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Feedback Control Techniques for a Discrete Dynamic Macroeconomic Model with Extra Taxation: An Algebraic Algorithmic Approach

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Abstract: In this paper, a model matching feedback law design technique is applied to a macroeconomical model. We calculate, using computational algebra methodology, which paths of government expenditure and extra taxation will lead the system to a desired dynamic behavior. The solution is based on algebraic methods and the development, in computer algebra software, of appropriate symbolic algorithms that produce a class of feedback laws as solutions. A method for solving a linear algebraic system of polynomials equations is provided, as well as its application to the feedback law design.

Keywords: computational algebra; linear discrete dynamical systems; linear feedback; model matching; macroeconomical dynamics

MSC: 93B51; 93C95

1. Introduction

The most important objective of the mathematical system theory is to ensure that the policymaker is able to modify the time path of a system in such a way that certain targets are achieved. That is, a certain reference sequence is set and the inputs of the system are modified in an appropriate way, by means of a feedback law design, so that these reference values are matched by the outputs of the closed-loop system. The whole theme has been studied exhaustively, for both linear and nonlinear systems, too [1,2].

The importance of applying this objective to dynamical economic systems became apparent in the aftermath of the global financial crisis of 2008 and the debt crisis, wherein most of the policy programs proposed for overcoming the crisis were designed on the basis of feedback analysis [3].

The aim of this paper is to provide a framework for the design of fiscal policy that is based on the algorithmic linear feedback control methodology and the model-matching technique. It is an extension of the results presented in [4]. There, the existence of extra taxation devoted to wealth is studied, while here an extra general taxation is considered. This is in line with the concept of applying feedback control techniques to economics. These include, among others, the controllability of economic systems, optimal control and feedback design [5] and nonlinear control applications [6,7]. In particular, we use a variant of one of the most heavily used models in economics, the multiplier–accelerator model established by Samuelson in 1939 [8]. This model in an attempt to provide an explanation for the business cycle, i.e., the fluctuations that national income exhibits over time.

In this context, feedback laws (policy rules) are calculated for the policy instruments that will allow the policymaker to shape the future behavior of both national income and public debt, so that they simultaneously follow a predetermined trajectory, by using the tools of government expenditure and extra taxation. That is, we want to solve an exact tracking problem.

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This approach is entirely different from other methods used in macroeconomics, such as the optimal control method, wherein the usual objective is to maximize some function over time, dynamic programming methods, game theory, etc. [9]. Its advantages are as follows:

1. It does not rely on utility functions.
2. It can describe the concept of positive intervention in the economy.
3. It provides solutions into a short-term frame.
4. It is more practical for econometric applications.
5. It can be implemented computationally using suitable algorithms.

From the mathematical point of view, a linear deterministic discrete multiple input multiple output (MIMO) system is used, which describes the dynamic behavior of the aggregate economy and is asked to shape its future behavior so that it tracks a desired path for each of the outputs. In order to solve this tracking problem, the model matching approach is followed, which amounts to calculating feedback laws that produce a closed-loop system that is dynamically identical to a given desired one. In other words, their outputs are equal.

This desired system has the property that its output is identical to the predetermined time path of the target variables.

The whole methodology is based on the solution of an algebraic system of equations, involving polynomial matrices. This system has an infinite set of rational solutions. In order to obtain polynomial solutions, additional conditions must be satisfied. These conditions ensure that all the rational solutions can be simplified. This is accomplished by solving a set of Diophantine polynomial equations and by executing certain polynomial divisions. Finally, a symbolic algorithm was constructed, systematizing the whole procedure. A symbolic code for this algorithm was written by means of a proper symbolic software.

One of the advantages of our approach is that it is completely parameterized, thus allowing for proper computational algorithms to be developed. Most importantly, based on these algorithms, a whole class of feedback laws appropriate for solving the problem at hand can be calculated.

From this class of solutions, we can select some in order for additional conditions to be satisfied, such as, for example, maximizing tax revenues. The discovery of these specific solutions is done on a case-by-case basis and through specialized techniques.

In conclusion, summarizing, the innovations of this work are as follows:

1. It employs feedback control techniques in economics to quantitatively describe the concept of intervention.
2. The entire method can be implemented through suitable software.
3. It outlines conditions for solving specific polynomial algebraic equations that arise from the particular problem.

This paper is organized as follows: In Section 2, the necessary algebraic preliminaries are presented; in Section 3, the model is described and in Section 4 the problem is described; we can find theorems devoted to the solutions and the said algorithm in Sections 5 and 6; and finally in Section 7 an artificial example for the clarification of the approach is presented.

2. Preliminaries

Before we enter to the main notions of this paper, some algebraic preliminaries are presented. They are going to be used throughout our calculations. We present them here shortly for the sake of the completeness of our work. Extensive relative results are common in the literature [10,11].

- The lag operator q is defined as follows: $q^m f(t) = f(t-m)$, for any sequence $f(t)$, $t = 0, 1, 2, \ldots$ and any $m \in \mathbb{N}$.
- An expression of the form $A(q) = \sum_{i=0}^{k} a_i q^i$, $a_i \in \mathbb{R}$ is called a real q-polynomial of k degree with real coefficients.
- The set of all real q-polynomials is denoted by $\mathbb{R}[q]$.

- An expression of the form $A(q)/B(q)$, with $A(q), B(q) \in \mathbb{R}[q]$ is called a real q-fraction.
- The set of all real fractions is denoted by $\mathbb{R}[[q]]$.
- By $\mathbb{R}[q]^n$ we denote the set of all n-tuples with real q-polynomials as elements.
- The set of all $n \times m$ matrices, with real q-polynomials as entries, is denoted by $M_{n\times m}(\mathbb{R}[q])$.
- The set of all $n \times m$ matrices, with real q-fractions as entries, is denoted by $M_{n\times m}(\mathbb{R}[[q]])$.

If $A(q), B(q), C(q), D(q)$ are known real q-polynomials, then the expression

$$A(q)X(q) + B(q)Y(q) + C(q)Z(q) = D(q),$$

where the unknown q-polynomials $X(q), Y(q), Z(q)$ has to be determined, is called a polynomial Diophantine equation of third order. It is solvable, if and only if

$$\gcd(A(q), B(q), C(q)) | D(q)$$

(Details can be found in [11].)

3. The Model

The model we have chosen to work with has been extensively used in the economics literature [8] and many variations have appeared [6,7,12,13], as well as stochastic versions [14]. It consists of a set of linear discrete recurrent relations describing the dynamic evolution of certain economic quantities. Specifically,

$$Y(t) = C(t) + I(t) + \lambda_0 G(t) + \lambda_1 G(t-1) + \lambda_2 G(t-2) \tag{1}$$

$$C(t) = (1-s)Y^d(t-1) + sY^d(t-2) \tag{2}$$

$$Y^d(t) = Y(t) - T(t) \tag{3}$$

$$T(t) = \tau Y(t-1) + E(t) \tag{4}$$

$$I(t) = \nu(Y(t-1) - Y(t-2)) \tag{5}$$

$$B(t) = (1+r)B(t-1) + G(t) - T(t) \tag{6}$$

with

$$t \in \mathbb{N} \quad \text{and} \quad \lambda_0, \lambda_1, \lambda_2, \tau, \nu, s, r \in (0,1)$$

where the involved sequences have the following meanings:

- $Y(t) : \mathbb{N} \to \mathbb{R}$ denotes the national nominal income path or flow (GDP).
- $C(t) : \mathbb{N} \to \mathbb{R}^+$ denotes the consumption path
- $I(t) : \mathbb{N} \to \mathbb{R}^+$ denotes the investment path.
- $G(t) : \mathbb{N} \to \mathbb{R}^+$ stands for government expenditures.
- $Y^d(t) : \mathbb{N} \to \mathbb{R}$ is the disposal income.
- $T(t) : \mathbb{N} \to \mathbb{R}^+$ denotes the tax receipts.
- $E(t) : \mathbb{N} \to \mathbb{R}^+$ is a form of extra taxation.
- $B(t) : \mathbb{N} \to \mathbb{R}$ denotes the debt outstanding.

The parameters have the following meaning:

- The λ_i, $i = 0, 1, 2$, indicate the percentage of government's decisions to spend in the current and in the past two periods.
- s is the propensity to save.

- τ is the constant tax rate.
- ν denotes the standard investment accelerator.
- r is the constant interest on public debt.

$E(t)$ is a distinct form of taxation, assumed to represent extra tax, for instance on wealth or education. The λ_i parameters indicate that the decision regarding government expenditures in period t affects the level of GDP in period $t+i$; that is, there is a delay in the realization of the effects of changes in government expenditures on GDP levels. Equations (1)–(5) constitute a variant of the standard multiplier–accelerator model introduced in [8], whereas Equation (6) is the government budget constraint. Regarding the parameters of the system, $s \in (0,1)$ is the marginal propensity to save, $\tau \in (0,1)$ is the (constant) tax rate, $\nu > 0$ is the accelerator and $r \in (0,1)$ is the constant interest on previously issued public debt. Apart from the λ_i parameters, delays are also introduced into the model through Equations (2) and (5), since it is assumed that the current levels of $C(t), I(t)$, depend upon changes in lagged values of $Y(t)$.

Now, we are going to present another expression of the initial model, involving polynomial matrices.

Proposition 1. *The original model can be written in the algebraic form:*

$$\mathcal{D}(q)\vec{z}(t) = \mathcal{K}(q)\vec{u}(t) \tag{7}$$

where $\vec{z}(t) = (Y(t), B(t))^T$, $\vec{u}(t) = (G(t), E(t))^T$ *and* $\mathcal{D}, \mathcal{K}$ *are the q-polynomial matrices:*

$$\mathcal{D}(q) = \begin{bmatrix} D_{11}(q) & D_{12}(q) \\ D_{21}(q) & D_{22}(q) \end{bmatrix} = \begin{bmatrix} 1 - a_1 q - a_2 q^2 - a_3 q^3 & 0 \\ -\tau q & 1-(1+r)q \end{bmatrix}$$

$$\mathcal{K}(q) = \begin{bmatrix} K_{11}(q) & K_{12}(q) \\ K_{21}(q) & K_{22}(q) \end{bmatrix} = \begin{bmatrix} \lambda_0 + \lambda_1 q + \lambda_2 q^2 & a_4 q + a_5 q^2 \\ 1 & -1 \end{bmatrix}$$

Proof. By executing all the substitutions among the above Equations (1)–(6) and after the necessary algebra, we will end up with the following pair of equations:

$$\begin{aligned} &Y(t) - a_1 Y(t-1) - a_2 Y(t-2) + a_3 Y(t-3) = \lambda_0 G(t) + \lambda_1 G(t-1) + \\ &\quad + \lambda_2 G(t-2) - a_4 E(t-1) - a_5 E(t-2) \\ &B(t) - (1+r)B(t-1) - \tau Y(t-1) = G(t) - E(t) \end{aligned} \tag{8}$$

where

$$a_1 = 1 + \nu - s, \quad a_2 = s - \nu - \tau(1-s), \quad a_3 = s\tau, \quad a_4 = 1 - s, \quad a_5 = s$$

Then, the matrix expression comes in a straightforward manner after using the definition of the q-operator. □

This is the main form of the model we are going to work with throughout this paper. It follows the dynamic systems language. It is a discrete input–output control system, with $G(t)$ and $E(t)$ being the inputs, and $Y(t)$ and $B(t)$ the outputs. It consists of two recurrent relations. Certain initial conditions are assigned to the system, thus permitting the calculation of all future values of the outputs for any given input sequences.

4. The Formulation of the Problem

The problem under consideration is to calculate feedback laws for the inputs which, whenever they are fed back to the open-loop system (8), will modify its behavior so that predetermined fixed targets for the future values of income and public debt, say $Y^*(t)$ and $B^*(t)$, $t = 0, 1, 2, \ldots$, will be matched.

Let us consider a given linear system

$$\mathcal{D}^d(q)\vec{z}^*(t) = \mathcal{K}^d(q)\vec{u}_c(t) \quad , \quad \mathcal{D}^d(q), \mathcal{K}^d(q) \in M_{2\times 2}(\mathbb{R}[q]) \tag{9}$$

with

$$\mathcal{D}^d = \begin{pmatrix} D_{11}^d & D_{12}^d \\ D_{21}^d & D_{22}^d \end{pmatrix}, \quad \mathcal{K}^d = \begin{pmatrix} K_{11}^d & K_{12}^d \\ K_{21}^d & K_{22}^d \end{pmatrix}$$

The output vector contains the reference sequences, that is, $\vec{z}^*(t) = (Y(t)^*, B(t)^*)^T$. The input vector $\vec{u}_c(t)$ contains proper arbitrary values so that the desired output values are achieved. This is the "desired" system. It is an artificial system which represents an "ideal" economy whose dynamic behavior we want to match.

The problem upon discussion is that of determining polynomial matrices $\mathcal{R}(q), \mathcal{T}(q), \mathcal{S}(q)$, so that the feedback law is as follows:

$$\mathcal{R}(q)\vec{u}(t) = \mathcal{T}(q)\vec{u}_c(t) - \mathcal{S}(q)\vec{z}(t) \tag{10}$$

$$\mathcal{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \quad , \quad \mathcal{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad , \quad \mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{R}[q])$$

with $\vec{u}(t)$, $\vec{z}(t)$ as in Proposition 1 and $\vec{u}_c(t)$ as before, gives an input vector $\vec{u}(t)$, which, whenever it is fed back to the nominal plant, produces outputs identical equal to the desired one. That is, it produces income and debt paths, $Y(t)$ and $B(t)$, so that $Y(t) = Y^*(t)$ and $B(t) = B^*(t)$, for $t = 0, 1, 2, \ldots$. We have to solve the problem theoretically, by finding conditions for the solution, as well as algorithmically, by developing an algorithm for the calculation of the requested quantities.

5. Theoretical Results

In this section, we present the main theorems, describing the solutions of the above problem.

Theorem 1. *Let us suppose that a feedback law of the form (10) is connected to the nominal plant (7). The obtained closed-loop system will give the same output than that of the desired system (9) under the same initial conditions and the same input, if and only if the following relations hold:*

$$\mathcal{R}(q)\mathcal{D}(q) + \mathcal{K}(q)\mathcal{S}(q) = \mathcal{D}^d(q) \tag{11}$$

$$\mathcal{K}(q)\mathcal{T}(q) = \mathcal{K}^d(q) \tag{12}$$

$$\mathcal{R}(q)\mathcal{K}(q) = \mathcal{K}(q)\mathcal{R}(q) \tag{13}$$

$$\mathcal{R} = I + \overline{\mathcal{R}} \quad , \quad \mathcal{D} = I + \overline{\mathcal{D}} \tag{14}$$

(I the identity matrix).

Proof. For the sake of presentation, we drop the q symbol.

Let us firstly suppose that the output of the closed-loop system is uniquely defined and equal to the output of the desired system. The output of the closed-loop system is produced by feeding the system (7) with the input $\vec{u}(t)$, given by the feedback law (10). Both relations (7) and (10) are able to give unique quantities $\vec{z}(t)$ and $\vec{u}(t)$, at any time instant, if the following conditions hold:

$$\mathcal{R} = I + \overline{\mathcal{R}} \quad , \quad \mathcal{D} = I + \overline{\mathcal{D}}$$

To obtain the closed-loop system, we post-multiply (7) by $\mathcal{R}$ and we obtain $\mathcal{R}\mathcal{D}\vec{z}(t) = \mathcal{R}\mathcal{K}\vec{u}(t)$; then, in order for the feedback law to be involved, the matrices $\mathcal{R}$, $\mathcal{K}$ must be permuted, that is, $\mathcal{RK} = \mathcal{KR}$. That is the first assumption we must adopt. If this is valid, then successively we obtain

$$\mathcal{RD}\vec{z}(t) = \mathcal{RK}\vec{u}(t) = \mathcal{KR}\vec{u}(t) = \mathcal{K}(\mathcal{T}\vec{u}_c(t) - \mathcal{S}\vec{z}(t)) = \mathcal{KT}\vec{u}_c(t) - \mathcal{KS}\vec{z}(t) \Rightarrow$$

$$\Rightarrow \mathcal{RD}\vec{z}(t) + \mathcal{KS}\vec{z}(t) = \mathcal{KT}\vec{u}_c(t) \Leftrightarrow (\mathcal{RD} + \mathcal{KS})\vec{z}(t) = \mathcal{KT}\vec{u}_c(t)$$

For this system, to have $\vec{z}(t) = \vec{z}^*(t)$, where $\mathcal{D}^d(q)\vec{z}^*(t) = \mathcal{K}^d(q)\vec{u}_c(t)$, when the input sequence and the initial conditions are identical, we must obtain

$$\mathcal{RD} + \mathcal{KS} = \mathcal{D}^d \quad , \quad \mathcal{KT} = \mathcal{K}^d$$

Summarizing, we see that all the said conditions of the theorem are fulfilled.

We address the inverse now. Let us suppose that the relations (11)–(14) hold and that $\vec{z}(t) = \vec{z}^*(t)$, for $t = 0, 1, ..., m-1$, where m is the largest delay, found in the equations of the system. In other words, we have identical initial conditions. We shall prove that $\vec{z}(m) = \vec{z}^*(m)$. For $t = m$, we have

$$\mathcal{D}\vec{z}(m) = \mathcal{K}\vec{u}(m) \Rightarrow (I + \overline{\mathcal{D}})\vec{z}(m) = \mathcal{K}\vec{u}(m) \Rightarrow \vec{z}(m) = \mathcal{K}\vec{u}(m) - \overline{\mathcal{D}}\vec{z}(m)$$

But $\mathcal{R}\vec{u}(t) = \mathcal{T}\vec{u}_c(t) - \mathcal{S}\vec{z}(t)$ and

$$\mathcal{R} = I + \overline{\mathcal{R}} \Rightarrow \vec{u}(t) = \mathcal{T}\vec{u}_c(t) - \mathcal{S}\vec{z}(t) - \overline{\mathcal{R}}\vec{u}(t)$$

By setting $t = m$ and substituting, we obtain

$$\vec{z}(m) = \mathcal{KT}\vec{u}_c(m) - \mathcal{KS}\vec{z}(m) - \mathcal{K}\overline{\mathcal{R}}\vec{u}(m) - \overline{\mathcal{D}}\vec{z}(m) \tag{15}$$

All the quantities at the right-hand side of (15) are delays of the sequences with values at time instants less than m; thus, we can substitute the sequence $\vec{z}(t)$ with $\vec{z}^*(t)$, since $\vec{z}(t) = \vec{z}^*(t)$, for $t = 0, 1, 2, ..., m-1$.

From Equations (13) and (14), we obtain

$$(I + \overline{\mathcal{R}})\mathcal{K} = \mathcal{K}(I + \overline{\mathcal{R}}) \quad \Rightarrow \quad \overline{\mathcal{R}}\mathcal{K} = \mathcal{K}\overline{\mathcal{R}}$$

From Equations (11) and (14), we obtain

$$(I + \overline{\mathcal{R}})(I + \overline{\mathcal{D}}) + \mathcal{KS} = \mathcal{D}^d \text{ or } \mathcal{KS} = \mathcal{D}^d - I - \overline{\mathcal{R}} - \overline{\mathcal{D}} - \overline{\mathcal{RD}}$$

Substituting all the above into (15), we have

$$\vec{z}(m) = \mathcal{KT}\vec{u}_c(m) - \mathcal{D}^d\vec{z}^*(m) + \vec{z}^*(m) + \overline{\mathcal{R}}\vec{z}^*(m) +$$

$$+ \overline{\mathcal{RD}}\vec{z}^*(m) - \mathcal{K}\overline{\mathcal{R}}\vec{u}(m) \tag{16}$$

Post-multiplying (7) by $\overline{\mathcal{R}}$, we obtain

$$\overline{\mathcal{R}}\mathcal{D}\vec{z}^*(m) = \overline{\mathcal{R}}\mathcal{K}\vec{u}(m) \Leftrightarrow \overline{\mathcal{R}}(I + \overline{\mathcal{D}})\vec{z}^*(m) = \overline{\mathcal{R}}\mathcal{K}\vec{u}(m) = \mathcal{K}\overline{\mathcal{R}}\vec{u}(m) \Leftrightarrow$$

$$\Leftrightarrow (\overline{\mathcal{R}} + \overline{\mathcal{RD}})\vec{z}^*(m) = \mathcal{K}\overline{\mathcal{R}}\vec{u}(m) \Leftrightarrow \overline{\mathcal{RD}}\vec{z}^*(m) = \mathcal{K}\overline{\mathcal{R}}\vec{u}(m) - \overline{\mathcal{R}}\vec{z}^*(m)$$

where we have used relations (13) and (14). Using also the facts that $\mathcal{KT} = \mathcal{K}^d$, $\mathcal{D}^d\vec{z}^*(m) = \mathcal{K}^d\vec{u}_c(m)$, and substituting, once more, all the above into (16), the latter becomes

$$\vec{z}(m) = \mathcal{K}^d\vec{u}_c(m) - \mathcal{K}^d\vec{u}_c(m) + \vec{z}^*(m) + \overline{\mathcal{R}}\vec{z}^*(m) +$$

$$+ \mathcal{K}\overline{\mathcal{R}}\vec{u}(m) - \overline{\mathcal{R}}\vec{z}^*(m) - \mathcal{K}\overline{\mathcal{R}}\vec{u}(m) = \vec{z}^*(m)$$

as requested. By induction, we obtain the final result $\vec{z}(t) = \vec{z}^*(t)$, for any $t = 0, 1, 2, ...$, and the theorem is proved. □

The next theorem gives an existing theorem for the solution of the system.

Theorem 2. *Let us consider the equations of Theorem 1. If, for the elements of the matrices $\mathcal{D}, \mathcal{K}$, $\mathcal{D}^d, \mathcal{K}^d$, the following conditions hold:*

$$\gcd\left(D_{2\nu}K_{11}K_{12} - D_{1\nu}K_{12}, D_{1\nu} + D_{2\nu}K_{12}, K_{11} + K_{12}\right)\Big|\left(D^d_{2\nu}K_{12} + D^d_{1\nu}\right)$$

$$\gcd\left(- D_{2\nu}K_{11}K_{11} - D_{2\nu}K_{12} - D_{2\nu}K_{11} + D_{1\nu}K_{11}, -D_{1\nu} + D_{2\nu}K_{11}, K_{11} + K_{12}\right)\Big|\left(D^d_{2\nu}K_{11} - D^d_{1\nu}\right)$$

for $\nu = 1, 2$ and

$$(K_{11} + K_{12})\Big|\left(K^d_{1\rho} + (-1)^{\nu}K_{1\nu}K^d_{2\rho}\right) \quad , \quad \rho = 1, 2, \quad \nu = 1, 2$$

then this system of equations accepts polynomial solutions, that is, it is solvable in the set $\mathbb{R}[q]$. Furthermore, the set of solutions is infinite.

Proof. After executing the operations in Equations (11)–(13) and equating the corresponding elements, we take:

$$\begin{aligned}
R_{11}D_{11} + R_{12}D_{21} + K_{11}S_{11} + K_{12}S_{21} &= D^d_{11} \\
R_{11}D_{12} + R_{12}D_{22} + K_{11}S_{12} + K_{12}S_{22} &= D^d_{12} \\
R_{21}D_{11} + R_{22}D_{21} + K_{21}S_{11} + K_{22}S_{21} &= D^d_{21} \\
R_{21}D_{12} + R_{22}D_{22} + K_{21}S_{12} + K_{22}S_{22} &= D^d_{22} \\
K_{11}T_{11} + K_{12}T_{21} &= K^d_{11} \\
K_{11}T_{12} + K_{12}T_{22} &= K^d_{12} \\
K_{21}T_{11} + K_{22}T_{21} &= K^d_{21} \\
K_{21}T_{12} + K_{22}T_{22} &= K^d_{22} \\
R_{11}K_{11} + R_{12}K_{21} - K_{11}R_{11} - K_{21}R_{12} &= 0 \\
R_{11}K_{12} + R_{12}K_{22} - K_{12}R_{11} - K_{22}R_{12} &= 0 \\
R_{21}K_{11} + R_{22}K_{21} - K_{11}R_{21} - K_{21}R_{22} &= 0 \\
R_{21}K_{12} + R_{22}K_{22} - K_{12}R_{21} - K_{22}R_{22} &= 0
\end{aligned}$$

This is a polynomial system with, as unknowns, the polynomials S_{ij}, T_{ij}, R_{ij}. Let us denote by Sol_R the solution set of the last four equations of the above system, by Sol_T the solution set of the next four equations, by Sol_S the solution set of the first four equations and by Sol the solution set of all equations. Evidently,

$$Sol = Sol_R \cap Sol_T \cap Sol_S$$

Taking into consideration the fact that $K_{21} = 1$, $K_{22} = -1$, the Sol_R becomes

$$Sol_R = \left\{ \begin{pmatrix} R_{11} \\ R_{12} \\ R_{21} \\ R_{22} \end{pmatrix} = \begin{pmatrix} R_{11} \\ R_{12} \\ \dfrac{R_{12}}{K_{12}} \\ -\dfrac{K_{11}R_{12}}{K_{12}} - \dfrac{R_{12}}{K_{12}} + R_{11} \end{pmatrix}, R_{11}, R_{12} \in \mathbb{R}[q], \right\} \subset \mathbb{R}[[q]]^4$$

The Sol_T becomes

$$Sol_T = \left\{ \begin{pmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{K_{12}\mathrm{K}_{21}^d + \mathrm{K}_{11}^d}{K_{11} + K_{12}} \\ \dfrac{\mathrm{K}_{11}^d - K_{11}\mathrm{K}_{21}^d}{K_{11} + K_{12}} \\ \dfrac{K_{12}\mathrm{K}_{22}^d + \mathrm{K}_{12}^d}{K_{11} + K_{12}} \\ \dfrac{\mathrm{K}_{12}^d - K_{11}\mathrm{K}_{22}^d}{K_{11} + K_{12}} \end{pmatrix} \right\} \subset \mathbb{R}[[q]]^4$$

The Sol_S becomes

$$Sol_S = \left\{ \begin{pmatrix} S_{11} \\ S_{12} \\ S_{21} \\ S_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{-D_{11}K_{12}R_{21} - D_{21}K_{12}R_{22} - D_{11}R_{11} - D_{21}R_{12} + \mathrm{D}_{21}^d K_{12} + \mathrm{D}_{11}^d}{K_{11} + K_{12}} \\ \dfrac{-D_{12}K_{12}R_{21} - D_{22}K_{12}R_{22} - D_{12}R_{11} - D_{22}R_{12} + \mathrm{D}_{22}^d K_{12} + \mathrm{D}_{12}^d}{K_{11} + K_{12}} \\ \dfrac{D_{11}K_{11}R_{21} + D_{21}K_{11}R_{22} - D_{11}R_{11} - D_{21}R_{12} - \mathrm{D}_{21}^d K_{11} + \mathrm{D}_{11}^d}{K_{11} + K_{12}} \\ \dfrac{D_{12}K_{11}R_{21} + D_{22}K_{11}R_{22} - D_{12}R_{11} - D_{22}R_{12} - \mathrm{D}_{22}^d K_{11} + \mathrm{D}_{12}^d}{K_{11} + K_{12}} \end{pmatrix} \right\} \subset \mathbb{R}[[q]]^4$$

Let us first work with the set Sol_R. This is an infinite set of rational solutions. In order to restrict them to polynomial ones, extra conditions must be met, which guarantee that all the fractions included into Sol_R will be simplified into polynomials. This is accomplished by choosing $R_{12} = M \cdot K_{12}$, $M \in \mathbb{R}[q]$, an arbitrary q-polynomial, and the solution set Sol_R becomes

$$Sol_R = \left\{ \begin{pmatrix} R_{11} \\ R_{12} \\ R_{21} \\ R_{22} \end{pmatrix} = \begin{pmatrix} R_{11} \\ M \cdot K_{12} \\ M \\ -K_{11}M - M + R_{11} \end{pmatrix}, R_{11}, M \in \mathbb{R}[q], \right\} \subset \mathbb{R}[q]^4$$

Substituting these values into Sol_S, as well as the value $D_{12} = 0$, we will take

$$Sol_S = \left\{ \begin{pmatrix} S_{11} \\ S_{12} \\ S_{21} \\ S_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{-D_{11}K_{12}M + D_{21}K_{11}K_{12}M - D_{21}K_{12}R_{11} - D_{11}R_{11} + \mathrm{D}_{21}^d K_{12} + \mathrm{D}_{11}^d}{K_{11} + K_{12}} \\ \dfrac{-D_{12}K_{12}M + D_{22}K_{11}K_{12}M - D_{22}K_{12}R_{11} - D_{12}R_{11} + \mathrm{D}_{22}^d K_{12} + \mathrm{D}_{12}^d}{K_{11} + K_{12}} \\ \dfrac{D_{21}K_{11}^2(-M) + D_{11}K_{11}M - D_{21}K_{11}M - D_{21}K_{12}M + D_{21}K_{11}R_{11} - D_{11}R_{11} - \mathrm{D}_{21}^d K_{11} + \mathrm{D}_{11}^d}{K_{11} + K_{12}} \\ \dfrac{D_{22}K_{11}^2(-M) + D_{12}K_{11}M - D_{22}K_{11}M - D_{22}K_{12}M + D_{22}K_{11}R_{11} - D_{12}R_{11} - \mathrm{D}_{22}^d K_{11} + \mathrm{D}_{12}^d}{K_{11} + K_{12}} \end{pmatrix} \right\} \subset \mathbb{R}[[q]]^4$$

By asking these fractions to be simplified into polynomialsm we must ensure that the numerators are multipliers of the denominators. The truth of this assumption is equivalent to the solution of the following Diophantine equations:

$$(D_{2\nu}K_{11}K_{12} - D_{1\nu}K_{12})M + (-D_{1\nu} - D_{2\nu}K_{12})R_{11} + P_{1\nu}(K_{11} + K_{12}) = -D_{2\nu}^{d}K_{12} - D_{1\nu}^{d}$$

$$(-D_{2\nu}K_{11}K_{11} - D_{2\nu}K_{12} - D_{2\nu}K_{11} + D_{1\nu}K_{11})M + (-D_{1\nu} + D_{2\nu}K_{11})R_{11} + P_{2\nu}(K_{11} + K_{12}) =$$
$$= D_{2\nu}^{d}K_{11} - D_{1\nu}^{d}$$

for $\nu = 1,2$. The assumptions of the theorem guarantee the solution of these equations. For the set Sol_T, the assumption that numerators must be a multiplier of denominators is also satisfied by the conditions of the theorem and it implies the fact that there are polynomials $F_{\nu\rho} \in \mathbb{R}[q]$ such that

$$K_{1\rho}^{d} + (-1)^{\nu}K_{1\nu}K_{2\rho}^{d} = F_{\nu\rho}(K_{11} + K_{12}) \quad , \quad \rho = 1,2, \quad \nu = 1,2$$

or equivalently

$$(K_{11} + K_{12}) \Big| \left(K_{1\rho}^{d} + (-1)^{\nu}K_{1\nu}K_{2\rho}^{d}\right) \quad , \quad \rho = 1,2, \quad \nu = 1,2$$

which is valid. Therefore, we have proved the necessary conditions for solving the system. Now, collecting all the above results, we see that

$$Sol = \{(R_{11}, R_{12}, R_{21}, R_{22}, S_{11}, S_{12}, S_{21}, S_{22}, T_{11}, T_{12}, T_{21}, T_{22}) =$$
$$= (R_{11}, M \cdot K_{12}, M, -K_{11}M - M + R_{11}, F_{11}, F_{12}, F_{21}, F_{22}, P_{11}, P_{12}, P_{21}, P_{22})\}$$

which is an infinite set, since $M, R_{11}, F_{ij}, P_{ij} \in \mathbb{R}[q]$ is arbitrary. The theorem is proved. □

The next last theorem is devoted to the set of solutions.

Theorem 3. *Let us suppose that the polynomial matrices $\mathcal{R}_0$, $\mathcal{S}_0$, $\mathcal{T}_0$, solve the system (11)–(13); then, the polynomial matrices*

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{Q}\mathcal{K} \quad , \quad \mathcal{S} = \mathcal{S}_0 - \mathcal{Q}\mathcal{D} \quad , \quad \mathcal{T} = \mathcal{T}_0$$

are solutions too, for any arbitrary polynomial matrix $\mathcal{Q} \in M_{2\times 2}(\mathbb{R}[q])$, such that $\mathcal{Q}\mathcal{K} = \mathcal{K}\mathcal{Q}$.

Proof. Substituting the above expressions of $\mathcal{R}, \mathcal{S}, \mathcal{T}$ into (11), we obtain

$$(\mathcal{R}_0 + \mathcal{Q}\mathcal{K})\mathcal{D} + \mathcal{K}(\mathcal{S}_0 - \mathcal{Q}\mathcal{D}) = \mathcal{R}_0\mathcal{D} + \mathcal{Q}\mathcal{K}\mathcal{D} + \mathcal{K}\mathcal{S}_0 - \mathcal{K}\mathcal{Q}\mathcal{D}$$

Using the facts that $\mathcal{R}_0$ and $\mathcal{S}_0$ are solutions of the system (11) and that $\mathcal{Q}\mathcal{K} = \mathcal{K}\mathcal{Q}$, we conclude that the above quantity is equal to $\mathcal{D}^d$.

Furthermore, we have

$$(\mathcal{R}_0 + \mathcal{Q}\mathcal{K})\mathcal{K} = \mathcal{K}(\mathcal{R}_0 + \mathcal{Q}\mathcal{K})$$
$$\Leftrightarrow \mathcal{R}_0\mathcal{K} + \mathcal{Q}\mathcal{K}\mathcal{K} = \mathcal{K}\mathcal{R}_0 + \mathcal{K}\mathcal{Q}\mathcal{K}$$

Using again the facts that $\mathcal{Q}\mathcal{K} = \mathcal{K}\mathcal{Q}$ and $\mathcal{R}_0\mathcal{K} = \mathcal{K}\mathcal{R}_0$, we conclude that the relation (13) is valid, too. From all the above, we obtain that the new matrices also solve the problem. □

6. The Algorithm

Now, in order to organize the solution of the problem of designing appropriate feedback law, we develop a symbolic algorithmic procedure for producing the requested polynomial matrices $\mathcal{R}, \mathcal{S}$ and $\mathcal{T}$.

The Algorithm
Input:

- The parameters ν, s, τ, r and $\lambda_0, \lambda_1, \lambda_2$.
- The initial conditions: $Y(0), B(0), G(0), E(0), \ldots\ Y(3), B(3), G(3), E(3)$.
- The input sequence: $\vec{u}_c(t)$, $t = 0, 1, 2, ...$
- The amount k of the reference values.
- The reference values: $Y^*(t)$, $B^*(t)$, $t = 0, 1, 2, ..., k$.
- The degrees n_{11}, n_{12}, n_{21}, n_{22} of the polynomials D^d_{11}, D^d_{12}, D^d_{21}, D^d_{22}.
- The degrees m_{11}, m_{12}, m_{21}, m_{22} of the polynomials K^d_{11}, K^d_{12}, K^d_{21}, K^d_{22}.

Output: The polynomial matrices: $\mathcal{R}(q)$, $\mathcal{S}(q)$, $\mathcal{T}(q)$

Step 1: Construct and solve the linear system of algebraic equations for $t = 0, 1, \ldots, k$:

$$\sum_{i=0}^{n_{11}} d_i^{(11)} Y^*(t-i) + \sum_{i=0}^{n_{12}} d_i^{(12)} B^*(t-i) = \sum_{i=0}^{m_{11}} k_i^{(11)} u_{c1}(t-i) + \sum_{i=0}^{m_{12}} k_i^{(12)} u_{c2}(t-i)$$

$$\sum_{i=0}^{n_{21}} d_i^{(21)} Y^*(t-i) + \sum_{i=0}^{n_{22}} d_i^{(22)} B^*(t-i) = \sum_{i=0}^{m_{21}} k_i^{(21)} u_{c1}(t-i) + \sum_{i=0}^{m_{22}} k_i^{(22)} u_{c2}(t-i)$$

with unknowns the quantities $d_i^{(\nu\rho)}$, and $k_i^{(\nu\rho)}$, $\nu = 1, 2$, $\rho = 1, 2$.

Step 2: Form the polynomial matrices $\mathcal{D}^d$ and $\mathcal{K}^d$.

$$\mathcal{D}^d = \begin{pmatrix} d_i^{(11)} & d_i^{(12)} \\ d_i^{(21)} & d_i^{(22)} \end{pmatrix} \quad , \quad \mathcal{K}^d = \begin{pmatrix} k_i^{(11)} & k_i^{(12)} \\ k_i^{(21)} & k_i^{(22)} \end{pmatrix}$$

Step 3: Form the polynomial matrices $\mathcal{D}$ and $\mathcal{K}$.

Step 4: Find polynomials M, R, $P_{\nu\rho}$, $F_{\nu\rho}$, such that

$$(D_{2\nu}K_{11}K_{12} - D_{1\nu}K_{12})M + (-D_{1\nu} - D_{2\nu}K_{12})R + P_{1\nu}(K_{11} + K_{12}) = -D^d_{2\nu}K_{12} - D^d_{1\nu}$$

$$(-D_{2\nu}K_{11}K_{11} - D_{2\nu}K_{12} - D_{2\nu}K_{11} + D_{1\nu}K_{11})M + (-D_{1\nu} + D_{2\nu}K_{11})R_{11} + P_{2\nu}(K_{11} + K_{12}) = \\ = D^d_{2\nu}K_{11} - D^d_{1\nu}$$

$$K^d_{1\rho} + (-1)^\nu K_{1\nu}K^d_{2\rho} = F_{\nu\rho}(K_{11} + K_{12}) \quad , \quad \rho = 1, 2, \quad \nu = 1, 2$$

Step 5: Define:

$$\mathcal{R} = \begin{pmatrix} R & M \cdot K_{12} \\ M & -K_{11}M - M + R \end{pmatrix} \quad , \quad \mathcal{S} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \quad , \quad \mathcal{T} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

Theorem 4. *The output of the Algorithm solves the national income–debt tracking problem discussed here.*

Proof. It is a straightforward result of Theorems 1 and 2. □

Remark 1. *We can obtain an infinite number of outputs, due either to the arbitrary selection of the input sequence $\vec{u}_c(t)$, $t = 0, 1, 2, ...$, the solution of the system at Step 1 or the choice of polynomials M, R and P_{ij}, F_{ij}.*

7. An Example

In order to comprehend the effects of the proposed methodology on the macroeconomic system under consideration, we run some simulations for a model of artificial economy. In particular, let us suppose that $s = 0.2$, $\tau = 0.4$, $v = 1$, $r = 5\%$, $\lambda_0 = 0.4$, $\lambda_1 = 0.3$, $\lambda_2 = 0.3$, so that the open-loop system becomes

$$\begin{aligned} &Y(t) = 1.8Y(t-1) - 1.12Y(t-2) - 0.08Y(t-3) + 0.4G(t) + \\ &0.3G(t-1) + 0.3G(t-2) - 0.8E(t-1) - 0.8E(t-2) \\ &B(t) = 1.05B(t-1) - 0.4Y(t-1) + G(t)\text{ˇ}E(t) \end{aligned} \tag{17}$$

and the initial conditions are $Y(3) = 100$, $B(3) = 215$. For the purposes of our simulations, we assume that the policymaker aims for 1% growth in the levels of Y and 1% decrease in the levels of B, per period. Then, by applying the algorithm, we obtain a desired system; in this case, for the sake of simplification, we prefer to obtain a system without input:

$$Y^*(t) = 0.495977Y^*(t-1) + 0.503464Y^*(t-2) - 0.751468B^*(t-1) + 0.751208B^*(t-2)$$

$$B^*(t) = 1.9856Y^*(t-1) - 2.00585Y^*(t-2) + 1.03952B^*(t-1) - 0.0489441B^*(t-2)$$

The algorithm will provide us with a class of feedback laws. Such a causal feedback law giving values $G_1(t), E_1(t)$ is

$$\begin{aligned} &G_1(t) = 1.44G1(t-1) - 0.366G1(t-2) + 1.36Y(t-1) - 2.48Y(t-2) \\ &-0.645Y(t-3) + 2.27B(t-1)\text{ˇ}1.66B(t-2) + 0.234B(t-3) \\ &E_1(t) = 1.44E1(t-1) - 0.366E1(t-2) + 1.07Y(t-1)\text{ˇ}1.707Y(t-2) \\ &\text{ˇ}1.113Y(t-3) + 4.049B(t-1)\text{ˇ}3.53B(t-2) + 0.35B(t-3) \end{aligned} \tag{18}$$

whereas another feedback law of the class finding the same result is

$$\begin{aligned} &G_2(t) = 1.44G2(t-1) - 0.366G2(t-2) + 1.33Y(t-1) - 2.45Y(t-2) \\ &-0.642Y(t-3) + 2.26B(t-1)\text{ˇ}1.63B(t-2) + 0.236B(t-3) \\ &E_2(t) = 1.44E2(t-1) - 0.366E2(t-2) + 1.05Y(t-1)\text{ˇ}1.68Y(t-2) \\ &\text{ˇ}1.11Y(t-3) + 4.046B(t-1)\text{ˇ}3.51B(t-2) + 0.354B(t-3) \end{aligned} \tag{19}$$

Applying both of them to the system (17), we will take the outputs identically equal to the output of the desired system. In the following Table 1, we present the time paths for the inputs, resulting from the above two of the (infinitely many) feedback laws:

Table 1. The two feedbacks

t	$G_1(t)$	$E_1(t)$	$G_2(t)$	$E_2(t)$
4	31.54	4.44	35.5	8.45
5	30.54	2.91	40.3	12.68
6	32.92	4.76	49.5	21.3
7	33.19	4.49	50.4	28.7

As we can see from the table, under feedback law (19), the values of the inputs necessary for simultaneously reaching the reference sequences are much higher compared to the values under feedback law (18). This implies that, since the size of the policy interventions under feedback law (19) are essentially politically infeasible, the policymaker should choose the feedback law which provides a smoother transition path for the inputs.

8. Concluding Remark

The aim of this paper was to present an application of the model-matching approach, arising from mathematical system theory, for the design of economic policies, i.e., for the purpose of shaping the time path of a macroeconomic model, so that a desired behavior is accomplished. From a mathematical control theory perspective, the advantages of this approach are that it is based on algebraic tools and that it provides a general class of feedback laws as a solution.

Theoretical results have been provided describing the solvability of the problem, as well as a symbolic algorithm which incorporates the steps we must follow in order to face the problem.

A limitation of this procedure is that certain algebraic conditions must be valid in order to be able to obtain the sought feedback law.Future research will contain the extension of the methodology to more complicated systems like nonlinear or stochastic ones. Applications including real data must also be included into future works.

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