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Pontryagin Maximum Principle for Incommensurate Fractional-Orders Optimal Control Problems

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Abstract: We introduce a new optimal control problem where the controlled dynamical system depends on multi-order (incommensurate) fractional differential equations. The cost functional to be maximized is of Bolza type and depends on incommensurate Caputo fractional-orders derivatives. We establish continuity and differentiability of the state solutions with respect to perturbed trajectories. Then, we state and prove a Pontryagin maximum principle for incommensurate Caputo fractional optimal control problems. Finally, we give an example, illustrating the applicability of our Pontryagin maximum principle.

Keywords: incommensurate fractional-orders derivatives; fractional optimal control; continuity and differentiability of state trajectories; needle-like variations

MSC: 26A33; 49K15

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1. Introduction

The celebrated Pontryagin maximum principle was first formulated in 1956, being widely regarded as the central result of optimal control theory [1]. The significance of the maximum principle lies in the fact that rather than maximizing over a function space, the problem is converted to a pointwise optimization. Various generalizations of the Pontryagin maximum principle are still being investigated nowadays [2–4].

Fractional optimal control theory, as a branch of applied mathematics, deals with optimization issues for controlled fractional dynamics coupled with a performance index functional [5,6]. The development of the theory started shortly at the beginning of 21th century with some pioneering works [7,8], where necessary optimality conditions are derived using techniques from variational analysis. A major contribution from those works is weak versions of the Pontryagin maximum principle that consists in reducing the fractional optimal control problem (FOCP) to a fractional boundary value problem together with an optimality condition. Later, the theory emerged with the work of [9], by establishing a strong maximum principle of Pontryagin type, which introduces a Hamiltonian maximality condition in the set of necessary optimality conditions instead of the optimality condition from weak versions. Indeed, since this seminal work of 2014, the subject of maximum principle for FOCPs gained more interest. For instance, a new approach to the Pontryagin maximum principle for a problem involving a Lagrangian cost of Riemann–Liouville fractional integral subject to Caputo fractional dynamics is considered in [10]. The authors of [11], derived a Pontryagin maximum principle for FOCP defined by a general Bolza cost of fractional integral type with terminal constraints on the Caputo fractional dynamics. More recently, in 2022, Kamocki investigated an optimal control problem of multi-order fractional systems under a Lagrange-type functional, proving two existence results [12]. In contrast with Kamocki, here we are interested in proving the necessary optimality conditions of Pontryagin type.

Another important development in control theory is the so-called Bellman's dynamic programming principle, which gives a necessary and sufficient condition of optimality [13]. In this direction, the approach for solving an optimal control problem can be achieved by determining a certain value function that might happen to be a viscosity solution to a Hamilton–Jacobi–Bellman (HJB) equation. The dynamic programming principle has been extended for fractional discrete-time systems by the authors of references [14,15]. Furthermore, an attempt to derive a fractional version of the Hamilton–Jacobi–Bellman (HJB) equation has been investigated in [16]. In [17], Gomoyunov studies an extension of the dynamic programming principle to the case of fractional-order dynamical systems. A new approach is required for such problems so that for every intermediate time $t \in (0, T)$, it is necessary to introduce an auxiliary optimal control problem (sub-problem) with this time t considered as the initial one. Also, a derivation of the fractional version of the HJB equation is studied deeply [17]. To sum up, it is important to mention that the Pontryagin maximum principle, along with Bellman's dynamic programming principle, is one of the most effective and fundamental tools for investigating solutions to various optimal control problems.

Nowadays, there are many different definitions of fractional-order integrals or derivatives [18] and, in some sense, it is possible to consider a single broad class of fractional-order operators that include existing ones as particular cases [19]. This is important in analyzing a single operator rather than focusing on each individual one separately. Also, it enriches the subject of Pontryagin's maximum principle to handle a more general and wide class of fractional-order operators. For instance, a maximum principle is obtained for a combined fractional operator with a general analytic kernel in [20]. Also, some recent results for the Pontryagin maximum principle are investigated for fractional stochastic delayed systems with non-instantaneous impulses [21], for a degenerate fractional differential equation [22], and for distributed-order fractional derivatives [23]. In contrast, the main aim of our current study is to utilize the idea of multi-order or incommensurate orders of derivatives in the definition of optimal control problems and then analyze their solutions. In doing so, we start with the most popular fractional model, which is the Caputo.

The structure of the article is as follows. In Section 2, we present some basic definitions and properties from fractional calculus. In Section 3, our contribution is given: we start by introducing the incommensurate non-local fractional optimal control problem, then we prove the continuity of solutions (Lemma 3), differentiability of perturbed trajectories (Lemma 4), and, finally, the proof of the Pontryagin maximum principle (Theorem 1). In Section 4, we give an example illustrating the application of the new Pontryagin maximum principle. We end with Section 5, summarizing our work and giving some perspectives for future work.

2. Preliminaries

In this section, we briefly recall the necessary notions and results from fractional calculus. For more on the subject, we refer the interested readers to the books [24–26].

Let $(\alpha_i)_{i=1,\ldots,n} \in (0,1)$ be a multi-order of real numbers. In the sequel, we use the following notation:

$$L^{(\alpha)}([a,b],\mathbb{R}^n) := \left\{ x \in L^1([a,b],\mathbb{R}^n) : I^{\alpha_i}_{a^+} x_i, I^{\alpha_i}_{b^-} x_i \in AC([a,b],\mathbb{R}) \right\},$$

where $I^{\alpha_i}_{a^+}$ and $I^{\alpha_i}_{b^-}$ represent, respectively, the left and right Riemann–Liouville integrals of order α_i. We also use the notation $AC^{(\alpha)}([a,b],\mathbb{R}^n)$ to represent the set of absolutely continuous functions that can be represented as

$$x_i(t) = x_i(a) + I^{\alpha_i}_{a^+} f(t) \text{ and } x_i(t) = x_i(b) + I^{\alpha_i}_{b^-} f(t),$$

for some functions $f \in L^{\alpha_i} := \{ x \in L^1([a,b],\mathbb{R}) : I^{\alpha_i}_{a^+} x_i, I^{\alpha_i}_{b^-} x_i \in AC([a,b],\mathbb{R}) \}$.

Definition 1. *The left- and right-sided Riemann–Liouville incommensurate fractional derivative of orders* $(\alpha_i)_{i=1,\ldots,n} \in (0,1)$ *of a function* $x \in L^{(\alpha)}$ *are defined, respectively, by*

$$D_{a^+}^{(\alpha)}x(t) = \begin{cases} D_{a^+}^{\alpha_1}x_1(t), \\ D_{a^+}^{\alpha_2}x_2(t), \\ \vdots \quad \vdots \\ D_{a^+}^{\alpha_n}x_n(t), \end{cases} ; \qquad D_{b^-}^{(\alpha)}x(t) = \begin{cases} D_{b^-}^{\alpha_1}x_1(t), \\ D_{b^-}^{\alpha_2}x_2(t), \\ \vdots \quad \vdots \\ D_{b^-}^{\alpha_n}x_n(t), \end{cases}$$

with

$$D_{a^+}^{\alpha_i}x_i(t) = \frac{d}{dt}\left(I_{a^+}^{1-\alpha_i}x_i(t)\right),\ D_{b^-}^{\alpha_i}x_i(t) = -\frac{d}{dt}\left(I_{b^-}^{1-\alpha_i}x_i(t)\right),$$

where $I_{a^+}^{1-\alpha_i}$ *and* $I_{b^-}^{1-\alpha_i}$ *represent, respectively, the left- and right-sided Riemann–Liouville fractional integrals of order* $1-\alpha_i$*.*

Definition 2. *The left- and right-sided Caputo incommensurate fractional derivatives of order* $(\alpha_i)_{i=1,\ldots,n} \in (0,1)$ *of a function* $x \in AC^{(\alpha)}$ *are defined, respectively, by*

$${}^{C}D_{a^+}^{(\alpha)}x(t) = \begin{cases} {}^{C}D_{a^+}^{\alpha_1}x_1(t), \\ {}^{C}D_{a^+}^{\alpha_2}x_2(t), \\ \vdots \quad \vdots \\ {}^{C}D_{a^+}^{\alpha_n}x_n(t), \end{cases} ; \qquad {}^{C}D_{b^-}^{(\alpha)}x(t) = \begin{cases} {}^{C}D_{b^-}^{\alpha_1}x_1(t), \\ {}^{C}D_{b^-}^{\alpha_2}x_2(t), \\ \vdots \quad \vdots \\ {}^{C}D_{b^-}^{\alpha_n}x_n(t), \end{cases}$$

where

$$D_{a^+}^{\alpha_i}x_i(t) = I_{a^+}^{1-\alpha_i}\left(\frac{d}{dt}x_i(t)\right), \quad D_{b^-}^{\alpha_i}x_i(t) = -I_{b^-}^{1-\alpha_i}\left(\frac{d}{dt}x_i(t)\right).$$

Note that integration by parts is a powerful tool when two functions are multiplied together, being useful for our purposes in the proof of the Pontryagin maximum principle. In the sequel, the dot $\cdot$ is used for indicating scalar products.

Lemma 1 (Integration by parts formula [25]). *Let* $x \in L^{(\alpha)}$ *and* $y \in AC^{(\alpha)}$*. Then,*

$$\int_a^b x(t) \cdot {}^{C}D_{a^+}^{(\alpha)}y(t)dt = \left[y(t) \cdot I_{b^-}^{1-(\alpha)}x(t)\right]_a^b + \int_a^b y(t) \cdot D_{b^-}^{(\alpha)}x(t)dt,$$

where

$$I_{b^-}^{1-(\alpha)}x(t) = \begin{cases} I_{b^-}^{1-\alpha_1}x_1(t), \\ I_{b^-}^{1-\alpha_2}x_2(t), \\ \vdots \quad \vdots \\ I_{b^-}^{1-\alpha_n}x_n(t). \end{cases}$$

In what follows, we recall a generalized Gronwall inequality that is useful to prove the continuity and differentiability of perturbed trajectories.

Lemma 2 (Generalized Gronwall inequality [27]). *Let* α *be a positive real number and let* $p(\cdot)$, $q(\cdot)$, *and* $u(\cdot)$ *be non-negative continuous functions on* $[a,b]$ *with* $q(\cdot)$ *monotonic increasing on* $[a,b)$*. If*

$$u(t) \leq p(t) + q(t)\int_a^t (t-s)^{\alpha-1}u(s)ds,$$

then

$$u(t) \leq p(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(q(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}p(s)\right]ds$$

for all $t \in [a, b)$.

3. Main Results

In this section, our main concern is to find a control function $u \in L^{\infty}([a,b],\mathbb{R}^m)$ and its corresponding state trajectory $x \in AC^{(\alpha)}([a,b],\mathbb{R}^n)$, solution to the following incommensurate non-local fractional optimal control problem:

$$
\begin{gathered}
J[x(\cdot),u(\cdot)] = \varphi(b,x(b)) + \int_a^b L(t,x(t),u(t))dt \longrightarrow \max, \\
{}^{C}D_{a+}^{(\alpha)}x(t) = f(t,x(t),u(t)), \quad t \in [a,b] \text{ a.e.}, \\
x(\cdot) \in AC^{(\alpha)}, \quad u(\cdot) \in L^{\infty}, \\
x(a) = x_a \in \mathbb{R}^n, \quad u(t) \in \Omega,
\end{gathered} \tag{1}
$$

where Ω is a closed subset of $\mathbb{R}^m$. The data functions $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are subject to the following assumptions:

- The function φ is of class $\mathcal{C}^1$.
- Functions L and f are continuous in all its three arguments and of class $\mathcal{C}^1$ with respect to x and, in particular, locally Lipschitz-continuous, that is, for every compact $B \subset \mathbb{R}^n$ and for all $x, y \in B$ there is $K > 0$ such that $|L(t,x,u) - L(t,y,u)| \leq K\|x-y\|$ and $\|f(t,x,u) - f(t,y,u)\| \leq K\|x-y\|$.
- There exists also $N > 0$, such that $\left\|\frac{\partial L(t,x,u)}{\partial x}\right\| \leq N$ and $\left\|\frac{\partial f(t,x,u)}{\partial x}\right\| \leq N$.
- With respect to the control u, there exists $M > 0$ such that

$$
|L(t,x,u)| \leq M, \quad \| f(t,x,u) \| \leq M, \quad \forall (t,x) \in [a,b] \times \mathbb{R}^n.
$$

3.1. Needle-like Perturbation of the Optimal Control

We will prove a first-order necessary optimality condition of the Pontryagin type, which is only sufficient in very particular cases [28]. Any proof of a necessary optimality condition, e.g., Fermat theorem about stationary points or the classical Pontryagin maximum principle [1], begins by assuming the existence of a solution. We do the same here: we assume $u^*(t) \in \Omega$ to be an optimal control to problem (1) for $t \in [a,b]$. The reader who is interested in the question of the existence of u^* is referred to the recent paper [12]. Our aim, in this section, is to derive continuity and differentiability properties of perturbed trajectories, which are crucial to proving the necessary optimality condition for the optimal control problem (1). One way of achieving this is to perturb the optimal control by a needle-like variation and study the behavior of the corresponding state with respect to the optimal curve.

Denote by $\mathcal{L}[F(\cdot)]$ the set of all Lebesgue points in $[a,b)$ of the essentially bounded functions $t \mapsto f(t,x(t),u(t))$ and $t \mapsto L(t,x(t),u(t))$. Then, for $(\tau,v) \in \mathcal{L}[F(\cdot)] \times \Omega$, and for every $\theta \in [0, b-\tau)$, let us consider the needle-like variation $u^{\theta} \in L^{\infty}([a,b],\mathbb{R}^n)$ associated to the optimal control u^*, which is given by

$$
u^{\theta}(t) = \begin{cases} u^*(t) & \text{if} \quad t \notin [\tau, \tau+\theta), \\ v & \text{if} \quad t \in [\tau,\tau+\theta), \end{cases} \tag{2}
$$

for almost every $t \in [a,b]$.

Lemma 3 (Continuity of solutions)**.** *For any $(\tau,v) \in \mathcal{L}[F(\cdot)] \times \Omega$, denote by x^{θ} the corresponding state trajectory to the needle-like variation u^{θ}, that is, the state solution of*

$$
{}^{C}D_{a+}^{(\alpha)}x^{\theta}(t) = f\left(t, x^{\theta}(t), u^{\theta}(t)\right), \quad x^{\theta}(a) = x_a. \tag{3}
$$

Then, the state x^θ converges uniformly to the optimal state trajectory x^* whenever θ tends to zero.

Proof. By definition of incommensurate Caputo derivative, we have

$$ {}^{C}D^{(\alpha)}_{a+}\left(x^\theta(t) - x^*(t)\right) = f(t, x^\theta(t), u^\theta(t)) - f(t, x^*(t), u^*(t)). $$

Then, the integral representation is obtained as

$$ x^\theta(t) - x^*(t) = \int_0^t \operatorname{diag}\left(\frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\right) \cdot \left(f(t, x^\theta(t), u^\theta(t)) - f(t, x^*(t), u^*(t))\right) ds. $$

Moreover, the function $\alpha_i \mapsto \dfrac{c^{\alpha_i-1}}{\Gamma(\alpha_i)}$ is continuous on $[\min\limits_{1\le i\le n} \alpha_i, 1]$, where c is a non-zero constant. Thus, by the extreme value theorem due to Weierstrass, it attains a maximum. Hence, there exists $\bar{\alpha} \in [\min\limits_{1\le i\le n} \alpha_i, 1]$ such that $\dfrac{c^{\alpha_i-1}}{\Gamma(\alpha_i)} \le \dfrac{c^{\bar{\alpha}-1}}{\Gamma(\bar{\alpha})}$. This leads to

$$ \begin{aligned} x^\theta(t) - x^*(t) &\le \int_0^t \frac{(t-s)^{\bar{\alpha}-1}}{\Gamma(\bar{\alpha})} \cdot \left(f(t, x^\theta(t), u^\theta(t)) - f(t, x^*(t), u^*(t))\right) ds \\ &= I^{\bar{\alpha}}_{a+}\left(f(t, x^\theta(t), u^\theta(t)) - f(t, x^*(t), u^*(t))\right). \end{aligned} $$

Further, the following relation holds for function f,

$$ \begin{aligned} f(t, x^\theta(t), u^\theta(t)) - f(t, x^*(t), u^*(t)) = \{&f(t, x^\theta(t), u^\theta(t)) - f(t, x^*(t), u^\theta(t))\} \\ &+ \{f(t, x^*(t), u^\theta(t)) - f(t, x^*(t), u^*(t))\}. \end{aligned} $$

Using the triangular inequality, and noticing that u^θ and u^* are different only on $[\tau, \tau+\theta)$, we obtain

$$ \begin{aligned} \| x^\theta(t) - x^*(t) \| \le\ & I^{\bar{\alpha}}_{a^+}\left(\| f(t, x^\theta(t), u^\theta(t)) - f(t, x^*(t), u^\theta(t)) \|\right) \\ &+ I^{\bar{\alpha}}_{\tau^+}\left(\| f(t, x^*(t), u^\theta(t)) - f(t, x^*(t), u^*(t)) \|\right). \end{aligned} $$

Next, from data assumptions, K is a Lipschitz constant of f and M is an upper bound of f with respect to the control function. Thus, it follows that

$$ \| x^\theta(t) - x^*(t) \| \le K I^{\bar{\alpha}}_{a^+}\left(\| x^\theta(t) - x^*(t) \|\right) + M\frac{\theta^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)}. $$

Now, by applying the generalized Gronwall inequality of Lemma 2, it follows that

$$ \begin{aligned} \| x^\theta(t) - x^*(t) \| &\le \frac{M\theta^{\bar{\alpha}}}{\Gamma(\alpha+1)}\left[1 + \int_a^t \sum_{n=1}^{\infty} \frac{K^n}{\Gamma(n\bar{\alpha})}(t-s)^{n\bar{\alpha}-1} ds\right] \\ &\le \frac{M\theta^{\bar{\alpha}}}{\Gamma(\alpha+1)} E_{\alpha,1}(K(b-a)^\alpha), \end{aligned} $$

where $E_{\alpha,1}$ is the Mittag–Leffler function of parameter $\bar{\alpha}$ [29]. Hence, we obtain that $\left\|x^\theta - x^*\right\|$ converges to zero on $[a, b]$, which ends the proof. □

Lemma 4 (Differentiability of the perturbed trajectory)**.** *Suppose that (x^*, u^*) is an optimal pair to problem (1). Then, for $(\tau, v) \in \mathcal{L}[F(\cdot)] \times \Omega$, the quotient variational trajectory*

$\dfrac{x^{\theta}(\cdot)-x^{*}(\cdot)}{\theta}$ *converges uniformly on* $[\tau+\theta,b]$ *to* $\eta(\cdot)$ *when* θ *tends to zero, where* $\eta(\cdot)$ *is the unique solution to the incommensurate left Caputo fractional Cauchy problem*

$$\begin{cases} {}^{C}D^{(\alpha)}_{\tau+}\eta(t)=\dfrac{\partial f(t,x^{*}(t),u^{*}(t))}{\partial x}\cdot\eta(t), \quad t\in]\tau,b], \\ I^{1-\bar{\alpha}}_{\tau^{+}}\eta(\tau)=(f(\tau,x^{*}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau))), \end{cases} \tag{4}$$

with $\bar{\alpha}\in[\min\limits_{1\le i\le n}\alpha_i,1]$*, such that* $\dfrac{c^{\alpha_i-1}}{\Gamma(\alpha_i)}\le\dfrac{c^{\bar{\alpha}-1}}{\Gamma(\bar{\alpha})}$*, for some non-zero constant c and* $i=1,\dots,n$*.*

Proof. Set $z^{\theta}(t)=\dfrac{x^{\theta}(t)-x^{*}(t)}{\theta}-\eta(t)$ for all $t\in[\tau+\theta,b]$. Our aim is to prove that z^{θ} converges uniformly to zero whenever $\theta\to 0$.

The integral representation of z^{θ} is obtained as follows:

$$\begin{aligned} z^{\theta}(t)=&\int_{\tau}^{\tau+\theta}\operatorname{diag}\left(\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\cdot\frac{f(s,x^{\theta}(s),v)-f(s,x^{*}(s),v)}{\theta}ds \\ &+\int_{\tau}^{\tau+\theta}\operatorname{diag}\left(\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\cdot\frac{f(s,x^{*}(s),v)-f(s,x^{*}(s),u^{*}(s))}{\theta}ds \\ &-\frac{1}{\Gamma(\bar{\alpha})}(t-\tau)^{\bar{\alpha}-1}(f(\tau,x^{*}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau))) \\ &+\int_{\tau+\theta}^{t}\operatorname{diag}\left(\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\cdot\left(\frac{f(s,x^{\theta}(s),u^{*}(s))-f(s,x^{*}(s),u^{*}(s))}{\theta}\right. \\ &\left.-\frac{\partial f(s,x^{*}(s),u^{*}(s))}{\partial x}\times\frac{x^{\theta}(s)-x^{*}(s)}{\theta}\right)ds \\ &\int_{\tau}^{\tau+\theta}\operatorname{diag}\left(\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\cdot\frac{\partial f(s,x^{*}(s),u^{*}(s))}{\partial x}\times\eta(s)ds \\ &+\int_{\tau+\theta}^{t}\operatorname{diag}\left(\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\right)\cdot\frac{\partial f(s,x^{*}(s),u^{*}(s))}{\partial x}\times z^{\theta}(s)ds. \end{aligned}$$

Note that by the existence property of $\bar{\alpha}$, we have that $\dfrac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\le\dfrac{(t-s)^{\bar{\alpha}-1}}{\Gamma(\bar{\alpha})}$ for all $i=1,\dots,n$. Thus, we can deduce that

$$\begin{aligned} z^{\theta}(t)\le&\frac{1}{\Gamma(\bar{\alpha})}\int_{\tau}^{\tau+\theta}(t-s)^{\bar{\alpha}-1}\frac{f(s,x^{\theta}(s),v)-f(s,x^{*}(s),v)}{\theta}ds \\ &+\frac{1}{\Gamma(\bar{\alpha})}\int_{\tau}^{\tau+\theta}(t-s)^{\bar{\alpha}-1}\frac{f(s,x^{*}(s),v)-f(s,x^{*}(s),u^{*}(s))}{\theta}ds \\ &-\frac{1}{\Gamma(\bar{\alpha})}(t-\tau)^{\bar{\alpha}-1}(f(\tau,x^{*}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau))) \\ &+\frac{1}{\Gamma(\bar{\alpha})}\int_{\tau+\theta}^{t}(t-s)^{\bar{\alpha}-1}\left(\frac{f(s,x^{\theta}(s),u^{*}(s))-f(s,x^{*}(s),u^{*}(s))}{\theta}\right. \\ &\left.-\frac{\partial f(s,x^{*}(s),u^{*}(s))}{\partial x}\times\frac{x^{\theta}(s)-x^{*}(s)}{\theta}\right)ds \\ &-\frac{1}{\Gamma(\bar{\alpha})}\int_{\tau}^{\tau+\theta}(t-s)^{\bar{\alpha}-1}\frac{\partial f(s,x^{*}(s),u^{*}(s))}{\partial x}\times\eta(s)ds \\ &+\frac{1}{\Gamma(\bar{\alpha})}\int_{\tau+\theta}^{t}(t-s)^{\bar{\alpha}-1}\frac{\partial f(s,x^{*}(s),u^{*}(s))}{\partial x}\times z^{\theta}(s)ds. \end{aligned} \tag{5}$$

In line with the work of [11], precisely the proof of Proposition 3.3 of this reference, we obtain that each term appearing in the right hand side of (5) is bounded, which yields the following estimate:

$$\|z^{\theta}(t)\| \leq \Theta_1^{\theta}(t-(\tau+\theta))^{\bar{\alpha}-1}E_{\bar{\alpha},\bar{\alpha}'}\left(N(b-a)^{\bar{\alpha}}\right)+\Theta_2^{\theta}(t-(\tau+\theta))^{\bar{\alpha}'-1}E_{\bar{\alpha},\bar{\alpha}'}\left(N(b-a)^{\bar{\alpha}}\right),$$

where functions Θ_1^{θ} and Θ_2^{θ} both converge uniformly to zero whenever θ tends to zero. Hence, we conclude that $z^{\theta}(t)$ converges uniformly to zero as θ goes to zero, which is the desired result. □

3.2. Pontryagin's Maximum Principle for Problem (1)

The fractional Pontryagin maximum principle has many applications, for example in Engineering, Economics and Health [30,31]. Here, we state and prove the main result of our work: a Pontryagin maximum principle for the incommensurate fractional-order optimal control problem (1).

Theorem 1 (Pontryagin Maximum Principle for (1))**.** *If $(x^*(\cdot),u^*(\cdot))$ is an optimal pair for* (1)*, then there exists $\lambda \in L^{(\alpha)}$, called the adjoint function variable, such that the following conditions hold for all t in the interval $[a,b]$:*

- *the maximality condition*
$$H(t,x^*(t),u^*(t),\lambda(t)) = \max_{\omega\in\Omega} H(t,x^*(t),\omega,\lambda(t)); \tag{6}$$
- *the adjoint system*
$$D_{b-}^{(\alpha)}\lambda(t) = \frac{\partial H}{\partial x}(t,x^*(t),u^*(t),\lambda(t)); \tag{7}$$
- *the transversality condition*
$$I_{b^-}^{1-(\alpha)}\lambda(b) = \frac{\partial \varphi}{\partial x}(b,x^*(b)), \tag{8}$$

where the Hamiltonian function H is defined by

$$H(t,x,u,\lambda) = L(t,x,u)+\lambda\cdot f(t,x,u). \tag{9}$$

Proof. Let $x^{\theta}(t)$ be the corresponding state trajectory to the needle-like variation u^{θ} defined by (2). Observe that, by integration by parts, we have for $\lambda(\cdot)\in L^{(\alpha)}$,

$$\int_a^b \lambda(t)\cdot {}^{C}D_{a+}^{(\alpha)}x^{\theta}(t)dt = \left[x^{\theta}(t)\cdot I_{b-}^{1-(\alpha)}\lambda(t)\right]_a^b + \int_a^b x^{\theta}(t)\cdot D_{b-}^{(\alpha)}\lambda(t)dt. \tag{10}$$

This relation can be added to the objective functional at (x^{θ},u^{θ}) defined by

$$J(x^{\theta},u^{\theta}) = \varphi(b,x^{\theta}(b)) + \int_a^b L\left(t,x^{\theta}(t),u^{\theta}(t)\right)ds,$$

meaning that

$$J(x^{\theta},u^{\theta}) = \varphi(b,x^{\theta}(b)) + \int_a^b \left[L\left(t,x^{\theta}(t),u^{\theta}(t)\right)+\lambda(t)\cdot{}^{C}D_{a+}^{(\alpha)}x^{\theta}(t)-x^{\theta}(t)\cdot D_{b-}^{(\alpha)}\lambda(t)\right]dt$$
$$x^{\theta}(b)\cdot I_{b-}^{1-(\alpha)}\lambda(b)+x^{\theta}(a)\cdot I_a^{1-(\alpha)}\lambda(a), \tag{11}$$

which, by substituting (3) to this latter expression, leads to

$$J(x^\theta,u^\theta)=\varphi(b,x^\theta(b))+\int_a^b\Big[H\Big(t,x^\theta(t),u^\theta(t),\lambda(t)\Big)-x^\theta(t)\cdot D_{b-}^{(\alpha)}\lambda(t)\Big]dt \\ -x^\theta(b)\cdot I_{b-}^{1-(\alpha)}\lambda(b)+x_a\cdot I_a^{1-(\alpha)}\lambda(a),$$

where $H(t,x,u,\lambda)=L(t,x,u)+\lambda\cdot f(t,x,u)$. Next, we write down the Taylor expansion

$$\varphi(b,x^\theta(b))=\varphi(b,x^*(b))+\Big(x^\theta(b)-x^*(b)\Big)\cdot\frac{\partial\varphi}{\partial x}(b,x^*(b))+\circ\Big(\|\ x^\theta-x^*\ \|\Big);$$
$$H\Big(t,x^\theta(t),u^\theta(t),\lambda(t)\Big)=H\Big(t,x^*(t),u^\theta(t),\lambda(t)\Big)+\Big(x^\theta(t)-x^*(t)\Big)\cdot\frac{\partial H}{\partial x}\Big(t,x^*(t),u^\theta(t),\lambda(t)\Big) \\ +o\Big(\|\ x^\theta-x^*\ \|\Big).$$

Note that, by the continuity Lemma 3, we have the uniform convergence of $\|x^\theta-x^*\|\to 0$ whenever $\theta\to 0$. Thus, the residue term in the Taylor expansion can be expressed as a function of θ. Therefore, we can evaluate the quotient $\dfrac{J(x^\theta,u^\theta)-J(x^*,u^*)}{\theta}:=\delta J$ as follows:

$$\delta J=\frac{x^\theta(b)-x^*(b)}{\theta}\cdot\frac{\partial\varphi}{\partial x}(b,x^*(b))+\int_a^b\frac{H\big(t,x^*(t),u^\theta(t),\lambda(t)\big)-H(t,x^*(t),u^*(t),\lambda(t))}{\theta}dt \\ +\int_a^b\Big(\frac{\partial H}{\partial x}\Big(t,x^*(t),u^\theta(t),\lambda(t)\Big)-D_{b-}^{(\alpha)}\lambda(t)\Big)\cdot\frac{x^\theta(t)-x^*(t)}{\theta}dt-\frac{x^\theta(b)-x^*(b)}{\theta}\cdot I_{b-}^{1-(\alpha)}\lambda(b) \\ +o(\theta)(1+b-a).$$

Now, by the differentiability Lemma 4, we have that $\frac{x^\theta(t)-x^*(t)}{\theta}$ converges uniformly to $\eta(t)$ when θ tends to zero. Therefore, the limit of δJ when θ tends to zero can be expressed as

$$\lim_{\theta\to 0}\delta J=\eta(b)\cdot\frac{\partial\varphi}{\partial x}(b,x^*(b))+\lim_{\theta\to 0}\int_a^b\frac{H\big(t,x^*(t),u^\theta(t),\lambda(t)\big)-H(t,x^*(t),u^*(t),\lambda(t))}{\theta}dt \\ +\int_a^b\Big(\frac{\partial H}{\partial x}(t,x^*(t),u^*(t),\lambda(t))-D_{b-}^{(\alpha)}\lambda(t)\Big)\cdot\eta(t)dt-\eta(b)\cdot I_{b-}^{1-(\alpha)}\lambda(b).$$

Next, we fix

$$D_{b-}^{(\alpha)}\lambda(t)=\frac{\partial H}{\partial x}(t,x^*(t),u^*(t),\lambda(t))\quad\text{with}\quad I_{b-}^{1-(\alpha)}\lambda(b)=\frac{\partial\varphi}{\partial x}(b,x^*(b)),$$

that is, the adjoint Equation (7) and the transversality condition (8). Thus, we are left with

$$\lim_{\theta\to 0}\delta J=\lim_{\theta\to 0}\int_a^b\frac{H\big(t,x^*(t),u^\theta(t),\lambda(t)\big)-H(t,x^*(t),u^*(t),\lambda(t))}{\theta}dt.$$

Moreover, recalling that $u^\theta(t)=\begin{cases}u^*(t) & \text{if}\quad t\notin[\tau,\tau+\theta);\\ v & \text{if}\quad t\in[\tau,\tau+\theta),\end{cases}$ for almost every $t\in[a,b]$, it follows that

$$\lim_{\theta\to 0}\delta J=\lim_{\theta\to 0^+}\frac{1}{\theta}\int_\tau^{\tau+\theta}[H(s,x^*(s),v,\lambda(s))-H(s,x^*(s),u^*(s),\lambda(s))]ds.$$

However, notice that τ is a Lebesgue point of

$$H(s,x^*(s),v,\lambda(s))-H(s,x^*(s),u^*(s),\lambda(s)):=\psi(s).$$

Thus, from the Lebesgue differentiation property,

$$\left|\frac{1}{\theta}\int_{\tau}^{\tau+\theta}\psi(s)ds-\psi(\tau)\right|=\left|\frac{1}{\theta}\int_{\tau}^{\tau+\theta}(\psi(s)-\psi(\tau))ds\right|\leq\frac{1}{\theta}\int_{\tau}^{\tau+\theta}|\psi(s)-\psi(\tau)|ds,$$

and we have that the right-hand side tends to zero for almost every point τ. As a consequence,

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{1}{\theta}\int_{\tau}^{\tau+\theta}&[H(s,x^*(s),v,\lambda(s))-H(s,x^*(s),u^*(s),\lambda(s))]ds\\&=H(\tau,x^*(\tau),v,\lambda(\tau))-H(\tau,x^*(\tau),u^*(\tau),\lambda(\tau)).\end{aligned}\tag{12}$$

Further, by the optimality assumption of the pair

$$(x^*,u^*),\text{ one has also }\lim_{\theta\to 0}\delta J\leq 0,$$

altogether, we obtain that

$$H(\tau,x^*(\tau),v,\lambda(\tau))-H(\tau,x^*(\tau),u^*(\tau),\lambda(\tau))\leq 0.$$

Finally, because τ is an arbitrary Lebesgue point of the control u^* and v is an arbitrary element of the set Ω, it follows that the relation

$$H(t,x^*(t),u^*(t),\lambda(t))=\max_{\omega\in\Omega}H(t,x^*(t),\omega,\lambda(t))$$

holds at all Lebesgue points, which ends the proof. □

Theorem 1 gives a necessary optimality condition that provides an algorithm that can be used to solve general Bolza-type fractional optimal control problems that depend on multi-order Caputo fractional derivatives: given a problem (1),

(i) one writes the associated Hamiltonian (9);
(ii) we use the maximality condition (6) to obtain an expression of the optimal controls in terms of the state and adjoint variables;
(iii) we substitute the expressions obtained in step (ii) in the adjoint system (7);
(iv) finally, we solve the system obtained in step (iii) together with the initial conditions $x(a)=x_a$ and the transversality condition (8).

A simple example of the usefulness of our result is given in Section 4.

4. An Illustrative Example

Let us study the following optimal control multi-order fractional Caputo problem:

$$\begin{gathered}x_2(5)+\int_1^5(1+\exp(2u(t)))dt\longrightarrow\max,\\x=(x_1,x_2)\in AC^{(\frac{1}{3},\frac{1}{2})}\left([1,5],\mathbb{R}^2\right),\quad u\in L^\infty([1,5],\mathbb{R}),\\\begin{cases}{}^{C}D_{1^+}^{\frac{1}{3}}x_1(t)=1-\exp(2u(t)),\\{}^{C}D_{1^+}^{\frac{1}{2}}x_2(t)=x_1(t),\end{cases}\\x(1)=(1,1),\quad u(t)\in[-2,7].\end{gathered}\tag{13}$$

The Hamiltonian function and the running cost of this problem (13) are given, respectively, by

$$H(t,x,u,\lambda)=1+\exp(2u)+\lambda_1(1-\exp(2u))+\lambda_2x_1,\quad\text{and }\varphi(b,x(b))=x_2(5).$$

We do not know if the problem has a solution or not but, if the problem has a solution, then it must satisfy the necessary optimality condition is given by our Theorem 1. Precisely, if (x^*, u^*) is an optimal pair solution to problem (13), then by application of our Theorem 1, there exists an adjoint function $\lambda \in L^{(\frac{1}{3},\frac{1}{2})}$, satisfying

$$\begin{cases} D^{\frac{1}{3}}_{5^-}\lambda_1(t) = \frac{\partial H}{\partial x_1} = \lambda_2, & I^{1-\frac{1}{3}}_{5^-} = \frac{\partial \varphi}{\partial x_1} = 0; \\ D^{\frac{1}{2}}_{5^-}\lambda_2(t) = \frac{\partial H}{\partial x_2} = 0, & I^{1-\frac{1}{2}}_{5^-} = \frac{\partial \varphi}{\partial x_1} = 1. \end{cases}$$

We integrate, to obtain that

$$\lambda_1(t)\frac{(5-t)^{\frac{2}{3}} - 1}{\Gamma(\frac{2}{3})}, \quad \text{and} \quad \lambda_2(t) = \frac{(5-t)^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})}.$$

Moreover, from the maximality condition (6), it yields that

$$u^*(t) \in \arg_{v\in[-2,7]} \left\langle \begin{pmatrix} 1-\exp(2v(t)) \\ 1+\exp(2v(t)) \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \right\rangle,$$

for almost $t \in [1,5]$. Therefore, using the classical Cauchy–Schwartz inequality, we obtain

$$\begin{pmatrix} 1-\exp(2u^*(t)) \\ 1+\exp(2u^*(t)) \end{pmatrix} = \frac{1}{|(\lambda_1(t),1)|_{\mathbb{R}^2}} \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}.$$

This leads to

$$\tanh(u^*(t)) = \frac{1}{\lambda_1(t)} = \Gamma\left(\frac{2}{3}\right)(5-t)^{\frac{1}{3}}.$$

Finally, we obtain that u^* is given by

$$u^*(t) = \arctan h\left(\Gamma\left(\frac{2}{3}\right)(5-t)^{\frac{1}{3}}\right), \text{ for almost } t \in [1,5]. \tag{14}$$

Note that (14) is just a Pontryagin extremal (candidate).

For practical applications, the problems are difficult and one needs to use numerical methods. Many software packages for computing fractional-order derivatives and solving fractional differential systems are now available. We refer the interested reader to [32] and references therein.

5. Conclusions

In this paper, we have studied a general Bolza-type fractional optimal control problem that depends on multi-order Caputo fractional derivatives. We have established a Pontryagin maximum principle for this incommensurate fractional-order problem. Our approach starts with a sensitivity analysis from which we prove the continuity and differentiability of perturbed trajectories to the optimal state. An illustrative example shows the applicability of our main result (Theorem 1), which provides a Pontryagin maximum principle for incommensurate fractional-order optimal control problems.

Recent applications of fractional mathematical modeling have shown the importance of considering incommensurate orders in infectious disease dynamics [33]. This might permit greater flexibility in capturing the heterogeneous nature of disease dynamics, accounting for factors such as population demographics or social behaviors of individuals [34]. Further, it has an advantage when introducing a control function in such models, by the fact that the more accurate the model, the better the control. Therefore, for future work, it would be interesting to emphasize numerical methods for incommensurate-order problems in order to handle applications on existing models of multi-order derivatives.

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