{W}_1,\ \mathrm{W}_2,\ \ldots,\ \mathrm{W}_n) = \frac{\underset{i=1}{\overset{n}{\bigoplus}}\ (\tilde{\omega}_j(1+T(\mathrm{W}_i)\mathrm{W}_i)}{\sum_{i=1}^{n}\tilde{\omega}_j(1+T(\mathrm{W}_i))}$$
$$= \left(1 - \prod_{i=1}^{n}(1-(m_i)^{\frac{\tilde{\omega}_{j((1+T(\mathrm{W}_i))}}{\sum_{i=1}^{n}\omega_j(1+T(\mathrm{W}_i))}},\ \prod_{i=1}^{n}(n_i)^{\frac{\tilde{\omega}_{j(1+T(\mathrm{W}_i))}}{\sum_{i=1}^{n}\tilde{\omega}_{j(1+T(\mathrm{W}_i))}}}\right)$$

or

$$IFPWG_{\omega}(\mathrm{W}_1,\ \mathrm{W}_2,\ \ldots,\ \mathrm{W}_n) = \frac{\underset{i=1}{\overset{n}{\otimes}}\ (\tilde{\omega}_j(1+T(\mathrm{W}_i)\mathrm{W}_i)}{\sum_{i=1}^{n}\tilde{\omega}_j(1+T(\mathrm{W}_i))}$$
$$= \left(\prod_{i=1}^{n}(m_i)^{\frac{\tilde{\omega}_{j(1+T(\mathrm{W}_i))}}{\sum_{j=1}^{n}\tilde{\omega}_{j(1+T(\mathrm{W}_i))}}},\ 1 - \prod_{i=1}^{n}(1-(n_i)^{\frac{\tilde{\omega}_{j((1+T(\mathrm{W}_i))}}{\sum_{i=1}^{n}\omega_j(1+T(\mathrm{W}_i))}}\right)$$

The outcomes are presented in Table 5.

Table 5. Aggregated outcomes of attributes of all alternatives.

Alternatives	$IFPWA_{\tilde{\omega}}$	$IFPWG_{\tilde{\omega}}$
A_1	(0.0203, 0.458)	(0.151, 0.470)
A_2	(0.466, 0.374)	(0.450, 0.393)
A_3	(0.429, 0.281)	(0.347, 0.306)
A_4	(0.500, 0.261)	(0.333, 0.292)
A_5	(0.409, 0.214)	(0.389, 0.217)

Table 5. *Cont.*

Alternatives	$IFPWA_{\tilde{\omega}}$	$IFPWG_{\tilde{\omega}}$
A_6	$(0.545, 0.124)$	$(0.507, 0.142)$
A_7	$(0.471, 0.177)$	$(0.401, 0.183)$
A_8	$(0.452, 0.261)$	$(0.400, 0.275)$
A_9	$(0.581, 0.213)$	$(0.541, 0.226)$
A_{10}	$(0.523, 0.201)$	$(0.372, 0.227)$
A_{11}	$(0.703, 0.205)$	$(0.615, 0.236)$
A_{12}	$(0.507, 0.384)$	$(0, 0.618)$
A_{13}	$(0.535, 0.220)$	$(0.420, 0.239)$
A_{14}	$(0.411, 0.252)$	$(0.302, 0.262)$
A_{15}	$(0.395, 0.276)$	$(0.259, 0.312)$

Step 3. Calculate the interval-valued equivalence classes based on the proposed approach and for step size $n = 5$ represented in Table 6.

Table 6. Interval-valued equivalence classes.

$IFPWA_{\tilde{\omega}}$	$[A_1] = \{A_1\}$ $[A_2] = \{A_2, A_3, A_4, A_5, A_7, A_8, A_{14}\}$ $[A_6] = \{A_6, A_9, A_{10}, A_{12}, A_{13}\}$ $[A_{11}] = \{A_{11}\}$ $[A_{15}] = \{A_{15}\}$
$IFPWG_{\tilde{\omega}}$	$[A_1] = \{A_1\}$ $[A_2] = \{A_2, A_5, A_7, A_8, A_{10}, A_{13}\}$ $[A_3] = \{A_3, A_4, A_{14}, A_{15}\}$ $[A_6] = \{A_6, A_9, A_{11}\}$ $[A_{12}] = \{A_{12}\}$

Step 4. The set of states for Yes is $X = \{A_1, A_2, A_4, A_{15}, A_{11}\}$ and for No is $X' = \{A_3, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{12}, A_{13}, A_{14}\}$. Now calculate the membership values, non-membership values, and error values in the following Table 7.

Table 7. Membership values, non-membership values, and error values.

	$IFPWA_{\tilde{\omega}}$			$IFPWG_{\tilde{\omega}}$		
Alternatives	**Membership Values**	**Non-Membership Values**	**Error Values**	**Membership Values**	**Non-Membership Values**	**Error Values**
A_1	1	0	0	1	0	0
A_2	0.28	0.72	0	0.16	0.83	0.01
A_3	0.28	0.72	0	0.50	0.50	0
A_4	0.28	0.72	0	0.50	0.50	0
A_5	0.28	0.72	0	0.16	0.83	0.01
A_6	0	1	0	0.33	0.66	0.01
A_7	0.28	0.72	0	0.16	0.83	0.01
A_8	0.28	0.72	0	0.16	0.83	0.01
A_9	0	1	0	0.33	0.66	0.01
A_{10}	0	1	0	0.16	0.83	0.01
A_{11}	1	0	0	0.33	0.66	0.01
A_{12}	0	1	0	0	1	0
A_{13}	0	1	0	0.16	0.83	0.01
A_{14}	0.28	0.72	0	0.50	0.50	0
A_{15}	1	0	0	0.50	0.50	0

Step 5. The cost parameter matrix is given in Table 8 and the aggregation of the thresholds by Equations (4)–(6) is represented in Table 9.

Table 8. Intuitionistic fuzzy cost parameter matrix.

	X	X'
$\mathscr{A}_{\mathscr{P}}$	$(0, 1)$	$(0.8, 0.1)$
$\mathscr{A}_{\mathscr{B}}$	$(0.3, 0.7)$	$(0.5, 0.4)$
$\mathscr{A}_{\mathscr{N}}$	$(0.9, 0.1)$	$(0.05, 0.8)$

Table 9. Thresholds for all elements.

	$IFPWA_{\tilde{\omega}}$			$IFPWG_{\tilde{\omega}}$		
Alternatives	$\chi_1(e)$	$\psi_1(e)$	$\omega_1(e)$	$\chi_1(e)$	$\psi_1(e)$	$\omega_1(e)$
A_1	0.719808	0.248034	0.403588	0.719808	0.248034	0.403588
A_2	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_3	0.719808	0.248034	0.403588	0.719808	0.248034	0.403588
A_4	0.719808	0.248034	0.403588	0.719808	0.248034	0.403588
A_5	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_6	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_7	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_8	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_9	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_{10}	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_{11}	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_{12}	0.719808	0.248034	0.403588	0.719808	0.248034	0.403588
A_{13}	0.719808	0.248034	0.403588	0.71261	0.245554	0.399552
A_{14}	0.719808	0.248034	0.403588	0.719808	0.248034	0.403588
A_{15}	0.719808	0.248034	0.403588	0.719808	0.248034	0.403588

Step 6. Finally, the classification of the elements based on the decision rules presented in Equations (16)–(18) for POS, NEG, and BND regions is shown in Table 10,

Table 10. Classification of alternatives accordingly.

$IFPWA\tilde{\omega}$	$IFPWG\tilde{\omega}$
$POS(X)=\{A_1,A_{15},A_{11}\}$ $NEG(X)=\{A_{12},A_6,A_{13},A_9,A_{10}\}$ $BND(X)=\{A_2,A_3,A_4,A_8,A_7,A_{10}\}$	$POS(X) = \{A_1\}$ $NEG(X) = \{A_2, A_5, A_7, A_8, A_{10}, A_{12}, A_{13}\}$ $BND(X) = \{A_3, A_4, A_6, A_9, A_{11}, A_{14}, A_{15}\}$

The results show that the alternatives in the POS zone have confirmed the presence of coronavirus disease; in the NEG region alternatives are safe and in the BND region alternatives are not confirmed. In addition, for new alternatives, we can classify them based on the descriptions of the already evaluated alternatives. Figure 3 shows the effects on the alternative due to the $IFPWA_{\omega}$ and $IFPWG_{\omega}$ operators.

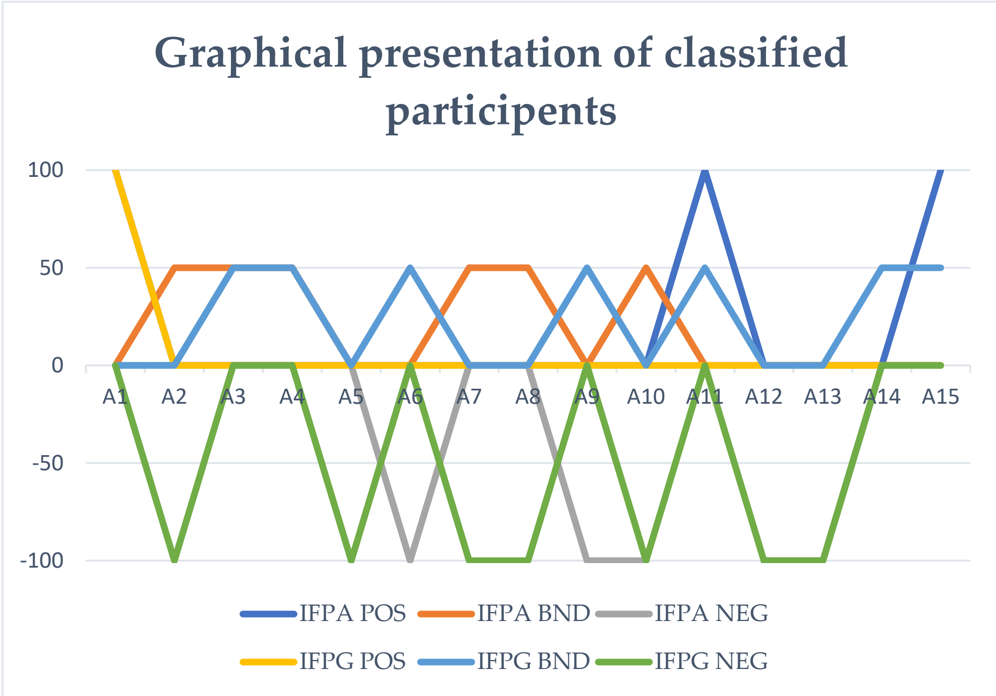

Figure 3. Graphical presentation of classification.

5.2. Benefits of the Proposed Model

In the proposed approach, there are benefits which are disclosed in the following:

(1) The most attractive and significant role of this approach is that it is a more generalized form. This approach is a generalized form of IFSs. If the NMGs are reduced to zero then the IFSs are converted into fuzzy sets;
(2) The power aggregation operators are very suitable and simple operators to cope with the problem of decision making under a fuzzy environment especially; these operators help to conclude the attribute's values of elements. To consider the importance, these operators are designed for novel data and used to aggregate the information;
(3) The existing approaches in the literature for TWD consist of the theories of Yao [37] and are very traditional. In this approach, we used some new steps for TWD, such as power aggregation operators which are designed. Moreover, interval-valued classes are developed to classify the participants;
(4) In this medical case, diagnosing the disease is a very big issue for experts as well as patients. To cope with this challenge, we created a model made up of many patients with their disease's attributes. Finally, the experts calculated the decisions.

6. Conclusions and Future Work

In the article, we firstly reviewed the basic idea of intuitionistic fuzzy sets and power aggregation operators. Moreover, we revised the model of three-way decision based on the Bayesian theory introduced by Yao [25]. In classical TWD models, equivalence classes play a vital role in discretizing the information system. In this paper, we developed a novel approach to discretize the information table. To classify the participants, interval-valued classes are used and three zones on the bases of those classes. The Bayesian model for minimizing risk is also revised for decision taking. Aggregation operators are used to aggregate the results and compose the attributes values into single values. Considering the importance of operators, we utilized power aggregation operators. Moreover, an algorithm to identify the disease using the proposed approach was produced. We disclosed the benefits of the approach: this approach is more general than the existing TWD model. Next, the findings of this study will be enlarged to the extension of the fuzzy and rough data and some new aggregation operators to cope with real-life problems will be developed. Moreover, we will utilize the established approach towards the existing literature [41–45].

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