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Pricing European Vulnerable Options with Jumps and Stochastic Default Obstacles Barrier under Regime Switching

Xiangdong Liu *  and Zanbin Zhang

School of Economics, Jinan University, Guangzhou 510632, China; zzb1998@stu2021.jnu.edu.cn

* Correspondence: tliuxd@jnu.edu.cn; Tel.: +86-20-8522-3026

Abstract: In this paper, we propose an enhanced model for pricing vulnerable options. Specifically, our model assumes that parameters such as interest rates, jump intensity, and asset value volatility are governed by an observable continuous-time finite-state Markov chain. We take into account European vulnerable options that are exposed to both default risk and rare shocks from underlying and counterparty assets. We also consider stochastic default barriers driven by a regime-switching model and geometric Brownian motion, thus improving upon the assumption of fixed default barriers. The risky assets follow a related jump-diffusion process, whereas the default barriers are influenced by a geometric Brownian motion correlated with the risky assets. Within the framework of our model, we derive an explicit pricing formula for European vulnerable options. Furthermore, we conduct numerical simulations to examine the effects of default barriers and other related parameters on option prices. Our findings indicate that stochastic default barriers increase credit risk, resulting in a decrease in option prices. By considering the aforementioned factors, our research contributes to a better understanding of pricing vulnerable options in the context of counterparty credit risk in over-the-counter trading.

Keywords: regime-switching; jump; default barrier; european vulnerable option; credit risk

MSC: 91G40; 91G80



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1. Introduction

The over-the-counter (OTC) market represents a significant portion of the financial market, with many financial institutions actively trading options on it. Unlike exchange-traded markets, OTC transactions are unorganized and lack transparency, which increases the risk of counterparty default. This means that one party may fail to fulfill its contractual obligations. Since the global financial crisis of 2007–2008, participants in the OTC market have become increasingly concerned about the credit default risk associated with options. The structural model, also known as the firm value model, is a key tool for pricing options with credit risk. This article explores option pricing within the framework of the firm value model. Johnson and Stulz [1] were the first to define options with credit risk as vulnerable options.

There are several models for vulnerable option pricing in the literature. Klein [2,3] proposed a model that allows for correlation between the underlying asset of the option and the counterparty's asset and assumes that the default barrier is the fixed liability of the option writer. He also proposed an improved method that includes both the fixed liabilities of the option writer and losses from vulnerable option shorts in the default barrier. Hui [4] proposed a European vulnerable option pricing model with a stochastic default barrier representing the option writer's liability. Wang [5] assumed that both the counterparty's assets and related assets were driven by a related jump-diffusion process and derived a pricing formula for vulnerable options. These studies suggest that the default barrier has an impact on the pricing of European vulnerable options. However, they only consider the

impact of barriers on options and do not take into account how a country's economic status at different times affects option prices.

Since the dynamics of risky asset values are not continuous, some literature introduces jumping processes, compound Poisson processes, or self-exciting Hawkes processes to better describe their value models. These studies discuss the pricing formula for European vulnerable options (see XU [6], TIAN [7], Ma [8], and Zhou [9]). Wang [10] proposed decomposing the underlying asset risk and counterparty assets into systemic and non-systemic risks and used the obtained pricing formula to discuss the difference between vulnerable option prices. In addition, many empirical analyses have concluded that the volatility of risky assets is not fixed. Therefore, some literature assumes that the underlying asset is driven by stochastic volatility models such as the Heston model and mean reversion process and discusses the pricing formula for European vulnerable options (see Liu [11], Yang [12], Lee [13], Wang [14]). Yoon [15] uses double Merlin transforms to obtain closed-form pricing formulas for European vulnerable options under constant and Hull–White interest rate models (see Liu [16]). Huang [17] investigated the pricing of European-style vulnerable options in the presence of non-affine stochastic volatility and double exponential jumps in the underlying asset price process.

Most option pricing models are within the framework of Brownian motion, which makes it difficult to capture external factors such as the state of the national economy. The literature also incorporates vulnerable option pricing under the regime-switching model. Elliott [18,19] was the first to consider introducing regime switching into the option pricing model. Wang [20] uses the local risk minimization method to obtain the optimal hedging strategy for European vulnerable call options. When the option writer's asset value is depressed by other financial institutions due to poor sales, Yang [21] used polynomial change numerical techniques to obtain a semi-analytical pricing formula for European vulnerable options. Assuming that the default barrier in the vulnerable option model is fixed, Wang [22] assumes that risk assets are driven by regime-switching geometric Brownian motion and obtains a European vulnerable option pricing formula. NIU [23] assumes that the value of risk assets is driven by a relevant regime-switching double exponential jump-diffusion model and uses two-dimensional Laplace transforms to obtain a European vulnerable option pricing formula. Han [24] assumes that the value of risk assets is driven by a relevant regime-switching jump-diffusion model and uses the Esscher transform to select an equivalent martingale measure to obtain a European vulnerable option pricing formula and discusses the impact of jump risk on European vulnerable option pricing. Capponi [25] presents an efficient method for pricing vulnerable contingent claims in a regime-switching market using a suitable change of probability measure and a Poisson series representation. Damircheli [26] investigated the modeling of default probability for publicly traded companies using a synchronous-jump regime-switching model and incorporating tempered stable processes. They proposed an efficient meshfree collocation method to solve the partial integro-differential equations associated with the credit risk model.

However, based on the above literature, we can see that the factors affecting the pricing of European vulnerable options include default barriers, risk asset value models, and national economic conditions. We found that there is currently no study on the pricing of European vulnerable options under a regime-switching model with stochastic default barriers. In light of this, we naturally consider proposing a model that is closer to the actual situation. This article assumes that risky assets are driven by a relevant regime-switching jump-diffusion model and that the default barrier follows a regime-switching geometric Brownian motion. We obtain an explicit solution for the European vulnerable option pricing formula. In addition, we use numerical simulation to explore the impact of default barriers and parameters on option prices and find that stochastic default barriers increase credit risk, leading to lower option prices. We differ from Han [24] in that they assume that the default barrier is constant. We assume that the dynamics of the default barrier

are regime-switching geometric Brownian motions related to the underlying assets and counterparty assets.

The structure of this paper is as follows: Section 2 establishes a regime-switching model for risk assets and default barriers. Section 3 uses the Esscher transform to select an equivalent martingale measure. Section 4 uses the risk-neutral pricing principle to obtain a pricing formula for European vulnerable options. Section 5 presents a numerical analysis of the proposed model using Monte Carlo simulation methods. The conclusion is given in Section 6.

2. Model Description

We construct a continuous-time financial market on a probability space (Ω, \mathcal{F}, P) . We assume that all risky assets can be continuously traded in time $[0, T]$. Let $X = \{X_t; t \geq 0\}$ denote a continuous-time Markov process on probability space (Ω, \mathcal{F}, P) with a finite state space $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots, 1, \dots, 0)' \in R^N$ with the i -th element is '1'. We assume that the states of the economy are modelled by a continuous-time Markov process $X = \{X_t; t \geq 0\}$. According to Elliott [27] semi-martingale decomposition for X , we have $X_t = X_0 + \int_0^t AX_s ds + M_t$, where $A = (q_{ij})_{N \times N}$ is Q- matrix of X and M_t is a R^N -valued martingale with respect to the natural filtration generated by $\{X_t; t \geq 0\}$.

Assumption 1. *The risk-free rate r_t is related to the state of the economy. Let $r_t = \langle r, X_t \rangle$, where $r = \{r_1, r_2, \dots, r_N\}' \in R^N$. For any $i = 1, 2, 3, \dots, N$, we have $r_i > 0$; $\langle \cdot, \cdot \rangle$ denotes the inner product of the vector. When $i \neq j$, $\langle e_i, e_j \rangle = 0$; otherwise, $\langle e_i, e_j \rangle = 1$.*

Assumption 2. *Suppose that S_t is the market value of the underlying assets at time t , and V_t denote the value of the counterparty's company assets at time t . V_t and S_t are driven by the following jump-diffusion processes:*

$$\frac{dS_t}{S_{t-}} = [\alpha_t - k_1(\lambda_t + \lambda_{1t})]dt + \sigma_{1t}dW_{1t} + (e^{Z_{1t}^i} - 1)dN_t + (e^{Z_{1t}^i} - 1)dN_{1t} \tag{1}$$

$$\frac{dV_t}{V_{t-}} = [\beta_t - k_2(\lambda_t + \lambda_{2t})]dt + \sigma_{2t}dW_{2t} + (e^{Z_{2t}^i} - 1)dN_t + (e^{Z_{2t}^i} - 1)dN_{2t} \tag{2}$$

where W_{1t} , W_{2t} are standard Brownian motions. N_{1t} and N_{2t} , respectively, indicate the individual shocks to the value of the underlying asset S_t and the individual shocks to the value of the counterparty's company assets V_t , where N_{1t} and N_{2t} are pure scalar Poisson process with intensities λ_{1t} and λ_{2t} , respectively. N_t refers to the common shocks to the corresponding assets, where N_t are pure scalar Poisson process with intensity λ_t .

For $i = 1, 2$, we suppose that Z_{t-}^i denotes the underlying asset's jump amplitude when the jump happens, following a normal distribution with mean μ_i and variance δ_i^2 . At any time, $s \neq t$, Z_{t-}^i and Z_{s-}^i are independent of each other; $k_i = E(e^{Z_{t-}^i} - 1) = e^{(\mu_i + \frac{1}{2}\delta_i^2)} - 1$ represents the average jump of the asset. α_t and σ_{1t} are the expected return rate of the underlying asset S_t and the volatility of the underlying asset S_t , respectively. β_t and σ_{2t} are the expected return rate of the value of the counterparty's company assets V_t and the volatility of the value of the counterparty's company assets V_t , respectively. We assume that $\alpha_t, \beta_t, \sigma_{1t}, \sigma_{2t}, \lambda_{1t}, \lambda_{2t}$ and λ_t all depend on the economic state X . Let $\alpha_t = \langle \alpha, X_t \rangle, \beta_t = \langle \beta, X_t \rangle, \sigma_{1t} = \langle \sigma_1, X_t \rangle, \sigma_{2t} = \langle \sigma_2, X_t \rangle, \lambda_{1t} = \langle \lambda_1, X_t \rangle, \lambda_{2t} = \langle \lambda_2, X_t \rangle, \lambda_t = \langle \lambda, X_t \rangle$, where $\alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_N)' \in R^N, \beta = (\beta_1, \dots, \beta_i, \dots, \beta_N)' \in R^N, \sigma_1 = (\sigma_{11}, \dots, \sigma_{1i}, \dots, \sigma_{1N})' \in R^N, \sigma_2 = (\sigma_{21}, \dots, \sigma_{2i}, \dots, \sigma_{2N})' \in R^N, \lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_N)' \in R^N, \lambda_1 = (\lambda_{11}, \dots, \lambda_{1i}, \dots, \lambda_{1N})' \in R^N$, and $\lambda_2 = (\lambda_{21}, \dots, \lambda_{2i}, \dots, \lambda_{2N})' \in R^N$. For any $j = 1, 2, 3, \dots, N$, and $i = 1, 2$, there are $\sigma_{ij} > 0$.

Assumption 3. The value of the counterparty’s company liabilities, also called the default barrier, is represented by D_t , and D_t is driven by the following geometric Brownian motion:

$$\frac{dD_t}{D_t} = \gamma_t dt + \sigma_{3t} dW_{3t} \tag{3}$$

where W_{3t} is the standard Brownian motion, and σ_{3t} is the volatility of the counterparty’s company liabilities. We assume that γ_t and σ_{3t} are related to the economic state X . Let $\gamma_t = \langle \gamma, X_t \rangle$, $\sigma_{3t} = \langle \sigma_3, X_t \rangle$, where $\gamma = (\gamma_1, \dots, \gamma_i, \dots, \gamma_N)' \in R^N$, $\sigma_3 = (\sigma_{31}, \dots, \sigma_{3i}, \dots, \sigma_{3N})' \in R^N$. For any $i = 1, 2, 3, \dots, N$, there are $\sigma_{3i} > 0$.

Assumption 4. We assume that the covariance matrix of the Brownian motion (W_{1t}, W_{2t}, W_{3t}) on probability space (Ω, \mathcal{F}, P) is as follows:

$$\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} t \tag{4}$$

We assume that $N_{1t}, N_{2t}, N_t, Z_{t-}^1$ and Z_{t-}^2 are independent of each other. Moreover, we assume that $N_{1t}, N_{2t}, N_t, Z_{t-}^1$ and Z_{t-}^2 are independent of W_{1t}, W_{2t}, W_{3t} . In addition, for the convenience of calculation in the following chapters, we express the relevant three-dimensional Brownian motion W_{1t}, W_{2t} , and W_{3t} as

$$\begin{pmatrix} W_{1t} \\ W_{2t} \\ W_{3t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}} & \sqrt{\frac{(1 - \rho_{12}^2)(1 - \rho_{13}^2) - (\rho_{23} - \rho_{12}\rho_{13})^2}{1 - \rho_{12}^2}} \end{pmatrix} \begin{pmatrix} \bar{W}_{1t} \\ \bar{W}_{2t} \\ \bar{W}_{3t} \end{pmatrix} \tag{5}$$

where W_{1t}, \bar{W}_{2t} and \bar{W}_{3t} are independent Brownian motions.

Assumption 5. At the option maturity T , if the value of the counterparty’s company assets is not greater than the value of the counterparty’s company liabilities, we believe that the option has defaulted. The option holder will receive $\frac{(1 - \omega)D_T}{V_T}$ times the notional amount from the option writer, where ω is the recovery rate. Therefore, at the maturity T , the payment function of the vulnerable European call option with a strike price of K is as follows:

$$(S_T - K)^+ \left\{ I_{\{V_T \geq D_T\}} + \frac{(1 - \omega)V_T}{D_T} I_{\{V_T < D_T\}} \right\} \tag{6}$$

where $I_{\{\cdot\}}$ is represented as an indicator function.

3. Selection of Equivalent Martingale Measures

Due to the presence of jumps and the uncertainty introduced by regime switching, the market is incomplete and there are infinitely many equivalent martingale measures. Furthermore, it is not easy to obtain an equivalent martingale measure for two jump-diffusion processes with correlated jumps. It is important to note that while systemic risks can be hedged, non-systematic risks are specific to individual risk assets and cannot be hedged, so their premiums need not be considered. In this paper, we assume that the jump component represents systemic risk and its risk premium must be taken into account. We should measure changes in the jump component. In this section, we apply the regime-switching Esscher transform to select equivalent martingale measures for three stochastic processes with associated jump risks under regime switching.

Using Ito’s theorem, the solutions of Equations (1)–(3) are given by

$$\frac{S_t}{S_0} = e^{L_{1t}} = \exp\left\{\int_0^t \left[\alpha_s - k_1(\lambda_s + \lambda_{1s}) - \frac{1}{2}\sigma_{1s}^2\right] ds + \int_0^t \sigma_{1s} dW_{1s} + \int_0^t Z_{s-}^1 dN_{1s} + \int_0^t Z_{s-}^1 dN_s\right\} \tag{7}$$

$$\frac{V_t}{V_0} = e^{L_{2t}} = \exp\left\{\int_0^t \left[\beta_s - k_2(\lambda_s + \lambda_{2s}) - \frac{1}{2}\sigma_{2s}^2\right] ds + \int_0^t \sigma_{2s} dW_{2s} + \int_0^t Z_{s-}^2 dN_{2s} + \int_0^t Z_{s-}^2 dN_s\right\} \tag{8}$$

$$\frac{D_t}{D_0} = e^{L_{3t}} = \exp\left\{\int_0^t \left[\gamma_s - \frac{1}{2}\sigma_{3s}^2\right] ds + \int_0^t \sigma_{3s} dW_{3s}\right\} \tag{9}$$

where L_{1t} is the return process of the underlying asset S_t , L_{2t} is the return process of the underlying asset V_t , and L_{3t} is the return process of the underlying asset D_t . According to the above Equations (7)–(9), we have the three process L_{1t} , L_{2t} , and L_{3t} as follows:

$$L_{1t} = \int_0^t \left[\alpha_s - k_1(\lambda_s + \lambda_{1s}) - \frac{1}{2}\sigma_{1s}^2\right] ds + \int_0^t \sigma_{1s} dW_{1s} + \int_0^t Z_{s-}^1 dN_{1s} + \int_0^t Z_{s-}^1 dN_s = C_{1t} + J_{1t} + J_t \tag{10}$$

$$L_{2t} = \int_0^t \left[\beta_s - k_2(\lambda_s + \lambda_{2s}) - \frac{1}{2}\sigma_{2s}^2\right] ds + \int_0^t \sigma_{2s} dW_{2s} + \int_0^t Z_{s-}^2 dN_{2s} + \int_0^t Z_{s-}^2 dN_s = C_{2t} + J_{2t} + J_t' \tag{11}$$

$$L_{3t} = \int_0^t \left[\gamma_s - \frac{1}{2}\sigma_{3s}^2\right] ds + \int_0^t \sigma_{3s} dW_{3s} = C_{3t} \tag{12}$$

Let $\{\mathcal{F}_t^X\}_{t \in [0, T]}$ and $\{\mathcal{F}_t^i\}_{t \in [0, T]}$ ($i = 1, 2, 3$) be the natural σ -filtrations generated by $\{X_t\}_{t \in [0, T]}$ and $\{L_{it}\}_{t \in [0, T]}$ ($i = 1, 2, 3$), respectively. For any $i = 1, 2, 3$ and $t \in [0, T]$, define a filtration $\{\mathcal{H}_t^i\}_{t \in [0, T]}$ such that \mathcal{H}_t^i is the σ - algebra with $\mathcal{H}_t^i = \mathcal{F}_t^X \vee \mathcal{F}_t^i$. For any $t \in [0, T]$, let $\mathcal{R}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2 \vee \mathcal{H}_t^3$. Let $V_{1t}^{c_1}, V_{2t}^{c_2}, V_{3t}^{c_3}, V_{1t}^{j_1}, V_{2t}^{j_2}, V_t^j$ and $V_t^{j'}$ be the regime-switching Esscher transform parameters, and they are all related to the economic state of X . For any $m \in \{c_1, j_1\}, l \in \{j, j'\}$, and $n \in \{c_2, j_2\}$, we assume that $V_{1t}^m = \langle V_1^m, X_t \rangle, V_{2t}^n = \langle V_2^n, X_t \rangle$, and $V_t^l = \langle V^l, X_t \rangle$, where $V_1^m = (V_{11}^m, \dots, V_{1i}^m, \dots, V_{1N}^m)' \in R^N, V_2^n = (V_{21}^n, \dots, V_{2i}^n, \dots, V_{2N}^n)' \in R^N$, and $V^l = (V_1^l, \dots, V_i^l, \dots, V_N^l)' \in R^N$. For the regime-switching Esscher transform parameter $V_{3t}^{c_3}$, we assume that $V_{3t}^{c_3} = \langle V_3^{c_3}, X_t \rangle$, where $V_3^{c_3} = (V_{31}^{c_3}, \dots, V_{3i}^{c_3}, \dots, V_{3N}^{c_3})' \in R^N$.

Under the given condition \mathcal{F}_t^X , we use the regime-switching Esscher transform proposed by Elliott [18] to select an equivalent martingale measure. Define the equivalent measure of \bar{P} as P , the Radon-Nikodym derivative of the regime-switching Esscher transform can be written as follows:

$$\frac{d\bar{P}}{dP} = \frac{\exp\left\{\int_0^T V_{1t}^{c_1} dC_{1t} + \int_0^T V_{2t}^{c_2} dC_{2t} + \int_0^T V_{3t}^{c_3} dC_{3t}\right\}}{E_P\left\{\exp\left[\int_0^T V_{1t}^{c_1} dC_{1t} + \int_0^T V_{2t}^{c_2} dC_{2t} + \int_0^T V_{3t}^{c_3} dC_{3t}\right] \middle| \mathcal{F}_T^X\right\}} \times \frac{\exp\left\{\int_0^T V_{1t}^{j_1} dJ_{1t} + \int_0^T V_{2t}^{j_2} dJ_{2t} + \int_0^T V_t^j dJ_t + \int_0^T V_t^{j'} dJ_t'\right\}}{E_P\left\{\exp\left[\int_0^T V_{1t}^{j_1} dJ_{1t} + \int_0^T V_{2t}^{j_2} dJ_{2t} + \int_0^T V_t^j dJ_t + \int_0^T V_t^{j'} dJ_t'\right] \middle| \mathcal{F}_T^X\right\}} \tag{13}$$

By Equation (5) and calculation, we have

$$\begin{aligned} \frac{d\bar{P}}{\bar{P}} = & \exp \left\{ \int_0^T V_{1t}^{c_1} \sigma_{1t} dW_{1t} + \int_0^T V_{2t}^{c_2} \sigma_{2t} dW_{2t} + \int_0^T V_{3t}^{c_3} \sigma_{3t} dW_{3t} \right\} \\ \times \exp \left\{ -\frac{1}{2} \int_0^T \left[(V_{1t}^{c_1})^2 \sigma_{1t}^2 + (V_{2t}^{c_2})^2 \sigma_{2t}^2 + (V_{3t}^{c_3})^2 \sigma_{3t}^2 \right] dt + \int_0^T \left(V_{1t}^j Z_{1t}^1 + V_{2t}^j Z_{2t}^2 \right) dN_t \right\} \\ \times \exp \left\{ -\int_0^T \left[\rho_{12} V_{1t}^{c_1} \sigma_{1t} V_{2t}^{c_2} \sigma_{2t} + \rho_{13} V_{1t}^{c_1} \sigma_{1t} V_{3t}^{c_3} \sigma_{3t} + \rho_{23} V_{2t}^{c_2} \sigma_{2t} V_{3t}^{c_3} \sigma_{3t} \right] dt \right\} \\ \times \exp \left\{ -\int_0^T \lambda_{1t} \left[\exp \left(\mu_1 V_{1t}^{j_1} + \frac{1}{2} (\delta_1 V_{1t}^{j_1})^2 \right) - 1 \right] dt + \int_0^T V_{1t}^{j_1} Z_{1t}^1 dN_{1t} \right\} \\ \times \exp \left\{ -\int_0^T \lambda_{2t} \left[\exp \left(\mu_2 V_{2t}^{j_2} + \frac{1}{2} (\delta_2 V_{2t}^{j_2})^2 \right) - 1 \right] dt + \int_0^T V_{2t}^{j_2} Z_{2t}^2 dN_{2t} \right\} \\ \times \exp \left\{ -\int_0^T \lambda_t \left[\exp \left(\mu_1 V_t^j + \mu_2 V_t^{j'} + \frac{1}{2} (\delta_1 V_t^j)^2 + \frac{1}{2} (\delta_2 V_t^{j'})^2 \right) - 1 \right] dt \right\} \end{aligned} \tag{14}$$

Delbaen [28] showed that the absence of arbitrage is essentially equivalent to the existence of equivalent martingale measures. Under an equivalent martingale measure, the price process of a risky asset is a martingale, which is known as the martingale condition. Due to the uncertainty introduced by the economic state X, the discount process must be defined on a larger filtration R. From the principles of asset pricing and no-arbitrage, it follows that under the equivalent martingale measures \bar{P} , the discount process of risky assets is a martingale.

Theorem 1. *The discounting process of risk assets satisfies the following formula:*

$$E_{\bar{P}} \left\{ \exp \left[-\int_0^t r_s ds \right] S_t | \mathcal{R}_u \right\} = \exp \left(-\int_0^u r_s ds \right) S_u \tag{15}$$

$$E_{\bar{P}} \left\{ \exp \left[-\int_0^t r_s ds \right] V_t | \mathcal{R}_u \right\} = \exp \left(-\int_0^u r_s ds \right) V_u \tag{16}$$

$$E_{\bar{P}} \left\{ \exp \left[-\int_0^t r_s ds \right] D_t | \mathcal{R}_u \right\} = \exp \left(-\int_0^u r_s ds \right) D_u \tag{17}$$

if and only if the regime-switching Esscher transform parameters satisfy the following conditions:

$$\alpha_s - k_1(\lambda_s + \lambda_{1s}) - r_s + V_{1s}^{c_1} \sigma_{1s}^2 + \rho_{12} \sigma_{1s} V_{2s}^{c_2} \sigma_{2s} + \rho_{13} V_{3s}^{c_3} \sigma_{1s} \sigma_{3s} + \tilde{\lambda}_{1s} k_1^{j_1} + \tilde{\lambda}_s k_1^j = 0 \tag{18}$$

$$\beta_s - k_2(\lambda_s + \lambda_{2s}) - r_s + V_{2s}^{c_2} \sigma_{2s}^2 + \rho_{12} \sigma_{1s} V_{1s}^{c_1} \sigma_{2s} + \rho_{23} V_{3s}^{c_3} \sigma_{2s} \sigma_{3s} + \tilde{\lambda}_{2s} k_2^{j_2} + \tilde{\lambda}_s k_2^{j'} = 0 \tag{19}$$

$$\gamma_s - r_s + V_{3s}^{c_3} \sigma_{3s}^2 + \rho_{13} \sigma_{1s} V_{1s}^{c_1} \sigma_{3s} + \rho_{23} V_{2s}^{c_2} \sigma_{2s} \sigma_{3s} = 0 \tag{20}$$

where $\tilde{\lambda}_{1s} = \lambda_{1s} \exp \left(\mu_1 V_{1s}^{j_1} + \frac{1}{2} (\delta_1 V_{1s}^{j_1})^2 \right)$, $\tilde{\lambda}_{2s} = \lambda_{2s} \exp \left(\mu_2 V_{2s}^{j_2} + \frac{1}{2} (\delta_2 V_{2s}^{j_2})^2 \right)$, $\tilde{\lambda}_s = \lambda_s \exp \left(\mu_1 V_s^j + \mu_2 V_s^{j'} + \frac{1}{2} (\delta_1 V_s^j)^2 + \frac{1}{2} (\delta_2 V_s^{j'})^2 \right)$, $k_1^{j_1} = (k_1 + 1) e^{\delta_1^2 V_{1s}^{j_1}} - 1$, $k_2^{j_2} = (k_2 + 1) e^{\delta_2^2 V_{2s}^{j_2}} - 1$, $k_1^j = (k_1 + 1) e^{\delta_1^2 V_{1s}^j} - 1$, $k_2^{j'} = (k_2 + 1) e^{\delta_2^2 V_{2s}^{j'}} - 1$.

Proof. By Bayes' rule, we have

$$\begin{aligned} E_{\bar{P}} \left\{ e^{-\int_0^t r_s ds} S_t | \mathcal{R}_u \right\} &= E_P \left\{ e^{-\int_0^t r_s ds} S_t \frac{d\bar{P}}{dP} | \mathcal{R}_u \right\} = \frac{E_P \left\{ e^{-\int_0^t r_s ds} S_t \frac{d\bar{P}}{dP} | \mathcal{R}_u \right\}}{E_P \left\{ \frac{d\bar{P}}{dP} | \mathcal{R}_u \right\}} \\ &= \exp \left(-\int_0^u r_s ds \right) S_u E_P \left\{ \exp \left[M_S^A + M_S^B + M_S^C + M_S^J \right] \right\} = \exp \left(-\int_0^u r_s ds \right) S_u \end{aligned} \tag{21}$$

So, if and only if the following formula holds, the risk asset discounting process is martingale

$$\alpha_s - k_1(\lambda_s + \lambda_{1s}) - r_s + V_{1s}^{c_1} \sigma_{1s}^2 + \rho_{12} \sigma_{1s} V_{2s}^{c_2} \sigma_{2s} + \rho_{13} V_{3s}^{c_3} \sigma_{1s} \sigma_{3s} + \tilde{\lambda}_{1s} k_1^j + \tilde{\lambda}_s k_1^j = 0$$

Similarly, for the risk asset V_t , the following conditions can be proved:

$$\beta_s - k_2(\lambda_s + \lambda_{2s}) - r_s + V_{2s}^{c_2} \sigma_{2s}^2 + \rho_{12} \sigma_{1s} V_{1s}^{c_1} \sigma_{2s} + \rho_{23} V_{3s}^{c_3} \sigma_{2s} \sigma_{3s} + \tilde{\lambda}_{2s} k_2^j + \tilde{\lambda}_s k_2^j = 0$$

The following proves the conditions for the equivalent martingale measures of the risk asset D_t . According to Baye’s rule, we also know

$$\begin{aligned} E_{\bar{P}} \left\{ e^{-\int_0^t r_s ds} D_t | \mathcal{R}_u \right\} &= E_P \left\{ e^{-\int_0^t r_s ds} D_t \frac{d\bar{P}}{dP} | \mathcal{R}_u \right\} = \frac{E_P \left\{ e^{-\int_0^t r_s ds} D_t \frac{d\bar{P}}{dP} | \mathcal{R}_u \right\}}{E_P \left\{ \frac{d\bar{P}}{dP} | \mathcal{R}_u \right\}} \\ &= E_P \left\{ \exp \left[\int_u^t \left[\gamma_s - r_s + V_{3s}^{c_3} \sigma_{3s}^2 + \rho_{13} \sigma_{1s} V_{1s}^{c_1} \sigma_{3s} + \rho_{23} V_{2s}^{c_2} \sigma_{2s} \sigma_{3s} \right] ds \right] \right\} \exp \left(- \int_0^u r_s ds \right) D_u \\ &= \exp \left(- \int_0^u r_s ds \right) D_u \end{aligned} \tag{22}$$

We have $\gamma_s - r_s + V_{3s}^{c_3} \sigma_{3s}^2 + \rho_{13} \sigma_{1s} V_{1s}^{c_1} \sigma_{3s} + \rho_{23} V_{2s}^{c_2} \sigma_{2s} \sigma_{3s} = 0$.
The proof of Theorem 1 is complete. \square

Theorem 2. Under the equivalent martingale measure \bar{P} , S_t , V_t and D_t are driven by the following formula:

$$\log \left(\frac{S_t}{S_0} \right) = \int_0^t \left[r_s - \tilde{\lambda}_s k_1^j - \tilde{\lambda}_{1s} k_1^j - \frac{1}{2} \sigma_{1s}^2 \right] ds + \int_0^t \sigma_{1s} d\tilde{W}_{1s} + \int_0^t \tilde{Z}_{s-}^1 d\tilde{N}_{1s} + \int_0^t \hat{Z}_{s-}^1 d\tilde{N}_s \tag{23}$$

$$\log \left(\frac{V_t}{V_0} \right) = \int_0^t \left[r_s - \tilde{\lambda}_s k_2^j - \tilde{\lambda}_{2s} k_2^j - \frac{1}{2} \sigma_{2s}^2 \right] ds + \int_0^t \sigma_{2s} d\tilde{W}_{2s} + \int_0^t \tilde{Z}_{s-}^2 d\tilde{N}_{2s} + \int_0^t \hat{Z}_{s-}^2 d\tilde{N}_s \tag{24}$$

$$\log \left(\frac{D_t}{D_0} \right) = \int_0^t \left[r_s - \frac{1}{2} \sigma_{3s}^2 \right] ds + \int_0^t \sigma_{3s} d\tilde{W}_{3s} \tag{25}$$

where \tilde{W}_{1s} , \tilde{W}_{2s} and \tilde{W}_{3s} are the standard Brownian motion under the equivalent martingale measure \bar{P} , and satisfies the following relationship:

$$\tilde{W}_{1t} = W_{1t} - \int_0^t V_{1s}^{c_1} \sigma_{1s} ds - \int_0^t (\rho_{12} V_{2s}^{c_2} \sigma_{2s} + \rho_{13} V_{3s}^{c_3} \sigma_{3s}) ds \tag{26}$$

$$\tilde{W}_{2t} = W_{2t} - \int_0^t V_{2s}^{c_2} \sigma_{2s} ds - \int_0^t (\rho_{12} V_{1s}^{c_1} \sigma_{1s} + \rho_{23} V_{3s}^{c_3} \sigma_{3s}) ds \tag{27}$$

$$\tilde{W}_{3t} = W_{3t} - \int_0^t V_{3s}^{c_3} \sigma_{3s} ds - \int_0^t (\rho_{13} V_{1s}^{c_1} \sigma_{1s} + \rho_{23} V_{2s}^{c_2} \sigma_{2s}) ds \tag{28}$$

$$\log \left(\frac{D_t}{D_0} \right) = \int_0^t \left[r_s - \frac{1}{2} \sigma_{3s}^2 \right] ds + \int_0^t \sigma_{3s} d\tilde{W}_{3s}$$

The covariance matrix of the Brownian motion $(\tilde{W}_{1t}, \tilde{W}_{2t}, \tilde{W}_{3t})$ is the same as that of (W_{1t}, W_{2t}, W_{3t}) .

\tilde{N}_{1t} , \tilde{N}_{2t} and \tilde{N}_t are Poisson processes with respective intensities $\tilde{\lambda}_{1t} = \lambda_{1t} \exp \left(\mu_1 V_{1t}^{j_1} + \frac{1}{2} (\delta_1 V_{1t}^{j_1})^2 \right)$, $\tilde{\lambda}_{2t} = \lambda_{2t} \exp \left(\mu_2 V_{2t}^{j_2} + \frac{1}{2} (\delta_2 V_{2t}^{j_2})^2 \right)$ and $\tilde{\lambda}_t = \lambda_t \exp \left(\mu_1 V_t^j + \mu_2 V_t^{j'} + \frac{1}{2} (\delta_1 V_t^j)^2 + \frac{1}{2} (\delta_2 V_t^{j'})^2 \right)$, respectively. \tilde{Z}_{t-}^1 , \tilde{Z}_{t-}^2 , \hat{Z}_{t-}^1 and \hat{Z}_{t-}^2 are normal random variables, and $\tilde{Z}_{t-}^1 \sim$

$N(\mu_1 + V_{1t}^j \delta_1^2, \delta_1^2)$, $\tilde{Z}_{t-}^2 \sim N(\mu_2 + V_{2t}^j \delta_2^2, \delta_2^2)$, $\hat{Z}_{t-}^1 \sim N(\mu_1 + V_{1t}^j \delta_1^2, \delta_1^2)$ and $\hat{Z}_{t-}^2 \sim N(\mu_2 + V_{2t}^j \delta_2^2, \delta_2^2)$. $k_1^j = (k_1 + 1)e^{\delta_1^2 V_{1t}^j} - 1$, $k_2^j = (k_2 + 1)e^{\delta_2^2 V_{2t}^j} - 1$, $k_1^j = (k_1 + 1)e^{\delta_1^2 V_{1t}^j} - 1$ and $k_2^j = (k_2 + 1)e^{\delta_2^2 V_{2t}^j} - 1$ are expressed as the average jumps of \tilde{Z}_{t-}^1 , \tilde{Z}_{t-}^2 , \hat{Z}_{t-}^1 and \hat{Z}_{t-}^2 , respectively.

Proof. According to the multi-dimensional Girsanov theorem, under the given conditions \mathcal{F}_t^X and the equivalent martingale measure \bar{P} , we know that \tilde{W}_{1t} , \tilde{W}_{2t} and \tilde{W}_{3t} are Brownian motion such that Equations (29)–(31) are established.

Using the theorem T10 [29] (p. 241), we observe that \tilde{Z}_{t-}^1 , \tilde{Z}_{t-}^2 , \hat{Z}_{t-}^1 and \hat{Z}_{t-}^2 are normal random variables, i.e., $\tilde{Z}_{t-}^1 \sim N(\mu_1 + V_{1t}^j \delta_1^2, \delta_1^2)$, $\tilde{Z}_{t-}^2 \sim N(\mu_2 + V_{2t}^j \delta_2^2, \delta_2^2)$, $\hat{Z}_{t-}^1 \sim N(\mu_1 + V_{1t}^j \delta_1^2, \delta_1^2)$ and $\hat{Z}_{t-}^2 \sim N(\mu_2 + V_{2t}^j \delta_2^2, \delta_2^2)$. We can calculate $k_1^j = E(e^{\tilde{Z}_{t-}^1} - 1) = (k_1 + 1)e^{\delta_1^2 V_{1t}^j} - 1$, $k_2^j = E(e^{\tilde{Z}_{t-}^2} - 1) = (k_2 + 1)e^{\delta_2^2 V_{2t}^j} - 1$, $k_1^j = E(e^{\hat{Z}_{t-}^1} - 1) = (k_1 + 1)e^{\delta_1^2 V_{1t}^j} - 1$ and $k_2^j = E(e^{\hat{Z}_{t-}^2} - 1) = (k_2 + 1)e^{\delta_2^2 V_{2t}^j} - 1$.

The proof of Theorem 2 is complete. \square

According to Theorem 1, we obtain the martingale condition, but from algebraic knowledge, we know that there are infinite solutions to Equations (18)–(20). In order to facilitate the numerical simulation in Section 5, we provide a set of special solutions. Let

$$\left\{ \begin{array}{l} \alpha_s - k_1(\lambda_s + \lambda_{1s}) - r_s + V_{1s}^{c_1} \sigma_{1s}^2 + \rho_{12} \sigma_{1s} V_{2s}^{c_2} \sigma_{2s} + \rho_{13} V_{3s}^{c_3} \sigma_{1s} \sigma_{3s} = 0 \\ \beta_s - k_2(\lambda_s + \lambda_{1s}) - r_s + V_{2s}^{c_2} \sigma_{2s}^2 + \rho_{12} \sigma_{1s} V_{1s}^{c_1} \sigma_{2s} + \rho_{23} V_{3s}^{c_3} \sigma_{2s} \sigma_{3s} = 0 \\ \gamma_s - r_s + V_{3s}^{c_3} \sigma_{3s}^2 + \rho_{13} \sigma_{1s} V_{1s}^{c_1} \sigma_{3s} + \rho_{23} V_{2s}^{c_2} \sigma_{2s} \sigma_{3s} = 0 \\ \lambda_{1s} \exp\left(\mu_1 V_{1s}^j + \frac{1}{2}(\delta_1 V_{1s}^j)^2\right) \left((k_1 + 1) \exp(\delta_1^2 V_{1s}^j) - 1\right) = 0 \\ \lambda_{2s} \exp\left(\mu_2 V_{2s}^j + \frac{1}{2}(\delta_2 V_{2s}^j)^2\right) \left((k_2 + 1) \exp(\delta_2^2 V_{2s}^j) - 1\right) = 0 \\ \lambda_s \exp\left(\mu_1 V_s^j + \mu_2 V_s^{j'} + \frac{1}{2}(\delta_1 V_s^j)^2 + \frac{1}{2}(\delta_2 V_s^{j'})^2\right) \left((k_1 + 1) e^{\delta_1^2 V_s^j} - 1\right) = 0 \\ \lambda_s \exp\left(\mu_1 V_s^j + \mu_2 V_s^{j'} + \frac{1}{2}(\delta_1 V_s^j)^2 + \frac{1}{2}(\delta_2 V_s^{j'})^2\right) \left((k_1 + 1) e^{\delta_2^2 V_s^{j'}} - 1\right) = 0 \end{array} \right. \tag{29}$$

Solving the above equation can obtain the regime-switching Esscher transform parameters as

$$\begin{aligned} V_{1s}^{c_1} &= \frac{\pi_1(\rho_{23}^2 - 1)\sigma_{3s}\sigma_{2s} + \pi_2(\rho_{12} - \rho_{23}\rho_{13})\sigma_{1s}\sigma_{3s} + \pi_3(\rho_{13} - \rho_{23}\rho_{12})\sigma_{1s}\sigma_{2s}}{\sigma_{3s}\sigma_{2s}\sigma_{1s}^2 \left[(1 - \rho_{12}^2)(1 - \rho_{13}^2) - (\rho_{23} - \rho_{13}\rho_{12})^2 \right]} \\ V_{2s}^{c_2} &= \frac{\pi_1(\rho_{12} - \rho_{23}\rho_{13})\sigma_{3s}\sigma_{2s} + \pi_2(\rho_{13}^2 - 1)\sigma_{1s}\sigma_{3s} + \pi_3(\rho_{23} - \rho_{13}\rho_{12})\sigma_{1s}\sigma_{2s}}{\sigma_{3s}\sigma_{1s}\sigma_{2s}^2 \left[(1 - \rho_{12}^2)(1 - \rho_{13}^2) - (\rho_{23} - \rho_{13}\rho_{12})^2 \right]} \\ V_{3s}^{c_3} &= \frac{\pi_1(\rho_{13} - \rho_{23}\rho_{12})\sigma_{3s}\sigma_{2s} + \pi_2(\rho_{23} - \rho_{13}\rho_{12})\sigma_{1s}\sigma_{3s} + \pi_3(\rho_{12}^2 - 1)\sigma_{1s}\sigma_{2s}}{\sigma_{1s}\sigma_{2s}\sigma_{3s}^2 \left[(1 - \rho_{12}^2)(1 - \rho_{13}^2) - (\rho_{23} - \rho_{13}\rho_{12})^2 \right]} \\ V_{1s}^j &= V_s^j = -\frac{\mu_1 + \frac{1}{2}\delta_1^2}{\delta_1^2}, V_s^{j'} = V_{2s}^j = -\frac{\mu_2 + \frac{1}{2}\delta_2^2}{\delta_2^2}, \tilde{\lambda}_s = \lambda_s \exp\left(-\frac{\mu_1^2}{2\delta_1^2} + \frac{\delta_1^2}{8} - \frac{\mu_2^2}{2\delta_2^2} + \frac{\delta_2^2}{8}\right), \\ \tilde{\lambda}_{1s} &= \lambda_{1s} \exp\left(-\frac{\mu_1^2}{2\delta_1^2} + \frac{\delta_1^2}{8}\right), \tilde{\lambda}_{2s} = \lambda_{2s} \exp\left(-\frac{\mu_2^2}{2\delta_2^2} + \frac{\delta_2^2}{8}\right), k_1^j = k_2^j = k_1^{j'} = k_2^{j'} = 0, \text{ where} \\ \pi_1 &= \alpha_s - k_1(\lambda_s + \lambda_{1s}) - r_s, \pi_2 = \beta_s - k_2(\lambda_s + \lambda_{2s}) - r_s \text{ and } \pi_3 = \gamma_s - r_s. \end{aligned}$$

4. Pricing Vulnerable Options

In this section, we derive an explicit formula for pricing vulnerable options. Due to the similarity between call and put options, we only provide detailed calculations for vulnerable European call options.

Theorem 3. Under the proposed model framework, we obtain the pricing formula of European vulnerable call options as

$$C(S, V, D, X) = \int_{[0,T]^N} C^* \psi(J_1, J_2, \dots, J_N) dJ_1 dJ_2 \dots dJ_N \tag{30}$$

where $\psi(J_1, J_2, \dots, J_N)$ is the joint distribution of accumulated stay time $J = (J_1, J_2, \dots, J_N)$ given $X_0 = X$ under the equivalent martingale measure \bar{P} . Under the equivalent martingale measure \bar{P} , the characteristic function of $J = (J_1, J_2, \dots, J_N)$ satisfies the following formula:

$$E_{\bar{P}}\{exp[i\langle \xi, J \rangle] | X_0 = X\} = \langle X exp[AT + iGT], I \rangle \tag{31}$$

where $i^2 = -1$, $\xi = (\xi_1, \xi_2, \dots, \xi_N)' \in R^N$, $I = (1, 1, \dots, 1)' \in R^N$ and G is a N order diagonal matrix generated by $\xi_1, \xi_2, \dots, \xi_N$;

$$\begin{aligned} C^* &= E_{\bar{P}}\left\{e^{-\int_0^T r_s ds} (S_T - K)^+ \left\{ I_{\{H_T \geq 1\}} + (1 - \omega) H_T I_{\{H_T < 1\}} \right\} | \mathcal{F}_T^X \right\} \\ &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} \mathbb{Q}(\tilde{N}_s = n, \tilde{N}_{1s} = n_1, \tilde{N}_s = n_2) e^{-T \bar{R}_T} [A_1(m_1, m_2) + A_2(m_1, m_2) + A_3(m_1, m_2) + A_4(m_1, m_2)] \\ &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} \frac{(\bar{\lambda}_s T)^n}{n!} \frac{(\bar{\lambda}_{1s} T)^{m_1-n}}{(m_1-n)!} \frac{(\bar{\lambda}_{2s} T)^{m_2-n}}{(m_2-n)!} e^{-T(\bar{\lambda}_s + \bar{\lambda}_{1s} + \bar{\lambda}_{2s} + \bar{R}_T)} \\ &\quad \times [A_1(m_1, m_2) + A_2(m_1, m_2) + A_3(m_1, m_2) + A_4(m_1, m_2)] \end{aligned} \tag{32}$$

Proof. According to the principle of risk-neutral pricing, under the equivalent martingale measures \bar{P} , $S_0 = S$, $D_0 = D$, $V_0 = V$ and the initial state of the national economy $X_0 = X$, the European vulnerable options $C(S, V, D, X)$ can be written as follows:

$$\begin{aligned} C(S, V, D, X) &= E_{\bar{P}}\left\{exp\left[-\int_0^T r_s ds\right] (S_T - K)^+ \left\{ I_{\{V_T \geq D_T\}} + \frac{(1-\omega)V_T}{D_T} I_{\{V_T < D_T\}} \right\} \right\} \\ &= E_{\bar{P}}\left\{exp\left[-\int_0^T r_s ds\right] (S_T - K)^+ \left\{ I_{\{H_T \geq 1\}} + (1 - \omega) H_T I_{\{H_T < 1\}} \right\} \right\} \end{aligned} \tag{33}$$

where $H_T = \frac{V_T}{D_T}$. Under the equivalent martingale measure \bar{P} , the expression of V_T and D_T are expressed by Equations (22)–(24), and the expression of H_T is

$$\begin{aligned} \log\left(\frac{H_T}{H_0}\right) &= \int_0^T \left[\frac{1}{2} \sigma_{3s}^2 - \tilde{\lambda}_s k_2^j - \tilde{\lambda}_{2s} k_2^{j2} - \frac{1}{2} \sigma_{2s}^2 \right] ds + \int_0^T \sigma_{2s} d\tilde{W}_{2s} - \int_0^T \sigma_{3s} d\tilde{W}_{3s} \\ &\quad + \int_0^T \tilde{Z}_{s-}^2 d\tilde{N}_{2s} + \int_0^T \hat{Z}_{s-}^2 d\tilde{N}_s \end{aligned} \tag{34}$$

The model parameters we propose include the interest rate, jump intensity, and volatility of asset value, which are governed by an observable continuous-time finite-state Markov chain. To facilitate calculations, we need to determine the residence time of each state in the finite-state Markov chain (which can be interpreted as the duration of “good” and “bad” states in the market economy). We assume that J_i represents the staying time of the market economy state $\{X_t\}_{t \in [0,T]}$ in state i within $[0, T]$ time, we have

$$\begin{aligned} \bar{R}_T &= \frac{1}{T} \int_0^T r_s ds = \frac{1}{T} \sum_{i=0}^N r_i J_i, \quad \bar{U}_{1T} = \frac{1}{T} \int_0^T \sigma_{1s}^2 ds = \frac{1}{T} \sum_{i=0}^N \sigma_{1i}^2 J_i, \\ \bar{U}_{2T} &= \frac{1}{T} \int_0^T \sigma_{2s}^2 ds = \frac{1}{T} \sum_{i=0}^N \sigma_{2i}^2 J_i, \quad \bar{U}_{3T} = \frac{1}{T} \int_0^T \sigma_{3s}^2 ds = \frac{1}{T} \sum_{i=0}^N \sigma_{3i}^2 J_i, \end{aligned}$$

$$\begin{aligned} \bar{U}_{4T} &= \frac{1}{T} \int_0^T \sigma_{1s} \sigma_{2s} ds = \frac{1}{T} \sum_{i=0}^N \sigma_{1i} \sigma_{2i} J_i, \quad \bar{U}_{5T} = \frac{1}{T} \int_0^T \sigma_{1s} \sigma_{3s} ds = \frac{1}{T} \sum_{i=0}^N \sigma_{1i} \sigma_{3i} J_i, \\ \bar{U}_{6T} &= \frac{1}{T} \int_0^T \sigma_{2s} \sigma_{3s} ds = \frac{1}{T} \sum_{i=0}^N \sigma_{2i} \sigma_{3i} J_i, \quad \bar{\lambda}_s = \frac{1}{T} \int_0^T \tilde{\lambda}_s ds = \frac{1}{T} \sum_{i=0}^N \tilde{\lambda}_i J_i, \\ \bar{\lambda}_{1s} &= \frac{1}{T} \int_0^T \tilde{\lambda}_{1s} ds = \frac{1}{T} \sum_{i=0}^N \tilde{\lambda}_{1i} J_i, \quad \bar{\lambda}_{2s} = \frac{1}{T} \int_0^T \tilde{\lambda}_{2s} ds = \frac{1}{T} \sum_{i=0}^N \tilde{\lambda}_{2i} J_i. \end{aligned}$$

We need to calculate the European call vulnerable options $C(S, V, D, X)$ under condition $S_0 = S, D_0 = D, V_0 = V$ and $X_0 = X$, and first calculate the European call vulnerable option under the given conditions $\mathcal{F}_T^X, S_0 = S, D_0 = D, V_0 = V$ and $X_0 = X$, i.e.,

$$\begin{aligned} C^* &= C(S, V, D, \bar{R}_T, \bar{U}_{1T}, \bar{U}_{2T}, \bar{U}_{3T}, \bar{U}_{4T}, \bar{U}_{5T}, \bar{U}_{6T}, \bar{\lambda}_s, \bar{\lambda}_{1s}, \bar{\lambda}_{2s}) \\ &= E_{\bar{P}} \left\{ e^{-\int_0^T r_s ds} (S_T - K)^+ \left\{ I_{\{H_T \geq 1\}} + (1 - \omega) H_T I_{\{H_T < 1\}} \right\} \middle| \mathcal{F}_T^X \right\} \end{aligned} \tag{35}$$

In order to calculate the European call vulnerable option under conditions $S_0 = S, D_0 = D, V_0 = V$ and $X_0 = X$, we need to know the joint conditional distribution of $\bar{R}_T, \bar{U}_{1T}, \bar{U}_{2T}, \bar{U}_{3T}, \bar{U}_{4T}, \bar{U}_{5T}, \bar{U}_{6T}, \bar{\lambda}_s, \bar{\lambda}_{1s}$ and $\bar{\lambda}_{2s}$ under the equivalent martingale measure \bar{P} . But $\bar{R}_T, \bar{U}_{1T}, \bar{U}_{2T}, \bar{U}_{3T}, \bar{U}_{4T}, \bar{U}_{5T}, \bar{U}_{6T}, \bar{\lambda}_s, \bar{\lambda}_{1s}$ and $\bar{\lambda}_{2s}$ are regulated by the observable continuous-time finite state Markov chain, so the joint conditional distribution of $\bar{R}_T, \bar{U}_{1T}, \bar{U}_{2T}, \bar{U}_{3T}, \bar{U}_{4T}, \bar{U}_{5T}, \bar{U}_{6T}, \bar{\lambda}_s, \bar{\lambda}_{1s}$ and $\bar{\lambda}_{2s}$ under the equivalent martingale measure \bar{P} is determined by the conditional joint distribution of the cumulative stay time $J = (J_1, J_2, \dots, J_N)$ under conditions $S_0 = S, D_0 = D, V_0 = V$ and $X_0 = X$ under the equivalent martingale measure \bar{P} . Thus, we have

$$C(S, V, D, X) = \int_{[0, T]^N} C^* \psi(J_1, J_2, \dots, J_N) dJ_1 dJ_2 \dots dJ_N \tag{36}$$

where $\psi(J_1, J_2, \dots, J_N)$ is the joint distribution of cumulative stay time $J = (J_1, J_2, \dots, J_N)$ under the equivalent martingale measure \bar{P} and $X_0 = X$. Under the equivalent martingale measure \bar{P} , the characteristic function of $J = (J_1, J_2, \dots, J_N)$ satisfies the following formula:

$$E_{\bar{P}} \{ \exp[i \langle \xi, J \rangle] | X_0 = X \} = \langle X \exp[AT + iGT], I \rangle \tag{37}$$

where $i^2 = -1, \xi = (\xi_1, \xi_2, \dots, \xi_N)' \in R^N, I = (1, 1, \dots, 1)' \in R^N$ and G is an N order diagonal matrix generated by $\xi_1, \xi_2, \dots, \xi_N$. See Buffington [26] for proof of theoretical properties.

We next calculate C^* , given the condition $G_T^{(n, n_1, n_2)} = \{ \tilde{N}_s = n, \tilde{N}_{1s} = n_1, \tilde{N}_{2s} = n_2 \}$ and \mathcal{F}_T^X, S_T all jumps are $m_1 = n_1 + n$ and H_T all jumps are $m_2 = n_2 + n$, we have

$$\begin{aligned} C^* &= E_{\bar{P}} \left\{ e^{-\int_0^T r_s ds} (S_T - K)^+ \left\{ I_{\{H_T \geq 1\}} + (1 - \omega) H_T I_{\{H_T < 1\}} \right\} \middle| \mathcal{F}_T^X \right\} \\ &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} \mathbb{Q}(\tilde{N}_s = n, \tilde{N}_{1s} = n_1, \tilde{N}_{2s} = n_2) E_{\bar{P}} \left\{ e^{-\int_0^T r_s ds} (S_{T, m_1} - K)^+ \left\{ I_{\{H_{T, m_2} \geq 1\}} + (1 - \omega) H_{T, m_2} I_{\{H_{T, m_2} < 1\}} \right\} \right\} \end{aligned} \tag{38}$$

By calculation, we have

$$\begin{aligned} E_{\bar{P}} \left\{ e^{-\int_0^T r_s ds} (S_{T, m_1} - K)^+ \left\{ I_{\{H_{T, m_2} \geq 1\}} + (1 - \omega) H_{T, m_2} I_{\{H_{T, m_2} < 1\}} \right\} \right\} \\ = e^{-T \bar{R}_T} [A_1(m_1, m_2) + A_2(m_1, m_2) + A_3(m_1, m_2) + A_4(m_1, m_2)] \end{aligned} \tag{39}$$

By observing $A_1 - A_4$, we find that $\left(\log\left(\frac{S_{T,m_1}}{S_0}\right), \log\left(\frac{H_{T,m_2}}{H_0}\right)\right)$ is a bivariate normal distribution under the equivalent martingale measure \bar{P} . In order to calculate $A_1 - A_4$, we first obtain the properties of the bivariate normal distribution $\left(\log\left(\frac{S_{T,m_1}}{S_0}\right), \log\left(\frac{H_{T,m_2}}{H_0}\right)\right)$ as follows:

$$M_1(m_1) = E_{\bar{P}}\left[\log\left(\frac{S_{T,m_1}}{S_0}\right)\right] = \int_0^T \left[r_s - \frac{1}{2}\sigma_{1s}^2\right] ds - \frac{1}{2}m_1\delta_1^2 = T\left(\bar{R}_T - \frac{1}{2}\bar{U}_{1T}\right) - \frac{1}{2}m_1\delta_1^2$$

$$M_2(m_2) = E_{\bar{P}}\left[\log\left(\frac{H_{T,m_2}}{H_0}\right)\right] = \int_0^T \frac{1}{2}(\sigma_{3s}^2 - \sigma_{2s}^2) ds - \frac{1}{2}m_2\delta_2^2 = \frac{T}{2}(\bar{U}_{3T} - \bar{U}_{2T}) - \frac{1}{2}m_2\delta_2^2$$

$$H^2(m_1) = Var\left[\log\left(\frac{S_{T,m_1}}{S_0}\right)\right] = \int_0^T \sigma_{1s}^2 ds + m_1\delta_1^2 = T\bar{U}_{1T} + m_1\delta_1^2$$

$$H^2(m_2) = Var\left[\log\left(\frac{H_{T,m_2}}{H_0}\right)\right] = T(\bar{U}_{2T} + \bar{U}_{3T} - 2\bar{U}_{6T}\rho_{23}) + m_2\delta_2^2$$

$$Cov\left(\log\left(\frac{S_{T,m_1}}{S_0}\right), \log\left(\frac{H_{T,m_2}}{H_0}\right)\right) = \int_0^T (\sigma_{1s}\sigma_{2s}\rho_{12} - \sigma_{1s}\sigma_{3s}\rho_{13}) ds = T(\bar{U}_{4T}\rho_{12} + \bar{U}_{5T}\rho_{13})$$

$$\bar{\rho}_1(m_1, m_2) = corr\left(\log\left(\frac{S_{T,m_1}}{S_0}\right), \log\left(\frac{H_{T,m_2}}{H_0}\right)\right) = \frac{T(\bar{U}_{4T}\rho_{12} + \bar{U}_{5T}\rho_{13})}{\sqrt{H^2(m_1)}\sqrt{H^2(m_2)}}$$

In order to simplify the calculation, we can rewrite $\left(\log\left(\frac{S_{T,m_1}}{S_0}\right), \log\left(\frac{H_{T,m_2}}{H_0}\right)\right)$ as

$$\log\left(\frac{S_{T,m_1}}{S_0}\right) = M_1(m_1) + \sqrt{H^2(m_1)}\varepsilon_1 \tag{40}$$

$$\log\left(\frac{H_{T,m_2}}{H_0}\right) = M_2(m_2) + \sqrt{H^2(m_2)}\varepsilon_2 \tag{41}$$

where $(\varepsilon_1, \varepsilon_2)$ is a two-dimensional standard normal distribution, and the correlation coefficient is $\bar{\rho}_1(m_1, m_2)$. Next, we have

$$A_1(m_1, m_2) = S_0 \exp\left\{M_1(m_1) + \frac{1}{2}H^2(m_1)\right\} N_2(a_1(m_1), a_2(m_1, m_2), \bar{\rho}_1(m_1, m_2)) \tag{42}$$

$$A_2(m_1, m_2) = -KN_2(b_1(m_1), b_2(m_2), \bar{\rho}_1(m_1, m_2)) \tag{43}$$

$$A_3(m_1, m_2) = S_0(1 - \omega)H_0 \exp\left\{M_1(m_1) + M_2(m_2) + \frac{1}{2}(H^2(m_1) + H^2(m_2)) + \bar{\rho}_1(m_1, m_2)H(m_1)H(m_2)\right\} N_2(c_1(m_1, m_2), c_2(m_1, m_2), \bar{\rho}_1(m_1, m_2)) \tag{44}$$

$$A_4(m_1, m_2) = -K(1 - \omega)H_0 e^{\{M_2(m_2) + \frac{1}{2}H^2(m_2)\}} N_2(d_1(m_1, m_2), d_2(m_1, m_2), \bar{\rho}_1(m_1, m_2)) \tag{45}$$

Therefore, given the condition \mathcal{F}_T^X , we can obtain:

$$\begin{aligned}
 C^* &= E_{\mathbb{P}}\left\{e^{-\int_0^T r_s ds} (S_T - K)^+ \left\{ I_{\{H_T \geq 1\}} + (1 - \omega)H_T I_{\{H_T < 1\}} \right\} \middle| \mathcal{F}_T^X \right\} \\
 &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} \mathbb{Q}(\tilde{N}_s = n, \tilde{N}_{1s} = n_1, \tilde{N}_s = n_2) e^{-T \bar{R}_T} [A_1(m_1, m_2) + A_2(m_1, m_2) + A_3(m_1, m_2) + A_4(m_1, m_2)] \\
 &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} \frac{(\bar{\lambda}_s T)^n}{n!} \frac{(\bar{\lambda}_{1s} T)^{m_1-n}}{(m_1-n)!} \frac{(\bar{\lambda}_{2s} T)^{m_2-n}}{(m_2-n)!} e^{-T(\bar{\lambda}_s + \bar{\lambda}_{1s} + \bar{\lambda}_{2s} + \bar{R}_T)} \\
 &\quad \times [A_1(m_1, m_2) + A_2(m_1, m_2) + A_3(m_1, m_2) + A_4(m_1, m_2)]
 \end{aligned} \tag{46}$$

The proof of Theorem 3 is complete. \square

5. Numerical Simulation

In this section, we present the numerical results of European vulnerable call options prices given by Equations (1)–(3). To facilitate our analysis, we use the set of baseline parameters given in Tables 1 and 2, and we assume that there are 252 trading days in a year. To facilitate our simulation, we assume that for X_t there are only two regimes, i.e., $N = 2$. The first regime $e_1 = (1, 0)'$ can be interpreted as a good economic state. The second plan $e_2 = (0, 1)'$ can be interpreted as a bad economic state. We refer to relevant literature (such as Han [24]) to choose the initial state of the economy $X = X_0 = (1, 0)$, and the generation matrix of the Markov chain process X_t

$$A = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix} \tag{47}$$

Table 1. Parameter values of vulnerable European call options in the base case.

Parameter	Value	Parameter	Value
Strike price	$K = 10$	Mean jump size of V	$\mu_1 = 0$
Time to maturity	$T = 1$	Mean jump size of S	$\mu_2 = 0$
Initial price of S	$S_0 = 10$	correlation coefficient	$\rho_{12} = 0.7$
Initial price of D	$D_0 = 10$	correlation coefficient	$\rho_{23} = 0.6$
Initial price of V	$V_0 = 10$	correlation coefficient	$\rho_{13} = 0.5$
Volatility of jump size of V	$\delta_2 = 0.1$	Volatility of jump size of S	$\delta_1 = 0.1$

Table 2. Parameter values of vulnerable European call options in the base case.

Parameter	Value in State e_1	Value in State e_2
r	$r_1 = 0.02$	$r_1 = 0.02$
S's Volatility	$\sigma_{S1} = 0.3$	$\sigma_{S2} = 0.4$
V's Volatility	$\sigma_{V1} = 0.3$	$\sigma_{V2} = 0.4$
D's Volatility	$\sigma_{D1} = 0.3$	$\sigma_{D2} = 0.4$
common jump intensity	$\lambda_1 = 1$	$\lambda_2 = 2$
S jump intensity	$\lambda_{S1} = 1$	$\lambda_{S2} = 2$
V jump intensity	$\lambda_{V1} = 1$	$\lambda_{V2} = 2$

From Figure 1, it can be observed that as the initial price of the underlying asset, S_0 , increases, the price of the European fragile option also increases. This is because as the price of the underlying asset rises, the potential profit for the option holder increases, thereby increasing the value of the option.

Additionally, we can observe that under State 1 (good economy), the price of the call option is relatively low, whereas under State 2 (bad economy), the price of the call option is relatively high. This is mainly because in a bad economy, the volatility of the risky asset increases, leading to an increase in the risk value of the option. Therefore, investors need to pay a higher price to purchase options under adverse economic conditions.

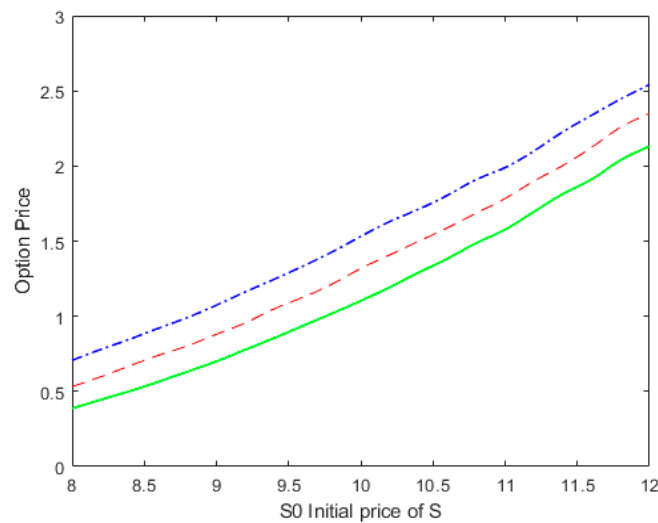


Figure 1. Vulnerable European call option price against the initial price of the underlying assets S. The solid, dashed, and dot-dashed lines correspond to State 1, the proposed model, and State 2, respectively.

Furthermore, when considering a mixed state of the economy, i.e., during a transitional period, the value of the option lies between State 1 and State 2. This is because in a mixed state, the economic conditions are uncertain, and the value of the option is influenced by different economic states. As a result, the price of the option falls between the prices observed in a good economy and a bad economy.

In light of these observations, we propose the switching model to better explain the impact of economic conditions on option valuation. In the switching model, the parameters of the risky asset value model evolve under two different economic states, allowing for a more accurate reflection of the changing economic conditions on option values. By introducing the switching model, our model can more flexibly capture changes in economic conditions, thereby better reflecting the pricing of fragile options in practice.

Now, let us turn to Figure 2. Figure 2 shows that as the initial value of the default barrier (counterparty’s debt), D_0 , increases, the price of the European fragile option exhibits a decreasing trend. This is because a larger D_0 increases the risk of credit default, reducing the demand for options from investors and leading to a decline in option prices.

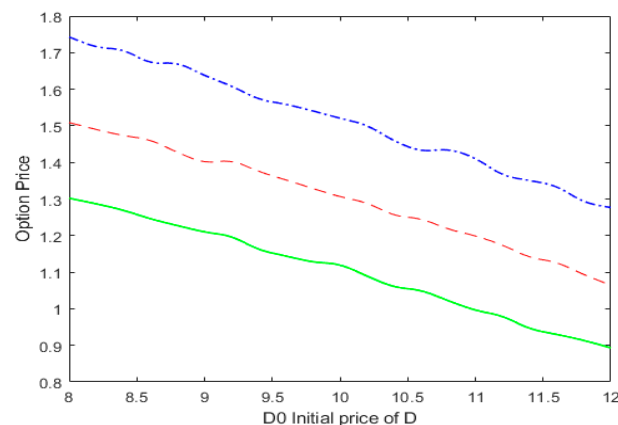


Figure 2. Vulnerable European call option price against the initial price of the default barrier D. The solid, dashed, and dot-dashed lines correspond to State 1, the proposed model, and State 2, respectively.

Furthermore, by examining Figure 3, we compare the prices of the European fragile option under four different models with the initial price of the underlying asset, S_0 . Compared to the Han [24] model, our proposed model has lower prices. This indicates that our model, which considers the dynamic changes in the default barrier, more accurately

reflects the increase in default risk, resulting in lower option values. Compared to the Hui [4] model, our model has higher option prices. This is because our model incorporates a jump process of the risky asset, increasing the prices of options and better capturing the impact of economic conditions on options. Compared to the Wang [22] model, when S_0 is between 8 and 9, our model has similar prices to the Wang [22] model. However, when S_0 is between 9 and 12, our model has higher prices.

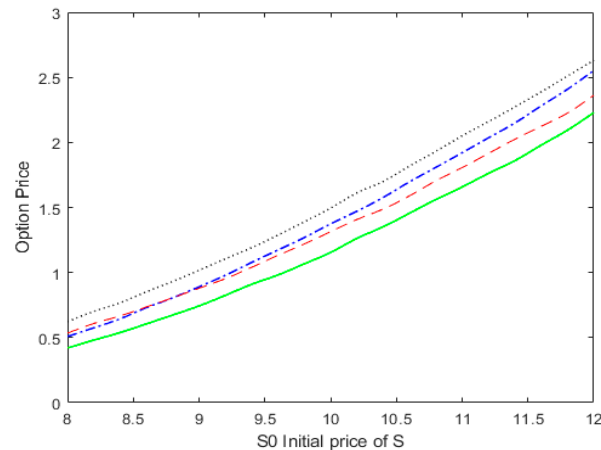


Figure 3. Vulnerable European call option price against the initial price of the underlying assets S . The solid, dashed, dotted, and dot-dashed lines correspond to Hui's [4] model, the proposed model, Han's [22] model, and Wang's [20] model, respectively.

Figure 4 shows the price of European Vulnerable options plotted against the initial value of the default barrier (D_0) for four different models. As D_0 increases, the price of European Vulnerable options decreases. Notably, our proposed model exhibits lower prices compared to the Han [24] model, indicating that the uncertainty of the default barrier increases over time, resulting in higher default risk and lower option prices. To provide a more comprehensive explanation of these findings, let us delve into the observed models, starting with the Han [24] and Wang [22] models. In these models, as the fixed default barrier (representing the initial value of the counterparty's liabilities) increases, the option holder assumes higher credit risk. This suggests that if the default barrier is significantly smaller than the counterparty's assets, the likelihood of default diminishes, leading to higher option prices. Next, let us consider our proposed model and the Hui [4] model. Both models exhibit a similar rate of decline in option prices, indicating that they share a comparable sensitivity to changes in the default barrier. As D_0 increases, the prices of options in both models decline proportionally. Lastly, we compare our proposed model with the Wang [22] model. When D_0 is small, we find that the Wang [22] model yields lower prices compared to our proposed model. This discrepancy arises because, in such instances, the impact of the jump process on credit risk outweighs the effect of the default barrier. These observations provide a detailed explanation of the relationship between the price of European Vulnerable options and the initial value of the default barrier across different models. As D_0 increases, default risk amplifies, leading to a decrease in option prices. Moreover, the influence of the default barrier and the jump process on option prices and credit risk varies among the different models. These findings contribute to a better understanding and assessment of pricing and risk management for vulnerable options.

Figures 5–8 show the impact of basic parameters on vulnerable option prices, including time to maturity, recovery rate, the correlation coefficient between the underlying asset and the counterparty's assets, and common jump intensity. These figures display the option prices calculated using our proposed model plotted against the initial price of the underlying asset S_0 from 8 to 12 with alternative basic parameters.

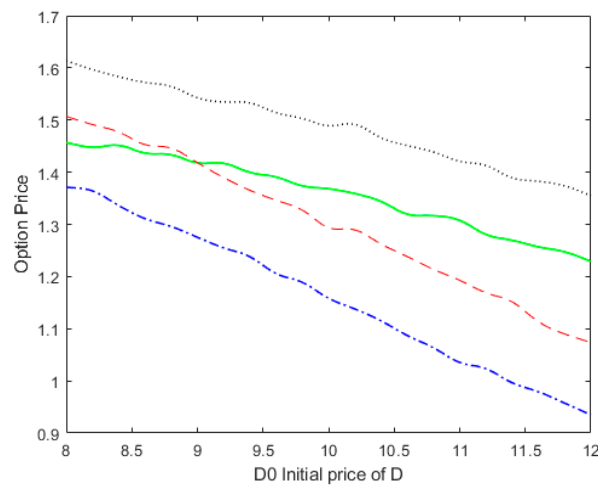


Figure 4. Vulnerable European call option price against the initial price of the default barrier D. The solid, dashed, dotted, and dot-dashed lines correspond to Wang’s [22] model, the proposed model, Han’s [24] model, and Hui’s [4] model, respectively.

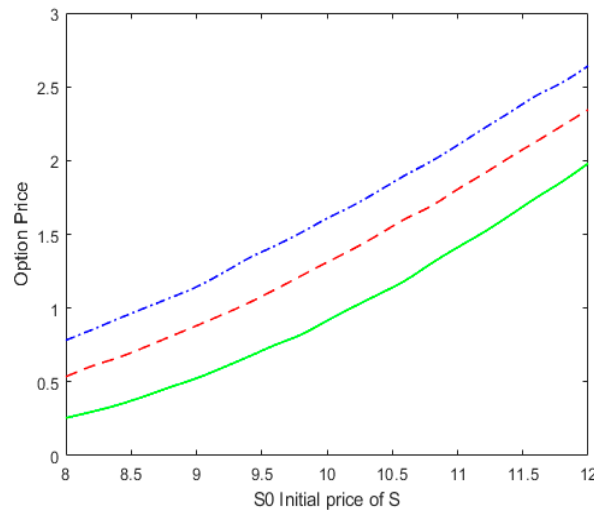


Figure 5. The option price with different maturity T for $\omega = 0.5$, $\lambda_1 = 1$, $\lambda_2 = 2$ and $\rho_{13} = 0.5$. The solid, dashed, and dot-dashed lines correspond to T = 0.5, T = 1, and T = 1.5, respectively.

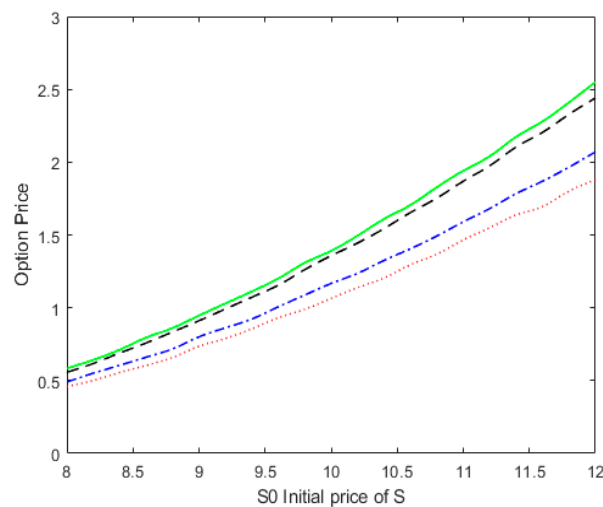


Figure 6. The option price with different recovery rate ω for T = 1, $\lambda_1 = 1$, $\lambda_2 = 2$ and $\rho_{13} = 0.5$. The solid, dashed, dotted, and dot-dashed lines correspond to $\omega = 0.3$, $\omega = 0.5$, $\omega = 1$, and $\omega = 0.8$, respectively.

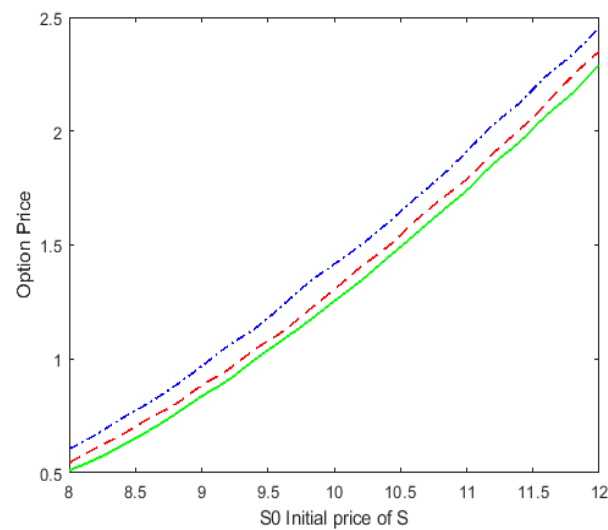


Figure 7. The option price with a correlation coefficient between the underlying asset and the assets of the counterparty for $\omega = 0.5$, $\lambda_1 = 1$, $\lambda_2 = 2$ and $T = 1$. The solid, dashed, and dot-dashed lines correspond to $\rho_{13} = 0.9$, $\rho_{13} = 0.5$, and $\rho_{13} = 1$, respectively.

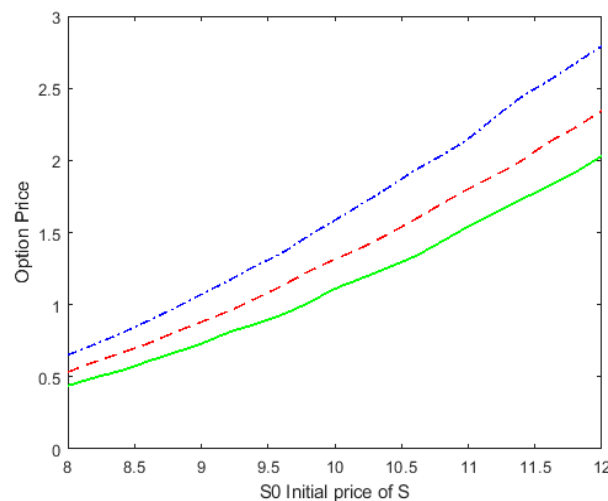


Figure 8. The option price with different common jump intensity λ for $T = 1$, $\omega = 0.5$, and $\rho_{13} = 0.5$. The solid, dashed, and dot-dashed lines correspond to $(\lambda_1, \lambda_2) = (0.5, 1)$, $(\lambda_1, \lambda_2) = (1, 2)$, and $(\lambda_1, \lambda_2) = (2, 4)$, respectively.

As shown in Figure 5, as the option’s time to maturity increases, the option price also increases. This indicates that a longer time to maturity increases the risk of counterparty default and, therefore, increases the value of the option. Figure 6 shows that the option price increases as the recovery rate ω increases. This is because a higher recovery rate results in a higher return at maturity. In Figure 7, we can see that the option price decreases as the correlation coefficient ρ_{13} increases. The correlation coefficient reflects the relationship between two assets that move together. This means that as the initial value of the underlying asset S_0 increases, the distance between the three lines also increases. A stronger correlation between the underlying assets and the default barrier means that default risk has a greater impact on vulnerable option prices and results in lower option prices. Figure 8 shows that the option price increases as common jump intensity increases. This result is consistent with our intuitive understanding.

6. Conclusions

This paper discusses the pricing of vulnerable European options using a regime-switching jump-diffusion model. In the structured model, it is assumed that the counterparty's asset-liability structure (or default barrier) is driven by a geometric Brownian motion under a regime-switching model, and there exists a correlation among all risk assets. By applying the Esscher transform to select an equivalent martingale measure, a pricing formula for vulnerable options is derived. The results of numerical simulations show that the European vulnerable option prices in the proposed model are lower than those in the model assuming a fixed counterparty's asset-liability structure. This result is intuitive because the counterparty's liability level is stochastic, increasing the risk of option default and thus lowering the option prices.

This paper mainly investigates the pricing problem of options with random default barriers and credit risk under a regime-switching model. Several research findings are obtained. However, there are still some gaps between the proposed model and the real financial market. Further research can be conducted in the following aspects:

1. More data on OTC option trading can be collected to discuss the parameter estimation issues of the relevant pricing models for vulnerable options, referring to existing literature.
2. As both models in this paper assume fixed volatility of risk assets and risk-free interest rates, further research directions can consider introducing stochastic volatility and risk-free interest rates into the proposed models to make them adhere to more general models. Additionally, with the advancement of computer performance, alternative methods can be considered to numerically solve the option pricing formulas.

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