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Multi-Shell Models of Celestial Bodies with an Intermediate Layer of Fluid: Dynamics in the Case of the Large Values of the Ekman Number

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Abstract: We consider a mechanical system that is comprised of three parts: a rigid outer shell with a spherical cavity, a spherical core inside this cavity, and an intermediate layer of liquid between the core and the shell. Such a model provides an adequate description of the behavior of a wide variety of celestial bodies. The centers of the inner and outer liquid's spherical boundaries are assumed to coincide. Assuming that the viscosity of the liquid is high, we obtained an approximate solution to the Navier–Stokes equations that describes a so called *creeping* flow of the liquid, which sets on after all transient processes die out. We note that the effect of the liquid on the rotational motion of the system can be modeled as a special torque acting upon the system with “solidified” fluid.

Keywords: Stokes flow; fluid–body interaction; subsurface ocean

MSC: 74F10; 76D05; 76U05; 76M45



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1. Introduction

It is customary to model a celestial object as a three-component structure comprised of an inner core, an intermediate layer of liquid, and an outer solid shell [1–8]. We will deal with the simplest, but fairly common version of such an approach, which does not take into account that in reality the interfaces between liquid and solid components represent a mushy transition zone [9,10]. The mass proportion of each component strongly depends on the object's type. For example, the core of an Earth-like planet is considerably smaller than the outer shell (mantle) (Figure 1a). At the same time, the icy crust, covering the hypothetical subsurface oceans of some satellites of Jupiter and Saturn, is relatively thin as compared to the satellites' radii (Figure 1b) [11]. Studying the dynamics of the multi-component object it is convenient to fix a coordinate frame to the most dominant component (either outer shell or inner core) and investigate the behavior of the other components relative to this frame.

In the theory of the rotating mass of liquid, the key role plays Ekman number [12], which reads

$$E = \frac{\mu}{\omega_{rot} \rho_f R_c^2}.$$

Here, ω_{rot} is the characteristic angular velocity of system's components, R_c is the characteristic size (e.g., the radius of the outer boundary of the liquid layer), ρ_f is the density of the liquid, and μ is the liquid's viscosity.

If $E \gg 1$, the flow of the liquid within the layer is known to be Stokesian or creeping [13]. In this case, the approximate relations for the liquid's velocities can be obtained from the Navier–Stokes equations with the relative-acceleration terms neglected. Stokes

flow in the gap between two concentric spheres has been considered by many experts. However, in the best part of the studies the motion of the bounding surfaces was assumed to be known, and the effect the liquid invokes on their dynamics was not studied (e.g., see [14,15]).

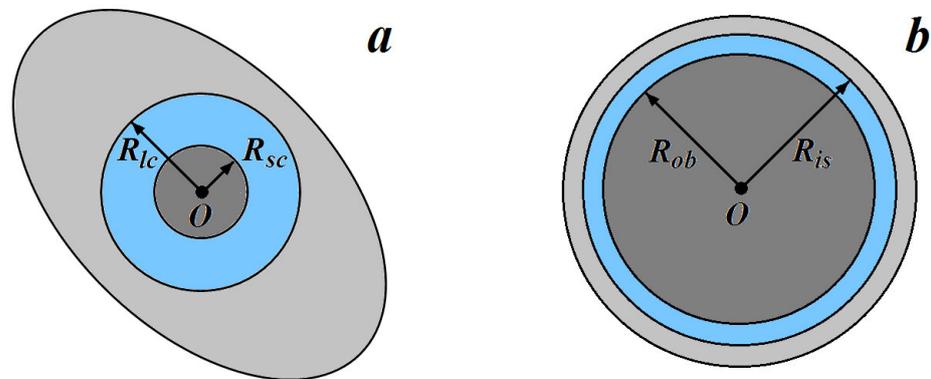


Figure 1. Celestial objects modeled as multi-shell structures: (a) “planet” (small inner core), (b) “icy satellite” with thin outer layer of ice, covering the subsurface ocean. In the case of a “planet,” the radii of the inner and outer boundaries of the liquid layer are denoted by R_{sc} and R_{lc} ; in the case of an icy satellite, they are denoted by R_{ob} and R_{is} , respectively. The visible contours of a body can have any shape; the only thing that matters is that the center of mass of the outer component coincides with the center of the cavity.

In [16], F.L. Chernousko showed that when $E \gg 1$ the analysis of the dynamics of a body containing a cavity filled with liquid can be simplified: the liquid in the cavity is considered “solidified”, and the liquid’s effect on the body is modeled as an additional torque applied to the body. Various aspects of the Chernousko approach are discussed in [17,18].

In the current paper, we show that in the case $E \gg 1$, a similar approach can be used for analyzing the rotation of celestial objects containing liquid layers. A useful byproduct of such an approach is an approximate formula for the velocity field of the liquid particles.

As far as we know, there are no multi-component celestial bodies with $E \gg 1$. Therefore, at least at present, our results seem to have only a peripheral relationship to the dynamics of actual astrophysical objects. At the same time, our results may be used, for example, for testing software that simulates the dynamics of such objects when it is important to be able to compare numerical results with a known analytical solution, regardless of whether this solution represents a real phenomena [19].

The paper is organized as follows. In Section 2, the governing equations for a three-component system mantle+liquid+core are presented. An approximate formula for the liquid velocity field after Stokes (creeping) flow sets in is obtained in Section 3. A condition of “dynamical agreement” between the motion of the core and the motion of the liquid near the core’s surface is presented in Section 4. This condition helps to establish all of the parameters of the liquid flow. Section 5 discusses some aspects of this flow. In Section 6, the approximate relations describing the regular precession of the mantle relative to the core are given. Some issues associated with the introduction of the additional torque that replicates the action of the liquid (the Chernousko model) are covered in Section 7. Modifications to the formulas for the liquid velocity to suit the case where the thin outer shell moves relative to the massive core are discussed in Section 8. A brief outline of the results obtained is presented in Section 9.

2. Equations of Motion of a “Planet”

We assume that the external forces and torques applied to the system allow for a motion in the course of which the center of mass of the outer shell (referred to further as *mantle*) coincides with the center of mass of the inner core. Such a regime (at least

approximately) occurs when a “planet” orbits a star. Under this assumption, the motion of the core relative to a mantle-fixed frame $Oxyz$ is a rotation with angular velocity ω_{sc}^r . For simplicity, we assume that the motion of the centers of mass is known (i.e., \mathbf{r}_O and $\mathbf{v}_O = d\mathbf{r}_O/dt$ are known functions of time). Relative to the frame $Oxyz$, the motion of the liquid is governed by the Navier–Stokes equations

$$\frac{1}{\nu} \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} + 2\boldsymbol{\omega} \times \mathbf{u} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \right] - \Delta \mathbf{u} + \nabla q = 0, \tag{1}$$

$$q = \frac{1}{\nu} \left[\frac{p}{\rho_f} + \Pi - \frac{1}{2} |\boldsymbol{\omega} \times \mathbf{r}|^2 - \left(\mathbf{r}, \frac{d\mathbf{v}_O}{dt} \right) \right].$$

Here, $\mathbf{u}(\mathbf{r}, t)$ is the liquid velocity, $p(\mathbf{r}, t)$ is the pressure in the liquid layer, Π is the potential of the gravity attraction forces exerted on the liquid particles, $\boldsymbol{\omega}$ is the angular velocity of the frame $Oxyz$, and $\nu = \mu/\rho_f$ is the liquid’s kinematic viscosity. The velocity $\mathbf{u}(\mathbf{r}, t)$ satisfies the standard incompressibility condition

$$\text{div } \mathbf{u} = 0 \tag{2}$$

and the kinematic relations

$$\mathbf{u} |_{|\mathbf{r}|=R_{lc}} = 0. \tag{3}$$

$$\mathbf{u} |_{|\mathbf{r}|=R_{sc}} = \boldsymbol{\omega}_{sc}^r \times \mathbf{r}, \tag{4}$$

where R_{lc} and R_{sc} stand for the radius of the cavity and the core, respectively.

Within the core, the distribution of the matter’s density is assumed to be spherically symmetric. In the frame $Oxyz$, the evolution of its rotational speed is governed by the equation

$$I_{sc} \frac{d\boldsymbol{\omega}_{sc}^r}{dt} = \mathbf{M}_{fluid} - I_{sc} \frac{d\boldsymbol{\omega}}{dt}. \tag{5}$$

Here, \mathbf{M}_{fluid} is the friction torque that the liquid exerts on the core. To find \mathbf{M}_{fluid} , we integrate the tangential stress components over the core’s surface S_{sc} , that is,

$$\mathbf{M}_{fluid} = \int_{S_{sc}} \mathbf{r} \times (T \cdot \mathbf{n}) dS. \tag{6}$$

Here, T is the stress tensor and \mathbf{n} is the unit outward normal to the surface of the core.

The total angular momentum \mathbf{K}_p^Σ relative to the center of mass can be decomposed as

$$\mathbf{K}_p^\Sigma = \mathbf{K}_p^s + \mathbf{K}_{fluid}^r + \mathbf{K}_{sc}^r.$$

Here, \mathbf{K}_p^s is the angular momentum of the “solidified” planet in which the liquid and the inner core are fixed relative to the mantle; the internal momentum of the liquid can be written as $\mathbf{K}_{fluid}^r = \rho_f \int_{V_{lc}} \mathbf{r} \times \mathbf{u} dV$; the attitude momentum of the core reads $\mathbf{K}_{sc}^r = I_{sc} \boldsymbol{\omega}_{sc}^r$; and $\mathbf{K}_p^s = I_p \boldsymbol{\omega}$, where I_p stands for the tensor of inertia of the “solidified” planet. Let \mathbf{M}_Σ be the net external torque applied to the system; then, the time evolution of the total angular momentum is described by the Euler equation

$$\frac{d}{dt} \left(\mathbf{K}_p^s + \mathbf{K}_{fluid}^r + \mathbf{K}_{sc}^r \right) + \boldsymbol{\omega} \times \left(\mathbf{K}_p^s + \mathbf{K}_{fluid}^r + \mathbf{K}_{sc}^r \right) = \mathbf{M}_\Sigma.$$

The governing equations should be augmented with kinematic relations of the form

$$\frac{d\mathbf{z}}{dt} = \mathbf{k}(\mathbf{z}, \boldsymbol{\omega}), \tag{7}$$

where the parameters \mathbf{z} quantify the orientation of the mantle-fixed frame (e.g., the Euler angles).

Hereafter, the units of mass, length, and time are assumed to be so chosen that $\rho_f = 1$, $R_{lc} \sim R_{sc} \sim 1$ and therefore $\nu \sim E \gg 1$.

3. Stokes Flow in the Liquid Layer

If $E \gg 1$, after an initial transient process, the flow in the cavity becomes creeping flow [13]. Omitting the inertial terms in (1) yields the following equation governing the creeping flow:

$$\Delta \mathbf{u} = \nabla q + \frac{d\boldsymbol{\omega}}{dt} \times \frac{\mathbf{r}}{\nu}, \tag{8}$$

where the velocity \mathbf{u} satisfies the incompressibility condition (2) and the kinematic relations (3) and (4).

We seek a solution to (8) of the form

$$\mathbf{u} = \nu^{-1} \left(\frac{d\boldsymbol{\omega}}{dt} \times \nabla f \right), \tag{9}$$

which upon substitution into (8) provides the following equation in the function $f(\mathbf{r})$

$$\Delta \nabla f = \mathbf{r}. \tag{10}$$

The velocity field satisfies the boundary condition $\mathbf{u} |_{|\mathbf{r}|=R_{lc}} = 0$ provided that

$$\nabla f |_{|\mathbf{r}|=R_{lc}} = 0. \tag{11}$$

To meet the boundary condition (3), the rotation of the inner core and the velocity (9) must agree kinematically, that is,

$$\boldsymbol{\omega}_{sc}^r = -\frac{\alpha}{\nu} \frac{d\boldsymbol{\omega}}{dt}, \tag{12}$$

$$\nabla f |_{|\mathbf{r}|=R_{sc}} = -\alpha \mathbf{r}. \tag{13}$$

Moreover, Equation (5) implies a certain “dynamical” agreement between the motion of the liquid and the motion of the inner core. In Section 4, we will derive the value of α for which (12) provides an approximate solution to (5), thereby revealing the dynamics of all of the system’s components. For simplicity, put

$$f |_{|\mathbf{r}|=R_{lc}} = 0. \tag{14}$$

By symmetry, it is quite natural to assume that the solution to the boundary-value problems (10)–(12) and (14) lies among functions that depend solely on $r = |\mathbf{r}|$. For such functions, this problem reduces to the boundary-value problem for the third-order ODE

$$\frac{d}{dr} \left(r^2 \frac{d^2 f}{dr^2} \right) - 2 \frac{df}{dr} = r^3 \tag{15}$$

of the form

$$f(R_{lc}) = \frac{df}{dr}(R_{lc}) = 0, \quad \frac{df}{dr}(R_{sc}) = -\alpha R_{sc}. \tag{16}$$

The solution to the boundary-value problems (15) and (16) looks like

$$f(r) = \frac{r^4}{40} + \beta_1 r^2 + \frac{\beta_2}{r} + \beta_3,$$

where the constants $\beta_1, \beta_2, \beta_3$ satisfy the linear relations

$$\begin{aligned} \frac{R_{lc}^4}{40} + \beta_1 R_{lc}^2 + \frac{\beta_2}{R_{lc}} + \beta_3 &= 0, \\ \frac{R_{lc}^2}{10} + 2\beta_1 - \frac{\beta_2}{R_{lc}^3} &= 0, \quad \frac{R_{sc}^2}{10} + 2\beta_1 - \frac{\beta_2}{R_{sc}^3} = 0. \end{aligned}$$

It is straightforward to obtain

$$f(r) = \frac{1}{40} (r^2 - R_{lc}^2)^2 + \beta_2 \left[\left(\frac{1}{r} - \frac{1}{R_{lc}} \right) + \frac{1}{2R_{lc}^3} (r^2 - R_{lc}^2) \right] \tag{17}$$

and

$$\alpha = \frac{1}{10} (R_{lc}^2 - R_{sc}^2) - \beta_2 \left(\frac{1}{R_{lc}^3} - \frac{1}{R_{sc}^3} \right)$$

4. “Dynamical” Agreement between the Motion of the Inner Core and the Liquid

From Equation (5), it follows that after the creeping flow onset one has

$$\mathbf{M}_{fluid} \approx I_{sc} \frac{d\omega}{dt} \tag{18}$$

To obtain \mathbf{M}_{fluid} , one should calculate the integral (6). Following [14], introduce spherical coordinates whose polar axis is aligned with $\frac{d\omega}{dt}$. The velocity components of a liquid particle are

$$u_r = u_\theta = 0, u_\varphi = -v^{-1} \left(\frac{r^3}{10} + 2\beta_1 r - \frac{\beta_2}{r^2} \right) \left| \frac{d\omega}{dt} \right| \sin \theta.$$

The force per unit area of the surface of the inner core can be determined with the help of the $\tau_{r\varphi}$ -component of the stress tensor T

$$\tau_{r\varphi} = v \left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} \right)_{r=R_{sc}} = - \left(\frac{R_{sc}^2}{5} + \frac{3\beta_2}{R_{sc}^3} \right) \left| \frac{d\omega}{dt} \right| \sin \theta.$$

Integrating over the sphere’s surface yields

$$\mathbf{M}_{fluid} = \frac{8\pi}{3} \rho R_{sc}^3 \left(\frac{R_{sc}^2}{5} + \frac{3\beta_2}{R_{sc}^3} \right) \frac{d\omega}{dt} = I_{fluid} \left(1 + \frac{15\beta_2}{R_{sc}^5} \right) \frac{d\omega}{dt}, \tag{19}$$

where I_{fluid} is the central moment of inertia of the cavity of radius R_{sc} filled with liquid with density $\rho_f = 1$.

Equating (18) and (19) we obtain

$$\beta_2 = \frac{R_{sc}^2}{15} \left(\frac{I_{sc}}{I_{fluid}} - 1 \right). \tag{20}$$

With this expression for β_2 at our disposal, it is easy to find that

$$\begin{aligned} \alpha &= R_{sc}^2 \Xi_{sc} \left(\frac{R_{sc}}{R_{lc}}, \frac{I_{fluid}}{I_{sc}} \right) \\ \Xi_{sc}(\xi, \eta) &= \frac{1}{15} \cdot \left(\frac{1}{\eta} - 1 \right) (1 - \xi^3) + \frac{1}{10} \cdot \left(\frac{1}{\xi^2} - 1 \right) \end{aligned}$$

The graph of Ξ_{sc} is given in Figure 2. It can be easily checked that $\alpha > 0$ when $I_{fluid} < I_{sc}$, which is quite natural since usually the inner core is more dense than the liquid.

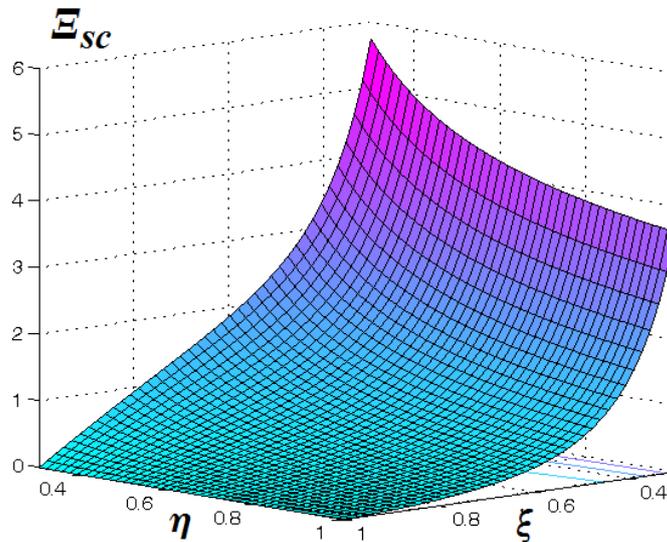


Figure 2. The graph of the function $\Xi_{sc}(\xi, \eta)$. The argument ξ characterizes the ratio of the radii of the inner and outer boundaries of the liquid layer, and the argument η represents the ratio of the densities of the liquid and the substance of the inner core (assumed homogeneous).

5. Common Properties of the Flow in the Liquid Layer

Using (9), (17) and (20), it is possible to draw the lines of constant velocity magnitude in a plane through the axis of relative rotation of the inner core (Figure 3). Obviously, the magnitude acquires its maximum in the “equatorial” plane, which is perpendicular to the relative angular velocity of the core and passes through its center.

The most trivial distribution of the velocity magnitude occurs when

$$I_{sc} = I_{fluid}. \tag{21}$$

Assuming that the core is uniform in density, Equation (21) implies that its density is equal to the density of the ambient liquid. So, if (21) is fulfilled, then in the equatorial plane

$$u_{\varphi} = \frac{1}{10\nu} (r^2 - R_{ic}^2)r. \tag{22}$$

Thus, if $R_{sc} \geq \frac{R_{lc}}{\sqrt{3}}$ the maximum of the magnitude is reached on the surface of the inner core, whereas for $R_{sc} < \frac{R_{lc}}{\sqrt{3}}$ the maximum occurs at $r = \frac{R_{lc}}{\sqrt{3}}$.

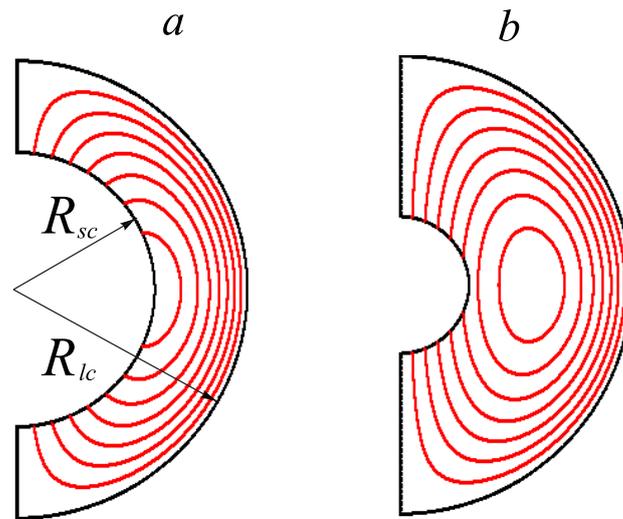


Figure 3. Velocity level lines in a plane through the instantaneous axis of relative rotation of the inner core (which is also the flow’s axis of symmetry) when $I_{sc}/I_{fluid} = 1$: (a) $R_{sc}/R_{lc} = 0.6 > 1/\sqrt{3}$ (the velocity of the liquid reaches its maximum on the surface of the inner core), (b) $R_{sc}/R_{lc} = 0.3 < 1/\sqrt{3}$ (the maximal velocity of the flow is attained inside the layer).

6. Motion of the Core Relative to the Mantle

The attitude dynamics of the core is governed by Equation (5). Let us examine the dynamics in greater detail assuming that the mantle is a dynamically symmetric rigid body that is in a precession motion with parameters ω_ψ , ω_ϕ and θ ; as usual, ω_ψ is the precession angular velocity, ω_ϕ stands for the intrinsic angular velocity, and θ is the angle between the mantle’s axis of symmetry and the precession axis. Suppose that the Oz -axis of the mantle-fixed frame is aligned with the symmetry axis; then, in this frame the angular velocity of the mantle and the relative angular velocity of the core (up to an inessential time shift) can be written as

$$\omega = (\omega_\psi \sin \theta \sin \omega_\phi t, \omega_\psi \sin \theta \cos \omega_\phi t, \omega_\psi \cos \theta + \omega_\phi)^T,$$

$$\omega_{sc}^r \approx -\frac{\alpha}{\nu} \frac{d\omega}{dt} = -\frac{\alpha}{\nu} (\omega_\psi \omega_\phi \sin \theta \cos \omega_\phi t, -\omega_\psi \omega_\phi \sin \theta \sin \omega_\phi t, 0)^T.$$

Let three orthogonal unit vectors $\mathbf{e}_{x'}^{sc}$, $\mathbf{e}_{y'}^{sc}$, $\mathbf{e}_{z'}^{sc}$ be fixed to the core. The kinematic equations

$$\frac{d\mathbf{e}_{x'}^{sc}}{dt} = \omega_{sc}^r \times \mathbf{e}_{x'}^{sc}, \quad \frac{d\mathbf{e}_{y'}^{sc}}{dt} = \omega_{sc}^r \times \mathbf{e}_{y'}^{sc}, \quad \frac{d\mathbf{e}_{z'}^{sc}}{dt} = \omega_{sc}^r \times \mathbf{e}_{z'}^{sc}, \tag{23}$$

admit a solution

$$\mathbf{e}_{x'}^{sc} = \left(1 + O(\nu^{-2}), O(\nu^{-2}), -\frac{\alpha\omega_\psi}{\nu} \sin \theta \cos \omega_\phi t + O(\nu^{-2}) \right)^T$$

$$\mathbf{e}_{y'}^{sc} = \left(O(\nu^{-2}), 1 + O(\nu^{-2}), -\frac{\alpha\omega_\psi}{\nu} \sin \theta \sin \omega_\phi t + O(\nu^{-2}) \right)^T \tag{24}$$

$$\mathbf{e}_{z'}^{sc} = \left(-\frac{\alpha\omega_\psi}{\nu} \sin \theta \cos \omega_\phi t + O(\nu^{-2}), \frac{\alpha\omega_\psi}{\nu} \sin \theta \sin \omega_\phi t + O(\nu^{-2}), 1 + O(\nu^{-2}) \right)^T$$

This solution can be characterized as the core “swaying” with zero secular drift with respect to the mantle. The other solutions to (23) can be obtained from (24) by the action of the rotation group.

Remark. The secular drift of the core still can occur in the course of the mantle’s moderate despinning or acceleration. For example, suppose $\alpha > 0$ and the mantle gradually decelerates; then, with respect to the frame $Oxyz$, the inner core rotates with angular velocity

$\omega_{sc}^r \approx -\frac{\alpha}{\nu} \left| \frac{d\omega}{dt} \right|$ in the direction opposite the vector of angular acceleration. In a sense, this mimics the superrotation of Earth’s inner core [20]. However, the current concept is that quantitatively reliable estimates of the superrotation can only be obtained with due regard for dynamo effects and turbulent convection within the liquid layer.

7. Chernousko’s Approach to Modelling Rotational Motion of the System

The total attitude angular momentum reads

$$\mathbf{K}^r = \nu^{-1} \int_{V_{lc}} \mathbf{r} \times \left(\frac{d\omega}{dt} \times \nabla f \right) dV + I_{sc} \omega_{sc}^r \tag{25}$$

The integral evaluates as follows:

$$\begin{aligned} \int_{V_{lc}} \mathbf{r} \times \left(\frac{d\omega}{dt} \times \nabla f \right) dV &= \int_{V_{lc}} \left[(r, \nabla f) \frac{d\omega}{dt} - \nabla f \cdot \left(\mathbf{r}, \frac{d\omega}{dt} \right) \right] dV = \\ &= \int_{V_{lc}} \left\{ [(r, \nabla f) + f] \frac{d\omega}{dt} - \nabla \left[f \cdot \left(\mathbf{r}, \frac{d\omega}{dt} \right) \right] \right\} dV = -2 \int_{V_{lc}} f dV \cdot \frac{d\omega}{dt} = -\kappa_{lc} \frac{d\omega}{dt}. \end{aligned} \tag{26}$$

Here,

$$\begin{aligned} \kappa_{lc} &= \frac{\pi}{525} \cdot \left\{ 15(R_{lc}^7 - R_{sc}^7) - 42R_{lc}^2 (R_{lc}^5 - R_{sc}^5) + 30R_{lc}^4 (R_{lc}^3 - R_{sc}^3) + \right. \\ &\left. + 24R_{sc}^5 \left(\frac{I_{sc}}{I_{fluid}} - 1 \right) \left[5(R_{lc}^2 - R_{sc}^2) + \frac{1}{R_{lc}^3} (R_{lc}^5 - R_{sc}^5) + \frac{5}{R_{lc}} (R_{lc}^3 - R_{sc}^3) \right] \right\} \end{aligned}$$

From (12) and (26), it follows that

$$\mathbf{K}^r = -\frac{\kappa_{\Sigma}}{\nu} \frac{d\omega}{dt}, \quad \kappa_{\Sigma} = \kappa_{lc} + I_{sc} R_{sc}^2 \Xi_{sc}.$$

The approach proposed by F.L. Chernousko to modeling the dynamics of a rigid body with a cavity entirely filled with a highly viscous liquid is as follows: the liquid is assumed to be frozen or “solidified”; the influence of the liquid is then represented as a special torque acting upon the body with solidified liquid [16]. In the context of this approach, the Euler equations can be rewritten as

$$\frac{d\mathbf{K}_p^r}{dt} + \omega \times \mathbf{K}_p^r = \mathbf{M}_{\Sigma} + \mathbf{M}_c, \tag{27}$$

where

$$\mathbf{M}_c = -\frac{d\mathbf{K}^r}{dt} - \omega \times \mathbf{K}^r \approx \frac{\kappa_{\Sigma}}{\nu} \left(\frac{d^2\omega}{dt^2} + \omega \times \frac{d\omega}{dt} \right). \tag{28}$$

Suppose that the derivatives $d\omega/dt$ and $d^2\omega/dt^2$ in (28) are found for the “zero approximation” model, that is, when the “planet” is assumed to be a one-piece uniform rigid body; then, Equation (27), together with the kinematic relations (7), forms a *finite-dimensional* dynamical system that governs our originally infinite-dimensional system. Such models (obtained within Chernousko’s approach) are as accurate as $\sim \nu^{-1}$ during time ν [17,18].

8. Simple Dynamical Model of an “Icy Satellite” with Subsurface Ocean

We now turn to the dynamics of the so-called membrane worlds encountered in the satellite systems of the giant planets [11]. Consider a satellite in the shape of a sphere of radius R_{ob} about which a thin icy shell (crust) is floating on a liquid layer (subsurface ocean). The inner radius of the icy crust will be further denoted as R_{is} .

By a similar argument as in Sections 3 and 4, it can be shown that the creeping flow of the ocean is described by formula (9), in which

$$f(r) = \frac{1}{40} (r^2 - R_{ob}^2)^2 - \frac{R_{is}^5}{15} \left(1 + \frac{I_{is}}{I_{fluid}} \right) \left[\left(\frac{1}{r} - \frac{1}{R_{ob}} \right) + \frac{1}{2R_{ob}^3} (r^2 - R_{ob}^2) \right].$$

Here, I_{is} is the moment of inertia of the icy crust, and $I_{fluid} = \frac{8}{15} \pi R_{is}^5$ is the moment of inertia of a ball of radius R_{is} , whose density is equal to the density of the liquid; both moments are about an axis through the center of mass.

Lines of equal velocity magnitude in a plane through the axis of relative rotation of the crust are shown in Figure 4.

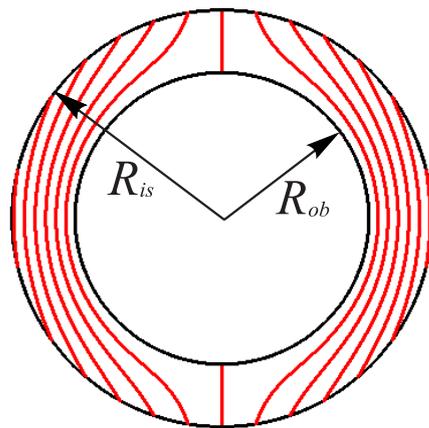


Figure 4. Distribution of velocities in the ocean: lines of equal velocity magnitude in a plane through the instantaneous axis of relative rotation of the icy crust (which is the flow’s axis of symmetry) for $R_{ob}/R_{is} = 0.7$, $I_{is}/I_{fluid} = 0.05$, $\alpha \approx 0.08308$.

The relative angular velocity of the crust is $\omega_{is}^r = -\frac{\alpha}{\nu} \frac{d\omega}{dt}$, where

$$\alpha = R_{is}^2 \Xi_{is} \left(\frac{R_{ob}}{R_{is}}, \frac{I_{is}}{I_{fluid}} \right)$$

$$\Xi_{is}(\xi, \eta) = \frac{1}{15} \cdot (1 + \eta) \left(\frac{1}{\xi^3} - 1 \right) - \frac{1}{10} \cdot (1 - \xi^2).$$

It is easy to verify that for $\xi = 1 - \Delta$, $0 < \Delta \ll 1$, the following relation holds

$$\Xi_{is}(1 - \Delta, \eta) \approx \frac{\eta \Delta}{5} > 0. \tag{29}$$

This means that for a relatively shallow ocean ($R_{is} - R_{ob} \ll R_{is}$), the coefficient α is positive. Thus, in this case when a satellite moves with angular acceleration (or deceleration), the icy crust will drift relative to the core with angular velocity $\omega_b^r = \frac{|\alpha|}{\nu} \left| \frac{d\omega}{dt} \right|$ in the direction opposite the vector $\frac{d\omega}{dt}$. The drift of the icy crust of some giant-planet satellites is discussed in, for example, [21,22]. Since for these objects the condition $E \gg 1$ does not hold, our results can hardly help to obtain new observationally confirmed features of their behavior.

9. Conclusions

We studied the motion of the liquid in the layer between two concentric spherical surfaces. In the case where the motion of one surface is solely determined by the moment of the tangential stress, an approximate expression for the Stokes flow of the liquid is derived. Using this expression, we identified an important class of motions of the three-component system mantle+liquid+core, namely, where the motion of one of the bodies and the liquid are subordinate to the motion imparted by the other body. We showed that such a subordination can be used to obtain a finite dimensional model of the system (the effect of intrinsic motion of liquid particles is modelled as a torque applied to “solidified” system [16]).

Although our study was inspired by real models used to investigate the dynamics of astrophysical objects, our main results were obtained under assumptions that can hardly be met within these models’ framework. Nevertheless, it is worth noting that in the classical problem of describing a creeping flow between two concentric spherical surfaces, we have found a new class of approximate solutions that is valuable in its own right. For example, such solutions can be useful for testing special flow analysis software (as mentioned in the Introduction).

Future research may include the application of perturbation theory for obtaining similar results when the surfaces bounding the flow are not spherical [23,24]. Additionally, our approach may find its application in the study of porous bodies floating within liquid-filled cavities [25,26].

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