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Exploration of New Optical Solitons in Magneto-Optical Waveguide with Coupled System of Nonlinear Biswas–Milovic Equation via Kudryashov's Law Using Extended F-Expansion Method

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Abstract: Optical soliton solutions in a magneto-optical waveguide and other exact solutions are investigated for the coupled system of the nonlinear Biswas–Milovic equation with Kudryashov's law using the extended F-expansion method. Various types of solutions are extracted, such as dark soliton solutions, singular soliton solutions, a dark–singular combo soliton, singular combo soliton solutions, Jacobi elliptic solutions, periodic solutions, combo periodic solutions, hyperbolic solutions, rational solutions, exponential solutions and Weierstrass solutions. The obtained different types of wave solutions help in obtaining nonlinear optical fibers in the future. Furthermore, some selected solutions are described graphically to demonstrate the physical nature of the obtained solutions. The results show that the current method gives effectual and direct mathematical tools for resolving the nonlinear problems in the field of nonlinear wave equations.

Keywords: solitons; magneto-optical waveguide; Biswas–Milovic equation; extended F-expansion method

MSC: 35C08; 35C09; 35C07; 35J05

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1. Introduction

Nonlinear evolution equations play a major role in a variety of scientific and engineering fields. The studies of solitary wave solutions for a nonlinear evolution equation attracted many researchers (see [1–3]). The theory of optical solitons is the pioneer area of research in the field of nonlinear optics and the telecommunication industry. Studying optical solitons prorogation through waveguides and optical fibers has been of great interest in the research in the past few years (see [4–13]). Kudryashov's law of refractive index is used to explain the soliton transmission dynamics across optical fibers. Nikolay Kudryashov proposed six forms of nonlinear refractive index structures (see [14–18]). Recently, the Biswas–Milovic equation (BME) was a generalized version of the nonlinear Schrödinger's equation (NLSE) (see [19]). Many authors studied the BME by different methods (see [20–30]). Few authors studied the coupled system of a nonlinear BME (see [31,32]).

In the present study, we consider the coupled system of a nonlinear BME with Kudryashov's refractive index law as follows [31]:

$$i\,(u^m)_t + a_1(u^m)_{xx} + \left(\frac{b_1}{|u|^{2n}} + \frac{c_1}{|u|^n} + d_1|u|^n + e_1|u|^{2n} + \frac{f_1}{|v|^{2n}} + \frac{g_1}{|v|^n} + h_1|v|^n + k_1|v|^{2n}\right)u^m$$

$$= \varrho_1 v^m + i\left[\alpha_1\left(u^m|u|^{2n}\right)_x + \theta_1\,(|u|^{2n})_x\,u^m + \mu_1\,|u|^{2n}\,(u^m)_x + \xi_1(u^m)_x\right], \tag{1}$$

$$i\,(v^m)_t + a_2(v^m)_{xx} + \left(\frac{b_2}{|v|^{2n}} + \frac{c_2}{|v|^n} + d_2|v|^n + e_2|v|^{2n} + \frac{f_2}{|u|^{2n}} + \frac{g_2}{|u|^n} + h_2|u|^n + k_2|u|^{2n}\right)v^m$$

$$= \varrho_2 u^m + i\left[\alpha_2\left(v^m|v|^{2n}\right)_x + \theta_2\,(|v|^{2n})_x\,v^m + \mu_2\,|v|^{2n}\,(v^m)_x + \xi_2(v^m)_x\right], \tag{2}$$

where $u(x,t)$ and $v(x,t)$ are the profiles of the wave; $a_l\ (l=1,2)$ are the coefficients of the chromatic dispersion; $b_l,\ c_l,\ d_l$ and $e_l\ (l=1,2)$ are the coefficients of the self-phase modulation; $f_l,\ g_l,\ h_l$ and $k_l\ (l=1,2)$ are the cross-phase modulation (XPM); $\varrho_l\ (l=1,2)$ are the coefficients of a magneto-optic; $\alpha_l\ (l=1,2)$ are the coefficients of self-steepening terms; θ_l and $\mu_l\ (l=1,2)$ are the coefficients of the nonlinear dispersion; $\xi_l\ (l=1,2)$ are the coefficients of the inter-modal dispersion; and n and m are the power nonlinearity and the maximum intensity, respectively.

In this work, the extended F-expansion method scheme is introduced to obtain various and novel solutions for the proposed model. These solutions include dark solitons, singular solitons, dark–singular combo solitons, singular combo solitons, Jacobi elliptic solutions, periodic solutions, combo periodic solutions, hyperbolic solutions, rational solutions, exponential solutions and Weierstrass solutions. Moreover, for the physical illustration, some of the obtained solutions are represented graphically.

2. Extended F-Expansion Method

In this section, a brief description of the extended F-expansion method is introduced (see [33,34]). Assuming the nonlinear partial differential equation (NLPDE) as follows:

$$\mathfrak{F}(u, u_t, u_{x_1}, u_{x_1x_2}, u_{tx_1}, u_{tx_2}, \ldots) = 0. \tag{3}$$

Let

$$u\,(t,\ x_1,\ x_2,\ x_3, \ldots,\ x_n) = \mathcal{H}(\epsilon), \quad \epsilon = \sum_{j=1}^{n} x_j + \gamma\,t, \tag{4}$$

then, Equation (3) is reduced to:

$$R(\mathcal{H}, \mathcal{H}', \mathcal{H}'', \ldots) = 0. \tag{5}$$

The solution of (5) is written in the following form:

$$\mathcal{H}(\epsilon) = \alpha_0 + \sum_{j=1}^{N}\left(\alpha_j\,\chi^j(\epsilon) + \frac{\beta_i}{\chi^j(\epsilon)}\right), \tag{6}$$

where $\alpha_0,\ \alpha_j$ and β_j are constants, and χ achieves the following equation:

$$\chi'^2(\epsilon) = s_0 + s_1\,\chi(\epsilon) + s_2\,\chi^2(\epsilon) + s_3\,\chi^3(\epsilon) + s_4\,\chi^4(\epsilon), \tag{7}$$

where $s_l,\ (l = 0,\ 1,2,3,4)$ are constants.

The integer N in Equation (6) can be estimated from Equation (5) using the balance rule.

Substituting from Equation (6) into Equation (5) with the auxiliary Equation (7), then we obtain a polynomial in χ. In this polynomial, we combine all terms of the same forces and equalize them with zero.

Hence, we obtain a system of algebraic equations that can be solved by Mathematica software to obtain the unknown constants $\alpha_0,\ \alpha_j, \beta_j (j = 1, 2, \ldots, N)$, γ and $s_l\ (l = 0,\ 1, 2, 3, 4)$. Therefore, various forms of solutions can be raised for Equation (3).

3. Exact Wave Solutions

Assuming the exact traveling wave solutions of the Equations (1) and (2) are:

$$u(x,t) = \mathcal{F}_1(\epsilon)\, e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{8}$$

$$v(x,t) = \mathcal{F}_2(\epsilon)\, e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{9}$$

where

$$\epsilon = x + \gamma\, t. \tag{10}$$

The amplitude component of the soliton is $\mathcal{F}_j (j = 1, 2)$. $\gamma,\ \rho$ and $\Im$ are the velocity, frequency and wave number of the soliton, while ϑ is the phase constant represent.

Applying (8) and (9) in (1) and (2) then splitting the real and imaginary parts, we obtain:

$$a_1\, m\, \mathcal{F}_1^{m-1}\mathcal{F}_1'' + a_1\, m(m-1)\, \mathcal{F}_1^{m-2}\left(\mathcal{F}_1'\right)^2 - m\left(a_1 m\rho^2 - \rho\, \xi_1 + \Im\right) \mathcal{F}_1^m + \left(m\rho\, (\alpha_1 + \mu_1) + e_1\right) \mathcal{F}_1^{m+2\,n}$$

$$+b_1\mathcal{F}_1^{m-2n} + c_1\mathcal{F}_1^{m-n} + d_1\mathcal{F}_1^{m+n} + f_1\mathcal{F}_1^m\mathcal{F}_2^{-2n} + g_1\mathcal{F}_1^m\mathcal{F}_2^{-n} + h_1\mathcal{F}_1^m\mathcal{F}_2^n + k_1\mathcal{F}_1^m\mathcal{F}_2^{2n} - \varrho_1\mathcal{F}_2^m = 0, \tag{11}$$

$$a_2\, m\, \mathcal{F}_2^{m-1}\mathcal{F}_2'' + a_2\, m(m-1)\, \mathcal{F}_2^{m-2}\left(\mathcal{F}_2'\right)^2 - m\left(a_2 m\rho^2 - \rho\, \xi_2 + \Im\right) \mathcal{F}_2^m + \left(m\rho\, (\alpha_2 + \mu_2) + e_2\right) \mathcal{F}_2^{m+2\,n}$$

$$+b_2\, \mathcal{F}_2^{m-2n} + c_2\, \mathcal{F}_2^{m-n} + d_2\, \mathcal{F}_2^{m+n} + f_2\, \mathcal{F}_2^m\mathcal{F}_1^{-2n} + g_2\, \mathcal{F}_2^m\mathcal{F}_1^{-n} + h_2\, \mathcal{F}_2^m\mathcal{F}_1^n + k_2\, \mathcal{F}_2^m\mathcal{F}_1^{2n} - \varrho_2\, \mathcal{F}_1^m = 0, \tag{12}$$

and

$$\left[m(\gamma + 2a_1\, m\, \rho - \xi_1) - \left(m\, (\alpha_1 + \mu_1) + 2n\, (\alpha_1 + \theta_1)\right) \mathcal{F}_1^{2n}\right] \mathcal{F}_1^{m-1}\, \mathcal{F}_1' = 0, \tag{13}$$

$$\left[m(\gamma + 2a_2\, m\, \rho - \xi_2) - \left(m\, (\alpha_2 + \mu_2) + 2n\, (\alpha_2 + \theta_2)\right) \mathcal{F}_2^{2n}\right] \mathcal{F}_2^{m-1}\, \mathcal{F}_2' = 0. \tag{14}$$

When applying the linearly independent principle to Equations (13) and (14), we obtain:

$$m\left(\alpha_j + \mu_j\right) + 2n\left(\alpha_j + \theta_j\right) = 0, \quad (j = 1,\ 2), \tag{15}$$

$$\gamma = \xi_1 - 2a_1\, m\, \rho = \xi_2 - 2a_2\, m\, \rho. \tag{16}$$

Through Equation (16), we obtain:

$$\rho = \frac{\xi_1 - \xi_2}{2m\, (a_1 - a_2)}, \quad a_1 \neq a_2. \tag{17}$$

Setting

$$\mathcal{F}_1 = \mathcal{A}\, \mathcal{F}_2, \tag{18}$$

where $\mathcal{A} \neq 0,\ 1$, then Equations (11) and (12) are represented as follows:

$$a_1 m \mathcal{F}_1^{m-1}\mathcal{F}_1'' + a_1 m(m-1)\mathcal{F}_1^{m-2}(\mathcal{F}_1')^2 + (c_1 + g_1\mathcal{A}^{-n})\mathcal{F}_1^{m-n} - \Big[m\Big(a_1 m\rho^2 - \xi_1\rho + \Im\Big) + \varrho_1\mathcal{A}^m\Big]\mathcal{F}_1^m$$

$$+\Big(b_1 + f_1\mathcal{A}^{-2n}\Big)\mathcal{F}_1^{m-2n} + (d_1 + h_1\mathcal{A}^n)\mathcal{F}_1^{m+n} + \Big[m\rho(\alpha_1+\mu_1) + k_1\mathcal{A}^{2n} + e_1\Big]\mathcal{F}_1^{m+2n} = 0, \tag{19}$$

$$a_2 m\mathcal{A}^m\mathcal{F}_1^{m-1}\mathcal{F}_1'' + a_2 m(m-1)\mathcal{A}^m\mathcal{F}_1^{m-2}(\mathcal{F}_1')^2 + \mathcal{A}^m(c_2\mathcal{A}^{-n} + g_2)\mathcal{F}_1^{m-n} - [m\mathcal{A}^m(a_2 m\rho^2 - \xi_2\rho + \Im) + \varrho_2]\mathcal{F}_1^m$$

$$+\mathcal{A}^m\Big(f_2 + b_2\mathcal{A}^{-2n}\Big)\mathcal{F}_1^{m-2n} + \mathcal{A}^m(h_2 + d_2\mathcal{A}^n)\mathcal{F}_1^{m+n} + [k_2 + [m\rho(\alpha_2+\mu_2) + e_2]\mathcal{A}^{2n}]\mathcal{A}^m\mathcal{F}_1^{m+2n} = 0. \tag{20}$$

Equations (19) and (20) have the same form under the following constraints:

$$a_1 = a_2\,\mathcal{A}^m, \tag{21}$$

$$a_1\,m\,(m-1) = a_2\,m\,\mathcal{A}^m\,(m-1), \tag{22}$$

$$c_1 + g_1\,\mathcal{A}^{-n} = \mathcal{A}^m(c_2\,\mathcal{A}^{-n} + g_2), \tag{23}$$

$$b_1 + f_1\,\mathcal{A}^{-2n} = \mathcal{A}^m\Big(f_2 + b_2\,\mathcal{A}^{-2n}\Big), \tag{24}$$

$$\Big[m\Big(a_1\,m\,\rho^2 - \xi_1\,\rho + \Im\Big) + \varrho_1\,\mathcal{A}^m\Big] = \Big[m\,\mathcal{A}^m\Big(a_2\,m\,\rho^2 - \xi_2\,\rho + \Im\Big) + \varrho_2\Big], \tag{25}$$

$$d_1 + h_1\,\mathcal{A}^n = \mathcal{A}^m\,(h_2 + d_2\,\mathcal{A}^n), \tag{26}$$

$$\Big[m\,\rho(\alpha_1+\mu_1) + k_1\,\mathcal{A}^{2n} + e_1\Big] = \Big[k_2 + [m\,\rho(\alpha_2+\mu_2) + e_2]\,\mathcal{A}^{2n}\Big]\mathcal{A}^m. \tag{27}$$

Through the Equations (17), (21) and (25), we obtain:

$$\Im = \frac{(\xi_1-\xi_2)(\xi_2\,\mathcal{A}^m - \xi_1)}{2\,a_2\,m(1-\mathcal{A}^m)^2} + \frac{\varrho_2 - \varrho_1\,\mathcal{A}^m}{m(1-\mathcal{A}^m)}, \quad \mathcal{A}^m \neq 1 \text{ and } a_2 \neq 0. \tag{28}$$

When applying the balancing rule between the terms $\mathcal{F}_1^{m+2n}$ and $\mathcal{F}_1^{m-1}\mathcal{F}_1''$ in Equation (19), then $N = \frac{1}{n}$, so we can use the following transformation:

$$\mathcal{F}_1(\epsilon) = \mathcal{H}^{\frac{1}{n}}(\epsilon). \tag{29}$$

Therefore, Equation (19) can be represented as follows:

$$\mathcal{H}\,\mathcal{H}'' + \mathcal{L}_1\,(\mathcal{H}')^2 + \mathcal{L}_2 + \mathcal{L}_3\,\mathcal{H} - \mathcal{L}_4\,\mathcal{H}^2 + \mathcal{L}_5\,\mathcal{H}^3 + \mathcal{L}_6\,\mathcal{H}^4 = 0, \tag{30}$$

where $\mathcal{L}_1 = \frac{m-n}{n}$, $\mathcal{L}_2 = \frac{n(b_1+f_1\,\mathcal{A}^{-2n})}{a_1\,m}$, $\mathcal{L}_3 = \frac{n(c_1+g_1\,\mathcal{A}^{-n})}{a_1 m}$, $\mathcal{L}_4 = -\frac{n[m(a_1\,m\,\rho^2-\xi_1\,\rho+\Im)+\varrho_1\,\mathcal{A}^m]}{a_1\,m}$, $\mathcal{L}_5 = \frac{n(d_1+h_1\,\mathcal{A}^n)}{a_1\,m}$ and $\mathcal{L}_6 = \frac{n[m\,\rho(\alpha_1+\mu_1)+k_1\,\mathcal{A}^{2n}+e_1]}{a_1\,m}$.

So, we can now apply the extended F-expansion method to Equation (30) to obtain the traveling wave solutions of this equation as

$$\mathcal{H}(\epsilon) = \alpha_0 + \alpha_1\,\chi(\epsilon) + \frac{\beta_1}{\chi(\epsilon)}, \tag{31}$$

where α_1 or $\beta_1 \neq 0$.

Compensation by Equation (31) with Equation (7) in Equation (30), then we extract exact solutions of (1) and (2) as follows:

Case: 1 $s_1 = s_3 = 0$.

$$\mathcal{L}_3 = -\frac{(2\mathcal{L}_1^2+5\mathcal{L}_1+2)\mathcal{L}_5\left(\mathcal{L}_4\mathcal{L}_6(2\mathcal{L}_1+3)^2+(\mathcal{L}_1^2+3\mathcal{L}_1+2)\mathcal{L}_5^2\right)}{2(\mathcal{L}_1+1)(2\mathcal{L}_1+3)^3\mathcal{L}_6^2}, \qquad \alpha_0 = -\frac{(\mathcal{L}_1+2)\,\mathcal{L}_5}{2(2\mathcal{L}_1+3)\mathcal{L}_6},$$

(1.1) $\alpha_1 = \frac{\sqrt{-s_4\,(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}, \qquad \beta_1 = 0, \qquad s_2 = \frac{2\,\mathcal{L}_4\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2+3\,(\mathcal{L}_1^2+3\,\mathcal{L}_1+2)\mathcal{L}_5^2}{2\,(\mathcal{L}_1+1)(2\,\mathcal{L}_1+3)^2\,\mathcal{L}_6},$

$$\mathcal{L}_2 = \frac{\mathcal{L}_1(\mathcal{L}_1+2)\left(16s_0s_4(\mathcal{L}_1+1)\mathcal{L}_6^2(2\mathcal{L}_1+3)^4-4(\mathcal{L}_1+2)\mathcal{L}_4\mathcal{L}_5^2\mathcal{L}_6(2\mathcal{L}_1+3)^2-5(\mathcal{L}_1+1)(\mathcal{L}_1+2)^2\mathcal{L}_5^4\right)}{16(\mathcal{L}_1+1)(2\mathcal{L}_1+3)^4\mathcal{L}_6^3}.$$

(1.2) $\beta_1 = \frac{\sqrt{-s_0\,(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}, \qquad \alpha_1 = 0 \qquad s_2 = \frac{2\,\mathcal{L}_4\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2+3\,(\mathcal{L}_1^2+3\,\mathcal{L}_1+2)\mathcal{L}_5^2}{2\,(\mathcal{L}_1+1)(2\,\mathcal{L}_1+3)^2\,\mathcal{L}_6},$

$$\mathcal{L}_2 = \frac{\mathcal{L}_1(\mathcal{L}_1+2)\left(16s_0s_4(\mathcal{L}_1+1)\mathcal{L}_6^2(2\mathcal{L}_1+3)^4-4(\mathcal{L}_1+2)\mathcal{L}_4\mathcal{L}_5^2\mathcal{L}_6(2\mathcal{L}_1+3)^2-5(\mathcal{L}_1+1)(\mathcal{L}_1+2)^2\mathcal{L}_5^4\right)}{16(\mathcal{L}_1+1)(2\mathcal{L}_1+3)^4\mathcal{L}_6^3}.$$

(1.3) $\alpha_1 = \frac{\sqrt{-s_4(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}, \quad \beta_1 = \frac{\sqrt{-s_0(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}, \quad s_2 = \frac{2\mathcal{L}_6(2\mathcal{L}_1+3)^2(6\sqrt{s_0}\sqrt{s_4}(\mathcal{L}_1+1)+\mathcal{L}_4)+3\left(\mathcal{L}_1^2+3\mathcal{L}_1+2\right)\mathcal{L}_5^2}{2(\mathcal{L}_1+1)(2\mathcal{L}_1+3)^2\mathcal{L}_6},$

$$\mathcal{L}_2 = -\frac{\mathcal{L}_1(\mathcal{L}_1+2)\left(16\sqrt{s_0}\sqrt{s_4}\mathcal{L}_6(2\mathcal{L}_1+3)^2+(\mathcal{L}_1+2)\mathcal{L}_5^2\right)\left(4\mathcal{L}_6(2\mathcal{L}_1+3)^2(4\sqrt{s_0}\sqrt{s_4}(\mathcal{L}_1+1)+\mathcal{L}_4)+5\left(\mathcal{L}_1^2+3\mathcal{L}_1+2\right)\mathcal{L}_5^2\right)}{16(\mathcal{L}_1+1)(2\mathcal{L}_1+3)^4\mathcal{L}_6^3}.$$

From the case(1.1), under the conditions $\mathcal{L}_6(\mathcal{L}_1+2) < 0$ and $\mathcal{L}_5(2\mathcal{L}_1+3) > 0$, the exact solutions of (1) and (2) are obtained in the following forms:

(1.1,1) If $s_0 = 1,\ s_2 = -(r^2+1),\ s_4 = r^2$, and $0 < r \leq 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,1} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}}+r\,\mathrm{sn}[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{32}$$

$$v_{1.1,1} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}}+r\,\mathrm{sn}[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{33}$$

or

$$u_{1.1,2} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}}+r\,\mathrm{cd}[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{34}$$

$$v_{1.1,2} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}}+r\,\mathrm{cd}[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{35}$$

Special case. We extract the dark solitons when $r = 1$, as

$$u_{1.1,3} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}}+\tanh[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{36}$$

$$v_{1.1,3} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}}+\tanh[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{37}$$

(1.1,2) If $s_0 = r^2,\ s_2 = -\left(r^2+1\right),\ s_4 = 1$, and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,4} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \mathrm{ns}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{38}$$

$$v_{1.1,4} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \mathrm{ns}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{39}$$

or

$$u_{1.1,5} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \mathrm{dc}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{40}$$

$$v_{1.1,5} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \mathrm{dc}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{41}$$

Special case. We derive singular solitons or singular periodic solutions when $r = 1$ or $r = 0$, as

$$u_{1.1,6} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \coth[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{42}$$

$$v_{1.1,6} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \coth[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{43}$$

or

$$u_{1.1,7} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \csc[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{44}$$

$$v_{1.1,7} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \csc[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{45}$$

and

$$u_{1.1,8} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sec[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{46}$$

$$v_{1.1,8} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sec[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{47}$$

(1.1,3) If $s_0 = -r^2,\ s_2 = 2\,r^2 - 1,\ s_4 = 1 - r^2$, and $0 \le r < 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,9} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,\mathrm{nc}\,[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{48}$$

$$v_{1.1,9} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,\mathrm{nc}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{49}$$

(1.1,4) If $s_0 = 1,\ s_2 = 2 - r^2,\ s_4 = 1 - r^2$, and $0 \le r < 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,10} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,\mathrm{sc}\,[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{50}$$

$$v_{1.1,10} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,\mathrm{sc}[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \quad (51)$$

Special case. We obtain periodic solutions when $r = 0$, as

$$u_{1.1,11} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \tan[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (52)$$

$$v_{1.1,11} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \tan[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \quad (53)$$

(1.1,5) If $s_0 = 1 - r^2,\ s_2 = 2 - r^2,\ s_4 = 1$, and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,12} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{cs}\,[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (54)$$

$$v_{1.1,12} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{cs}[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \quad (55)$$

Special case. When $r = 1$ or $r = 0$, we derive singular solitons or periodic solutions as

$$u_{1.1,13} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{csch}[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (56)$$

$$v_{1.1,13} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{csch}[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (57)$$

or

$$u_{1.1,14} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \cot[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (58)$$

$$v_{1.1,14} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \cot[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \quad (59)$$

(1.1,6) If $s_0 = -r^2(1-r^2),\ s_2 = 2r^2 - 1,\ s_4 = 1$, and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,15} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{ds}\,[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (60)$$

$$v_{1.1,15} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{ds}[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \quad (61)$$

Special case. When $r = 1$ or $r = 0$, we obtain singular solitons or periodic solutions as

$$u_{1.1,16} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + csch[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (62)$$

$$v_{1.1,16} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + csch[x+\gamma\,t]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \quad (63)$$

or

$$u_{1.1,17}=\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\csc[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{64}$$

$$v_{1.1,17}=\mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\csc[x+\gamma\,t]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}.\tag{65}$$

(1.1,7) If $s_0=\frac{1}{4}$, $s_2=\frac{1-2r^2}{2}$, $s_4=\frac{1}{4}$, and $0\le r\le 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,18}=\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+(\mathrm{ns}[x+\gamma\,t]\pm\mathrm{cs}[x+\gamma\,t])\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{66}$$

$$v_{1.1,18}=\mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+(\mathrm{ns}[x+\gamma\,t]\pm\mathrm{cs}[x+\gamma\,t])\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}.\tag{67}$$

Special case. In the case of $r=1$ or $r=0$, we obtain singular solitons and dark solitons or periodic solutions as

$$u_{1.1,19}=\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\coth[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{68}$$

$$v_{1.1,19}=\mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\coth[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{69}$$

and

$$u_{1.1,20}=\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\tanh[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{70}$$

$$v_{1.1,20}=\mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\tanh[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{71}$$

or

$$u_{1.1,21}=\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\cot[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{72}$$

$$v_{1.1,21}=\mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\cot[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{73}$$

and

$$u_{1.1,22}=\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\tan[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)},\tag{74}$$

$$v_{1.1,22}=\mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\tan[\frac{x+\gamma\,t}{2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\,x+\Im\,t+\vartheta)}.\tag{75}$$

(1.1,8) If $s_0 = s_4 = \frac{1-r^2}{4}$, $s_2 = \frac{1+r^2}{2}$ and $0 \leq r < 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,23} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,(\mathrm{nc}[x+\gamma\, t] + \mathrm{sc}[x+\gamma\, t]) \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{76}$$

$$v_{1.1,23} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,(\mathrm{nc}[x+\gamma\, t] + \mathrm{sc}[x+\gamma\, t]) \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{77}$$

Special case. In the case of $r = 0$, the combo periodic solutions are obtained as follows:

$$u_{1.1,24} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + (\sec[x+\gamma\, t] + \tan[x+\gamma\, t]) \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{78}$$

$$v_{1.1,24} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + (\sec[x+\gamma\, t] + \tan[x+\gamma\, t]) \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}. \tag{79}$$

(1.1,9) If $s_0 = \frac{r^2}{4}$, $s_2 = \frac{r^2-2}{2}$, $s_4 = \frac{1}{4}$ and $0 \leq r \leq 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,25} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + (\mathrm{ns}[x+\gamma\, t] \pm \mathrm{ds}[x+\gamma\, t]) \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{80}$$

$$v_{1.1,25} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + (\mathrm{ns}[x+\gamma\, t] \pm \mathrm{ds}[x+\gamma\, t]) \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{81}$$

Special case. In the case of $r = 0$, we obtain singular periodic solutions as follows:

$$u_{1.1,26} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\,\csc[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{82}$$

$$v_{1.1,26} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\,\csc[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{83}$$

(1.1,10) If $s_0 = \frac{s_2^2}{4s_4}$, $s_2 < 0$ and $s_4 > 0$; thus, the dark soliton solutions are obtained as follows:

$$u_{1.1,27} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{-\frac{s_2}{2}}\tanh\left[(x+\gamma\, t)\sqrt{\frac{-s_2}{2}}\right] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{84}$$

$$v_{1.1,27} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{\frac{-s_2}{2}}\tanh\left[(x+\gamma\, t)\sqrt{\frac{-s_2}{2}}\right] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{85}$$

(1.1,11) If $s_0 = \frac{s_2^2}{4s_4}$, $s_2 > 0$ and $s_4 > 0$; thus, the periodic solutions are obtained as follows:

$$u_{1.1,28} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{\frac{s_2}{2}}\tan\left[(x+\gamma\, t)\sqrt{\frac{s_2}{2}}\right] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{86}$$

$$v_{1.1,28} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{\frac{s_2}{2}}\tan\left[(x+\gamma\,t)\sqrt{\frac{s_2}{2}}\right]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{87}$$

(1.1,12) If $s_0 = \frac{r^2\,s_2^2}{s_4\,(r^2+1)^2}$, $s_2 < 0$, $s_4 > 0$ and $0 < r \leq 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,29} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{\frac{-r^2 s_2}{r^2+1}}\,\mathrm{sn}\left[(x+\gamma t)\sqrt{\frac{-s_2}{r^2+1}}\right]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{88}$$

$$v_{1.1,29} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{\frac{-r^2 s_2}{r^2+1}}\,\mathrm{sn}\left[(x+\gamma t)\sqrt{\frac{-s_2}{r^2+1}}\right]\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{89}$$

(1.1,13) If $s_0 = s_4 = 1$, $s_2 = 2 - 4\,r^2$ and $0 \leq r \leq 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,30} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\mathrm{dn}[x+\gamma\,t]\,\mathrm{sn}[x+\gamma\,t]}{\mathrm{cn}[x+\gamma\,t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{90}$$

$$v_{1.1,30} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\mathrm{dn}[x+\gamma\,t]\,\mathrm{sn}[x+\gamma\,t]}{\mathrm{cn}[x+\gamma\,t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{91}$$

(1.1,14) If $s_0 = \frac{(r+1)^2}{4D_1^2}$, $s_2 = \frac{r^2-6r+1}{2}$, $s_4 = \frac{D_1^2(r+1)^2}{4}$ and $0 \leq r \leq 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,31} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(r+1)\,\mathrm{cn}[x+\gamma\,t]\,\mathrm{dn}[x+\gamma\,t]}{2(\mathrm{sn}[x+\gamma\,t]+1)(1-r\,\mathrm{sn}[x+\gamma\,t])}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{92}$$

$$v_{1.1,31} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(r+1)\,\mathrm{cn}[x+\gamma\,t]\,\mathrm{dn}[x+\gamma\,t]}{2(\mathrm{sn}[x+\gamma\,t]+1)(1-r\,\mathrm{sn}[x+\gamma\,t])}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{93}$$

Special case. In the case of $r = 0$, we thus obtain combo periodic solutions as follows:

$$u_{1.1,32} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\cos[x+\gamma\,t]}{\sin[x+\gamma\,t]+1}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{94}$$

$$v_{1.1,32} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\cos[x+\gamma\,t]}{\sin[x+\gamma\,t]+1}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{95}$$

where D_1 is a constant.

(1.1,15) If $s_0 = r^4 + 2r^3 + r^2$, $s_2 = -\left(r^2 + 6r + 1\right)$, $s_4 = \frac{4}{r}$ and $0 < r \leq 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,33} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\sqrt{r}\,\mathrm{cn}[x+\gamma\,t]\mathrm{dn}[x+\gamma\,t]}{r\,\mathrm{sn}^2[x+\gamma\,t]-1}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{96}$$

$$v_{1.1,33} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\sqrt{r}\,\mathrm{cn}[x+\gamma t]\mathrm{dn}[x+\gamma t]}{r\,\mathrm{sn}^2[x+\gamma t]-1}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{97}$$

(1.1,16) If $s_0 = \frac{r^4}{4\left(D_2^2+D_3^2\right)}$, $s_2 = \frac{r^2-2}{2}$, $s_4 = \frac{D_2^2+D_3^2}{4}$ and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,34} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{D_2^2+D_3^2}\mathrm{dn}[x+\gamma t] + \sqrt{-D_3^2r^2+D_2^2+D_3^2}}{D_3\mathrm{cn}[x+\gamma t]+D_2\mathrm{sn}[x+\gamma t]}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{98}$$

$$v_{1.1,34} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{D_2^2+D_3^2}\mathrm{dn}[x+\gamma t] + \sqrt{-D_3^2r^2+D_2^2+D_3^2}}{D_3\mathrm{cn}[x+\gamma t]+D_2\mathrm{sn}[x+\gamma t]}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{99}$$

where D_2 and D_3 are constant.

Special case. In the case of $r = 1$ or $r = 0$, we extract singular combo solitons or combo periodic solutions as follows:

$$u_{1.1,35} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{D_2\,\cosh[x+\gamma\,t] + \sqrt{D_2^2+D_3^2}}{D_2\,\sinh[x+\gamma\,t]+D_3}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{100}$$

$$v_{1.1,35} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{D_2\,\cosh[x+\gamma\,t] + \sqrt{D_2^2+D_3^2}}{D_2\,\sinh[x+\gamma\,t]+D_3}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{101}$$

or

$$u_{1.1,36} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\sqrt{D_2^2+D_3^2}}{D_2\,\sin[x+\gamma\,t]+D_3\cos[x+\gamma\,t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{102}$$

$$v_{1.1,36} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\sqrt{D_2^2+D_3^2}}{D_2\,\sin[x+\gamma\,t]+D_3\cos[x+\gamma\,t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{103}$$

(1.1,17) If $s_0 = s_4 = \frac{1-r^2}{4}$, $s_2 = \frac{1+r^2}{2}$ and $0 \le r < 1$; thus, the Jacobi elliptic solutions are obtained as

$$u_{1.1,37} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{1-r^2}\,\mathrm{cn}[x+\gamma t]}{1+\mathrm{sn}[x+\gamma t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{104}$$

$$v_{1.1,37} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{1-r^2}\mathrm{cn}[x+\gamma t]}{1+\mathrm{sn}[x+\gamma t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{105}$$

or

$$u_{1.1,38} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,(\mathrm{nc}[x+\gamma t]+\mathrm{sc}[x+\gamma t])\right)\right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{106}$$

$$v_{1.1,38} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\sqrt{1-r^2}\,(\mathrm{nc}[x+\gamma t]+\mathrm{sc}[x+\gamma t])\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{107}$$

(1.1,18) If $s_0=\frac{1}{4}$, $s_2=\frac{1+r^2}{2}$, $s_4=\frac{(1-r^2)^2}{4}$ and $0\le r<1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.1,39} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\frac{(1-r^2)\,\mathrm{sn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]\pm\mathrm{cn}[x+\gamma\,t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{108}$$

$$v_{1.1,39} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\frac{(1-r^2)\,\mathrm{sn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]\pm\mathrm{cn}[x+\gamma\,t]}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{109}$$

(1.1,19) If $s_0=\frac{1}{4}$, $s_2=\frac{r^2-2}{2}$, $s_4=\frac{r^4}{4}$ and $0<r\le 1$, we derive the Jacobi elliptic solutions as follows:

$$u_{1.1,40} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\frac{r^2\mathrm{cn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]+\sqrt{1-r^2}}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{110}$$

$$v_{1.1,40} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\frac{r^2\mathrm{cn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]+\sqrt{1-r^2}}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{111}$$

or

$$u_{1.1,41} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\frac{r^2\,\mathrm{sn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]+1}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{112}$$

$$v_{1.1,41} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\frac{r^2\,\mathrm{sn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]+1}\right)\right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{113}$$

(1.1,20) If $s_0=0$, $s_2<0$, $s_4>0$ and $0\le r\le 1$, we extract the periodic solutions as follows:

$$u_{1.1,42} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\sqrt{-s_2}\,\sec\left[(x+\gamma\,t)\sqrt{-s_2}\right]\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im\,t+\vartheta)}, \tag{114}$$

$$v_{1.1,42} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\sqrt{-s_2}\,\sec\left[(x+\gamma\,t)\sqrt{-s_2}\right]\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im\,t+\vartheta)}, \tag{115}$$

or

$$u_{1.1,43} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\sqrt{-s_2}\,\csc\left[(x+\gamma\,t)\sqrt{-s_2}\right]\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im\,t+\vartheta)}, \tag{116}$$

$$v_{1.1,43} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}}+\sqrt{-s_2}\,\csc\left[(x+\gamma\,t)\sqrt{-s_2}\right]\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im\,t+\vartheta)}. \tag{117}$$

From the case(1.2), under the conditions $\mathcal{L}_6(\mathcal{L}_1+2)<0$ and $\mathcal{L}_5(2\mathcal{L}_1+3)>0$, the exact solutions of (1) and (2) are obtained in the following forms:

(1.2,1) If $s_0 = 1,\ s_2 = -(r^2+1),\ s_4 = r^2$, and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.2,1} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{ns}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{118}$$

$$v_{1.2,1} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{ns}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{119}$$

or

$$u_{1.2,2} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{dc}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{120}$$

$$v_{1.2,2} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{dc}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}. \tag{121}$$

Special case. In the case of $r = 1$ or $r = 0$, we extract singular solitons or periodic solutions as

$$u_{1.2,3} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \coth[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{122}$$

$$v_{1.2,3} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \coth[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{123}$$

or

$$u_{1.2,4} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \csc[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{124}$$

$$v_{1.2,4} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \csc[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{125}$$

and

$$u_{1.2,5} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sec[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{126}$$

$$v_{1.2,5} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sec[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}. \tag{127}$$

(1.2,2) If $s_0 = 1 - r^2,\ s_2 = 2r^2 - 1,\ s_4 = -r^2$, and $0 \le r < 1$, we establish Jacobi elliptic solutions as

$$u_{1.2,6} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\ \mathrm{nc}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{128}$$

$$v_{1.2,6} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\ \mathrm{nc}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}. \tag{129}$$

(1.2,3) If $s_0 = r^2,\ s_2 = -\left(r^2+1\right),\ s_4 = 1$, and $0 < r \leq 1$, we obtain Jacobi elliptic solutions as

$$u_{1.2,7} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + r\,\mathrm{sn}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{130}$$

$$v_{1.2,7} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + r\,\mathrm{sn}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{131}$$

or

$$u_{1.2,8} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + r\,\mathrm{cd}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{132}$$

$$v_{1.2,8} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + r\,\mathrm{cd}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{133}$$

Special case. In the case of $r = 1$, we derive dark solitons as

$$u_{1.2,9} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \tanh[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{134}$$

$$v_{1.2,9} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \tanh[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{135}$$

(1.2,4) If $s_0 = 1,\ s_2 = 2 - r^2,\ s_4 = 1 - r^2$, and $0 \leq r \leq 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.2,10} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{cs}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{136}$$

$$v_{1.2,10} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{cs}[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{137}$$

Special case. In the case of $r = 1$ or $r = 0$, we extract singular solitons or periodic solutions as follows:

$$u_{1.2,11} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + csch[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{138}$$

$$v_{1.2,11} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + csch[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{139}$$

or

$$u_{1.2,12} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \cot[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{140}$$

$$v_{1.2,12} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \cot[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{141}$$

(1.2,5) If $s_0 = 1,\ s_2 = 2r^2 - 1,\ s_4 = -r^2(1 - r^2)$, and $0 \le r \le 1$, we obtain Jacobi elliptic solutions as follows:

$$u_{1.2,13} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{ds}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{142}$$

$$v_{1.2,13} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathrm{ds}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{143}$$

(1.2,6) If $s_0 = 1 - r^2,\ s_2 = 2 - r^2,\ s_4 = 1$, and $0 \le r < 1$, we obtain Jacobi elliptic solutions as follows:

$$u_{1.2,14} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,\mathrm{sc}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{144}$$

$$v_{1.2,14} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\,\mathrm{sc}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{145}$$

Special case. In the case of $r = 0$, we obtain periodic solutions as follows:

$$u_{1.2,15} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \tan[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{146}$$

$$v_{1.2,15} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \tan[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{147}$$

(1.2,7) If $s_0 = s_4 = \frac{1-r^2}{4},\ s_2 = \frac{1+r^2}{2}$ and $0 \le r < 1$; thus, the Jacobi elliptic solutions are obtained as

$$u_{1.2,16} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{1-r^2}}{\mathrm{nc}[x+\gamma\, t]+\mathrm{sc}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{148}$$

$$v_{1.2,16} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{1-r^2}}{\mathrm{nc}[x+\gamma\, t]+\mathrm{sc}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{149}$$

Special case. In the case of $r = 0$, we establish periodic solutions as

$$u_{1.2,17} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{1}{\tan[x+\gamma\, t]+\sec[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{150}$$

$$v_{1.2,17} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{1}{\tan[x+\gamma\, t]+\sec[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{151}$$

(1.2,8) If $s_0 = \frac{s_2^2}{4s_4}$, $s_2 > 0$ and $s_4 > 0$, we establish periodic solutions as

$$u_{1.2,18} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{\frac{s_2}{2}}\cot\left[(x+\gamma\, t)\sqrt{\frac{s_2}{2}}\right] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{152}$$

$$v_{1.2,18} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{\frac{s_2}{2}}\cot\left[(x+\gamma\, t)\sqrt{\frac{s_2}{2}}\right] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{153}$$

(1.2,9) If $s_0 = \frac{s_2^2}{4s_4}$, $s_2 < 0$ and $s_4 > 0$, we establish singular solitons as

$$u_{1.2,19} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sqrt{-\frac{s_2}{2}}\coth\left[(x+\gamma\,t)\,\sqrt{\frac{-s_2}{2}}\right] \right) \right)^{\frac{1}{n}} \; e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{154}$$

$$v_{1.2,19} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sqrt{\frac{-s_2}{2}}\coth\left[(x+\gamma\,t)\,\sqrt{\frac{-s_2}{2}}\right] \right) \right)^{\frac{1}{n}} \; e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{155}$$

(1.2,10) If $s_0 = s_4 = 1,\ s_2 = 2 - 4\,r^2$ and $0 \le r \le 1$, we establish Jacobi elliptic solutions as

$$u_{1.2,20} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\mathrm{cn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]\,\mathrm{sn}[x+\gamma\,t]} \right) \right)^{\frac{1}{n}} \; e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{156}$$

$$v_{1.2,20} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\mathrm{cn}[x+\gamma\,t]}{\mathrm{dn}[x+\gamma\,t]\,\mathrm{sn}[x+\gamma\,t]} \right) \right)^{\frac{1}{n}} \; e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{157}$$

(1.2,11) If $s_0 = \frac{r^4}{4\left(D_2^2+D_3^2\right)}$, $s_2 = \frac{r^2-2}{2}$, $s_4 = \frac{D_2^2+D_3^2}{4}$ and $0 < r \le 1$, we obtain Jacobi elliptic solutions as

$$u_{1.2,21} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{r^2(D_3\,\mathrm{cn}[x+\gamma\,t]+D_2\,\mathrm{sn}[x+\gamma\,t])}{\sqrt{D_2^2+D_3^2}\,\mathrm{dn}[x+\gamma\,t]+\sqrt{-D_3^2r^2+D_2^2+D_3^2}} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{158}$$

$$v_{1.2,21} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{r^2(D_3\,\mathrm{cn}[x+\gamma\,t]+D_2\,\mathrm{sn}[x+\gamma\,t])}{\sqrt{D_2^2+D_3^2}\,\mathrm{dn}[x+\gamma\,t]+\sqrt{-D_3^2r^2+D_2^2+D_3^2}} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{159}$$

where D_2 and D_3 are constant.

Special case. In the case of $r = 1$, we establish hyperbolic solutions as

$$u_{1.2,22} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{D_2\sinh[x+\gamma\,t]+D_3}{D_2\,\cosh[x+\gamma\,t]+\sqrt{D_2^2+D_3^2}} \right) \right)^{\frac{1}{n}} \; e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{160}$$

$$v_{1.2,22} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{D_2\sinh[x+\gamma\,t]+D_3}{D_2\,\cosh[x+\gamma\,t]+\sqrt{D_2^2+D_3^2}} \right) \right)^{\frac{1}{n}} \; e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{161}$$

(1.2,12) If $s_0 = \frac{1}{4}$, $s_2 = \frac{1+r^2}{2}$, $s_4 = \frac{(1-r^2)^2}{4}$ and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as

$$u_{1.2,23} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\mathrm{dn}[x+\gamma\,t]+\mathrm{cn}[x+\gamma t]}{\mathrm{sn}[x+\gamma t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta}, \tag{162}$$

$$v_{1.2,23} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\,\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\mathrm{dn}[x+\gamma\,t]+\mathrm{cn}[x+\gamma t]}{\mathrm{sn}[x+\gamma t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta}. \tag{163}$$

(1.2,13) If $s_0 = \frac{1}{4}$, $s_2 = \frac{r^2-2}{2}$, $s_4 = \frac{r^4}{4}$ and $0 \le r \le 1$, we extract Jacobi elliptic solutions as

$$u_{1.2,24} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\text{dn}[x+\gamma\, t] + \sqrt{1-r^2}}{\text{cn}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{164}$$

$$v_{1.2,24} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\text{dn}[x+\gamma\, t] + \sqrt{1-r^2}}{\text{cn}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{165}$$

or

$$u_{1.2,25} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\text{dn}[x+\gamma\, t] + 1}{\text{sn}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{166}$$

$$v_{1.2,25} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \frac{\text{dn}[x+\gamma\, t] + 1}{\text{sn}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{167}$$

From the case(1.3), if $\mathcal{L}_6(\mathcal{L}_1+2) < 0$ and $\mathcal{L}_5(2\mathcal{L}_1+3) > 0$, the exact solutions of (1) and (2) are obtained in the following forms:

(1.3,1) If $s_0 = 1,\ s_2 = -\left(r^2+1\right),\ s_4 = r^2$ or $s_0 = r^2,\ s_2 = -\left(r^2+1\right),\ s_4 = 1$ and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as follows:

$$u_{1.3,1} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + r\,\text{sn}[x+\gamma\, t] + \text{ns}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{168}$$

$$v_{1.3,1} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + r\,\text{sn}[x+\gamma\, t] + \text{ns}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{169}$$

$$u_{1.3,2} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + r\,\text{cd}[x+\gamma\, t] + \frac{1}{\text{cd}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{170}$$

$$v_{1.3,2} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + r\,\text{cd}[x+\gamma\, t] + \text{dc}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{171}$$

Special case. In the case of $r = 1$ or $r = 0$, we derive singular solitons or periodic solutions as

$$u_{1.3,3} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\ \coth[2(x+\gamma\, t)] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{172}$$

$$v_{1.3,3} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\ \coth[2(x+\gamma\, t)] \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im\, t+\vartheta)}, \tag{173}$$

or

$$u_{1.3,4} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \csc[x+\gamma t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{174}$$

$$v_{1.3,4} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \csc[x+\gamma t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{175}$$

$$u_{1.3,5} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sec[x+\gamma t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{176}$$

$$v_{1.3,5} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sec[x+\gamma t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{177}$$

(1.3,2) If $s_0 = 1$, $s_2 = 2 - r^2$, $s_4 = 1 - r^2$ or $s_0 = 1 - r^2$, $s_2 = 2 - r^2$, $s_4 = 1$ and $0 \le r \le 1$, we obtain Jacobi elliptic solutions as

$$u_{1.3,6} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\ \ \mathrm{sc}[x+\gamma\, t] + \mathrm{cs}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{178}$$

$$v_{1.3,6} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\,\sqrt{\mathcal{L}_6}} + \sqrt{1-r^2}\ \ \mathrm{sc}[x+\gamma\, t] + \mathrm{cs}[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}. \tag{179}$$

Special case. In the case of $r = 1$ or $r = 0$, we establish singular solitons or periodic solutions as

$$u_{1.3,7} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + csch[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{180}$$

$$v_{1.3,7} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + csch[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{181}$$

or

$$u_{1.3,8} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \csc[2(x+\gamma t)] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{182}$$

$$v_{1.3,8} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \csc[2(x+\gamma t)] \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im\, t+\vartheta)}. \tag{183}$$

(1.3,3) If $s_0 = s_4 = \frac{1}{4}$, $s_2 = \frac{1-2r^2}{2}$ and $0 \le r \le 1$, we derive Jacobi elliptic solutions as

$$u_{1.3,9} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(\mathrm{ns}[x+\gamma\, t] + \mathrm{cs}[x+\gamma\, t])^2 + 1}{\mathrm{ns}[x+\gamma\, t] + \mathrm{cs}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im\, t+\vartheta)}, \tag{184}$$

$$v_{1.3,9} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(\mathrm{ns}[x+\gamma\, t] + \mathrm{cs}[x+\gamma\, t])^2 + 1}{\mathrm{ns}[x+\gamma\, t] + \mathrm{cs}[x+\gamma\, t]} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}. \tag{185}$$

Special case. In the case of either $r = 1$ or $r = 0$, we extract singular solitons or periodic solutions as

$$u_{1.3,10} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\ \coth[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{186}$$

$$v_{1.3,10} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\,\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\ \coth[x+\gamma\, t] \right) \right)^{\frac{1}{n}} e^{i(\rho\, x+\Im\, t+\vartheta)}, \tag{187}$$

or

$$u_{1.3,11} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\,\cot[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{188}$$

$$v_{1.3,11} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\,\cot[x+\gamma\,t] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{189}$$

(1.3,4) If $s_0 = s_4 = \frac{1-r^2}{4}$, $s_2 = \frac{1+r^2}{2}$ and $0 \le r \le 1$, we obtain Jacobi elliptic solutions as

$$u_{1.3,12} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{1-r^2}\,(\mathrm{nc}[x+\gamma\,t]+\mathrm{sc}[x+\gamma\,t])^2+1}{\mathrm{nc}[x+\gamma\,t]+\mathrm{sc}[x+\gamma\,t]} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im\,t+\vartheta)}, \tag{190}$$

$$v_{1.3,12} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{\sqrt{1-r^2}\,(\mathrm{nc}[x+\gamma\,t]+\mathrm{sc}[x+\gamma\,t])^2+1}{\mathrm{nc}[x+\gamma\,t]+\mathrm{sc}[x+\gamma\,t]} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im\,t+\vartheta)}. \tag{191}$$

Special case. In the case of $r = 0$, we derive combo periodic solutions as

$$u_{1.3,13} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(\sec[x+\gamma\,t]+\tan[x+\gamma\,t])^2+1}{\sec[x+\gamma\,t]+\tan[x+\gamma\,t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{192}$$

$$v_{1.3,13} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(\sec[x+\gamma\,t]+\tan[x+\gamma\,t])^2+1}{\sec[x+\gamma\,t]+\tan[x+\gamma\,t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{193}$$

(1.3,5) If $s_0 = \frac{s_2^2}{4s_4}$, $s_2 < 0$ and $s_4 > 0$, we derive singular solitons as

$$u_{1.2,14} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{-2\,s_2}csch\Big[(x+\gamma\,t)\,\sqrt{-2\,s_2}\Big] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{194}$$

$$v_{1.3,14} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{-2\,s_2}csch\Big[(x+\gamma\,t)\,\sqrt{-2\,s_2}\Big] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{195}$$

(1.3,6) If $s_0 = \frac{s_2^2}{4s_4}$, $s_2 > 0$ and $s_4 > 0$, we obtain periodic solutions as

$$u_{1.2,15} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{2\,s_2}\,\csc\Big[(x+\gamma\,t)\,\sqrt{2\,s_2}\Big] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{196}$$

$$v_{1.3,15} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \sqrt{2\,s_2}\,\csc\Big[(x+\gamma\,t)\,\sqrt{2\,s_2}\Big] \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{197}$$

(1.3,7) If $s_0 = \frac{(r\pm1)^2}{4D_1^2}$, $s_2 = \frac{r^2\mp6r+1}{2}$, $s_4 = \frac{D_1^2(r\pm1)^2}{4}$ and $0 \le r < 1$, we establish Jacobi elliptic solutions as

$$u_{1.3,16} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + |r\pm1|\,\mathcal{W}(x,t) + \frac{|r\pm1|}{\mathcal{W}(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{198}$$

$$v_{1.3,16} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5 \sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + |r\pm1|\,\mathcal{W}(x,t) + \frac{|r\pm1|}{\mathcal{W}(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{199}$$

where $\mathcal{W}(x,t) = \frac{\text{cn}[x+\gamma\ t]\ \text{dn}[x+\gamma\ t]}{(\text{sn}[x+\gamma\ t]+1)(\pm r\ \text{sn}[x+\gamma\ t]+1)}$ and D_1 is a constant.

Special case. In the case of $r = 0$, we derive periodic solutions as

$$u_{1.3,17} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\ \sec[x+\gamma\ t] \right) \right)^{\frac{1}{n}} e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{200}$$

$$v_{1.3,17} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\ \sec[x+\gamma\ t] \right) \right)^{\frac{1}{n}} e^{i(\rho\ x+\Im\ t+\vartheta)}. \tag{201}$$

(1.3,8) If $s_0 = r^4 + 2r^3 + r^2,\ s_2 = -\left(r^2+6r+1\right),\ s_4 = \frac{4}{r}$ and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as

$$u_{1.3,18} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\sqrt{r}\,\mathcal{S}_1(x,t) + \frac{r+1}{\mathcal{S}_1(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{202}$$

$$v_{1.3,18} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{2\,(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + 2\sqrt{r}\,\mathcal{S}_1(x,t) + \frac{r+1}{\mathcal{S}_1(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{203}$$

where $\mathcal{S}_1(x,t) = \frac{\text{cn}[x+\gamma\ t]\ \text{dn}[x+\gamma\ t]}{r\ \text{sn}^2[x+\gamma\ t]-1}$.

(1.3,9) If $s_0 = \frac{r^4}{4\left(D_2^2+D_3^2\right)},\ s_2 = \frac{r^2-2}{2},\ s_4 = \frac{D_2^2+D_3^2}{4}$ and $0 \le r \le 1$; thus, the Jacobi elliptic solutions are obtained as

$$u_{1.3,19} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_2(x,t) + \frac{r^2}{\mathcal{S}_2(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{204}$$

$$v_{1.3,19} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_2(x,t) + \frac{r^2}{\mathcal{S}_2(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{205}$$

where $\mathcal{S}_2(x,t) = \frac{\sqrt{D_2^2+D_3^2}\ \text{dn}[x+\gamma\ t]+\sqrt{-D_3^2r^2+D_2^2+D_3^2}}{D_3\ \text{cn}[x+\gamma\ t]+D_2\ \text{sn}[x+\gamma\ t]}$, while D_2 and D_3 are constant.

Special case. In the case of either $r = 1$ or $r = 0$, the singular–bright combo solitons or combo periodic solutions are thus obtained as

$$u_{1.3,20} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_3(x,t) + \frac{1}{\mathcal{S}_3(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{206}$$

$$v_{1.3,20} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_3(x,t) + \frac{1}{\mathcal{S}_3(x,t)} \right) \right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \tag{207}$$

or

$$u_{1.3,21} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\sqrt{D_2^2+D_3^2}}{D_2\ \sin[x+\gamma\ t]+D_3\ \cos[x+\gamma\ t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{208}$$

$$v_{1.3,21} = \mathcal{A} \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}} \left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\sqrt{D_2^2+D_3^2}}{D_2\ \sin[x+\gamma\ t]+D_3\ \cos[x+\gamma\ t]} \right) \right)^{\frac{1}{n}} e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{209}$$

where $\mathcal{S}_3(x,t) = \frac{D_2\ \cosh[x+\gamma\ t]+\sqrt{D_2^2+D_3^2}}{D_2\ \sinh[x+\gamma\ t]+D_3}$.

(1.3,10) If $s_0 = s_4 = \frac{1}{4}$, $s_2 = \frac{1-2\,r^2}{2}$ and $0 \le r \le 1$, we obtain Jacobi elliptic solutions as follows:

$$u_{1.3,22} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_4(x,t) + \frac{1}{\mathcal{S}_4(x,t)}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \quad (210)$$

$$v_{1.3,22} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_4(x,t) + \frac{1}{\mathcal{S}_4(x,t)}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \quad (211)$$

or

$$u_{1.3,23} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_5(x,t) + \frac{1}{\mathcal{S}_5(x,t)}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \quad (212)$$

$$v_{1.3,23} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \mathcal{S}_5(x,t) + \frac{1}{\mathcal{S}_5(x,t)}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \quad (213)$$

where $\mathcal{S}_4(x,t) = \frac{\mathrm{sn}[x+\gamma\ t]}{1+\mathrm{cn}[x+\gamma\ t]}$, and $\mathcal{S}_5(x,t) = \frac{\mathrm{cn}[x+\gamma\ t]}{\sqrt{1-r^2}\ \mathrm{sn}[x+\gamma\ t]+\mathrm{dn}[x+\gamma\ t]}$.

(1.3,11) If $s_0 = \frac{1}{4}$, $s_2 = \frac{1+\,r^2}{2}$, $s_4 = \frac{(1-r^2)^2}{4}$ and $0 \le r \le 1$, we obtain Jacobi elliptic solutions as

$$u_{1.3,24} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(\mathrm{dn}[x+\gamma\ t]+\mathrm{cn}[x+\gamma\ t])^2+(1-r^2)\mathrm{sn}^2[x+\gamma\ t]}{\mathrm{sn}[x+\gamma\ t](\mathrm{dn}[x+\gamma\ t]+\mathrm{cn}[x+\gamma\ t])}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \quad (214)$$

$$v_{1.3,24} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{(\mathrm{dn}[x+\gamma\ t]+\mathrm{cn}[x+\gamma\ t])^2+(1-r^2)\mathrm{sn}^2[x+\gamma\ t]}{\mathrm{sn}[x+\gamma\ t](\mathrm{dn}[x+\gamma\ t]+\mathrm{cn}[x+\gamma\ t])}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}. \quad (215)$$

Special case. In the case of $r = 0$, we derive combo periodic solutions as

$$u_{1.3,25} = \left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\ \cot\left[\frac{x+\gamma\ t}{2}\right]}{\cos[x+\gamma\ t]+1}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}, \quad (216)$$

$$v_{1.3,25} = \mathcal{A}\left(\frac{\sqrt{-(\mathcal{L}_1+2)}}{2\sqrt{\mathcal{L}_6}}\left(\frac{\mathcal{L}_5\sqrt{-(\mathcal{L}_1+2)}}{(2\,\mathcal{L}_1+3)\sqrt{\mathcal{L}_6}} + \frac{2\ \cot\left[\frac{x+\gamma\ t}{2}\right]}{\cos[x+\gamma\ t]+1}\right)\right)^{\frac{1}{n}} e^{i(\rho x+\Im t+\vartheta)}. \quad (217)$$

Case: 2 $s_0 = s_1 = 0$.

$$\alpha_0 = -\frac{(\mathcal{L}_1+2)\mathcal{L}_5}{2(2\,\mathcal{L}_1+3)\mathcal{L}_6} + \frac{\sqrt{-s_3^2\,s_4\,(\mathcal{L}_1+2)\,\mathcal{L}_6}}{4\,s_4\,\mathcal{L}_6}, \quad \alpha_1 = \frac{\sqrt{-s_3^2\,s_4\,(\mathcal{L}_1+2)\mathcal{L}_6}}{s_3\,\mathcal{L}_6}, \quad \beta_1 = 0,$$

$$s_2 = \frac{3\,s_3^2\,(\mathcal{L}_1+1)\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2 + 4\,s_4\left(2\,\mathcal{L}_4\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2 + 3\left(\mathcal{L}_1^2+3\,\mathcal{L}_1+2\right)\mathcal{L}_5^2\right)}{8\,s_4\,(\mathcal{L}_1+1)\,(2\,\mathcal{L}_1+3)^2\mathcal{L}_6},$$

$$\begin{aligned}\mathcal{L}_2 = {} & \frac{\mathcal{L}_1\,(\mathcal{L}_1+2)}{256\,s_4^2\,(\mathcal{L}_1+1)\,(2\,\mathcal{L}_1+3)^5\,\mathcal{L}_6^4}\Big(-16\,s_4^2\,\mathcal{L}_5^2\,\mathcal{L}_6\,(2\,\mathcal{L}_1^2+7\mathcal{L}_1+6)(4\,\mathcal{L}_4\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2 \\ & +5\,(\mathcal{L}_1^2+3\,\mathcal{L}_1+2)\mathcal{L}_5^2)) + s_3^2\,\mathcal{L}_6\,(\mathcal{L}_1+1)(2\,\mathcal{L}_1+3)^2(3\,s_3^2\,\mathcal{L}_6^2\,(2\,\mathcal{L}_1+3)^3 \\ & +8\,\mathcal{L}_5\sqrt{-s_3^2\,s_4\,(\mathcal{L}_1+2)(2\,\mathcal{L}_1+3)^4\,\mathcal{L}_6^3}) + 8\,s_4\,(2\,\mathcal{L}_4\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2 \\ & +3\,(\mathcal{L}_1^2+3\,\mathcal{L}_1+2)\mathcal{L}_5^2)(s_3^2\,\mathcal{L}_6^2\,(2\,\mathcal{L}_1+3)^3 + 4\,\mathcal{L}_5\sqrt{-s_3^2\,s_4\,(\mathcal{L}_1+2)(2\,\mathcal{L}_1+3)^4\,\mathcal{L}_6^3}\Big),\end{aligned}$$

$$\mathcal{L}_3 = \frac{(2\mathcal{L}_1+1)}{32 s_4^2(\mathcal{L}_1+1)(2\mathcal{L}_1+3)^4\mathcal{L}_6^3}\Big(-16 s_4^2(2\mathcal{L}_1^2+7\mathcal{L}_1+6)\mathcal{L}_5\mathcal{L}_6(\mathcal{L}_4\mathcal{L}_6(2\mathcal{L}_1+3)^2+(\mathcal{L}_1^2+3\mathcal{L}_1+2)\mathcal{L}_5^2)$$
$$+4\,s_4\,(2\,\mathcal{L}_4\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2+3\,(\mathcal{L}_1^2+3\,\mathcal{L}_1+2)\,\mathcal{L}_5^2)\sqrt{-s_3^2\,s_4\,(\mathcal{L}_1+2)(2\,\mathcal{L}_1+3)^4\mathcal{L}_6^3}+$$
$$s_3^2(\mathcal{L}_1+1)(2\mathcal{L}_1+3)^2\mathcal{L}_6\sqrt{-s_3^2 s_4(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4\mathcal{L}_6^3}\Big).$$

Through this case, if $(\mathcal{L}_1+2)<0$ and $(2\mathcal{L}_1+3)$, $\mathcal{L}_5$, s_4, s_3 and $\mathcal{L}_6>0$, the exact solutions of (1) and (2) are in the following forms:

(2.1) If $s_2<0$; thus, the combo periodic solutions are obtained as

$$u_{2.1} = \left(\frac{\sqrt{-s_3^2\,s_4\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{\mathcal{L}_6}\left(\frac{(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}+2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)}}{4\,s_4(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}}-\frac{s_2\sec^2\left[\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]}{s_3\left(s_3+2\sqrt{-s_2\,s_4}\,\tan\left[\left(\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]\right)\right.}\right)\right)^{\frac{1}{n}}$$
$$\times\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{218}$$

$$v_{2.1} = \mathcal{A}\left(\frac{\sqrt{-s_3^2\,s_4\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{\mathcal{L}_6}\left(\frac{(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}+2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)}}{4\,s_4(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}}-\frac{s_2\sec^2\left[\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]}{s_3\left(s_3+2\sqrt{-s_2\,s_4}\,\tan\left[\left(\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]\right)\right.}\right)\right)^{\frac{1}{n}}$$
$$\times\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{219}$$

and

$$u_{2.2} = \left(\frac{\sqrt{-s_3^2\,s_4\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{\mathcal{L}_6}\left(\frac{(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}+2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)}}{4\,s_4(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}}-\frac{s_2\csc^2\left[\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]}{s_3\left(s_3+2\sqrt{-s_2\,s_4}\,\cot\left[\left(\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]\right)\right.}\right)\right)^{\frac{1}{n}}$$
$$\times\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{220}$$

$$v_{2.2} = \mathcal{A}\left(\frac{\sqrt{-s_3^2\,s_4\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{\mathcal{L}_6}\left(\frac{(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}+2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)}}{4\,s_4(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}}-\frac{s_2\csc^2\left[\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]}{s_3\left(s_3+2\sqrt{-s_2\,s_4}\,\cot\left[\left(\frac{(x+\gamma\,t)\,\sqrt{-s_2}}{2}\right]\right)\right.}\right)\right)^{\frac{1}{n}}$$
$$\times\ e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{221}$$

(2.2) If $s_2>0$, we obtain singular combo solitons as

$$u_{2.3} = \left(\frac{\sqrt{-s_3^2\,s_4\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{\mathcal{L}_6}\left(\frac{(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}+2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)}}{4\,s_4(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}}+\frac{s_2 csch^2\left[\frac{(x+\gamma\,t)\,\sqrt{s_2}}{2}\right]}{s_3\left(s_3+2\sqrt{s_2\,s_4}\,\coth\left[\left(\frac{(x+\gamma\,t)\,\sqrt{s_2}}{2}\right]\right)\right.}\right)\right)^{\frac{1}{n}}$$
$$\times\ e^{i(\rho\,x+\Im\,t+\vartheta)}, \tag{222}$$

$$v_{2.3} = \mathcal{A}\left(\frac{\sqrt{-s_3^2\,s_4\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{\mathcal{L}_6}\left(\frac{(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}+2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)}}{4\,s_4(2\,\mathcal{L}_1+3)\sqrt{s_3^2\,\mathcal{L}_6}}+\frac{s_2 csch^2\left[\frac{(x+\gamma\,t)\,\sqrt{s_2}}{2}\right]}{s_3\left(s_3+2\sqrt{s_2\,s_4}\,\coth\left[\left(\frac{(x+\gamma\,t)\,\sqrt{s_2}}{2}\right]\right)\right.}\right)\right)^{\frac{1}{n}}$$
$$\times\ e^{i(\rho\,x+\Im\,t+\vartheta)}. \tag{223}$$

Case: 3 $s_0=s_1=s_2=0.$

$$\alpha_0 = -\frac{\mathcal{L}_5(\mathcal{L}_1+2)}{2\mathcal{L}_6(2\mathcal{L}_1+3)}+\frac{\sqrt{-s_3^2 s_4\mathcal{L}_6(\mathcal{L}_1+2)}}{4 s_4\mathcal{L}_6}, \quad \alpha_1 = \frac{\sqrt{-s_3^2 s_4\mathcal{L}_6(\mathcal{L}_1+2)}}{s_3\mathcal{L}_6}, \quad \beta_1 = 0,$$

$$\mathcal{L}_2 = \frac{\mathcal{L}_1(\mathcal{L}_1+2)}{256 s_4^2 \mathcal{L}_6^3 (2\mathcal{L}_1+3)^4}\Big(16 s_4^2\,\mathcal{L}_5^4(\mathcal{L}_1+2)^2 + 24 s_3^2 s_4 \mathcal{L}_6 \mathcal{L}_5^2 (\mathcal{L}_1+2)(2\mathcal{L}_1+3)^2$$
$$- s_3^2(2\mathcal{L}_1+3)(3 s_3^2 \mathcal{L}_6^2 (2\mathcal{L}_1+3)^3 + 16\mathcal{L}_5\sqrt{-s_3^2 s_4 \mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4})\Big),$$
$$\mathcal{L}_3 = \frac{(2\mathcal{L}_1+1)}{16 s_4^2(2\mathcal{L}_1+3)^3\mathcal{L}_6^2}\Big(4 s_4^2(\mathcal{L}_1+2)^2\mathcal{L}_5^3 + 3 s_3^2 s_4(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^2\mathcal{L}_6\mathcal{L}_5$$
$$- s_3^2(2\mathcal{L}_1+3)\sqrt{-s_3^2 s_4(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4\mathcal{L}_6^3}\Big),$$
$$\mathcal{L}_4 = -\frac{3(\mathcal{L}_1+1)\big(s_3^2\mathcal{L}_6(2\mathcal{L}_1+3)^2 + 4 s_4(\mathcal{L}_1+2)\mathcal{L}_5^2\big)}{8 s_4(2\mathcal{L}_1+3)^2\mathcal{L}_6}.$$

Through this case, if $s_4 \neq 0$, we obtain rational solutions to (1) and (2), under the conditions s_4 and $(\mathcal{L}_1+2) > 0$, s_3, $\mathcal{L}_6$, $\mathcal{L}_5$ and $(2\mathcal{L}_1+3) < 0$, and $s_3^2(x+\gamma t)^2 > 4 s_4$, in the following forms:

$$u_3 = \left(\frac{\sqrt{-s_4\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{4 s_4 \mathcal{L}_6}\left(\frac{s_3(2\mathcal{L}_1+3)\mathcal{L}_6 + 2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)\mathcal{L}_6}}{(2\mathcal{L}_1+3)\mathcal{L}_6} + \frac{16 s_3\, s_4}{s_3^2(x+\gamma t)^2 - 4 s_4}\right)\right)^{\frac{1}{n}} e^{i(\rho\, x + \Im\, t + \vartheta)}, \tag{224}$$

$$v_3 = \mathcal{A}\left(\frac{\sqrt{-s_4\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4 s_4 \mathcal{L}_6}\left(\frac{s_3(2\mathcal{L}_1+3)\mathcal{L}_6 + 2\mathcal{L}_5\sqrt{-s_4(\mathcal{L}_1+2)\mathcal{L}_6}}{(2\mathcal{L}_1+3)\mathcal{L}_6} + \frac{16 s_3\, s_4}{s_3^2(x+\gamma t)^2 - 4 s_4}\right)\right)^{\frac{1}{n}} e^{i(\rho x + \Im t + \vartheta)}. \tag{225}$$

Case: 4 $s_3 = s_4 = 0$.

$$\alpha_0 = -\frac{\mathcal{L}_5\,(\mathcal{L}_1+2)}{2\,\mathcal{L}_6(2\,\mathcal{L}_1+3)} + \frac{\sqrt{-s_0\, s_1^2\,\mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4}}{4\, s_0\,\mathcal{L}_6^2(2\mathcal{L}_1+3)^2}, \quad \alpha_1 = 0,$$

$$\beta_1 = \frac{\sqrt{-s_0\, s_1^2\,\mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4}}{s_1\,\mathcal{L}_6^2(2\,\mathcal{L}_1+3)^2},$$

$$\mathcal{L}_2 = \frac{\mathcal{L}_1(\mathcal{L}_1+2)}{256 s_0^2\mathcal{L}_6^3(2\mathcal{L}_1+3)^4}\Big(16 s_0^2\mathcal{L}_5^2(\mathcal{L}_1+2)\big[\mathcal{L}_5^2(\mathcal{L}_1+2) - 4 s_2\mathcal{L}_6(2\mathcal{L}_1+3)^2\big]$$
$$- s_1^2(2\mathcal{L}_1+3)(3 s_1^2\mathcal{L}_6^2(2\mathcal{L}_1+3)^3 + 16\mathcal{L}_5\sqrt{-s_0 s_1^2\mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4})$$
$$+ 8 s_0(2\mathcal{L}_1+3)(s_1^2\mathcal{L}_6(2\mathcal{L}_1+3)(2 s_2\mathcal{L}_6(2\mathcal{L}_1+3)^2 + 3(\mathcal{L}_1+2)\,\mathcal{L}_5^2)$$
$$+ 8 s_2\mathcal{L}_5\sqrt{-s_0 s_1^2\mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4})\Big),$$
$$\mathcal{L}_3 = \frac{2\mathcal{L}_1+1}{16 s_0^2\mathcal{L}_6^2(2\mathcal{L}_1+3)^3}\Big(4 s_0^2(\mathcal{L}_1+2)\mathcal{L}_5(\mathcal{L}_5^2(\mathcal{L}_1+2) - 2 s_2\mathcal{L}_6(2\mathcal{L}_1+3)^2)$$
$$- s_1^2(2\,\mathcal{L}_1+3)\sqrt{-s_0 s_1^2\mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4}$$
$$+ s_0(2\,\mathcal{L}_1+3)(3\, s_1^2\,\mathcal{L}_5\,\mathcal{L}_6(2\,\mathcal{L}_1^2 + 7\,\mathcal{L}_1 + 6) + 4 s_2\sqrt{-s_0\, s_1^2\,\mathcal{L}_6^3\,(\mathcal{L}_1+2)\,(2\,\mathcal{L}_1+3)^4})\Big),$$
$$\mathcal{L}_4 = \frac{(\mathcal{L}_1+1)\Big[-3\, s_1^2\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2 - 4\, s_0\,\Big(3\,(\mathcal{L}_1+2)\,\mathcal{L}_5^2 - 2\, s_2\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2\Big)\Big]}{8\, s_0\,\mathcal{L}_6(2\,\mathcal{L}_1+3)^2}.$$

Through this case, if $(\mathcal{L}_1+2) < 0$, $(2\mathcal{L}_1+3)$, $\mathcal{L}_5$ and $\mathcal{L}_6 > 0$, then the exact solutions of (1) and (2) are in the following forms:

(4.1) If $s_1 < 0$, $s_2 > 0$ and $s_0 = \frac{s_1^2}{4 s_2}$, we obtain exponential solutions as

$$u_{4.1} = \left(\frac{\sqrt{-4\, s_1^2\, s_2\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4\, s_1^2\,\mathcal{L}_6}\left(\frac{2\mathcal{L}_5\sqrt{-s_1^2\,\mathcal{L}_6(\mathcal{L}_1+2)} + \mathcal{L}_6\,(2\mathcal{L}_1+3)\sqrt{4 s_2\, s_1^2}}{\mathcal{L}_6\,(2\,\mathcal{L}_1+3)\sqrt{4 s_2}} + \frac{2\, s_1^2}{-s_1 + 2\, s_2\,\exp\left[(x+\gamma\, t)\sqrt{s_2}\right]}\right)\right)^{\frac{1}{n}}$$

$$\times\ e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{226}$$

$$v_{4.1}=\mathcal{A}\left(\frac{\sqrt{-4\,s_1^2\,s_2\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4\,s_1^2\,\mathcal{L}_6}\left(\frac{2\mathcal{L}_5\sqrt{-s_1^2\,\mathcal{L}_6(\mathcal{L}_1+2)}+\ \mathcal{L}_6\,(2\mathcal{L}_1+3)\sqrt{4s_2\,s_1^2}}{\mathcal{L}_6\,(2\,\mathcal{L}_1+3)\sqrt{4s_2}}+\frac{2\,s_1^2}{-s_1+2\,s_2\,\exp\left[(x+\gamma\,t)\sqrt{s_2}\right]}\right)\right)^{\frac{1}{n}}$$

$$\times\ e^{i(\rho\ x+\Im\ t+\vartheta)}. \tag{227}$$

(4.2) If $s_2<0$, $s_1=0$ and $s_0>0$, we derive periodic solutions as

$$u_{4.2}=\left(\frac{\sqrt{-s_0\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4s_0\,\mathcal{L}_6}\left(\frac{2\mathcal{L}_5\sqrt{-s_0\mathcal{L}_6(\mathcal{L}_1+2)}}{\mathcal{L}_6(2\,\mathcal{L}_1+3)}+4\sqrt{-s_2\,s_0}\ \csc\left[(x+\gamma\,t)\ \sqrt{-s_2}\right]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{228}$$

$$v_{4.2}=\mathcal{A}\left(\frac{\sqrt{-s_0\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4s_0\,\mathcal{L}_6}\left(\frac{2\mathcal{L}_5\sqrt{-s_0\mathcal{L}_6(\mathcal{L}_1+2)}}{\mathcal{L}_6(2\,\mathcal{L}_1+3)}+4\sqrt{-s_2\,s_0}\ \csc\left[(x+\gamma\,t)\ \sqrt{-s_2}\right]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\ x+\Im\ t+\vartheta)}. \tag{229}$$

(4.3) If $s_2>0$, $s_1=0$ and $s_0>0$, we extract singular solitons as

$$u_{4.3}=\left(\frac{\sqrt{-s_0\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4s_0\,\mathcal{L}_6}\left(\frac{2\mathcal{L}_5\sqrt{-s_0\mathcal{L}_6(\mathcal{L}_1+2)}}{\mathcal{L}_6(2\,\mathcal{L}_1+3)}+4\sqrt{s_2\,s_0}\ csch[(x+\gamma\,t)\ \sqrt{s_2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{230}$$

$$v_{4.3}=\mathcal{A}\left(\frac{\sqrt{-s_0\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4s_0\,\mathcal{L}_6}\left(\frac{2\mathcal{L}_5\sqrt{-s_0\mathcal{L}_6(\mathcal{L}_1+2)}}{\mathcal{L}_6(2\,\mathcal{L}_1+3)}+4\sqrt{s_2\,s_0}\ csch[(x+\gamma\,t)\ \sqrt{s_2}]\right)\right)^{\frac{1}{n}}\ e^{i(\rho\ x+\Im\ t+\vartheta)}. \tag{231}$$

Case: 5 $s_2=s_4=0,\ \ s_3>0,\ \ s_0$ and $s_1\neq 0$.

$$\alpha_0=-\frac{\mathcal{L}_5\,(\mathcal{L}_1+2)}{2\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)}+\frac{s_1\,\sqrt{-s_0\,s_1^2\,\mathcal{L}_6(\mathcal{L}_1+2)}}{4\,s_0\,s_1\,\mathcal{L}_6},\qquad \alpha_1=0,\qquad \beta_1=\frac{\sqrt{-s_0\,s_1^2\,\mathcal{L}_6(\mathcal{L}_1+2)}}{s_1\,\mathcal{L}_6},$$

$$\begin{aligned}\mathcal{L}_2=&-\tfrac{\mathcal{L}_1(\mathcal{L}_1+2)}{256\,s_0^2\,s_1\,\mathcal{L}_6^3(2\,\mathcal{L}_1+3)^4}\Big(s_1^3\,(2\,\mathcal{L}_1+3)(3\,s_1^2\,\mathcal{L}_6^2(2\,\mathcal{L}_1+3)^3\\&+16\,\mathcal{L}_5\,\sqrt{-s_0\,s_1^2\,\mathcal{L}_6^3\,(\mathcal{L}_1+2)(2\,\mathcal{L}_1+3)^4})\\&-24s_0s_1^3\mathcal{L}_5^2\mathcal{L}_6(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^2+16s_0^2(4s_1^2s_3\mathcal{L}_6^2(2\mathcal{L}_1+3)^4\\&+8s_3\mathcal{L}_5(2\mathcal{L}_1+3)\sqrt{-s_0\,s_1^2\,\mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4}-s_1\mathcal{L}_5^4(\mathcal{L}_1+2)^2)\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{L}_3=&\tfrac{2\mathcal{L}_1+1}{16\,s_0^2\,s_1\,\mathcal{L}_6^2(2\mathcal{L}_1+3)^3}\Big(3s_0\,s_1^3\,\mathcal{L}_5\,\mathcal{L}_6\,(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^2\\&-s_1^3(2\mathcal{L}_1+3)\sqrt{-s_0\,s_1^2\,\mathcal{L}_6^3(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4}+\\&4s_0^2(s_1\,\mathcal{L}_5^3(\mathcal{L}_1+2)^2-2s_3(2\mathcal{L}_1+3)\sqrt{-s_0\,s_1^2\,\mathcal{L}_6^3\,(\mathcal{L}_1+2)(2\mathcal{L}_1+3)^4})\Big).\end{aligned}$$

$$\mathcal{L}_4=-\frac{3\,(\mathcal{L}_1+1)\Big(s_1^2\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2+4\,s_0\,\mathcal{L}_5^2\,(\mathcal{L}_1+2)\Big)}{8\,s_0\,\mathcal{L}_6\,(2\,\mathcal{L}_1+3)^2}.$$

Through this case, if $(\mathcal{L}_1+2)<0,\ \mathcal{L}_5$ and $(2\mathcal{L}_1+3)>0,\ \mathcal{L}_6$ and $s_i>0,\ i=0,\ 1,$ then we obtain Weierstrass solutions of (1) and (2) in the following forms:

$$u_5=\left(\frac{\sqrt{-s_0\,s_1^2\,\mathcal{L}_6\,(\mathcal{L}_1+2)}}{4\,s_0\,s_1\,\mathcal{L}_6}\left(\frac{s_1^2\ \ \mathcal{L}_6(2\mathcal{L}_1+3)+2\,\mathcal{L}_5\,\sqrt{-s_0\,s_1^2\,\mathcal{L}_6(\mathcal{L}_1+2)}}{s_1\,\mathcal{L}_6(2\,\mathcal{L}_1+3)}+\frac{4\,s_0}{\wp\left[\frac{(x+\gamma\,t)\,\sqrt{s_3}}{2},\mathcal{E}_2,\mathcal{E}_3\right]}\right)\right)^{\frac{1}{n}}$$

$$\times e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{232}$$

$$v_5 = \mathcal{A}\left(\frac{\sqrt{-s_0\ s_1^2\ \mathcal{L}_6\ (\mathcal{L}_1+2)}}{4\ s_0\ s_1\ \mathcal{L}_6}\left(\frac{s_1^2\ \ \mathcal{L}_6(2\mathcal{L}_1+3)+2\ \mathcal{L}_5\ \sqrt{-s_0\ s_1^2\ \mathcal{L}_6(\mathcal{L}_1+2)}}{s_1\ \mathcal{L}_6(2\ \mathcal{L}_1+3)}+\frac{4\ s_0}{\wp\left[\frac{(x+\gamma\ t)\ \sqrt{s_3}}{2},\mathcal{E}_2,\mathcal{E}_3\right]}\right)\right)^{\frac{1}{n}}$$

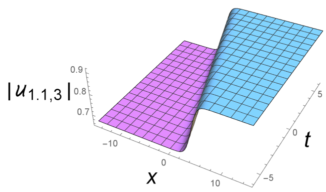
$$\times e^{i(\rho\ x+\Im\ t+\vartheta)}, \tag{233}$$

where $\mathcal{E}_2 = -\frac{4\ s_1}{s_3}$ and $\mathcal{E}_3 = -\frac{4\ s_0}{s_3}$.

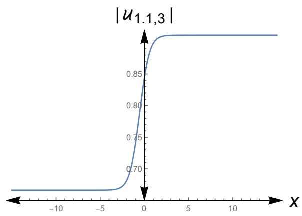
4. Results and Discussion

The obtained wave solutions in this paper using the extended F-expansion method differ from the gained results of various researchers by some other existing methods. Different families of solutions were obtained from Equation (7) by giving specific values to the parameters. Dark soliton solutions, singular soliton solutions, the dark–singular combo soliton, singular combo soliton solutions, Jacobi elliptic solutions, periodic solutions, combo periodic solutions, hyperbolic solutions, rational solutions, exponential solutions and Weierstrass solutions were extracted. Zayed et al. [31] obtained solitons in magneto-optics waveguides for the nonlinear Biswas–Milovic equation with Kudryashov's law of refractive index using the unified auxiliary equation method. Kengne [32] commented on "Solitons in magneto-optics waveguides for the nonlinear Biswas–Milovic equation with Kudryashov's law of refractive index using the unified auxiliary equation method". So, several innovative and corrected results have been achieved in this work, which have not been specified before.

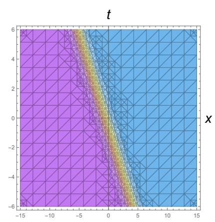
Figure 1 presents the dark soliton solutions of Equation (36), when $m = 3$, $n = 8$, $a_2 = 0.9$, $\mathcal{A} = -0.5$, $\xi_1 = 0.8$, $\xi_2 = 0.6$, $\varrho_2 = 0.5$, $\varrho_1 = 0.6$, $b_1 = 0.9$, $f_1 = 0.8$, $c_1 = 0.5$, $g_1 = 1.1$, $d_1 = -0.8$, $h_1 = 1.3$, $e_1 = 1.4$, $\alpha_1 = 0.9$, $\mu_1 = 1$, $k_1 = 0.5$, $\vartheta = 0.8$ and $-15 < x < 15$. Figure 2 presents the singular soliton solutions of Equation (42), when $m = 3$, $n = 3.5$, $a_2 = -1.9$, $\mathcal{A} = 1.5$, $\xi_1 = 2.8$, $\xi_2 = 1.6$, $\varrho_2 = 1.5$, $\varrho_1 = 1.6$, $b_1 = 1.9$, $f_1 = 1.8$, $c_1 = 1.5$, $g_1 = 2.1$, $d_1 = -2.8$, $h_1 = 0.6$, $e_1 = 1.4$, $\alpha_1 = 1.9$, $\mu_1 = 1.4$, $k_1 = 1.5$, $\vartheta = 1.8$ and $-15 < x < 15$. Figure 3 presents the periodic solutions of Equation (46), when $m = 3$, $n = 3.5$, $a_2 = -1.8$, $\mathcal{A} = 1.4$, $\xi_1 = 2.7$, $\xi_2 = 1.6$, $\varrho_2 = 1.5$, $\varrho_1 = 1.6$, $b_1 = 1.8$, $f_1 = 1.7$, $c_1 = 1.5$, $g_1 = 2.2$, $d_1 = -2.8$, $h_1 = 0.6$, $e_1 = 1.4$, $\alpha_1 = 1.8$, $\mu_1 = 1.5$, $k_1 = 1.4$, $\vartheta = 1.7$ and $-15 < x < 15$. Figure 4 presents the singular combo soliton solutions of Equation (100), when $m = 3$, $n = 3.5$, $a_2 = -1.7$, $\mathcal{A} = 1.4$, $\xi_1 = 2.6$, $\xi_2 = 1.5$, $\varrho_2 = 1.4$, $\varrho_1 = 1.5$, $b_1 = 1.6$, $f_1 = 1.4$, $c_1 = 1.4$, $g_1 = 2.2$, $d_1 = -2.8$, $h_1 = 0.6$, $e_1 = 1.4$, $\alpha_1 = 1.8$, $\mu_1 = 1.5$, $k_1 = 1.4$, $\vartheta = 1.7$, $D_2 = \sqrt{2}$, $D_3 = -\sqrt{2}$ and $-25 < x < 25$. Figure 5 presents the hyperbolic solution of Equation (160), when $m = 3$, $n = 3.5$, $a_2 = -1.6$, $\mathcal{A} = 1.5$, $\xi_1 = 2.7$, $\xi_2 = 1.5$, $\varrho_2 = 1.4$, $\varrho_1 = 1.5$, $b_1 = 1.6$, $f_1 = 1.5$, $c_1 = 1.5$, $g_1 = 2.1$, $d_1 = -2.8$, $h_1 = 0.6$, $e_1 = 1.5$, $\alpha_1 = 1.8$, $\mu_1 = 1.5$, $k_1 = 1.4$, $\vartheta = 1.7$, $D_2 = \sqrt{2}$, $D_3 = -\sqrt{2}$ and $-15 < x < 15$.



(**a**) 3D plot.

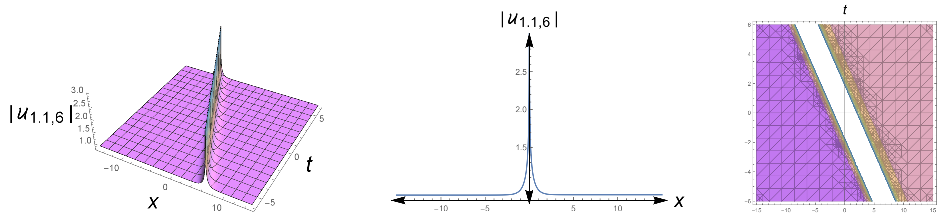


(**b**) 2D plot.



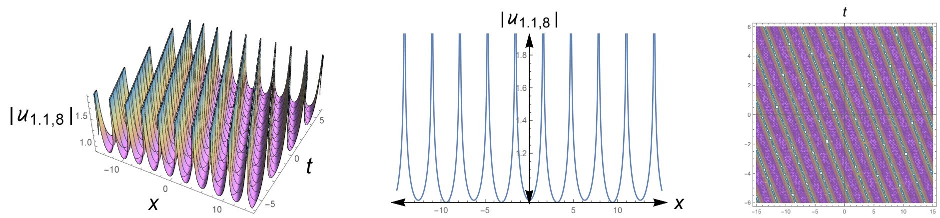
(**c**) Contour plot.

Figure 1. Graphical simulation of dark soliton solutions to Equation (36).



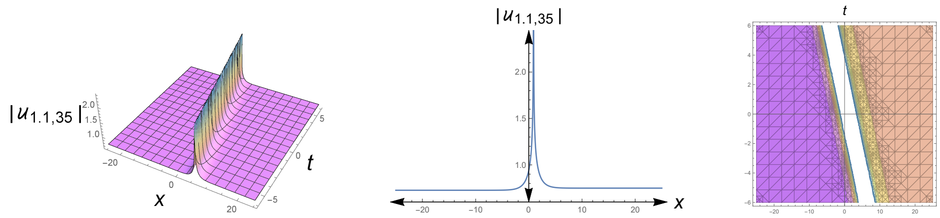
(**a**) 3D plot. (**b**) 2D plot. (**c**) Contour plot.

Figure 2. Graphical simulation of singular soliton solutions to Equation (42).



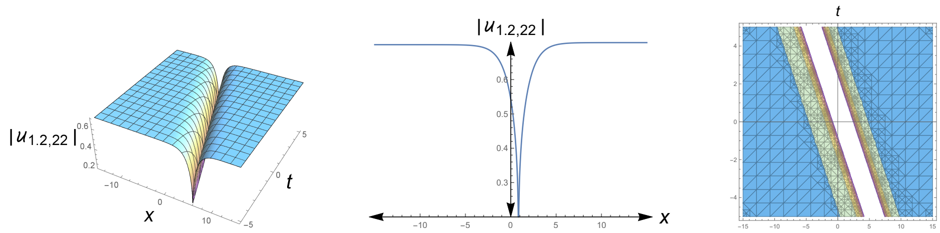
(**a**) 3D plot. (**b**) 2D plot. (**c**) Contour plot.

Figure 3. Graphical simulation of periodic solutions to Equation (46).



(**a**) 3D plot. (**b**) 2D plot. (**c**) Contour plot.

Figure 4. Graphical simulation of singular combo soliton solutions to Equation (100).



(**a**) 3D plot. (**b**) 2D plot. (**c**) Contour plot.

Figure 5. Graphical simulation of hyperbolic solutions to Equation (160).

5. Conclusions

In this work, the coupled system of the nonlinear Biswas–Milovic equation in a magneto-optical waveguide with Kudryashov's law was studied successfully by applying the extended F-expansion method. Various types of traveling wave solutions were extracted, such as the dark soliton solutions, singular soliton solutions, dark–singular combo soliton, singular combo soliton solutions, Jacobi elliptic solutions, periodic solutions, combo periodic solutions, hyperbolic solutions, rational solutions, exponential solutions and Weierstrass solutions. This paper shows that some of the obtained solutions by this effective method are new compared with the obtained solutions in [31,32]. For certain solutions, contour, three- and two-dimensional visualizations were provided for further illustration. Finally, our solutions were checked using Mathematica software by putting them back into the original equations.

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