mathematics

Article

Redheffer-Type Bounds of Special Functions

Reem Alzahrani † and Saiful R. Mondal *,†

Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa 31982, Saudi Arabia;
* Correspondence: smondal@kfu.edu.sa
† These authors contributed equally to this work.

Abstract: In this paper, we aim to construct inequalities of the Redheffer type for certain functions defined by the infinite product involving the zeroes of these functions. The key tools used in our proofs are classical results on the monotonicity of the ratio of differentiable functions. The results are proved using the n^{th} positive zero, denoted by $b_n(\nu)$. Special cases lead to several examples involving special functions, namely, Bessel, Struve, and Hurwitz functions, as well as several other trigonometric functions.

Keywords: Redheffer inequality; Bessel functions; Struve functions; Dini functions; Lommel functions; q-Bessel functions

MSC: 33B10; 33C10; 26D07; 26D05

check for updates

Citation: Alzahrani, R.; Mondal, S.R. Redheffer-Type Bounds of Special Functions. *Mathematics* **2023**, *11*, 379. https://doi.org/10.3390/math11020379

Academic Editor: Yamilet Quintana

Received: 30 November 2022
Revised: 4 January 2023
Accepted: 8 January 2023
Published: 11 January 2023

1. Introduction

Several famous inequalities for real functions have been proposed in the literature. One of them is the Redheffer inequality, which states that

$$\frac{\sin(x)}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad \text{for all} \quad x \in \mathbb{R}. \tag{1}$$

Inequality (1) was proposed by Redheffer [1] and proved by Williams [2]. This work motivated many researchers, regarding its generalization, refinement, and applications. A new (but relatively difficult) proof of (1) using the Lagrange mean value theorem in combination with induction was given in [3]. In 2015, Sándor and Bhayo [4] offered two new interesting proofs and established two converse inequalities. They also pointed out a hyperbolic analog. Other notable works related to the Redheffer inequality include [5–10]. Motivated by the inequality (1), C.P. Chen, J.W. Zhao, and F. Qi [8], using mathematical induction and infinite product representations of $\cos(x), \sinh(x), \cosh(x)$

$$\cos(x) = \prod_{n\geq 1}\left[1 - \frac{4x^2}{(2n-1)^2\pi^2}\right], \quad \cosh(x) = \prod_{n\geq 1}\left[1 + \frac{4x^2}{(2n-1)^2\pi^2}\right], \tag{2}$$

and

$$\frac{\sinh(x)}{x} = \prod_{n\geq 1}\left(1 + \frac{x^2}{n^2\pi^2}\right), \tag{3}$$

respectively, established the following Redheffer-type inequalities:

$$\cos(x) \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2} \quad \text{and} \quad \cosh(x) \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \quad \text{for all} \quad |x| \leq \frac{\pi}{2}. \tag{4}$$

A hyperbolic analog of inequality (1) has also been established [8], by proving that

$$\frac{\sinh(x)}{x} \le \frac{\pi^2 + x^2}{\pi^2 - x^2}, \quad \text{for all} \quad |x| < \pi. \tag{5}$$

In [6], inequalities (1) and (4) were extended and sharpened, and a Redheffer-type inequality for $\tan(x)$ was also established, as follows:

(i) Let $0 < x < \pi$. Then,

$$\left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\beta} \le \frac{\sin(x)}{x} \le \left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\alpha} \tag{6}$$

hold if and only if $\alpha \le \pi^2/12$ and $\beta \ge 1$.

(ii) Let $0 \le x \le \pi/2$. Then,

$$\left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\beta} \le \cos(x) \le \left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\alpha} \tag{7}$$

hold if and only if $\alpha \le \pi^2/16$ and $\beta \ge 1$.

(iii) Let $0 < x < \pi/2$. Then,

$$\left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\alpha} \le \frac{\tan(x)}{x} \le \left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\beta} \tag{8}$$

hold if and only if $\alpha \le \pi^2/24$ and $\beta \ge 1$.

(iv) Let $0 < x < r$. Then,

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \le \frac{\sinh(x)}{x} \le \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{9}$$

hold if and only if $\alpha \le 0$ and $\beta \ge r^2/12$.

(v) Let $0 < x < r$. Then,

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \le \cosh(x) \le \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{10}$$

hold if and only if $\alpha \le 0$ and $\beta \ge r^2/4$.

(vi) Let $0 < x < r$. Then,

$$\left(\frac{r^2 - x^2}{r^2 + x^2}\right)^{\beta} \le \frac{\tanh(x)}{x} \le \left(\frac{r^2 - x^2}{r^2 + x^2}\right)^{\alpha} \tag{11}$$

hold if and only if $\alpha \le 0$ and $\beta \ge r^2/6$.

The Bessel function J_ν of order ν is the solution of the differential equation:

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0. \tag{12}$$

The function $I_\nu(x) = -iJ_\nu(ix)$ is known as the modified Bessel function. It is well known that trigonometric functions are connected with Bessel and modified Bessel functions, as follows

$$\sin(x) = \sqrt{\frac{\pi x}{2}} J_{1/2}(x), \quad \cos(x) = \sqrt{\frac{\pi x}{2}} J_{-1/2}(x),$$
$$\sinh(x) = \sqrt{\frac{\pi x}{2}} I_{1/2}(x), \quad \cosh(x) = \sqrt{\frac{\pi x}{2}} I_{-1/2}(x).$$

Based on the relationship between trigonometric and Bessel functions as stated above, and as Bessel and modified Bessel functions have infinite product representations involving their zeros, the Redheffer inequality (1) has been generalized for modified Bessel functions in [7], and sharpened in [9]. There are several other special functions, such as Struve and q-Bessel functions, which have infinite product representations and are also related to trigonometric functions.

Motivated by the above facts, the aim of this study was to address the following problem:

Problem 1. *Construct the class of functions f that can be represented by an infinite product with the factors involving the zeroes of f, such that f exhibits a Redheffer-type inequality.*

To answer Problem 1, we consider a sequence $\{b_n(\nu)\}_{\nu \in \mathbb{R}, n \geq 1}$, such that

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2(\nu)} \mapsto l(\nu)$$

for $\nu \in I \subset \mathbb{R}$ and the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{b_n^2(\nu)}\right)$$

is also absolutely convergent to a function of x for $x \in I_x \subset \mathbb{R}$.

We study several properties of functions that are members of the following two classes:

$$\mathcal{F}_\nu := \left\{ \eta_\nu(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{b_n^2(\nu)}\right) \right\}, \tag{13}$$

$$\mathcal{G}_\nu := \left\{ \chi_\nu(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{b_n^2(\nu)}\right) \right\}. \tag{14}$$

It is easy to check that, for a fixed ν, $\{b_1(\nu), b_2(\nu), \ldots, b_n(\nu), \ldots\}$ is a set of zeroes of the functions in the class $\mathcal{F}_\nu$. Unless mentioned otherwise, throughout the article, we denote by $b_n(\nu)$ the n^{th} positive zero of the functions in the class $\mathcal{F}_\nu$. For $\lambda_\nu \in \mathcal{G}_\nu$ and $\eta_\nu \in \mathcal{F}_\nu$, it immediately follows that $\lambda_\nu(x) = \eta_\nu(ix)$, where $i = \sqrt{-1}$.

Using a similar concept as in [7,9], we derived the Redheffer inequality for the functions from both classes, $\mathcal{F}_\nu$ and $\mathcal{G}_\nu$. We also investigate the increasing/decreasing, log convexity, and convexity nature of the functions (or their products) from the above two classes. The main results are discussed in Section 2, while Section 3 provides several examples based on the main result in Section 2. In Section 4, we compare the obtained result with known results; especially the results given in [7,9–11].

The following lemma is required in the following.

Lemma 1 ([12]). *Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, where $a_k \in \mathbb{R}$ and $b_k > 0$ for all k. Furthermore, suppose that both series converge on $|x| < r$. If the sequence $\{a_k/b_k\}_{k\geq 0}$ is increasing (or decreasing), then the function $x \mapsto f(x)/g(x)$ is also increasing (or decreasing) on $(0, r)$.*

Lemma 2 (Lemma 2.2 in [13]). *Suppose that $-\infty < a < b < \infty$ and $p, q : [a, b) \mapsto \infty$ are differentiable functions, such that $q'(x) \neq 0$ for $x \in (a, b)$. If p'/q' is increasing (or decreasing) on (a, b), then so is $(p(x) - p(a))/(q(x) - q(a))$.*

2. Main Results

Theorem 1. *Suppose that $\lambda_\nu \in \mathcal{G}_\nu$ and $\eta_\nu \in \mathcal{F}_\nu$. Then, the following assertions are true:*

1. *The function $x \mapsto \lambda_\nu(x)$ is increasing on $(0, \infty)$.*
2. *The function $x \mapsto \lambda_\nu(x)$ is strictly log-convex on $I_\nu = (-b_1(\nu), b_1(\nu))$ and strictly geometric convex on $(0, \infty)$.*
3. *The function $x \mapsto \lambda_\nu(x)$ satisfies the sharp exponential Redheffer-type inequality*

$$\left(\frac{b_1^2(\nu) + x^2}{b_1^2(\nu) - x^2}\right)^{a_\nu} \leq \lambda_\nu(x) \leq \left(\frac{b_1^2(\nu) + x^2}{b_1^2(\nu) - x^2}\right)^{b_\nu} \tag{15}$$

   on I_ν. Here, $a_\nu = 0$ and $b_\nu = b_1^2(\nu) l(\nu)/2$ are the best possible constants.
4. *The function $x \mapsto \lambda_\nu(x)\eta_\nu(x)$ is increasing on $(-b_1(\nu), 0]$ and decreasing on $\big(0, b_1(\nu)\big]$*
5. *The function $x \mapsto \lambda_\nu(x)/\eta_\nu(x)$ is strictly log-convex on I_ν.*
6. *The function $x \mapsto \eta_\nu(x)$ satisfies the sharp Redheffer-type inequality.*

$$\left(\frac{b_1^2(\nu) - x^2}{b_1^2(\nu)}\right)^{a_\nu} \leq \eta_\nu(x) \leq \left(\frac{b_1^2(\nu) - x^2}{b_1^2(\nu)}\right)^{b_\nu} \tag{16}$$

   on I_ν. Here, $b_\nu = 1$ and $a_\nu = b_1^2(\nu) l(\nu)$ are the best possible constants.

Proof. As $\lambda_\nu \in \mathcal{G}_\nu$, from (14), it follows that

$$\lambda_\nu(x) = \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{b_n^2(\nu)}\right). \tag{17}$$

Similarly, as $\eta_\nu \in \mathcal{G}_\nu$, from (13), it follows that

$$\eta_\nu(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{b_n^2(\nu)}\right). \tag{18}$$

1. Logarithmic differentiation of (17) leads to

$$(log(\lambda_\nu(x)))' = \frac{\lambda_\nu'(x)}{\lambda_\nu(x)} = \sum_{n=1}^{\infty} \frac{2x}{b_n^2(\nu) + x^2} > 0 \tag{19}$$

   for $x \in (0, \infty)$. This implies that $log(\lambda_\nu(x))$ is increasing and, consequently, $\lambda_\nu(x)$ is also increasing.

2. Let $x \in I_\nu$. Differentiation of both sides of (19) gives

$$(log(\lambda_\nu(x)))'' = \sum_{n=1}^{\infty}\left(\frac{2}{b_n^2(\nu)+x^2} - \frac{4x^2}{\left(b_n^2(\nu)+x^2\right)^2}\right) = \sum_{n=1}^{\infty}\frac{2\left(b_n^2(\nu)-x^2\right)}{\left(b_n^2(\nu)+x^2\right)^2} > 0,$$

for $x \in I_\nu$ This is equivalent to the function $x \longmapsto \lambda_\nu(x)$ being log-convex on I_ν. From (19), we also have

$$\left(\frac{x\lambda_\nu'(x)}{\lambda_\nu(x)}\right)' = \sum_{n=1}^{\infty}\left(2 - \frac{b_n^2(\nu)}{b_n^2(\nu)+x^2}\right)' = \sum_{n=1}^{\infty}\frac{2xb_n^2(\nu)}{(b_n^2(\nu)+x^2)2}.$$

This implies that $x \longmapsto x\lambda_\nu'(x)/\lambda_\nu(x)$ is increasing on $x \in (0,\infty)$ and, as a consequence, we have that $x \longmapsto \lambda_\nu(x)$ is geometrically convex on $(0,\infty)$.

3. Consider the function

$$h_\nu(x) := \frac{log(\lambda_\nu(x))}{log(b_1^2(\nu)+x^2) - log(b_1^2(\nu)-x^2)}.$$

For $x \in [0,\infty)$, define

$$p(x) = log(\lambda_\nu(x)), \quad q(x) = log(b_1^2(\nu)+x^2) - log(b_1^2(\nu)-x^2).$$

From the calculation along with (19), it follows that

$$\frac{p'(x)}{q'(x)} = \frac{\frac{\lambda_\nu'(x)}{\lambda_\nu(x)}}{\frac{2x}{b_1^2(\nu)+x^2}+\frac{2x}{b_1^2(\nu)-x^2}} = \frac{\lambda_\nu'(x)}{2x\lambda_\nu(x)}.\frac{b_1^4(\nu)-x^4}{2b_1^2(\nu)} = \frac{1}{2b_1^2(\nu)}\sum_{n=1}^{\infty}\frac{b_1^4(\nu)-x^4}{b_n^2(\nu)+x^2}.$$

Then,

$$\frac{d}{dx}\left(\frac{p'(x)}{q'(x)}\right) = \frac{1}{2b_1^2(\nu)}\sum_{n=1}^{\infty}\frac{-4x^3(b_n^2(\nu)+x^2)-2x(b_1^4(\nu)-x^4)}{(b_n^2(\nu)+x^2)^2} = -\frac{x}{b_1^2(\nu)}\sum_{n=1}^{\infty}\frac{2x^2b_n^2(\nu)+x^4+b_1^2(\nu)}{(b_n^2(\nu)+x^2)^2} \le 0$$

on $x \in [0,\infty)$. Thus, $p'(x)/q'(x)$ is decreasing and, hence,

$$h_\nu(x) = \frac{p(x)}{q(x)} = \frac{p(x)-p(0)}{q(x)-q(0)}$$

is also decreasing on $[0, b_1(\nu)]$. Finally,

$$\lim_{x\to b_1(\nu)} h_\nu(x) < h_\nu(x) < \lim_{x\to 0} h_\nu(x),$$

where

$$a_\nu := \lim_{x\to b_1(\nu)} h_\nu(x) = \lim_{x\to b_1(\nu)} \frac{p(x)}{q(x)} = \lim_{x\to b_1(\nu)} \frac{p'(x)}{q'(x)} = 0,$$
$$b_\nu := \lim_{x\to 0} h_\nu(x) = \lim_{x\to 0} \frac{p(x)}{q(x)} = \lim_{x\to 0} \frac{p'(x)}{q'(x)} = \frac{b_1^2(\nu)}{2} l(\nu)$$

are the best possible constants and

$$l(\nu) = \sum_{n=1}^{\infty} \frac{1}{b_n^2(\nu)}.$$

4. As $\lambda_\nu \in \mathcal{G}_\nu$ and $\eta_\nu \in \mathcal{F}_\nu$, from (13) and (14), it follows that

$$\lambda_\nu(x)\eta_\nu(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^4}{b_n^4(\nu)}\right).$$

Logarithmic differentiation yields

$$\frac{(\lambda_\nu(x)\eta_\nu(x))'}{\lambda_\nu(x)\eta_\nu(x)} = -\sum_{n=1}^{\infty} \frac{4x^3}{b_n^4(\nu) - x^4},$$

which is negative for $x \in (0, b_1(\nu))$ and positive for $x \in (-b_1(\nu), 0)$. Hence, the result follows.

5. From part (2), it follows that $x \longmapsto \lambda_\nu(x)$ is strictly log-convex on I_ν. Now, consider the function $x \longmapsto (\eta_\nu(x))^{-1}$. From (2), it follows that

$$\left(log\left((\eta_\nu(x))^{-1}\right)\right)' = \sum_{n=1}^{\infty} \frac{2x}{b_n^2(\nu) - x^2}$$

and

$$\left(log\left((\eta_\nu(x))^{-1}\right)\right)'' = 2\sum_{n=1}^{\infty} \frac{b_n^2(\nu) + x^2}{(b_n^2(\nu) - x^2)^2} > 0.$$

This implies that $x \longmapsto (\eta_\nu(x))^{-1}$ is strictly log-convex on I_ν. Finally, being the product of two strictly log-convex functions, $x \longmapsto \lambda_\nu(x)/\eta_\nu(x)$ is strictly log-convex on I_ν.

6. To prove this result, we first need to set up a Rayleigh-type function for the Lommel function. Define the function

$$\alpha_n^{(2m)}(\nu) := \sum_{n=1}^{\infty} b_n^{-2m}(\nu), \quad m = 1, 2, \ldots. \tag{20}$$

Logarithmic differentiation of $\chi_\nu(x)$ yields

$$\frac{x\chi_\nu'(x)}{\chi_\nu(x)} = -2\sum_{n=1}^{\infty} \frac{x^2}{b_n^2(\nu) - x^2} = \sum_{n=1}^{\infty} \frac{x^2}{b_n^2(\nu)}\left(1 - \frac{x^2}{b_n^2(\nu)}\right)^{-1} = \sum_{n=1}^{\infty} \frac{x^2}{b_n^2(\nu)} \sum_{m=0}^{\infty} \frac{x^{2m}}{b_n^{2m}(\nu)}.$$

Interchanging the order of the summation, it follows that

$$\frac{x\chi_\nu'(x)}{\chi_\nu(x)} = -2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{x^{2m+2}}{b_n^{2m+2}(\nu)} = -2\sum_{m=1}^{\infty} \alpha_n^{(2m)}(\nu) x^{2m}. \tag{21}$$

Consider the function

$$\varphi_\mu(x) := \frac{\log(\chi_\nu(x))}{\log\left(1 - \frac{x^2}{b_1^2(\nu)}\right)} = \frac{\mathrm{p}_\mu(x)}{\mathrm{q}_\mu(x)}. \tag{22}$$

The binomial series, together with (21), gives the ratio of p'_μ and q'_μ as

$$\frac{\mathrm{p}'_\mu(x)}{\mathrm{q}'_\mu(x)} = \frac{\frac{x\chi'_\nu(x)}{\chi_\nu(x)}}{\frac{-2x^2}{b_1(\nu)^2}\left(1 - \frac{x^2}{b_1(\nu)^2}\right)^{-1}} = \frac{\sum_{m=1}^{\infty} \alpha_n^{(2m)}(\nu)x^{2m}}{\sum_{m=1}^{\infty} b_1^{-2m}(\nu)x^{2m}}. \tag{23}$$

Denote $d_m = b_1^{2m}(\nu)\alpha_n^{(2m)}(\nu)$. Then,

$$\begin{aligned} d_{m+1} - d_m &= b_1^{2m+2}(\nu)\alpha_n^{(2m+2)}n(\nu) - b_1^{2m}(\nu)\alpha_n^{(2m)}n(\nu) \\ &= \sum_{n=1}^{\infty} \frac{b_1^{2m}(\nu)}{b_n^{2m}(\nu)}\left(\frac{b_1^2(\nu)}{b_n^2(\nu)} - 1\right) < 0. \end{aligned}$$

This is equivalent to saying that the sequence $\{d_m\}$ is decreasing. Hence, by Lemma 1, it follows that the ratio p'_μ/q'_μ is decreasing. In view of Lemma 2, we have that $\tau_\mu = p_\mu/q_\mu$ is decreasing.
From (22) and (23), it can be shown that

$$\lim_{x\to 0} \tau_\mu(x) = \lim_{x\to 0} \frac{p'_\mu(x)}{q'_\mu(x)} = \lim_{x\to 0} \frac{p''_\mu(x)}{q''_\mu(x)} = \lim_{x\to 0} \frac{p''_\mu(x)}{q''_\mu(x)} = b_1^2(\nu)\alpha_n^{(2)}(\nu), \tag{24}$$

and

$$\lim_{x\to b_1(\nu)} \tau_\mu(x) = \lim_{x\to b_1^2(\nu)} \frac{p'_\mu(x)}{q'_\mu(x)} = \lim_{x\to b_1^2(\nu)} \sum_{n=1}^{\infty} \frac{b_1^2(\nu) - x^2}{b_n^2(\nu) - x^2} = 1. \tag{25}$$

It is easy to see that $b_1^2(\nu)\alpha_n^{(2)}(\nu) = b_1^2(\nu)\mathtt{l}(\nu) = b_\nu$.

This completes the proof of all of the results. □

In the next result, by approaching a similar proof as in Theorem 1, we prove a sharper upper bound for λ_ν, compared to that presented in Theorem 1 (Part 3).

Theorem 2. *If $r > 0$ and $|x| < r$, then the following inequality*

$$\left(\frac{r^2 - x^2}{r^2}\right)^{a_\nu} \le \lambda_\nu(x) \le \left(\frac{r^2 - x^2}{r^2}\right)^{b_\nu} \tag{26}$$

holds, where $a_\nu = 0$ and $b_\nu = -r^2\mathtt{l}(\nu)$ are the best possible constants.

Proof. Due to symmetry, it is sufficient to show the result for $[0, r)$. Define $\Psi : [0, r) \longrightarrow \mathbb{R}$ as

$$\Psi(x) := log(\lambda_\nu(x)) - r^2\mathtt{l}(\nu)log\left(\frac{r^2}{r^2 - x^2}\right).$$

Then,

$$\Psi'(x) = \frac{\lambda_\nu'(x)}{\lambda_\nu(x)} - \frac{2xr^2}{r^2-x^2}\,\mathfrak{l}(\nu) = \sum_{n=1}^{\infty}\frac{2x}{b_n^2(\nu)+x^2} - \sum_{n=1}^{\infty}\frac{2xr^2}{(r^2-x^2)b_n^2(\nu)}$$
$$= \sum_{n=1}^{\infty}\frac{2x(r^2-x^2)b_n^2(\nu) - 2xr^2(b_n^2(\nu)+x^2)}{b_n^2(\nu)(r^2-x^2)(b_n^2(\nu)+x^2)}$$
$$= -2x^3\sum_{n=1}^{\infty}\frac{b_n^2(\nu)+r^2}{b_n^2(\nu)(r^2-x^2)(b_n^2(\nu)+x^2)} \le 0,$$

for $x \in [0, r)$. This implies that Ψ is decreasing, and $\Psi(x) \le \Psi(0) = 0$. This is equivalent to

$$log(\lambda_\nu(x)) \le log\left(\frac{r^2}{r^2-x^2}\right)^{r^2\mathfrak{l}(\nu)} \implies \lambda_\nu(x) \le \left(\frac{r^2-x^2}{r^2}\right)^{-r^2\mathfrak{l}(\nu)}.$$

This completes the proof. Now, to show the $b_\nu = -r^2\mathfrak{l}(\nu)$ is the best possible constant, consider

$$\delta_\nu := \frac{log(\lambda_\nu(x))}{log\left(\frac{r^2-x^2}{r^2}\right)}.$$

Then, using the Bernoulli–L'Hôpital rule, we have

$$\lim_{x\searrow 0}\delta_\nu(x) = \lim_{x\searrow 0}\frac{log(\lambda_\nu(x))}{log\left(\frac{r^2}{r^2-x^2}\right)}$$
$$= \lim_{x\searrow 0}\left(\frac{\lambda_\nu'(x)}{\lambda_\nu(x)} - \frac{r^2-x^2}{2x}\right)$$
$$= \lim_{x\searrow 0}\sum_{n=1}^{\infty}\frac{-(r^2-x^2)}{b_n^2(\nu)+x^2} = -\sum_{n=1}^{\infty}\frac{r^2}{b_n^2(\nu)} = -r^2\mathfrak{l}(\nu) = b_\nu.$$

Thus, b_ν is the best possible constant. □

3. Application Examples

As stated before, the primary aim of this work is to find a Redheffer-type inequality for functions that are combinations of well-known functions. By constructing examples, we show that Theorem 1 not only covers known results but also covers a wide range of functions. We list each case as an example.

3.1. Example Involving Trigonometric Functions

Our very first example involves the well-known function $f(x) = sinc(x)$. In mathematics, physics, and engineering, there are two forms of the $sinc(x)$ function; namely, non-normalized and normalized sinc functions. In mathematics, the non-normalized *sinc* function is defined, for $x \neq 0$, as:

$$sinc(x) := \frac{\sin(x)}{x}.$$

On the other hand, in digital and communication systems, the normalized form is defined as:

$$sinc(x) := \frac{\sin(\pi x)}{\pi x}, \quad x \neq 0.$$

The scaling of the independent variable (the x-axis) by a factor of π is the only distinction between the two definitions. In both scenarios, it is assumed that the limit value 1 corresponds to the function's value at the removable singularity at zero. The *sinc* function is an entire function, as it is analytic everywhere.

The normalized *sinc* has the following infinite product representation:

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right). \tag{27}$$

It is well known that the infinite series $\sum_{n=1}^{\infty} n^{-2}$ is convergent and

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

We can conclude that $sinc(x) \in \mathcal{F}_\nu$. From Theorem 1, it follows that

$$(1-x^2)^{a_\nu} \le sinc(x) \le (1-x^2)^{b_\nu}$$

with $|x| < 1$, $b_\nu = 1$, and $a_\nu = \pi^2/6$.

Now, replacing x with ix in (27), we have

$$\frac{\sinh(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{n^2}\right). \tag{28}$$

Clearly, $\sinh(\pi x)/\pi x \in \mathcal{F}_\nu$. Hence, by Theorem 1 (part 3), it follows that

$$\left(\frac{1+x^2}{1-x^2}\right)^{\tau_\nu} \le \frac{\sinh(\pi x)}{\pi x} \le \left(\frac{1+x^2}{1-x^2}\right)^{\delta_\nu}$$

for $|x| < 1$. Here, $\tau_\nu = 0$ and $\delta_\nu = \pi^2/6$ are the best possible values of the constants.

On the other hand, from Theorem 2, it follows that

$$\frac{\sinh(\pi x)}{\pi x} \le \left(\frac{r}{r^2 - x^2}\right)^{\delta_\nu}$$

for $|x| < r$, where $\delta_\nu = \pi^2/6$ is the best possible constant.

Next, we consider the infinite product

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2 - \nu^2}\right), \ |\nu| < \pi. \tag{29}$$

Using the Mathematica software, we find that

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2 - \nu^2}\right) = \frac{\nu \csc(\nu) \sin\left(\sqrt{\nu^2 + x^2}\right)}{\sqrt{\nu^2 + x^2}} \tag{30}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2 - \nu^2} = \frac{1 - \nu\cot(\nu)}{2\nu^2}. \tag{31}$$

Clearly, $\nu \csc(\nu) \sin\left(\sqrt{\nu^2 + x^2}\right)/\sqrt{\nu^2 + x^2} \in \mathcal{F}_\nu$, and we have the following result, according to Theorem 1.

Corollary 1. *Let $0 \neq \nu \in (-\pi, \pi)$. Then, the following inequality*

$$\left(\frac{\pi^2-\nu^2+x^2}{\pi^2-\nu^2-x^2}\right)^{a_\nu} \leq \frac{\nu\csc(\nu)\sin\left(\sqrt{\nu^2+x^2}\right)}{\sqrt{\nu^2+x^2}} \leq \left(\frac{\pi^2-\nu^2+x^2}{\pi^2-\nu^2-x^2}\right)^{b_\nu}$$

holds for $|x| < \pi^2 - \nu^2$. Here, $a_\nu = 0$ and $b_\nu = (1-\nu\cot(\nu))/4\nu^2(\pi^2-\nu^2)$ are the best possible constants.

3.2. Examples Involving Hurwitz Zeta Functions

The Hurwitz zeta functions are zeta functions defined for the complex variable s, with $Re(s) > 0$ and $\nu \neq -1, -2, -3, \ldots$, defined by

$$\zeta(s,\nu) := \sum_{n=0}^{\infty} \frac{1}{(n+\nu)^s}. \tag{32}$$

This series is absolutely convergent for given values of s and ν, and can be extended to meromorphic functions defined for all $s \neq 1$. In particular, the Riemann zeta function is given by $\zeta(s,1)$. For our study in this section, we consider $s = m \in \mathbb{N} \setminus \{1\}$ and $\nu > -1$.

Now, consider the infinite product

$$\chi_{m,\nu}(x) := \prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n+\nu)^m}\right), \quad m \geq 2 \text{ and } \nu > -1, \tag{33}$$

for which the product is convergent. In the closed form of the product, we consider $m = 2, 3, 4$. Then, $\chi_{m,\nu}(x)$ have the forms

$$\begin{aligned}
\chi_{2,\nu}(x) &= \frac{\Gamma(\nu+1)^2}{\Gamma(-x+\nu+1)\Gamma(x+\nu+1)}\\
\chi_{3,\nu}(x) &= \frac{\Gamma(\nu+1)^3}{\Gamma\left(-x^{2/3}+\nu+1\right)\Gamma\left(\frac{1}{2}\left(\left(1-i\sqrt{3}\right)x^{2/3}+2(\nu+1)\right)\right)\Gamma\left(\frac{1}{2}\left(\left(1+i\sqrt{3}\right)x^{2/3}+2(\nu+1)\right)\right)}\\
\chi_{4,\nu}(x) &= \frac{\Gamma(\nu+1)^4}{\Gamma\left(\nu-\sqrt{-x}+1\right)\Gamma\left(\nu+\sqrt{-x}+1\right)\Gamma\left(\nu-\sqrt{x}+1\right)\Gamma\left(\nu+\sqrt{x}+1\right)}.
\end{aligned}$$

Next, we state a result related to the inequalities involving $\chi_{m,\nu}(x)$. Although the result is a direct consequence of Theorem 1 (Part 6), taking $b_n(\nu) = (n+\nu)^{m/2}$ for $m \geq 2$ and $\nu > -1$, we state it as a theorem due to its independent interest. Clearly,

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2(\nu)} = \sum_{n=1}^{\infty} \frac{1}{(n+\nu)^m} = \zeta(m,\nu) - \frac{1}{\nu^m}.$$

Theorem 3. *If $m \geq 2$, $\nu > -1$ and $|x| < (n+\nu)^m$, then the following sharp exponential inequality holds:*

$$\left(\frac{(1+\nu)^m - x^2}{(1+\nu)^m}\right)^{a_{m,\nu}} \leq \chi_{m,\nu}(x) \leq \left(\frac{(1+\nu)^m - x^2}{(1+\nu)^m}\right)^{b_{m,\nu}}, \tag{34}$$

with the best possible constants as $b_{m,\nu} = 1$ and $a_{m,\nu} = (1+\nu)^m(\zeta(m,\nu) - \nu^{-m})$.

Taking $\nu = 1$ in (34), it follows that

$$\left(1-\frac{x^2}{2^m}\right)^{a_{m,1}} \leq \chi_{m,1}(x) \leq \left(1-\frac{x^2}{2^m}\right)^{b_{m,1}}. \tag{35}$$

Now, by choosing $m = 2,3,4,5,6$ in (35), we have the following special cases of Theorem 3:

$$(i) \quad \left(1-\frac{x^2}{4}\right)^{a_{2,1}} \leq \chi_{2,1}(x) \leq \left(1-\frac{x^2}{4}\right)^{b_{2,1}} \quad \text{with} \quad a_{2,1} = \frac{2\pi^2}{3},$$

$$(ii) \quad \left(1-\frac{x^2}{8}\right)^{a_{3,1}} \leq \chi_{3,1}(x) \leq \left(1-\frac{x^2}{8}\right)^{b_{3,1}} \quad \text{with} \quad a_{3,1} = 8\zeta(3,1) = 9.61646,$$

$$(iii) \quad \left(1-\frac{x^2}{16}\right)^{a_{4,1}} \leq \chi_{4,1}(x) \leq \left(1-\frac{x^2}{16}\right)^{b_{4,1}} \quad \text{with} \quad a_{4,1} = \frac{8\pi^4}{45},$$

$$(iv) \quad \left(1-\frac{x^2}{16}\right)^{a_{5,1}} \leq \chi_{5,1}(x) \leq \left(1-\frac{x^2}{16}\right)^{b_{5,1}} \quad \text{with} \quad a_{5,1} = 32\,\zeta(5,1) = 33.1817,$$

$$(v) \quad \left(1-\frac{x^2}{16}\right)^{a_{6,1}} \leq \chi_{6,1}(x) \leq \left(1-\frac{x^2}{16}\right)^{b_{6,1}} \quad \text{with} \quad a_{6,1} = \frac{64\pi^6}{945}$$

where, in each of the cases ($m = 2,3,4,5,6$), the best values of $b_{m,1} = 1$ and $\chi_{m,1}(x)$ are listed below

$$\begin{aligned}
\chi_{2,1}(x) &= \frac{\sin(\pi x)}{\pi x - \pi x^3},\\
\chi_{3,1}(x) &= -\frac{1}{(x^2-1)\Gamma(1-x^{2/3})\Gamma\left(\frac{1}{2}\left(1-i\sqrt{3}\right)x^{2/3}+1\right)\Gamma\left(\frac{1}{2}\left(1+i\sqrt{3}\right)x^{2/3}+1\right)},\\
\chi_{4,1}(x) &= -\frac{\sin(\pi\sqrt{x})\sinh(\pi\sqrt{x})}{\pi^2(x^3-x)}\\
\chi_{5,1}(x) &= \frac{1}{(1-x^2)\Gamma(1-x^{2/5})\Gamma(\sqrt[5]{-1}x^{2/5}+1)\Gamma(1-(-1)^{2/5}x^{2/5})\Gamma((-1)^{3/5}x^{2/5}+1)\Gamma(1-(-1)^{4/5}x^{2/5})},\\
\chi_{6,1}(x) &= \frac{\sin(\pi\sqrt[3]{x})\left(\cos(\pi\sqrt[3]{x})-\cosh\left(\sqrt{3}\pi\sqrt[3]{x}\right)\right)}{2\pi^3 x(x^2-1)}.
\end{aligned}$$

3.3. Examples Involving Bessel Functions

In this part, we discuss the generalization of the Redheffer type bound in terms of Bessel and modified Bessel functions. In this regard, we consider the very first result given by Baricz [7], and later by Khalid [9], as well as Baricz and Wu [10].

From ([14], p. 498), it is known that the Bessel function J_ν has the infinite product

$$\mathcal{J}_\nu(x) = 2^\nu\Gamma(\nu+1)x^{-\nu}J_\nu(x) = \prod_{n\geq 1}\left(1-\frac{x^2}{j_{\nu,n}^2}\right) \tag{36}$$

for arbitrary x and $\nu \neq -1,-2,-3,\ldots$. It is also well known that ([14], P. 502)

$$\sum_{n=1}^{\infty}\frac{1}{j_{n,\nu}^2} = \frac{1}{4(\nu+1)}.$$

This implies $\mathcal{J}_\nu \in \mathcal{F}_\nu$. Similarly, $\mathcal{I}_\nu(x)$—the normalized form of the modified Bessel function I_ν—can be expressed as

$$\mathcal{I}_\nu(x) = 2^\nu\Gamma(\nu+1)x^{-\nu}I_\nu(x) = \prod_{n\geq 1}\left(1+\frac{x^2}{j_{\nu,n}^2}\right), \tag{37}$$

which indicates that $\mathcal{I}_\nu \in \mathcal{G}_\nu$. Now, from Theorem 1 (3) and Theorem 2, we have the following results.

Theorem 4. *Consider $\nu > -1$ and $\mathcal{I}_\nu \in \mathcal{G}_\nu$.*

1. *For $|x| < j_{\nu,1}$, we have*

$$\left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)^{a_\nu} \leq \mathcal{I}_\nu(x) \leq \left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)^{b_\nu}, \tag{38}$$

with the best possible constants as $a_\nu = 0$ and $b_\nu = j_{\nu,1}^2/8(\nu+1)$.

2. *For any $r > 0$ and $|x| < r$, we have*

$$\left(\frac{r^2 - x^2}{r^2}\right)^{a_\nu} \leq \mathcal{I}_\nu(x) \leq \left(\frac{r^2 - x^2}{r^2}\right)^{b_\nu}, \tag{39}$$

with the best possible constants as $a_\nu = 0$ and $b_\nu = -r^2/4(\nu+1)$.

Now, from Theorem 1 (6), the following inequality holds for normalized Bessel functions.

Theorem 5. *Consider $\nu > -1$ and $\mathcal{J}_\nu \in \mathcal{F}_\nu$. For $|x| < j_{\nu,1}$, we have*

$$\left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2}\right)^{a_\nu} \leq \mathcal{J}_\nu(x) \leq \left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2}\right)^{b_\nu}, \tag{40}$$

with the best possible constants as $b_\nu = 1$ and $a_\nu = j_{\nu,1}^2/4(\nu+1)$.

3.4. Examples Involving Struve Functions

One of the most well-known special functions is the solution to the non-homogeneous Bessel differential equation

$$z^2y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = z^{\mu+1},$$

called the Struve functions, $\mathtt{S}_\nu$. If $\mathtt{h}_{\nu,n}$ denotes the nth positive zero of $\mathtt{S}_\nu$, then, for $|\nu| \leq 1/2$, the function $\mathtt{S}_\nu$ can be expressed as (see [15])

$$\mathtt{S}_\nu(z) = \frac{z^{\nu+1}}{2^\nu\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\mathtt{h}_{\nu,n}^2}\right). \tag{41}$$

From [16] (Theorem 1), it is useful to note that $\mathtt{h}_{\nu,n} > \mathtt{h}_{\nu,1} > 1$ for $|\nu| < 1/2$. From (41), consider the normalized form

$$\mathcal{S}_\nu(z) := \sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)z^{-\nu}\mathtt{S}_\nu(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\mathtt{h}_{\nu,n}^2}\right). \tag{42}$$

From [17], it follows that for $|\nu| \leq 1/2$,

$$\sum_{n\geq 1}\frac{1}{h_{v,n}^2} = \frac{1}{3(2v+3)}.$$

Consider the modified form of the Struve function

$$\mathcal{L}_\nu(z) = \mathcal{S}_\nu(iz) = \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\mathtt{h}_{\nu,n}^2}\right).$$

Clearly, $\mathcal{S}_\nu \in \mathcal{F}_\nu$ and $\mathcal{L}_\nu \in \mathcal{G}_\nu$.

Now, from Theorem 1 (3) and Theorem 2, we have the following results.

Theorem 6. *Consider $|\nu| < 1/2$ and $\mathcal{L}_\nu \in \mathcal{G}_\nu$.*

1. *For $|x| < h_{\nu,1}$, we have*

$$\left(\frac{h_{\nu,1}^2 + x^2}{h_{\nu,1}^2 - x^2}\right)^{a_\nu} \leq \mathcal{L}_\nu(x) \leq \left(\frac{h_{\nu,1}^2 + x^2}{h_{\nu,1}^2 - x^2}\right)^{b_\nu}, \tag{43}$$

with the best possible constants as $a_\nu = 0$ and $b_\nu = h_{\nu,1}^2/6(2\nu+3)$.

2. *For any $r > 0$ and $|x| < r$, we have*

$$\left(\frac{r^2 - x^2}{r^2}\right)^{a_\nu} \leq \mathcal{L}_\nu(x) \leq \left(\frac{r^2 - x^2}{r^2}\right)^{b_\nu}, \tag{44}$$

with the best possible constants as $a_\nu = 0$ and $b_\nu = -r^2/3(2\nu+3)$.

Now, from Theorem 1 (6), the following inequality holds for normalized Bessel functions.

Theorem 7. *Consider $\nu > -1$ and $\mathcal{S}_\nu \in \mathcal{F}_\nu$. For $|x| < h_{\nu,1}$, we have*

$$\left(\frac{h_{\nu,1}^2 - x^2}{h_{\nu,1}^2}\right)^{a_\nu} \leq \mathcal{S}_\nu(x) \leq \left(\frac{h_{\nu,1}^2 - x^2}{h_{\nu,1}^2}\right)^{b_\nu}, \tag{45}$$

with the best possible constants as $b_\nu = 1$ and $a_\nu = h_{\nu,1}^2/3(2\nu+3)$.

3.5. Examples Involving Dini Functions

The Dini function $d_\nu : \Omega \subseteqq \mathbb{C} \longrightarrow \mathbb{C}$ is defined by

$$d_\nu(z) = (1-v)\mathrm{J}_\nu(z) + z\mathrm{J}'_\nu(z) = \mathrm{J}_\nu(z) - z\mathrm{J}_{\nu+1}(z).$$

The modified Bessel functions are related to the Bessel functions by $\mathrm{I}_\nu(z) = i^{-\nu}\mathrm{J}_\nu(iz)$, which gives the modified Dini function

$$\xi_\nu = \Omega \subseteqq \mathbb{C} \longrightarrow \mathbb{C},$$

defined by

$$\xi_\nu(z) = i^{-\nu} d_\nu(iz) = (1-\nu)I_\nu(z) + zI'_\nu(z) = I_\nu(z) - zI_{\nu+1}(z).$$

For an integer ν, the domain Ω can be taken as the whole complex plane, while Ω is the whole complex plane minus an infinite slit from the origin if ν is not an integer.

In view of the Weierstrassian factorization of $d_\nu(z)$

$$d_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n\geq 1}\left(1 - \frac{z^2}{\alpha_{\nu,n}^2}\right), \tag{46}$$

where $\nu > -1$ and the formula $\xi(z) = i^{-1} d_\nu(iz)$, we have the following Weierstrassian factorization of $\xi_\nu(z)$ for all $\nu > -1$ and $z \in \Omega$:

$$\xi_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n\geq 1}\left(1 + \frac{z^2}{\alpha_{\nu,n}^2}\right), \tag{47}$$

where the infinite product is uniformly convergent on each compact subset of the complex plane, where $\alpha_{\nu,n}$ is the n^{th} positive zero of the Dini function d_ν. The principal branches of $d_\nu(z)$ and $\xi_\nu(z)$ correspond to the principal value of $(z/2)^\nu$, and are analytic in the z-plane cut along the negative real axis from 0 to infinity; that is, the half line $(\infty, 0]$. Now for $\nu > -1$, define the function $\Lambda_\nu : \mathbb{R} \longrightarrow [1, \infty)$ as

$$\Lambda_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} \xi_\nu(x) = \prod_{n\geq 1} \left(1 + \frac{x^2}{\alpha_{\nu,n}^2}\right). \tag{48}$$

Furthermore, for $\nu > -1$, let us define the function $\mathcal{D}_\nu : \mathbb{R} \longrightarrow \mathbb{R}$

$$\mathcal{D}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} d_\nu(x) = \prod_{n\geq 1} \left(1 - \frac{x^2}{\alpha_{\nu,n}^2}\right). \tag{49}$$

From [18], it follows that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_{\nu,n}^2} = \frac{3}{4(\nu+1)}.$$

Comprehensive details of the properties of Dini functions can be found in [11,18] and the references therein.

From the definition of the classes $\mathcal{F}_\nu$ and $\mathcal{G}_\nu$, it is clear that $\Lambda_\nu \in \mathcal{G}_\nu$ and $\mathcal{D}_\nu \in \mathcal{G}_\nu$. Thus, we have the following results, by Theorems 1 and 2.

Theorem 8. *Consider $\nu > -1$ and $\Lambda_\nu \in \mathcal{G}_\nu$.*

1. *For $|x| < \alpha_{\nu,1}$, we have*

$$\left(\frac{\alpha_{\nu,1}^2 + x^2}{\alpha_{\nu,1}^2 - x^2}\right)^{a_\nu} \leq \Lambda_\nu(x) \leq \left(\frac{\alpha_{\nu,1}^2 + x^2}{\alpha_{\nu,1}^2 - x^2}\right)^{b_\nu}, \tag{50}$$

with the best possible constants as $a_\nu = 0$ and $b_\nu = 3\alpha_{\nu,1}^2/8(\nu+1)$.

2. *For any $r > 0$ and $|x| < r$, we have*

$$\left(\frac{r^2 - x^2}{r^2}\right)^{a_\nu} \leq \Lambda_\nu(x) \leq \left(\frac{r^2 - x^2}{r^2}\right)^{b_\nu}, \tag{51}$$

with the best possible constants as $a_\nu = 0$ and $b_\nu = -3r^2/4(\nu+1)$.

Further, Theorem 1 (6) gives the following result.

Theorem 9. *For $\nu > -1$ and $|x| < \alpha_{\nu,1}$, we have*

$$\left(\frac{\alpha_{\nu,1}^2 - x^2}{\alpha_{\nu,1}^2}\right)^{a_\nu} \leq \mathcal{D}_\nu(x) \leq \left(\frac{\alpha_{\nu,1}^2 - x^2}{\alpha_{\nu,1}^2}\right)^{b_\nu}, \tag{52}$$

with the best possible constants as $b_\nu = 1$ and $a_\nu = 3\alpha_{\nu,1}^2/4(\nu+1)$.

3.6. Examples Involving q-Bessel Functions

This section considers the Jackson and Hahn–Exton q-Bessel functions, respectively denoted by $J_\nu^{(2)}(z;q)$ and $J_\nu^{(3)}(z;q)$. For $z \in \mathbb{C}$, $\nu > -1$ and $q \in (0,1)$, both functions are defined by the series

$$J_\nu^{(2)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{n(n+\nu)} \tag{53}$$

$$J_\nu^{(3)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n z^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{\frac{n(n+1)}{2}}. \tag{54}$$

Here,

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}\left(1 - aq^{k-1}\right), and \quad (a;q)_\infty = \prod_{k\geq 1}\left(1 - aq^{k-1}\right)$$

are known as the q-Pochhammer symbol. For a fixed z and $q \to 1$, both of the above q-Bessel functions relate to the classical Bessel function J_ν as $\mathrm{J}_\nu^{(2)}((1-z)q;q) \to J_\nu(z)$ and $\mathrm{J}_\nu^{(3)}((1-z)q;q) \to J_\nu(2z)$. The q-extension of Bessel functions has been studied by several authors, notably, references [19–24] and the various references therein. The geometric properties of q-Bessel functions have been discussed in [25]. It is worth noting that abundant results are available in the literature, regarding the q-extension of Bessel functions; however, we limit ourselves to the requirements of this article. For this purpose, we recall the Hadamard factorization for the normalized q-Bessel functions:

$$z \to \mathcal{J}_\nu^{(2)}(z;q) = 2^\nu c_\nu(q) z^{-\nu} \mathrm{J}_\nu^{(2)}(z;q) \quad \text{and} \quad z \to \mathcal{J}_\nu^{(3)}(z;q) = c_\nu(q) z^{-\nu} \mathrm{J}_\nu^{(3)}(z;q),$$

where $c_\nu(q) = (q;q)_\infty / (q^{\nu+1};q)_\infty$.

Lemma 3 ([25])**.** *For $\nu > -1$, the functions $z \to \mathcal{J}_\nu^{(2)}(z;q)$ and $z \to \mathcal{J}_\nu^{(3)}(z;q)$ are entire functions of order* zero, *which have Hadamard factorization of the form*

$$\mathcal{J}_\nu^{(2)}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z^2}{j_{\nu,n}^2(q)}\right), \quad \mathcal{J}_\nu^{(3)}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z^2}{l_{\nu,n}^2(q)}\right), \tag{55}$$

where $j_{\nu,n}(q)$ and $l_{\nu,n}(q)$ are the nth positive zeros of the functions $\mathcal{J}_\nu^{(2)}(.;q)$ and $\mathcal{J}_\nu^{(3)}(.;q)$, respectively.

We recall that, from [25], the q-extension of the first Rayleigh sum for Bessel functions of the first kind is

$$\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}, \quad \text{is} \quad \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2(q)} = \frac{q^{\nu+1}}{4(q-1)(q^{\nu+1}-1)}. \tag{56}$$

The series form of $\mathcal{J}_\nu^{(3)}(z;q)$ is

$$\mathcal{J}_\nu^{(3)}(z;q) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{n(n+1)}{2}}}{(q,q)_n (q^{\nu+1},q)_n}. \tag{57}$$

Comparing the coefficients of z^2 in (55) and (57), it follows that

$$\sum_{n=1}^{\infty} \frac{1}{l_{\nu,n}^2(q)} = \frac{q}{(q-1)(q^{\nu+1}-1)}. \tag{58}$$

The above facts imply that $\mathcal{J}_\nu^{(i)}(z;q) \in \mathcal{F}_\nu$ for $i = \{1,2\}$. For $i = \{1,2\}$ and $\nu > -1$, denote the n^{th} zero of $\mathcal{J}_\nu^{(i)}(z;q)$ by $b_{i,n}(\nu)$. From (56) and (58), it follows that

$$l_i(\nu) := \sum_{n=1}^{\infty} \frac{1}{b_{i,n}^2(\nu)} = \begin{cases} \frac{q^{\nu+1}}{4(q-1)(q^{\nu+1}-1)} & i = 1, \\ \frac{q}{(q-1)(q^{\nu+1}-1)} & i = 2. \end{cases}$$

Now, we have the following result, by Theorem 1 (6).

Theorem 10. *The function* $x \mapsto \mathcal{J}_\nu^{(i)}(z;q) \in \mathcal{F}_\nu$ *for* $i = \{1,2\}$ *satisfies the sharp Redheffer-type inequality*

$$\left(\frac{b_{i,1}^2(\nu) - x^2}{b_{i,1}^2(\nu)}\right)^{a_\nu} \leq \mathcal{J}_\nu^{(i)}(z;q) \leq \left(\frac{b_{i,1}^2(\nu) - x^2}{b_{i,1}^2(\nu)}\right)^{b_\nu} \tag{59}$$

on I_ν*. Here,* $b_\nu = 1$ *and* $a_\nu = b_{i,1}^2(\nu)l_i(\nu)$ *are the best possible constants.*

4. Conclusions

In this article, we defined two classes of functions on the real domain, using the infinite products of factors involving the positive zeroes of the function. We assume that the infinite product is uniformly convergent, and it is also assumed that the sum of the square of zeroes is convergent. We illustrate several examples that ensure that these classes are non-empty. Functions starting from the most fundamental trigonometric functions (i.e., sin, cos) to special functions, such as Bessel and q-Bessel functions, Hurwitz functions, Dini functions, and their hyperbolic forms, are included in the classes. In conclusion, it follows that the results obtained in Section 2 are similar to the results available in the literature for each of the individual functions listed above. For example, Redheffer-type inequalities for Bessel and modified functions, as stated in Theorem 5 and Theorem 4, form part of the results given previously in [7,9,10], while the inequality obtained in Theorem 8 has also been obtained in ([11], Theorem 7). From Theorem 1 (part 4), it follows that the function $x \mapsto \Lambda_\nu(x)\mathcal{D}_\nu(x)$ is increasing on $(-\alpha_{\nu,n}, 0)$ and decreasing on $(0, \alpha_{\nu,n})$, which has also been obtained in ([11], Theorem 8 (i)). To the best of our knowledge, Theorems 3 and 10 have not been published in the existing literature. We finally conclude that the Redheffer-type inequalities obtained in this study cover a wide range of functions, regarding Theorems 1 and 2. Using the Rayleigh concepts provided in [26], more investigations into the zeroes of special functions may lead to more examples related to the work in this study, and we intend to follow this line of research for future investigations.

Author Contributions: Conceptualization, R.A. and S.R.M.; methodology, R.A. and S.R.M.; validation, R.A. and S.R.M.; formal analysis, R.A. and S.R.M.; investigation, R.A. and S.R.M.; resources, R.A. and S.R.M.; writing—original draft preparation, R.A. and S.R.M.; writing—review and editing, R.A. and S.R.M.; supervision, S.R.M. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (grant project no. 1734).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Redheffer, R. Problem 5642. *Am. Math. Mon.* **1969**, *76*, 422. [CrossRef]
2. Williams, J. Solution of problem 5642. *Am. Math. Mon.* **1969**, *10*, 1153–1154.
3. Li, L.; Zhang, J. A new proof on Redheffer-Williams' inequality. *Far East J. Math. Sci.* **2011**, *56*, 213–217.
4. Sándor, J.; Bhayo, B.A. On an inequality of Redheffer. *Miskolc Math. Notes* **2015**, *16*, 475–482. [CrossRef]
5. Zhu, L. Extension of Redheffer type inequalities to modified Bessel functions. *Appl. Math. Comput.* **2011**, *217*, 8504–8506. [CrossRef]
6. Zhu, L.; Sun, J. Six new Redheffer-type inequalities for circular and hyperbolic functions. *Comput. Math. Appl.* **2008**, *56*, 522–529. [CrossRef]
7. Baricz, Á. Redheffer type inequality for Bessel functions. *J. Inequal. Pure Appl. Math.* **2007**, *8*, 6.

8. Chen, C.-P.; Zhao, J.; Qi, F. Three inequalities involving hyperbolically trigonometric functions. *RGMIA Res. Rep. Coll.* **2003**, *6*, 437–443.
9. Mehrez, K. Redheffer type inequalities for modified Bessel functions. *Arab J. Math. Sci.* **2016**, *22*, 38–42. [CrossRef]
10. Baricz, Á.; Wu, S. Sharp exponential Redheffer-type inequalities for Bessel functions. *Publ. Math. Debrecen* **2009**, *74*, 257–278. [CrossRef]
11. Baricz, Á.; Ponnusamy, S.; Singh, S. Modified Dini functions: Monotonicity patterns and functional inequalities. *Acta Math. Hungar.* **2016**, *149*, 120–142. [CrossRef]
12. Biernacki, M.; Krzyż, J. On the monotonity of certain functionals in the theory of analytic functions. *Ann. Univ. Mariae Curie-Skłodowska. Sect. A* **1955**, *9*, 135–147.
13. Anderson, G.D.; Vamanamurthy, M.K.; Vuorinen, M. Inequalities for quasiconformal mappings in space. *Pac. J. Math.* **1993**, *160*, 1–18. [CrossRef]
14. Watson, G.N. *A Treatise on the Theory of Bessel Functions*; Cambridge University Press: Cambridge, UK, 1944.
15. Baricz, Á.; Ponnusamy, S.; Singh, S. Turán type inequalities for Struve functions. *J. Math. Anal. Appl.* **2017**, *445*, 971–984. [CrossRef]
16. Baricz, Á.; Szász, R. Close-to-convexity of some special functions and their derivatives. *Bull. Malays. Math. Sci. Soc.* **2016**, *39* 427–437. [CrossRef]
17. Baricz, Á.; Kokologiannaki, C.G.; Pogány, T.K. Zeros of Bessel function derivatives. *Proc. Am. Math. Soc.* **2018**, *146*, 209–222. [CrossRef]
18. Baricz, Á.; Pogány, T.K.; Szász, R. Monotonicity properties of some Dini functions. In Proceedings of the 9th IEEE International Symposium on Applied Computational Intelligence and Informatics, Timisoara, Romania, 15–17 May 2014; pp. 323–326.
19. Abreu, L.D. A q-sampling theorem related to the q-Hankel transform. *Proc. Am. Math. Soc.* **2005**, *133*, 1197–1203. [CrossRef]
20. Annaby, M.H.; Mansour, Z.S.; Ashour, O.A. Sampling theorems associated with biorthogonal q-Bessel functions. *J. Phys. A* **2010**, *43*, 295204. [CrossRef]
21. Ismail, M.E.H. The zeros of basic Bessel functions, the functions $J_{\nu+ax}(x)$, and associated orthogonal polynomials. *J. Math. Anal. Appl.* **1982**, *86*, 1–19. [CrossRef]
22. Ismail, M.E.H.; Muldoon, M.E. On the variation with respect to a parameter of zeros of Bessel and q-Bessel functions. *J. Math. Anal. Appl.* **1988**, *135*, 187–207. [CrossRef]
23. Koelink, H.T.; Swarttouw, R.F. On the zeros of the Hahn-Exton q-Bessel function and associated q-Lommel polynomials. *J. Math. Anal. Appl.* **1994**, *186*, 690–710. [CrossRef]
24. Koornwinder, T.H.; Swarttouw, R.F. On q-analogs of the Fourier and Hankel transforms. *Trans. Am. Math. Soc.* **1992**, *333*, 445–461.
25. Baricz, Á.; Dimitrov, D.K.; Mező, I. Radii of starlikeness and convexity of some q-Bessel functions. *J. Math. Anal. Appl.* **2016**, *435*, 968–985. [CrossRef]
26. Kishore, N. The Rayleigh function. *Proc. Am. Math. Soc.* **1963**, *14*, 527–533. [CrossRef]