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Jensen–Mercer and Hermite–Hadamard–Mercer Type Inequalities for GA-h-Convex Functions and Its Subclasses with Applications

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Abstract: Many researchers have been attracted to the study of convex analysis theory due to both facts, theoretical significance, and the applications in optimization, economics, and other fields, which has led to numerous improvements and extensions of the subject over the years. An essential part of the theory of mathematical inequalities is the convex function and its extensions. In the recent past, the study of Jensen–Mercer inequality and Hermite–Hadamard–Mercer type inequalities has remained a topic of interest in mathematical inequalities. In this paper, we study several inequalities for GA-h-convex functions and its subclasses, including GA-convex functions, GA-s-convex functions, GA-Q-convex functions, and GA-P-convex functions. We prove the Jensen–Mercer inequality for GA-h-convex functions and give weighted Hermite–Hadamard inequalities by applying the newly established Jensen–Mercer inequality. We also establish inequalities of Hermite–Hadamard–Mercer type. Thus, we give new insights and variants of Jensen–Mercer and related inequalities for GA-h-convex functions. Furthermore, we apply our main results along with Hadamard fractional integrals to prove weighted Hermite–Hadamard–Mercer inequalities for GA-h-convex functions and its subclasses. As special cases of the proven results, we capture several well-known results from the relevant literature.

Keywords: convex functions; h-convex functions; GA-h-convex functions; Jensen–Mercer inequality; Hermite–Hadamard–Mercer type inequalities; Hadamard fractional integral

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1. Introduction and Preliminaries

Many researchers have been attracted to and have studied convex analysis theory because it is widely used in optimization, economics, and other disciplines. The convex analysis has undergone numerous improvements and extensions over the years. Among several other developments, one recent breakthrough is the introduction of *cr*- convex functions [1] utilized when establishing equivalent optimality conditions for nonlinear optimization problems (constrained, unconstrained) via objective function (interval valued). Theoretically, the notion of convex function (CF) has been known to exist even before [2]. The convex functions partially possess several fundamental features such as continuity, differentiability, and monotonicity of derivatives, which makes them a potential family to be studied in different branches of mathematics. A key such direction is the study of mathematical inequalities, in which a large number of important inequalities have been studied for convex functions. These inequalities include Jensen's inequality (JI), the Hermite–Hadamard inequality (HHI), the Jensen–Mercer inequality (JMI), and several other inequalities (see [2–5]) and references therein. The notion of convexity has been modified and generalized, and corresponding significant inequalities have been investigated for new families such as P-convex and Q-convex functions [6–8]. Another generalization of the family of convex functions is known as the class of s-convex functions (s-CF), for which the corresponding inequalities have also been studied (see [9,10]). Another significant and vastly studied generalization of the convex functions was introduced in [11], known as h-convex function (h-CF). Several researchers investigated different variants of standard inequalities for this family (see [12–16]). In [17,18], Niculescu introduced another type of convexity, known as GA-convexity and studied its properties. Several inequalities, such as HHI, the Fejer-type integral inequalities and JMI, have been established for GA-convex functions (GA-CFs) and their generalizations such as GA-s-convex functions (GA-s-CFs) and GA-h-convex functions (GA-h-CF) [19–23]. The notion of GA-h-CF, introduced in [23], was studied extensively, as it is a generalized class of functions and contains the classes of GA-CFs and GA-s-CFs. Moreover, in [24], the notion of the Hadamard fractional integral (HFI) was introduced. Many researchers studied the convexity and fractional operator-based inequalities (see [25–29]). The study of JMI, HH-JMI, Jensen–Mercer and Hermite–Mercer-type inequalities (for different families of convex functions) and the use of fractional integral operators, have been among the modern trends in the area of mathematical inequlaities (see [30–34]). By keeping in view the significance of the modern trends toward the above-mentioned inequalities, convexities and fractional integral operators, in this paper, we study several inequalities for GA-h-convex functions and their subclasses, including GA-convex functions, GA-s-convex functions, GA-Q-convex functions, and GA-P-convex functions. We prove the Jensen–Mercer inequality for GA-h-convex functions and give weighted Hermite–Hadamard inequalities by applying the newly proven Jensen–Mercer inequality. We also establish inequalities of Hermite–Hadamard–Mercer type. Furthermore, we apply our main results along with Hadamard fractional integrals to prove weighted Hermite–Hadamard–Mercer inequalities for GA-h-convex functions and its subclasses. As a special case of the proven results, we capture several well-known results from [20,21,25,27].

Before proceeding further, we denote by J an interval $[\alpha, \beta]$ with $\alpha < \beta$ (unless mentioned otherwise) $h - CF(J)$, $GA - h - CF(J)$, $GA - s - CF(J)$, $GA - Q - CF(J)$, and $GA - P - CF(J)$ the class of h-convex functions, GA-h-convex functions, GA-s-convex functions, GA-Q-convex functions, and GA-P-convex functions on J, respectively. For other abbreviations, see the table at the end.

Now, we include necessary notions, connection between these notions and corresponding inequalities (see [2,3]). We start the section by including well-known notion of CF.

Definition 1 ([2])**.** *A function $\psi : J \to \Re$ is said to be CF if for each $0 \leq \lambda \leq 1$ and $u, v \in J$, we have*

$$\psi(\lambda u + (1-\lambda)v) \leq \lambda\psi(u) + (1-\lambda)\psi(v).$$

The HHI for CF was proven in [2].

Theorem 1. *For a CF $\psi : J \to \Re$,*

$$\psi\left(\frac{\alpha_1 + \beta_1}{2}\right) \leq \frac{1}{\beta_1 - \alpha_1}\int_{\alpha_1}^{\beta_1} \phi(u)du \leq \frac{\psi(\alpha_1) + \psi(\beta_1)}{2} \quad (1)$$

holds, where $\alpha_1, \beta_1 \in J$ and $\alpha_1 < \beta_1$.

The inequality in (1) is among the most studied inequalities, as its several variants have been proven for diverse classes of functions including CF, s-CF, Q-CF, P-CF, and h-CF (see [6,7,9–11]). Some of these are included in the sequel.

Definition 2 ([11]). *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \to \Re$, ψ is said to be h-CF, if*

$$\psi(\lambda u + (1-\lambda)v) \le h(\lambda)\psi(u) + h(1-\lambda)\psi(v)$$

for all $u, v \in J$ and $0 \le \lambda \le 1$.

The following version of (1) for h-CF was proven in [12].

Theorem 2. *Let $h : [0,1] \to [0,\infty)$ with $h(\frac{1}{2}) \neq 0$ and $\psi : J \to \Re$ be a h-CF. Then, for any $\alpha_1, \beta_1 \in J$ with $\alpha_1 < \beta_1$, we have*

$$\frac{1}{2h(\frac{1}{2})}\psi\left(\frac{\alpha_1+\beta_1}{2}\right) \le \frac{1}{\beta_1-\alpha_1}\int_{\alpha_1}^{\beta_1}\psi(u)du \le [\psi(\alpha_1)+\psi(\beta_1)]\int_0^1 h(t)dt.$$

Definition 3 ([17,18]). *A function $\psi : J \subseteq (0,\infty) \to \Re$ is called GA-CF, if*

$$\psi\left(u^{\lambda}v^{1-\lambda}\right) \le \lambda\psi(u) + (1-\lambda)\psi(v)$$

holds for any $0 \le \lambda \le 1$ and $u, v \in J$.

The following variant of (1) for GA-CF was proven in [19].

Theorem 3. *For any GA-CF $\psi : J \to \Re$, and for any $\alpha_1, \beta_1 \in J$ with $\alpha_1 < \beta_1$, we have*

$$\psi\left(\sqrt{\alpha_1\beta_1}\right) \le \frac{1}{\ln\beta_1 - \ln\alpha_1}\int_{\alpha_1}^{\beta_1}\frac{\psi(u)}{u}dx \le \frac{\psi(\alpha_1)+\psi(\beta_1)}{2}.$$

Some classes related to the GA-convex function and the corresponding analogue of (1) for these classes of functions have been proven in [22,23]. We only recall the following definitions.

Definition 4. *A function $\psi : J \subseteq (0,\infty) \to [0,\infty)$ is called*

- *GA-s-CF, if $\psi\left(u^{\lambda}v^{1-\lambda}\right) \le (\lambda)^s\psi(u) + (1-\lambda)^s\psi(v)$;*
- *GA-Q-CF, if $\psi\left(u^{\lambda}v^{1-\lambda}\right) \le \frac{\psi(u)}{\lambda} + \frac{\psi(v)}{1-\lambda}$; and*
- *GA-P-CF, if $\psi\left(u^{\lambda}v^{1-\lambda}\right) \le \psi(u) + \psi(v)$*

hold for any $\lambda \in [0,1]$ ($\lambda \neq 0, 1$ for GA-Q-CF) , $u, v \in J$ and $s \in (0,1]$.

Definition 5. *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \subseteq (0,\infty) \to [0,\infty)$. The function ψ is called GA-h-CF if*

$$\psi(u^{\lambda}v^{1-\lambda}) \le h(\lambda)\psi(u) + h(1-\lambda)\psi(v), \tag{2}$$

for any $0 \le \lambda \le 1$ and $u, v \in J$.

The following remark [23] demonstrates the relationship between GA-h-CF with GA-s-CF, GA-Q-CF, and GA-P-CF.

Remark 1. *Under this assumption, when $h(\lambda) = \lambda$ ($h(\lambda) = \lambda^s$, $h(\lambda) = \frac{1}{\lambda}$ or $h(\lambda) = 1$), the GA-h-CF satisfies the inequality required for GA-CF (GA-s-CF, GA-Q-CF or GA-P-CF).*

The inequality (2) is equivalent to $(\psi \circ \exp)(\lambda \ln u + (1-\lambda)\ln v) \le h(\lambda)(\psi \circ \exp)(\ln u)$ $+ h(1-\lambda)(\psi \circ \exp)(\ln v)$, which concludes that the necessary and sufficient condition for $\psi : J \to [0,\infty)$ to be GA-h-CF is that $(\psi \circ \exp)$ is h-CF on $\ln J$. Similarly, in particular, equivalent conditions in exponential form for the notions of GA-s, GA-Q and GA-P-CFs can be produced as well.

Now, we recall a variant of (1) for GA-h-CF from [23]:

Theorem 4. *Let $h:[0,1]\to[0,\infty)$ and $\psi:J\to[0,\infty)$. If $\psi\in GA-h-CF(J)$, then*

$$\frac{1}{2h(\frac{1}{2})}\psi\left(\sqrt{\alpha_1\beta_1}\right)\leq\frac{1}{\ln\beta_1-\ln\alpha_1}\int_{\alpha_1}^{\beta_1}\frac{\psi(u)}{u}dx\leq[\psi(\alpha_1)+\psi(\beta_1)]\int_0^1 h(z)dz$$

holds for $\alpha_1<\beta_1$ and $\alpha_1,\beta_1\in J$, where $h(\frac{1}{2})\neq 0$.

The Jensen's inequality (JI) for *h*-CF was proven in [13].

Theorem 5. *Let $h:[0,1]\to[0,\infty)$ and $\psi:J\to\Re$ be two functions. Then $\psi\in h-CF(J)$ if and only if*

$$\psi\left(\sum_{i=1}^{n}\lambda_i v_i\right)\leq\sum_{i=1}^{n}h(\lambda_i)\psi(v_i) \tag{3}$$

for any $v_i\in J$, $\lambda_i\geq 0$, $i=1,2,...,n$ and $\sum_{i=1}^{n}\lambda_i=1$.

Similarly, one may prove the following analogue of JI for GA-*h*-CF.

Theorem 6. *Let $h:[0,1]\to[0,\infty)$ and $\psi:J\to[0,\infty)$ be two functions. Then $\psi\in GA-h-CF(J)$ if and only if*

$$\psi\left(\prod_{i=1}^{n}v_i^{\lambda_i}\right)\leq\sum_{i=1}^{n}h(\lambda_i)\psi(v_i) \tag{4}$$

for all $v_i\in J$, $\lambda_i\geq 0$, $i=1,2,...,n$ and $\sum_{i=1}^{n}\lambda_i=1$.

As a special case, the Theorem 6 also yields the JIs for the functions in $GA-s-CF(J)$, $GA-Q-CF(J)$ and $GA-P-CF(J)$.

The following Jensen–Mercer inequality (JMI) for *h*-CF was proven in [15].

Theorem 7. *For $\psi\in h-CF(J)$, we have*

$$\psi\left(\alpha+\beta-\sum_{i=1}^{n}\lambda_i v_i\right)\leq\sum_{i=1}^{n}h(\lambda_i)[h(\lambda)+h(1-\lambda)][\psi(\alpha)+\psi(\beta)]-\sum_{i=1}^{n}h(\lambda_i)\psi(v_i), \tag{5}$$

where $v_i\in J$ and $\lambda_i\in[0,1]$ with $\sum_{i=1}^{n}\lambda_i=1$.

We conclude the section by including vastly studied fractional integrals, known as Hadamard fractional integrals (HFIs) [24].

Definition 6. *Hadamard Fractional Integrals. Let a integrable function ψ in $L[u,v]$, with $u,v\geq 0$ and $u<v$, then*
the left side of HFIs of order $\lambda>0$ are defined by

$$J_{v-}^{\lambda}\psi(x)=\frac{1}{\Gamma(\lambda)}\int_x^v(\ln\frac{z}{x})^{\lambda-1}\psi(z)dz.\quad x<v.$$

The right side of the HFIs of order $\lambda>0$ are defined by

$$J_{u+}^{\lambda}\psi(x)=\frac{1}{\Gamma(\lambda)}\int_u^x(\ln\frac{x}{z})^{\lambda-1}\psi(z)dz.\quad u<x.$$

2. The Jensen–Mercer Inequality for GA-*h*-Convex Functions and Its Subclasses

The current section is devoted to proof of Jensen–Mercer inequalities (JMIs) for GA-*h*-CF. Consequently, we acquire the Jensen–Mercer inequality (JMI) for GA-*s*-CF, GA-*Q*-CF, and GA-*P*-CF as well. We begin with the proof of lemma first.

Lemma 1. *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \subseteq (0,\infty) \to [0,\infty)$ be a GA-h-CF. Then,*

$$\psi\left(\frac{\alpha\beta}{u}\right) \le [h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \psi(u) \tag{6}$$

holds for any $u \in J$ and $\lambda \in [0,1]$.

Proof. For any $u \in J = [\alpha, \beta]$, then there exist $\lambda \in [0,1]$ with $u = \alpha^{\lambda}\beta^{1-\lambda}$, and we write $\frac{\alpha\beta}{u} = \alpha^{1-\lambda}\beta^{\lambda}$. By definition of ψ,

$$\psi\left(\frac{\alpha\beta}{u}\right) = \psi\left(\alpha^{1-\lambda}\beta^{\lambda}\right) \le h(1-\lambda)\psi(\alpha) + h(\lambda)\psi(\beta).$$

By adding and subtracting $h(\lambda)\psi(\alpha)$ and $h(1-\lambda)\psi(\beta)$, we have

$$\begin{aligned}
\psi\left(\frac{\alpha\beta}{u}\right) &= h(\lambda)\psi(\alpha) + h(\lambda)\psi(\beta) - h(\lambda)\psi(\alpha) - h(1-\lambda)\psi(\beta) + h(1-\lambda)\psi(\beta) + h(1-\lambda)\psi(\alpha) \\
&= h(\lambda)\psi(\alpha) + h(\lambda)\psi(\beta) - [h(\lambda)\psi(\alpha) + h(1-\lambda)\psi(\beta)] + h(1-\lambda)\psi(\alpha) + h(1-\lambda)\psi(\beta) \\
&\le [h(\lambda) + h(1-\lambda)]\psi(\alpha) + [h(\lambda) + h(1-\lambda)]\psi(\beta) - \psi\left(\alpha^{\lambda}\beta^{1-\lambda}\right) \\
&= [h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \psi(u).
\end{aligned}$$

Thus, we get $\psi\left(\frac{\alpha\beta}{u}\right) \le [h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \psi(u)$, which completes the proof. □

Now we prove JMI for GA-*h*-CF by using Lemma 1.

Theorem 8. *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \subseteq (0,\infty) \to [0,\infty)$ be two functions. If ψ is GA-h-CF, then*

$$\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) \le \sum_{i=1}^{n} h(\lambda_i)[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} h(\lambda_i)\psi(v_i)$$

holds for any $v_i \in J$, $0 \le \lambda_i \le 1$ with $\sum_{i=1}^{n} \lambda_i = 1$.

Proof. First Method. By Equation (4), we have

$$\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) = \psi\left(\prod_{i=1}^{n}\left(\frac{\alpha\beta}{v_i}\right)^{\lambda_i}\right) \le \sum_{i=1}^{n} h(\lambda_i)\psi\left(\frac{\alpha\beta}{v_i}\right).$$

By inequality (6) and Lemma 1, we get

$$\begin{aligned}
\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) &\le \sum_{i=1}^{n} h(\lambda_i)[[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \psi(v_i)] \\
&= \sum_{i=1}^{n} h(\lambda_i)[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} h(\lambda_i)\psi(v_i) \\
\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) &\le \sum_{i=1}^{n} h(\lambda_i)[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} h(\lambda_i)\psi(v_i),
\end{aligned}$$

which was required.

Second Method. Because $\psi : J = [\alpha, \beta] \to [0, \infty)$ is GA-h-CF, therefore $(\psi \circ \exp)$ is h-CF on $\ln([\alpha, \beta])$. Consequently, by Equation (5), we get

$$(\psi \circ \exp)\left(\ln(\alpha) + \ln(\beta) - \sum_{i=1}^{n} \lambda_i \ln(v_i)\right) = \psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right)$$
$$\leq \sum_{i=1}^{n} h(\lambda_i)[h(\lambda) + h(1-\lambda)][(\psi \circ \exp)\ln(\alpha) + (\psi \circ \exp)\ln(\beta)] - \sum_{i=1}^{n} h(\lambda_i)(\psi \circ \exp)\ln(v_i)$$
$$= \sum_{i=1}^{n} h(\lambda_i)[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} h(\lambda_i)\psi(v_i)$$
$$\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) \leq \sum_{i=1}^{n} h(\lambda_i)[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} h(\lambda_i)\psi(v_i).$$

□

The following consequences of Theorem 8 provide JMIs for the subclasses of GA-h-CF.

Corollary 1. *Let $\psi : J \subseteq (0, \infty) \to [0, \infty)$ be a GA-s-CF. Then, for any $v_i \in J$, $0 \leq \lambda_i \leq 1$ with $\sum_{i=1}^{n} \lambda_i = 1$,*

$$\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) \leq \sum_{i=1}^{n} (\lambda_i)^s [(\lambda)^s + (1-\lambda)^s][\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} (\lambda_i)^s \psi(v_i)$$

holds.

Corollary 2. *Let $\psi : J \subseteq (0, \infty) \to [0, \infty)$ be a GA-Q-CF. Then, for any $v_i \in J$, $0 < \lambda_i < 1$ with $\sum_{i=1}^{n} \lambda_i = 1$,*

$$\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) \leq \sum_{i=1}^{n} \left(\frac{1}{\lambda_i}\right) \left[\frac{1}{\lambda} + \frac{1}{1-\lambda}\right] [\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} \left(\frac{1}{\lambda_i}\right) \psi(v_i)$$

holds.

Corollary 3. *Let $\psi : J \subseteq (0, \infty) \to [0, \infty)$ be a GA-P-CF. Then, for any $v_i \in J$, $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$,*

$$\psi\left(\frac{\alpha\beta}{\prod_{i=1}^{n} v_i^{\lambda_i}}\right) \leq 2[\psi(\alpha) + \psi(\beta)] - \sum_{i=1}^{n} \psi(v_i)$$

holds.

3. Generalized Weighted Hermite–Hadamard–Mercer Inequalities for GA-*h*-Convex Functions and Its Subclasses

The current section is devoted to establishing the main results of the manuscript and developing connections with the inequalities in the recent literature. First, we prove the generalized weighted Hermite–Hadamard–Mercer inequality (wHHMI) for GA-h-CFs. The special cases of the proven results coincide with the inequalities proven as main results in [20,21]. Before proving the main theorem, we fix the notation. We emphasize again, in the sequel, that we denote by $L(J) = L[\alpha, \beta]$ the class of integrable functions on $J = [\alpha, \beta]$.

Theorem 9. *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \to [0,\infty)$ with $\psi \in L(J)$. If ψ is GA-h-CF, then for any nonnegative and integrable function $g : J \to \Re$, we have*

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz &\leq \frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g(z)\frac{dz}{z}+\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\}\\
&\leq \frac{1}{2}\left[\psi(\frac{\alpha\beta}{u})+\psi(\frac{\alpha\beta}{v})\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]g(z)\frac{dz}{z}\\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]^2 g(z)\frac{dz}{z}\\
&\quad-\left[\frac{\psi(u)+\psi(v)}{2}\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]g(z)\frac{dz}{z}
\end{aligned} \tag{7}$$

for all $u,v \in J$, where $h\left(\frac{1}{2}\right) \neq 0$.

Proof. The GA-h-convexity of ψ implies

$$\begin{aligned}
\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right) &= \psi\left(\sqrt{\left[\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right]\left[\left(\frac{\alpha\beta}{u}\right)^{1-\lambda}\left(\frac{\alpha\beta}{v}\right)^{\lambda}\right]}\right)\\
&\leq h\left(\frac{1}{2}\right)\psi\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right)+h\left(\frac{1}{2}\right)\psi\left(\left(\frac{\alpha\beta}{u}\right)^{1-\lambda}\left(\frac{\alpha\beta}{v}\right)^{\lambda}\right)\\
&= h\left(\frac{1}{2}\right)\left[\psi\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right)+\psi\left(\left(\frac{\alpha\beta}{u}\right)^{1-\lambda}\left(\frac{\alpha\beta}{v}\right)^{\lambda}\right)\right].
\end{aligned}$$

Thus, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right) \leq \left[\psi\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right)+\psi\left(\left(\frac{\alpha\beta}{u}\right)^{1-\lambda}\left(\frac{\alpha\beta}{v}\right)^{\lambda}\right)\right] \tag{8}$$

for any $u,v \in I$ and $0 \leq \lambda \leq 1$ and $h\left(\frac{1}{2}\right) \neq 0$.

Now, by multiplying (8) with $\frac{1}{2}g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right)$, integrating with respect to λ over [0,1] and by applying substitution with $z = \left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}$, we get

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz \leq \frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g(z)\frac{dz}{z}+\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\},$$

which yields the first inequality of (7). To obtain the second inequality of (7), the definition of ψ gives

$$\psi\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) \leq h(\lambda)\psi\left(\frac{\alpha\beta}{u}\right)+h(1-\lambda)\psi\left(\frac{\alpha\beta}{v}\right) \tag{9}$$

and

$$\psi\left(\left(\frac{\alpha\beta}{u}\right)^{1-\lambda}\left(\frac{\alpha\beta}{v}\right)^{\lambda}\right) \leq h(1-\lambda)\psi\left(\frac{\alpha\beta}{u}\right) + h(\lambda)\psi\left(\frac{\alpha\beta}{v}\right). \tag{10}$$

By adding (9) and (10), we get

$$\psi\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) + \psi\left(\left(\frac{\alpha\beta}{u}\right)^{1-\lambda}\left(\frac{\alpha\beta}{v}\right)^{\lambda}\right) \leq [h(\lambda)+h(1-\lambda)]\left[\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right)\right]. \tag{11}$$

By multiplying (11) with $\frac{1}{2}g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right)$ and integrating w.r.t. λ over 0 to 1, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \psi\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) d\lambda \\
&\quad + \frac{1}{2}\int_0^1 \psi\left(\left(\frac{\alpha\beta}{u}\right)^{1-\lambda}\left(\frac{\alpha\beta}{v}\right)^{\lambda}\right) g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) d\lambda \\
&\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right)\right]\int_0^1 [h(\lambda)+h(1-\lambda)] g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) d\lambda.
\end{aligned} \tag{12}$$

By substituting $\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda} = z$ in (12), we get

$$\begin{aligned}
&\frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}} \psi(z)g(z)\frac{dz}{z} + \int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}} \psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\} \\
&\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right) + h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right] g(z)\frac{dz}{z},
\end{aligned}$$

which gives the second inequality in (7). For the remaining inequality in (7), the inequality (6) gives

$$\psi\left(\frac{\alpha\beta}{u}\right) \leq [h(\lambda)+h(1-\lambda)][\psi(\alpha)+\psi(\beta)] - \psi(u), \tag{13}$$

and

$$\psi\left(\frac{\alpha\beta}{v}\right) \leq [h(\lambda)+h(1-\lambda)][\psi(\alpha)+\psi(\beta)] - \psi(v). \tag{14}$$

Now, by adding (13) and (14), we have

$$\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right) \leq 2[h(\lambda)+h(1-\lambda)][\psi(\alpha)+\psi(\beta)] - [\psi(u)+\psi(v)]. \tag{15}$$

By multiplying (15) with $\frac{1}{2}[h(\lambda)+h(1-\lambda)]g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right)$, we get

$$\begin{aligned}
&\frac{1}{2}\left\{\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right)\right\}[h(\lambda)+h(1-\lambda)]g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) \\
&\leq [\psi(\alpha)+\psi(\beta)][h(\lambda)+h(1-\lambda)]^2 g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right) \\
&\quad - \left[\frac{\psi(u)+\psi(v)}{2}\right][h(\lambda)+h(1-\lambda)]g\left(\left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}\right).
\end{aligned}$$

By integrating w.r.t. λ over 0 to 1 and by applying the substitution with $z = \left(\frac{\alpha\beta}{u}\right)^{\lambda}\left(\frac{\alpha\beta}{v}\right)^{1-\lambda}$, we get

$$\begin{aligned}
&\frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]g(z)\frac{dz}{z} \\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h(\frac{\ln(\frac{vz}{\alpha\beta})}{\ln(\frac{v}{u})})+h(\frac{\ln(\frac{\alpha\beta}{uz})}{\ln(\frac{v}{u})})\right]^2 g(z)\frac{dz}{z} \\
&\quad -\left[\frac{\psi(u)+\psi(v)}{2}\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]g(z)\frac{dz}{z},
\end{aligned}$$

which completes the proof. □

By taking $h(\lambda) = \lambda$ in the Theorem 9 we have the following.

Corollary 4. *Let ψ be a function from J to $[0,\infty)$, such that $\psi \in L(J)$. If ψ is GA-CF and $g : J \to [0,\infty)$ be integrable, then*

$$\begin{aligned}
\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz &\leq \frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g(z)\frac{dz}{z}+\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\} \\
&\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz \leq \left[\phi(\alpha)+\phi(\beta)-\frac{\psi(u)+\phi(v)}{2}\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz \qquad (16)
\end{aligned}$$

for all $u, v \in J$.

The inequality (16) is the same as the inequality in Theorem 2.2 of [21].
By taking $u = \alpha$ and $v = \beta$ in Theorem 9, we get the wHHI for GA-h-CF.

Corollary 5. *Assuming the conditions of Theorem 9, we have*

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\frac{g(z)}{z}dz &\leq \frac{1}{2}\left\{\int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z}+\int_{\alpha}^{\beta}\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}\right\} \\
&\leq \frac{1}{2}[\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)+h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]g(z)\frac{dz}{z} \\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)+h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]^2 g(z)\frac{dz}{z} \\
&\quad -\left[\frac{\psi(\alpha)+\psi(\beta)}{2}\right]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)+h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]g(z)\frac{dz}{z}. \qquad (17)
\end{aligned}$$

Corollary 6. *Under the assumption of Theorem 9 and assuming $g(\frac{\alpha\beta}{z}) = g(z)$ for any $z \in [\alpha, \beta]$ (that is g is geometrically symmetric), (17) implies:*

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\frac{g(z)}{z}dz &\leq \int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z} \\
&\leq \frac{1}{2}[\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)+h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]g(z)\frac{dz}{z} \\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)+h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]^2 g(z)\frac{dz}{z} \\
&\quad -\left[\frac{\psi(\alpha)+\psi(\beta)}{2}\right]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)+h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]g(z)\frac{dz}{z}
\end{aligned}$$

for all $u, v \in J$.

Remark 2. *1. By taking $h(\lambda) = \lambda$ in Corollary 5, we get*

$$\begin{aligned}
\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\frac{g(z)}{z}dz &\leq \frac{1}{2}\left\{\int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z}+\int_{\alpha}^{\beta}\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}\right\} \\
&\leq \frac{\psi(\alpha)+\psi(\beta)}{2}\int_{\alpha}^{\beta}g(z)\frac{dz}{z}
\end{aligned}$$

for all $u, v \in J$, which gives inequality (12) in [21].

2. By taking $h(\lambda) = \lambda$ in Corollary 6, we get

$$\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\frac{g(z)}{z}dz \leq \int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z} \leq \frac{\psi(\alpha)+\psi(\beta)}{2}\int_{\alpha}^{\beta}\frac{g(z)}{z}dz,$$

for all $u, v \in J$, which gives inequality of Theorem 2.2 in [20].

Now, we prove another main result of this section, which as a special case yields Theorem 2.3 in [21].

Theorem 10. *Let $h : [0, 1] \to [0, \infty)$ and $\psi : J \to [0, \infty)$ be two functions. If $\psi \in L(J)$ and ψ is GA-h-CF, then for any integrable $g : J \to [0, \infty)$, we have*

$$\begin{aligned}
\frac{1}{2h(\frac{1}{2})}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz &\leq \frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g(z)\frac{dz}{z}+\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\} \\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{u}^{v}\left[h\left(\frac{\ln(\frac{z}{v})}{\ln(\frac{u}{v})}\right)+h\left(\frac{\ln(\frac{u}{z})}{\ln(\frac{u}{v})}\right)\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \\
&\quad -\frac{1}{2}\left[\int_{u}^{v}\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}+\int_{u}^{v}\psi(z)g\left(\frac{\alpha\beta z}{uv}\right)\frac{dz}{z}\right] \\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{u}^{v}\left[h\left(\frac{\ln(\frac{z}{v})}{\ln(\frac{u}{v})}\right)+h\left(\frac{\ln(\frac{u}{z})}{\ln(\frac{u}{v})}\right)\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}-\frac{1}{2h(\frac{1}{2})}\psi(\sqrt{uv})\int_{u}^{v}g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \qquad (18)
\end{aligned}$$

for all $u, v \in J$ *and* $h\left(\frac{1}{2}\right) \neq 0$.

Proof. The first inequality in (18) has already been established in Theorem 9. For the second inequality in (18), by applying (6) and GA-h-convexity of ψ, we have

$$\psi((\frac{\alpha\beta}{u})^{\lambda}(\frac{\alpha\beta}{v})^{1-\lambda}) = \psi(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}) \leq [h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \psi(u^{\lambda}v^{1-\lambda}). \quad (19)$$

Then we have

$$\psi((\frac{\alpha\beta}{u})^{1-\lambda}(\frac{\alpha\beta}{v})^{\lambda}) = \psi(\frac{\alpha\beta}{u^{1-\lambda}v^{\lambda}}) \leq [h(1-\lambda) + h(\lambda)][\psi(\alpha) + \psi(\beta)] - \psi(u^{1-\lambda}v^{\lambda}). \quad (20)$$

By adding (19) and (20), we get

$$\psi\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right) + \psi\left(\frac{\alpha\beta}{u^{1-\lambda}v^{\lambda}}\right) \leq 2[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \left[\psi\left(u^{\lambda}v^{1-\lambda}\right) + \psi\left(u^{1-\lambda}v^{\lambda}\right)\right]. \quad (21)$$

By multiplying (21) with $\frac{1}{2}g\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)$ and integrating λ over 0 to 1, we get

$$\frac{1}{2}\left\{\int_0^1 \psi\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)g\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)d\lambda + \int_0^1 \psi\left(\frac{\alpha\beta}{u^{1-\lambda}v^{\lambda}}\right)g\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)d\lambda\right\}$$
$$\leq [\psi(\alpha) + \psi(\beta)]\int_0^1 [h(\lambda) + h(1-\lambda)]g\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)d\lambda$$
$$- \left[\int_0^1 \psi\left(u^{\lambda}v^{1-\lambda}\right)g\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)d\lambda + \int_0^1 \psi\left(u^{1-\lambda}v^{\lambda}\right)g\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)d\lambda\right]. \quad (22)$$

By applying substitution with $z = u^{\lambda}v^{1-\lambda}$, inequality (22) becomes

$$\frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}} \psi(z)g(z)\frac{dz}{z} + \int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}} \psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\}$$
$$\leq [\psi(\alpha) + \psi(\beta)]\int_u^v \left[h\left(\frac{\ln(\frac{z}{v})}{\ln(\frac{u}{v})}\right) + h\left(\frac{\ln(\frac{u}{z})}{\ln(\frac{u}{v})}\right)\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}$$
$$- \frac{1}{2}\left[\int_u^v \psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} + \int_u^v \psi(z)g\left(\frac{\alpha\beta z}{uv}\right)\frac{dz}{z}\right].$$

Now, for the last inequality, the GA-h-convexity of ψ yields

$$\psi(\sqrt{uv}) = \psi\left(\sqrt{(u^{\lambda}v^{1-\lambda})(u^{1-\lambda}v^{\lambda})}\right) \leq h(\frac{1}{2})\left[\psi\left(u^{\lambda}v^{1-\lambda}\right) + \psi\left(u^{1-\lambda}v^{\lambda}\right)\right].$$

By adding both side $2[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)]$, we get

$$2[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \left[\psi\left(u^{\lambda}v^{1-\lambda}\right) + \psi\left(u^{1-\lambda}v^{\lambda}\right)\right],$$
$$\leq 2[h(\lambda) + h(1-\lambda)][\psi(\alpha) + \psi(\beta)] - \frac{1}{h(\frac{1}{2})}\psi(\sqrt{uv}). \quad (23)$$

By multiplying (23) with $\frac{1}{2}g\left(\frac{\alpha\beta}{u^{\lambda}v^{1-\lambda}}\right)$ and integrating it with λ over 0 to 1, we get

$$[\psi(\alpha)+\psi(\beta)]\int_u^v\left[h\left(\frac{\ln\left(\frac{z}{v}\right)}{\ln\left(\frac{u}{v}\right)}\right)+h\left(\frac{\ln\left(\frac{u}{z}\right)}{\ln\left(\frac{u}{v}\right)}\right)\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}$$
$$-\frac{1}{2}\left[\int_u^v\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}+\int_u^v\psi(z)g\left(\frac{\alpha\beta z}{uv}\right)\frac{dz}{z}\right]$$
$$\leq[\psi(\alpha)+\psi(\beta)]\int_u^v\left[h\left(\frac{\ln\left(\frac{z}{v}\right)}{\ln\left(\frac{u}{v}\right)}\right)+h\left(\frac{\ln\left(\frac{u}{z}\right)}{\ln\left(\frac{u}{v}\right)}\right)\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}-\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\sqrt{uv}\right)\int_u^v g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z},$$

which was required. □

Corollary 7. *Let ψ be a function from J to $[0,\infty)$, such that $\psi\in L(J)$. If ψ is GA-CF on J, and then for any integrable $g:J\to[0,\infty)$, we get*

$$\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz\leq\frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g(z)\frac{dz}{z}+\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\}$$
$$\leq\frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_u^v\frac{g(z)}{z}dz-\frac{1}{2}\left\{\int_u^v\psi(z)g(z)\frac{dz}{z}+\int_u^v\psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\}$$
$$\leq\left[\psi(\alpha)+\psi(\beta)-\psi\left(\sqrt{uv}\right)\right]\int_u^v\frac{g(z)}{z}dz \quad (24)$$

for all $u,v\in J$.

Proof. This follows immediately by taking $h(\lambda)=\lambda$ in Theorem 10. □

The inequality in Corollary 7 coincides with Theorem 2.3 in [21]. Now, by taking the special case as $u=\alpha$ and $v=\beta$ in Theorem 10, we get the following w-HH inequality for GA-h-CF.

Corollary 8. *Let $h:[0,1]\to[0,\infty)$ and $\psi:J\to[0,\infty)$ be two functions. If $\psi\in L(J)$ and ψ is GA-h-CF on J, then for any integrable $g:J\to[0,\infty)$, we have*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\sqrt{\alpha\beta}\right)\int_\alpha^\beta\frac{g(z)}{z}dz\leq\frac{1}{2}\left\{\int_\alpha^\beta\psi(z)g(z)\frac{dz}{z}+\int_\alpha^\beta\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}\right\}$$
$$\leq[\psi(\alpha)+\psi(\beta)]\int_\alpha^\beta\left[h\left(\frac{\ln\left(\frac{z}{\beta}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha}{z}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}$$
$$-\frac{1}{2}\left[\int_\alpha^\beta\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}+\int_\alpha^\beta\psi(z)g(z)\frac{dz}{z}\right]$$
$$\leq[\psi(\alpha)+\psi(\beta)]\int_\alpha^\beta\left[h\left(\frac{\ln\left(\frac{z}{\beta}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha}{z}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}$$
$$-\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\sqrt{\alpha\beta}\right)\int_\alpha^\beta g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \quad (25)$$

for all $u,v\in J$ and $h\left(\frac{1}{2}\right)\neq 0$.

Special Cases for GA-s-Convex (GA-Q-Convex and GA-P-Convex) Functions

In the current subsection, we obtained the studied inequalities for the subclasses $GA-s-CF(J)$, $GA-Q-CF(J)$ and $GA-P-CF(J)$ of $GA-h-CF(J)$. We start with the next theorem.

Theorem 11. *Let ψ be a function from J to $[0,\infty)$ with $\psi \in L(J)$. If $\psi \in GA-s-CF(J)$, then for any nonnegative and integrable $g : J \to \Re$, we have*

$$\begin{aligned}
2^{s-1}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz &\leq \frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g(z)\frac{dz}{z}+\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g\left(\frac{(\alpha\beta)^2}{uvz}\right)\frac{dz}{z}\right\}\\
&\leq \frac{1}{2}\left[\psi(\frac{\alpha\beta}{u})+\psi(\frac{\alpha\beta}{v})\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)^s+\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)^s\right]g(z)\frac{dz}{z}\\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)^s+\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)^s\right]^2 g(z)\frac{dz}{z}\\
&\quad-\left[\frac{\psi(u)+\psi(v)}{2}\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)^s+\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)^s\right]g(z)\frac{dz}{z} \qquad (26)
\end{aligned}$$

for all $u, v \in J$.

Proof. If we take $h(\lambda) = (\lambda)^s$, where $s \in (0,1]$. Then, the inequality (7) in Theorem 9 yields required the inequality (26). □

Furthermore, if we assume $u = \alpha$ and $v = \beta$ in Theorem 11, we obtain the wHHI for GA-s-CF.

Corollary 9. *Let ψ be a function from J to $[0,\infty)$ with $\psi \in L(J)$. If $\psi \in GA-s-CF(J)$, then for any nonnegative and integrable $g : J \to \Re$, we have*

$$\begin{aligned}
2^{s-1}\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\frac{g(z)}{z}dz &\leq \frac{1}{2}\{\int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z}+\int_{\alpha}^{\beta}\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}\}\\
&\leq \frac{1}{2}[\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[\left(\frac{\ln(\frac{z}{\alpha})}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^s+\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^s\right]g(z)\frac{dz}{z}\\
&\leq [\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[\left(\frac{\ln(\frac{z}{\alpha})}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^s+\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^s\right]^2 g(z)\frac{dz}{z}\\
&\quad-\left[\frac{\psi(\alpha)+\psi(\beta)}{2}\right]\int_{\alpha}^{\beta}\left[\left(\frac{\ln(\frac{z}{\alpha})}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^s+\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^s\right]g(z)\frac{dz}{z} \qquad (27)
\end{aligned}$$

for all $u, v \in J$.

Remark 3. *If we take g as $g(\frac{\alpha\beta}{z}) = g(z)$ for any $z \in [\alpha, \beta]$ in inequality (27), we get the following inequality, and, furthermore, by taking $h(\lambda) = \lambda$ we get the inequalities from [20].*

$$
\begin{aligned}
2^{s-1}\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\frac{g(z)}{z}dz &\le \int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z} \\
&\le \frac{1}{2}[\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^{s}+\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^{s}\right]g(z)\frac{dz}{z} \\
&\le [\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^{s}+\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^{s}\right]^{2}g(z)\frac{dz}{z} \\
&\quad -\left[\frac{\psi(\alpha)+\psi(\beta)}{2}\right]\int_{\alpha}^{\beta}\left[\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^{s}+\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)^{s}\right]g(z)\frac{dz}{z}
\end{aligned}
$$

for all $u, v \in J$.

Now, we present the consequences of Theorem 10 for the class of GA-s-CF to establish the inequalities of type [20].

Theorem 12. *Let ψ be a function from J to $[0,\infty)$ with $\psi \in L(J)$. If ψ is GA-s-CF, then for any nonnegative and integrable $g : J \to \Re$, it follows that*

$$
\begin{aligned}
2^{s-1}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\frac{g(z)}{z}dz &\le \frac{1}{2}\left\{\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g(z)\frac{dz}{z}+\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\psi(z)g\left(\frac{(\alpha\beta)^{2}}{uvz}\right)\frac{dz}{z}\right\} \\
&\le [\psi(\alpha)+\psi(\beta)]\int_{u}^{v}\left[\left(\frac{\ln\left(\frac{z}{v}\right)}{\ln\left(\frac{u}{v}\right)}\right)^{s}+\left(\frac{\ln\left(\frac{u}{z}\right)}{\ln\left(\frac{u}{v}\right)}\right)^{s}\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \\
&\quad -\frac{1}{2}\left[\int_{u}^{v}\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}+\int_{u}^{v}\psi(z)g\left(\frac{\alpha\beta z}{uv}\right)\frac{dz}{z}\right] \\
\le [\psi(\alpha)+\psi(\beta)]\int_{u}^{v}&\left[\left(\frac{\ln\left(\frac{z}{v}\right)}{\ln\left(\frac{u}{v}\right)}\right)^{s}+\left(\frac{\ln\left(\frac{u}{z}\right)}{\ln\left(\frac{u}{v}\right)}\right)^{s}\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}-2^{s-1}\psi(\sqrt{uv})\int_{u}^{v}g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \qquad (28)
\end{aligned}
$$

holds for all $u, v \in J$.

Proof. If we take $h(\lambda) = (\lambda)^{s}$, where $s \in (0,1]$ the GA-h-CF becomes GA-s-CF. Consequently, the inequality (18) in Theorem 10 becomes the required inequality (28). □

We get the following w-HH inequality for GA-s-CF, if we take $u = \alpha$ and $v = \beta$ in Theorem 12.

Corollary 10. *Let ψ be a function from J to $[0,\infty)$ with $\psi \in L(J)$. If ψ is GA-s-CF, then for any nonnegative and integrable $g : J \to \Re$*

$$
\begin{aligned}
2^{s-1}\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}\frac{g(z)}{z}dz &\le \frac{1}{2}\left\{\int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z}+\int_{\alpha}^{\beta}\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}\right\} \\
&\le [\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[\left(\frac{\ln\left(\frac{z}{\beta}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)^{s}+\left(\frac{\ln\left(\frac{\alpha}{z}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)^{s}\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \\
&\quad -\frac{1}{2}\left[\int_{\alpha}^{\beta}\psi(z)g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}+\int_{\alpha}^{\beta}\psi(z)g(z)\frac{dz}{z}\right]
\end{aligned}
$$

$$\leq [\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[\left(\frac{\ln\left(\frac{z}{\beta}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)^{s}+\left(\frac{\ln\left(\frac{\alpha}{z}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)^{s}\right]g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} - 2^{s-1}\psi\left(\sqrt{\alpha\beta}\right)\int_{\alpha}^{\beta}g\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \quad (29)$$

holds for all $u, v \in J$.

Remark 4. *Similarly, one may employ the conditions* $h(\lambda) = 1$ ($h(\lambda) = \frac{1}{\lambda}$, *where* $\lambda \in (0,1)$), *to get the consequence of Theorem 9, Theorem 10 and related results for GA-P-convex (GA-Q-convex) functions.*

4. Hermite–Hadamard–Mercer Inequality for GA-*h*-Convex Functions via Hadamard Fractional Integrals (HFI)

If we assume $g(z) = \frac{1}{\Gamma(\lambda)}\ln^{\lambda-1}\left(\frac{\alpha\beta}{zu}\right)$ or $g(z) = \frac{1}{\Gamma(\lambda)}\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)$ in Theorem 9, then we obtain the following HHMI for GA-*h*-CG via HFI.

Theorem 13. *Let* $h : [0,1] \to [0,\infty)$ *and* $\psi : J \to [0,\infty)$ *be two functions such that* $\psi \in L(J)$. *If* $\psi \in GA-h-CF(J)$, *and then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right) \leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{v}{u}\right)}\left\{J^{\lambda}_{\frac{\alpha\beta}{v}+}\psi\left(\frac{\alpha\beta}{u}\right)+J^{\lambda}_{\frac{\alpha\beta}{u}-}\psi\left(\frac{\alpha\beta}{v}\right)\right\}$$
$$\leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{v}{u}\right)}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}dz$$
$$\leq \frac{\Gamma(\lambda+1)}{\ln^{\lambda}\left(\frac{v}{u}\right)}[\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]^{2}\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}dz$$
$$-\frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{v}{u}\right)}[\psi(u)+\psi(v)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}dz \quad (30)$$

holds for all $u, v \in J$ *and* $\lambda > 0$, *and* $h(\frac{1}{2}) \neq 0$. *Furthermore, if we take* $\lambda = 1$ *in (30), then we get*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right) \leq \frac{1}{\ln\left(\frac{v}{u}\right)}\int_{u}^{v}\psi\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z}$$
$$\leq \frac{1}{2\ln\left(\frac{v}{u}\right)}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{dz}{z}$$
$$\leq \frac{1}{\ln\left(\frac{v}{u}\right)}[\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]^{2}\frac{dz}{z}$$
$$-\frac{1}{2\ln\left(\frac{v}{u}\right)}[\psi(u)+\psi(v)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{dz}{z}$$

for all $u, v \in J$ *with* $u < v$.

By taking $h(\lambda) = \lambda$ in (30), we get the following consequence of Theorem 13.

Corollary 11. *Let $\psi : J \to [0, \infty)$ be a function such that $\psi \in L[\alpha, \beta]$. If $\psi \in GA - CF(J)$, then*

$$\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right) \leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{v}{u}\right)}\left\{J^{\lambda}_{\frac{\alpha\beta}{v}+}\psi\left(\frac{\alpha\beta}{u}\right) + J^{\lambda}_{\frac{\alpha\beta}{u}-}\psi\left(\frac{\alpha\beta}{v}\right)\right\}$$
$$\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right)\right] \leq \psi(\alpha) + \psi(\beta) - \frac{\psi(u) + \psi(v)}{2}, \tag{31}$$

for all $u, v \in J$ and $\lambda > 0$. Furthermore, if we take $\lambda = 1$ in (31), then we get

$$\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right) \leq \frac{1}{\ln\left(\frac{v}{u}\right)}\int_{u}^{v}\psi\left(\frac{\alpha\beta}{z}\right)\frac{dz}{z} \leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right)\right]$$
$$\leq \psi(\alpha) + \psi(\beta) - \frac{\psi(u) + \psi(v)}{2}$$

for all $u, v \in J$ with $u < v$.

The inequalities in Corollary 11 coincide with inequalities of Corollary 2.2 in [21]. Furthermore, if we put $u = \alpha$ and $v = \beta$ in inequality (30), we get the following HHMI for GA-h-convex function via HFIs.

Corollary 12. *Let $\psi : J \to [0, \infty)$ with $\psi \in L(J)$. If $\psi \in GA - h - CF(J)$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\sqrt{\alpha\beta}\right) \leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{\alpha}{\beta}\right)}\left\{J^{\lambda}_{\alpha+}\psi(\beta) + J^{\lambda}_{\beta-}\psi(\alpha)\right\}$$
$$\leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{\beta}{\alpha}\right)}[\psi(\alpha) + \psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right) + h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{\beta}{z}\right)}{\Gamma(\lambda)z}dz$$
$$\leq \frac{\Gamma(\lambda+1)}{\ln^{\lambda}\left(\frac{\beta}{\alpha}\right)}[\psi(\alpha) + \psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right) + h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]^2\frac{\ln^{\lambda-1}\left(\frac{\beta}{z}\right)}{\Gamma(\lambda)z}dz$$
$$- \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{\beta}{\alpha}\right)}[\psi(\alpha) + \psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right) + h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{\beta}{z}\right)}{z\Gamma(\lambda)}dz$$

for all $u, v \in J$ and $\lambda > 0$ and $h\left(\frac{1}{2}\right) \neq 0$.

Remark 5. *If we take $h(\lambda) = \lambda$, we get*

$$\psi\left(\sqrt{\alpha\beta}\right) \leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{\alpha}{\beta}\right)}\left\{J^{\lambda}_{\alpha+}\psi(\beta) + J^{\lambda}_{\beta-}\psi(\alpha)\right\} \leq \frac{\psi(\alpha) + \psi(\beta)}{2}$$

for all $u, v \in J$, $\lambda > 0$, which gives the inequality of Theorem 2.1 from [25].

Now, if we put $g(z) = \frac{1}{\Gamma(\lambda)}\ln^{\lambda-1}\left(\frac{\alpha\beta}{zu}\right)$ or $g(z) = \frac{1}{\Gamma(\lambda)}\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)$ in Theorem 10, we get the following.

Theorem 14. *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \to [0,\infty)$ be two functions with $\psi \in L(J)$. If ψ is GA-h-CF, then*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right) &\leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{v}{u}\right)}\{J^{\lambda}_{\frac{\alpha\beta}{v}+}\psi\left(\frac{\alpha\beta}{u}\right) + J^{\lambda}_{\frac{\alpha\beta}{u}-}\psi\left(\frac{\alpha\beta}{v}\right)\} \\ &\leq \frac{\Gamma(\lambda+1)}{\ln^{\lambda}\left(\frac{v}{u}\right)}[\psi(\alpha)+\psi(\beta)]\int_u^v\left[h\left(\frac{\ln\left(\frac{z}{v}\right)}{\ln\left(\frac{u}{v}\right)}\right)+h\left(\frac{\ln\left(\frac{u}{z}\right)}{\ln\left(\frac{u}{v}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{z}{u}\right)}{z\Gamma(\lambda)}dz \\ &\quad - \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{v}{u}\right)}\left[J^{\lambda}_{v-}\psi(u)+J^{\lambda}_{u+}\psi(v)\right] \\ &\leq \frac{\Gamma(\lambda+1)}{\ln^{\lambda}\left(\frac{v}{u}\right)}[\psi(\alpha)+\psi(\beta)]\int_u^v\left[h\left(\frac{\ln\left(\frac{z}{v}\right)}{\ln\left(\frac{u}{v}\right)}\right)+h\left(\frac{\ln\left(\frac{u}{z}\right)}{\ln\left(\frac{u}{v}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{z}{u}\right)}{z\Gamma(\lambda)}dz - \frac{1}{2h(\frac{1}{2})}\psi(\sqrt{uv}) \quad (32)\end{aligned}$$

for all $u, v \in J$ and $\lambda > 0$ and $h(\frac{1}{2}) \neq 0$.

Corollary 13. *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \to [0,\infty)$ be two functions with $\psi \in L(J)$. If ψ is GA-h-CF, then*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}\psi\left(\sqrt{\alpha\beta}\right) &\leq \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{\beta}{\alpha}\right)}\left\{J^{\lambda}_{\alpha+}\psi(\beta)+J^{\lambda}_{\beta-}\psi(\alpha)\right\} \\ &\leq \frac{\Gamma(\lambda+1)}{\ln^{\lambda}\left(\frac{\beta}{\alpha}\right)}[\psi(\alpha)+\psi(\beta)]\int_\alpha^\beta\left[h\left(\frac{\ln\left(\frac{z}{\beta}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha}{z}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{z}{\alpha}\right)}{z\Gamma(\lambda)}dz \\ &\quad - \frac{\Gamma(\lambda+1)}{2\ln^{\lambda}\left(\frac{\beta}{\alpha}\right)}\left[J^{\lambda}_{\beta-}\psi(\alpha)+J^{\lambda}_{\alpha+}\psi(\beta)\right] \\ &\leq \frac{\Gamma(\lambda+1)}{\ln^{\lambda}\left(\frac{\beta}{\alpha}\right)}[\psi(\alpha)+\psi(\beta)]\int_\alpha^\beta\left[h\left(\frac{\ln\left(\frac{z}{\beta}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha}{z}\right)}{\ln\left(\frac{\alpha}{\beta}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{z}{\alpha}\right)}{z\Gamma(\lambda)}dz - \frac{1}{2h(\frac{1}{2})}\psi(\sqrt{\alpha\beta})\end{aligned}$$

for all $u, v \in J$ and $\lambda > 0$ and $h(\frac{1}{2}) \neq 0$.

Remark 6. *By following a similar pattern as that provided in the previous section, we may also acquire, as a special case, the results of this section for the classes of GA-s-convex (GA-Q-convex and GA-P-convex) functions.*

5. Weighted Hermite–Hadamard–Mercer Inequality for GA-*h*-Convex Functions via Hadamard Fractional Integrals.

Let $w : J = [\alpha,\beta] \to \Re$ be a non-negative and integrable function. If we take $g(z) = \frac{1}{\Gamma(\lambda)}\ln^{\lambda-1}\left(\frac{\alpha\beta}{zu}\right)w(z)$ and $g(z) = \frac{1}{\Gamma(\lambda)}\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)w(z)$ in Theorem 9, then we establish the following wHHMI for GA-*h*-CF via HFI.

Theorem 15. *Let $\psi : J \to [0,\infty)$ with $\psi \in L(J)$. If $\psi \in GA-h-CF(J)$ then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)J^{\lambda}_{\frac{\alpha\beta}{v}+}w(\frac{\alpha\beta}{u}) \leq \frac{1}{2}\left\{J^{\lambda}_{\frac{\alpha\beta}{v}+}\psi\left(\frac{\alpha\beta}{u}\right)+J^{\lambda}_{\frac{\alpha\beta}{u}-}\psi\left(\frac{\alpha\beta}{v}\right)\right\}$$
$$\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}w(z)dz$$
$$\leq [\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]^2\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}w(z)dz$$
$$-\left[\frac{\psi(u)+\psi(v)}{2}\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}w(z)dz \quad (33)$$

and

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)J^{\lambda}_{\frac{\alpha\beta}{u}-}w(\frac{\alpha\beta}{v}) \leq \frac{1}{2}\left\{J^{\lambda}_{\frac{\alpha\beta}{v}+}\psi\left(\frac{\alpha\beta}{u}\right)+J^{\lambda}_{\frac{\alpha\beta}{u}-}\psi\left(\frac{\alpha\beta}{v}\right)\right\}$$
$$\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)}{\Gamma(\lambda)z}w(z)dz$$
$$\leq [\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]^2\frac{\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)}{\Gamma(\lambda)z}w(z)dz$$
$$-\left[\frac{\psi(u)+\psi(v)}{2}\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\frac{\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)}{\Gamma(\lambda)z}w(z)dz. \quad (34)$$

By adding inequalities (33) and (34) we get

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\left[J^{\lambda}_{\frac{\alpha\beta}{v}+}w(\frac{\alpha\beta}{u})+J^{\lambda}_{\frac{\alpha\beta}{u}-}w(\frac{\alpha\beta}{v})\right] \leq \left\{J^{\lambda}_{\frac{\alpha\beta}{v}+}\psi\left(\frac{\alpha\beta}{u}\right)+J^{\lambda}_{\frac{\alpha\beta}{u}-}\psi\left(\frac{\alpha\beta}{v}\right)\right\}$$
$$\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right)+\psi\left(\frac{\alpha\beta}{v}\right)\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\left[\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}+\frac{\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)}{\Gamma(\lambda)z}\right]$$
$$w(z)dz$$
$$\leq [\psi(\alpha)+\psi(\beta)]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]^2\left[\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}+\frac{\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)}{\Gamma(\lambda)z}\right]w(z)dz$$
$$-\left[\frac{\psi(u)+\psi(v)}{2}\right]\int_{\frac{\alpha\beta}{v}}^{\frac{\alpha\beta}{u}}\left[h\left(\frac{\ln\left(\frac{vz}{\alpha\beta}\right)}{\ln\left(\frac{v}{u}\right)}\right)+h\left(\frac{\ln\left(\frac{\alpha\beta}{uz}\right)}{\ln\left(\frac{v}{u}\right)}\right)\right]\left[\frac{\ln^{\lambda-1}\left(\frac{\alpha\beta}{uz}\right)}{\Gamma(\lambda)z}+\frac{\ln^{\lambda-1}\left(\frac{vz}{\alpha\beta}\right)}{\Gamma(\lambda)z}\right]w(z)dz \quad (35)$$

for all $u, v \in J$ *and* $\lambda > 0$ *and* $h(\frac{1}{2}) \neq 0$.

Remark 7. *If we take $h(\lambda) = \lambda$, then we get the following inequality, which coincides with Corollary 2.3 in [21],*

$$\psi\left(\frac{\alpha\beta}{\sqrt{uv}}\right)\left[J^{\lambda}_{\frac{\alpha\beta}{v}+}w(\frac{\alpha\beta}{u}) + J^{\lambda}_{\frac{\alpha\beta}{u}-}w(\frac{\alpha\beta}{v})\right] \leq \left\{J^{\lambda}_{\frac{\alpha\beta}{v}+}\psi\left(\frac{\alpha\beta}{u}\right) + J^{\lambda}_{\frac{\alpha\beta}{u}-}\psi\left(\frac{\alpha\beta}{v}\right)\right\}$$
$$\leq \frac{1}{2}\left[\psi\left(\frac{\alpha\beta}{u}\right) + \psi\left(\frac{\alpha\beta}{v}\right)\right]\left[J^{\lambda}_{\frac{\alpha\beta}{v}+}w(\frac{\alpha\beta}{u}) + J^{\lambda}_{\frac{\alpha\beta}{u}-}w(\frac{\alpha\beta}{v})\right]$$
$$\leq \left[\psi(\alpha) + \psi(\beta) - \frac{\psi(u)+\psi(v)}{2}\right]\left[J^{\lambda}_{\frac{\alpha\beta}{v}+}w(\frac{\alpha\beta}{u}) + J^{\lambda}_{\frac{\alpha\beta}{u}-}w(\frac{\alpha\beta}{v})\right],$$

for all $u, v \in J$ and $\lambda > 0$.

If we take $u = \alpha$ and $v = \beta$ in (35), we get the following inequality.

Corollary 14. *Let $h : [0,1] \to [0,\infty)$ and $\psi : J \to [0,\infty)$ be two functions. Let the function $\psi \in L[\alpha, \beta]$ be the class of integrable function for any $\alpha, \beta \in J$ with $\alpha < \beta$. If ψ is GA-h-CF on $[\alpha, \beta]$ and let a nonnegative integrable function $g : [\alpha, \beta] \to \Re$, and then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\sqrt{\alpha\beta}\right)\left[J^{\lambda}_{\alpha+}w(\beta) + J^{\lambda}_{\beta-}w(\alpha)\right] \leq \left\{J^{\lambda}_{\alpha+}w\psi(\beta) + J^{\lambda}_{\beta-}\psi(\alpha)\right\}$$
$$\leq \frac{1}{2}[\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right) + h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]\left[\frac{\ln^{\lambda-1}\left(\frac{\beta}{z}\right)}{\Gamma(\lambda)z} + \frac{\ln^{\lambda-1}\left(\frac{z}{\alpha}\right)}{\Gamma(\lambda)z}\right]w(z)dz$$
$$\leq [\psi(\alpha)+\psi(\beta)]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right) + h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]^2\left[\frac{\ln^{\lambda-1}\left(\frac{\beta}{z}\right)}{\Gamma(\lambda)z} + \frac{\ln^{\lambda-1}\left(\frac{z}{\alpha}\right)}{\Gamma(\lambda)z}\right]w(z)dz$$
$$-\left[\frac{\psi(\alpha)+\psi(\beta)}{2}\right]\int_{\alpha}^{\beta}\left[h\left(\frac{\ln\left(\frac{z}{\alpha}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right) + h\left(\frac{\ln\left(\frac{\beta}{z}\right)}{\ln\left(\frac{\beta}{\alpha}\right)}\right)\right]\left[\frac{\ln^{\lambda-1}\left(\frac{\beta}{z}\right)}{\Gamma(\lambda)z} + \frac{\ln^{\lambda-1}\left(\frac{z}{\alpha}\right)}{\Gamma(\lambda)z}\right]w(z)dz \quad (36)$$

for all $u, v \in J$ and $\lambda > 0$ and $h(\frac{1}{2}) \neq 0$.

Remark 8. *If we take $h(\lambda) = \lambda$, then we get the following inequality, which coincides with the inequality in ([27], Theorem 2.1):*

$$\psi\left(\sqrt{\alpha\beta}\right)\left[J^{\lambda}_{\alpha+}w(\beta) + J^{\lambda}_{\beta-}w(\alpha)\right] \leq \left\{J^{\lambda}_{\alpha+}w\psi(\beta) + J^{\lambda}_{\beta-}\psi(\alpha)\right\}$$
$$\leq \left[\frac{\psi(\alpha)+\psi(\beta)}{2}\right]\left[J^{\lambda}_{\alpha+}w(\beta) + J^{\lambda}_{\beta-}w(\alpha)\right], \quad (37)$$

for all $u, v \in J$ and $\lambda > 0$.

Remark 9. *By following a similar pattern as that provided in the previous section, we may also acquire, as a special case, results for the classes of GA-s-CF (GA-Q-CF and GA-P-CF.*

6. Conclusions

In this paper, we have established several inequalities for GA-h-convex functions and its subclasses including as GA-convex functions, GA-s-convex functions, GA-Q-convex functions, and GA-P-convex functions. We proved the Jensen–Mercer inequality for GA-

h-convex functions and give weighted Hermite–Hadamard inequalities by applying the newly proved Jensen–Mercer inequality. We also established inequalities of Hermite–Hadamard–Mercer type. Furthermore, we have applied our main results along with Hadamard fractional integrals to prove weighted Hermite–Hadamard–Mercer inequalities for GA-*h*-convex functions and its subclasses. As special case of the proven results, we captured several well-known results from [20,21,25,27].

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Abbreviations

The following abbreviations are used in this manuscript:

JI	Jensen's inequality
JMI	Jensen-Mercer inequality
CF	Convex function
HHI	Hermite-Hadamard inequality
s-CF	*s*-convex functions
h-CF	*h*-convex functions
GA-CF	GA-convex functions
GA-*s*-CF	GA-*s*-convex functions
GA-*h*-CF	GA-*h*-convex functions
HFI	Hadamard-Fractional integral

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