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Some Characterizations of Complete Hausdorff KM-Fuzzy Quasi-Metric Spaces

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Abstract: Gregori and Romaguera introduced, in 2004, the notion of a KM-fuzzy quasi-metric space as a natural asymmetric generalization of the concept of fuzzy metric space in the sense of Kramosil and Michalek. Ever since, various authors have discussed several aspects of such spaces, including their topological and (quasi-)metric properties as well as their connections with domain theory and their relationship with other fuzzy structures. In particular, the development of the fixed point theory for these spaces and other related ones, such as fuzzy partial metric spaces, has received remarkable attention in the last 15 years. Continuing this line of research, we here establish general fixed point theorems for left and right complete Hausdorff KM-fuzzy quasi-metric spaces, which are applied to deduce characterizations of these distinguished kinds of fuzzy quasi-metric completeness. Our approach, which mixes conditions of Suzuki-type with contractions of $\alpha - \phi$-type in the well-known proposal of Samet et al., allows us to extend and improve some recent theorems on complete fuzzy metric spaces. The obtained results are accompanied by illustrative and clarifying examples.

Keywords: KM-fuzzy quasi-metric space; left complete; right complete; Suzuki^{+} fuzzy $\alpha - \phi$-contractive mapping; fixed point

MSC: 54A40; 54H25; 54E50; 47H10

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1. Introduction

The notions of a KM-fuzzy quasi-metric space and of a GV-fuzzy quasi-metric space were introduced in [1] (see also [2]) as natural asymmetric generalizations of the concept of fuzzy metric space in the sense of Kramosil and Michalek [3], and of George and Veeramani [4,5], respectively. Since then, several authors have studied in detail the topological, metric, and uniform properties of these spaces, as well as their connections with domain theory and their relationship with other structures, such as, for instance, fuzzy partial metric spaces [1,2,6–12].

On the other hand, the study of the fixed point theory for KM-fuzzy quasi-metric spaces has its origin in [13], where the authors gave a version of the Banach contraction principle that was applied, via the domain of words, to the resolution of recurrence equations associated with Quicksort algorithms. Since then, several authors have obtained significant fixed point theorems for these spaces not only of a theoretical nature but also with applications to some fields of the theory of computation (cf. [14–20]). In the very recent article [21], the authors presented an application to nonlinear fractional differential equations via contractive mappings of fuzzy-type on GV-fuzzy (quasi-)metric spaces.

Continuing with this appealing line of research, we here obtain some fixed point results for left and right complete Hausdorff KM-fuzzy quasi-metric spaces that involve conditions of Suzuki-type [22–25] joint with $\alpha - \phi$-contractions in the style of Samet et al. [26]. We point out that while the authors of [26] worked in the context of the so-called (c)-comparison functions, which are usually symbolized with ψ, we do so here in the more general context of comparison functions, which we symbolize with ϕ (see [27,28] for notions, examples, and details about these kinds of functions; see also Definition 2 below). Our results provide extensions and improvements to the fuzzy quasi-metric framework of a recent fixed point theorem that characterizes the completeness of KM-fuzzy metric spaces, obtained in [29] (see [30–32] for other recent characterizations of complete KM-fuzzy metric spaces via fixed point results). A number of illustrative and clarifying examples are also provided. Finally, as a sign of the potential interest of our fixed point results, they are applied to obtain characterizations of left complete Hausdorff and right complete doubly Hausdorff KM-fuzzy quasi-metric spaces, respectively.

A precedent of our study appears in [33], where left complete Hausdorff quasi-metric spaces and right complete doubly Hausdorff quasi-metric spaces are characterized with the help of fixed point theorems stated for $\alpha - \psi$-contractive mappings in the more restrictive framework of (c)-comparison functions.

At this point, it is worth mentioning that the study of aggregation of fuzzy quasi-metric spaces has also recently received fruitful attention from some authors [34,35].

2. Preliminaries

Although the concepts and properties that we include in this section are well known to researchers working in the field of fuzzy (quasi-)metric spaces, we think it is pertinent to note them to help (possible) nonspecialist readers, and also in order to speed us through the rest of the exposition.

In the following, we designate by $\mathbb{N}$ and by $\mathbb{R}^+$ the set of all natural numbers and the set of all non-negative real numbers, respectively. Our main sources for quasi-metric spaces are [36,37], for continuous t-norms they are [38,39] , for fuzzy quasi-metric spaces they are [1,2], and for topological notions and their properties it is [40].

Let X be a (nonempty) set. Consider the following conditions for a function $q : X \times X \to \mathbb{R}^+$ and every $u, v, w \in X$:

(q1) $q(u,v) = q(v,u) = 0 \Leftrightarrow u = v$;

(q2) $q(u,v) = 0 \Leftrightarrow u = v$;

(q3) $q(u,v) \leq q(u,w) + q(w,v)$.

If q fulfills conditions (q1) and (q3), we say that it is a quasi-metric on X, and we say that it is a T_1 quasi-metric on X if it fulfills conditions (q2) and (q3).

By a (T_1) quasi-metric space, we mean a pair (X,q) such that X is a set and q is a (T_1) quasi-metric on X.

Given a quasi-metric q on a set X, we recap the following properties and concepts which will be employed in the rest of the paper.

- The function $q^{-1} : X \times X \to \mathbb{R}^+$ defined as $q^{-1}(u,v) = q(v,u)$ for all $u, v \in X$, is also a quasi-metric on X. If q is T_1, then q^{-1} is also T_1.
- The collection of all open balls $\{B_q(u,\varepsilon) : u \in X, \varepsilon > 0\}$ is a base for a T_0 topology $\tau(q)$ on X, where $B_q(u,\varepsilon) = \{v \in X : q(u,v) < \varepsilon\}$ for all $u \in X$ and $\varepsilon > 0$. If q is T_1, then the topology $\tau(q)$ is T_1. We say that (X,q) is a Hausdorff quasi-metric space if $\tau(q)$ is a Hausdorff (or T_2) topology, and we say that (X,q) is a doubly Hausdorff quasi-metric space if both $\tau(q)$ and $\tau(q^{-1})$ are Hausdorff topologies on X.
- A sequence $(u_n)_{n\in\mathbb{N}}$ in X is $\tau(q)$-convergent to a point $u \in X$ if and only if $q(u,u_n) \to 0$ as $n \to +\infty$. In this case, we write $u_n \to u$ for $\tau(q)$.
- A sequence $(u_n)_{n\in\mathbb{N}}$ in X is called left Cauchy (left K-Cauchy in the classical terminology [37,41,42]) if for each $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that $q(u_n,u_m) < \varepsilon$ whenever $n_\varepsilon \leq n \leq m$.
- We say that (X,q) is left complete (left K-sequentially complete in the classical terminology) if every left Cauchy sequence is $\tau(q)$-convergent.
- A sequence $(u_n)_{n\in\mathbb{N}}$ in X is called right Cauchy (right K-Cauchy in the classical terminology) if it is a left Cauchy sequence in (X,q^{-1}), i.e., if for each $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that $q(u_m,u_n) < \varepsilon$ whenever $n_\varepsilon \leq n \leq m$.
- We say that (X,q) is right complete (right K-sequentially complete in the classical terminology) if every right Cauchy sequence is $\tau(q)$-convergent.

There are many examples of Hausdorff quasi-metric spaces in the literature (see, e.g., [33,36,37]). Next, we provide three noteworthy instances of such spaces.

Example 1. *Let q be the quasi-metric on $\mathbb{N}$ given by $q(n,n) = 0$ for all $n \in \mathbb{N}$, and $q(n,m) = 1/n$ whenever $n \neq m$. Then $\tau(q)$ is the discrete topology on $\mathbb{N}$, so $(\mathbb{N}, q)$ is a Hausdorff quasi-metric space. Moreover, it is neither left complete nor right complete, while $\tau(q^{-1})$ is the cofinite topology on $\mathbb{N}$ and, thus, $(\mathbb{N}, q^{-1})$ is not Hausdorff but it is, clearly, left and right complete; in fact, every noneventually constant sequence is $\tau(q)$-convergent to any $n \in \mathbb{N}$.*

Example 2. *The Alexandroff (or the one-point) compactification of $\mathbb{N}$ is the metrizable topological space (ω, τ_ω) where $\omega := \mathbb{N} \cup \{\infty\}$ and the τ_ω-open sets are all subsets of $\mathbb{N}$, and the sets of the form $\{\infty\} \cup (\mathbb{N} \backslash F)$, where F is a finite subset of $\mathbb{N}$ (see [40] (page 170)). In [33], they constructed an interesting quasi-metric q_ω on ω given by $q_\omega(u,u) = 0$ for all $u \in \omega$, $q_\omega(\infty, n) = 2^{-n}$ for all $n \in \mathbb{N}$, $q_\omega(n,m) = 2^{-n}$ if $n, m \in \mathbb{N}$ with $n < m$, and $q_\omega(u,v) = 1$ otherwise.*

In fact, the topology $\tau(q_\omega)$ agrees with τ_ω, and the topology $\tau((q_\omega)^{-1})$ agrees with the discrete topology on ω. Hence, (ω, q_ω) is a doubly Hausdorff quasi-metric space. Moreover (ω, q_ω) is both left complete and right complete (see [33] (Example 5) for details).

Example 3. *Let $(\mathbb{R}, q_S)$ be the celebrated Sorgenfrey quasi-metric line (see, e.g., [37] (Example 1.1.6)), where q_S is the quasi-metric on the set $\mathbb{R}$ of all real numbers, given by $q_S(u,v) = v - u$ if $u \leq v$, and $q_S(u,v) = 1$ if $u > v$. It is well known that $(\mathbb{R}, q_S)$ is a right complete doubly Hausdorff quasi-metric space; however, it is not left complete because $(-1/n)_{n\in\mathbb{N}}$ is a left Cauchy sequence that is not $\tau(q_S)$-convergent.*

Remember [38,39] that a continuous triangular norm (continuous t-norm for short) is an associative and commutative binary operation $* : [0,1] \times [0,1] \to [0,1]$ such that (i) $x * 1 = x$ for all $x \in [0,1]$; (ii) $x \leq y$ implies $x * z \leq y * z$ for all $x, y, z \in [0,1]$; (iii) $*$ is continuous.

As usual, we denote by $\wedge$ the continuous t-norm given by $x \wedge y = \min\{x, y\}$ for all $x, y \in [0,1]$. It is well known, and easy to see, that $* \leq \wedge$ for every continuous t-norm $*$.

In [1] (see also [2]), the notions of a KM-fuzzy quasi-metric space and of a GV-fuzzy quasi-metric space were introduced and discussed as asymmetric generalizations of the classical notions of fuzzy metric space in the senses of Kramosil and Michalek [3] and George and Veeramani [4,5], respectively.

Let X be a (nonempty) set. Consider the following conditions for a fuzzy set $\mathcal{Q}$ in $X \times X \times \mathbb{R}^+$, a continuous t-norm $*$, and every $u, v, w \in X$:

(Q1) $\mathcal{Q}(u,v,0) = 0$;
(Q2) $u = v \Leftrightarrow \mathcal{Q}(u,v,t) = \mathcal{Q}(v,u,t) = 1$ for all $t > 0$;
(Q3) $u = v \Leftrightarrow \mathcal{Q}(u,v,t) = 1$ for all $t > 0$;
(Q4)$\mathcal{Q}(u,v,t+s) \geq \mathcal{Q}(u,w,t) * \mathcal{Q}(w,v,s)$ for all $t, s \geq 0$;
(Q5) $\mathcal{Q}(u,v,\cdot) : \mathbb{R}^+ \to [0,1]$ is left continuous.

If $\mathcal{Q}$ fulfills conditions (Q1), (Q2), (Q4), and (Q5) we say that the pair $(\mathcal{Q}, *)$ (or simply $\mathcal{Q}$) is a KM-fuzzy quasi-metric on X, and we say that it is a T_1 KM-fuzzy quasi-metric on X if it fulfills conditions (Q1), (Q3), (Q4), and (Q5).

By a (T_1) KM-fuzzy quasi-metric space, we mean a triple $(X, \mathcal{Q}, *)$ such that X is a set and $(\mathcal{Q}, *)$ is a (T_1) KM-fuzzy quasi-metric on X.

A KM-fuzzy quasi-metric $(\mathcal{Q}, *)$ on X fulfilling for every $u, v \in X$:

(Q6) $\mathcal{Q}(u,v,t) = \mathcal{Q}(v,u,t)$ for all $t > 0$, is said to be a KM-fuzzy metric on X.

In this case, we say that the triple $(X, \mathcal{Q}, *)$ is a KM-fuzzy metric space, which corresponds with the classical notion of fuzzy metric space due to Kramosil and Michalek [3], with the only exception of the condition $\mathcal{Q}(u,v,t) \to 1$ as $t \to +\infty$ for all $u, v \in X$, which is required in [3].

In some excerpts and if no confusion arises, KM-fuzzy metrics will be designated with $\mathcal{M}$ instead of $\mathcal{Q}$ and the corresponding KM-fuzzy metric spaces with $(X, \mathcal{M}, *)$.

Remark 1. *Since every GV-fuzzy (quasi-)metric space $(X, \mathcal{Q}, *)$ can be viewed as a KM-fuzzy (quasi-)metric space simply by defining $\mathcal{Q}(u, v, 0) = 0$ for all $u, v \in X$, we only consider KM-fuzzy (quasi-)metric spaces in the rest of the paper.*

Similar to the quasi-metric framework, given a KM-fuzzy quasi-metric $(\mathcal{Q}, *)$ on a set X, we recap the following properties and concepts which will be employed in the rest of the paper:

- For each $u, v \in X$, the function $\mathcal{Q}(u, v, \cdot)$ is nondecreasing.
- The pair $(\mathcal{Q}^{-1}, *)$ is also a KM-fuzzy quasi-metric on X, where the fuzzy set $\mathcal{Q}^{-1} : X \times X \times \mathbb{R}^+ \to [0, 1]$ is defined as $\mathcal{Q}^{-1}(u, v, t) = \mathcal{Q}(v, u, t)$ for all $u, v \in X$ and $t \geq 0$. If $(\mathcal{Q}, *)$ is T_1, then $(\mathcal{Q}^{-1}, *)$ is also T_1.
- The collection of all open balls
$$\{B_{\mathcal{Q}}(u, \varepsilon, t) : u \in X,\ \varepsilon \in (0, 1),\ t > 0\},$$
is a base of open sets for a T_0 topology $\tau(\mathcal{Q})$ on X, where $B_{\mathcal{Q}}(u, \varepsilon, t) = \{v \in X : \mathcal{Q}(u, v, t) > 1 - \varepsilon\}$ for all $u \in X, \varepsilon \in (0, 1)$ and $t > 0$. If $\mathcal{Q}$ is T_1, then the topology $\tau(\mathcal{Q})$ is T_1. We say that $(X, \mathcal{Q}, *)$ is a Hausdorff KM-fuzzy quasi-metric space if $\tau(\mathcal{Q})$ is a Hausdorff (or T_2) topology, and we say that $(X, \mathcal{Q}, *)$ is a doubly Hausdorff KM-fuzzy quasi-metric space if both $\tau(\mathcal{Q})$ and $\tau(\mathcal{Q}^{-1})$ are Hausdorff topologies on X.
- A sequence $(u_n)_{n \in \mathbb{N}}$ in X is $\tau(\mathcal{Q})$-convergent to a point $u \in X$ if and only if for each $t > 0$, $\mathcal{Q}(u, u_n, t) \to 1$ as $n \to +\infty$. In this case, we write $u_n \to u$ for $\tau(\mathcal{Q})$.
- A sequence $(u_n)_{n \in \mathbb{N}}$ in X is called left Cauchy if for each $t > 0$ and $\varepsilon \in (0, 1)$ there exists an $n_{t,\varepsilon} \in \mathbb{N}$ such that $\mathcal{Q}(u_n, u_m, t) > 1 - \varepsilon$ whenever $n_{t,\varepsilon} \leq n \leq m$.
- We say that $(X, \mathcal{Q}, *)$ is left complete if every left Cauchy sequence is $\tau(\mathcal{Q})$-convergent.
- A sequence $(u_n)_{n \in \mathbb{N}}$ in X is called right Cauchy if it is a left Cauchy sequence in $(X, \mathcal{Q}^{-1}, *)$, i.e., if for each $t > 0$ and $\varepsilon \in (0, 1)$ there exists an $n_{t,\varepsilon} \in \mathbb{N}$ such that $\mathcal{Q}(u_m, u_n, t) > 1 - \varepsilon$ whenever $n_{t,\varepsilon} \leq n \leq m$.
- We say that $(X, \mathcal{Q}, *)$ is right complete if every right Cauchy sequence is $\tau(\mathcal{Q})$-convergent.

The following is a basic but fundamental example of a KM-fuzzy quasi-metric space. In fact, it constitutes an important generator of KM-fuzzy quasi-metric spaces from quasi-metric spaces, where the main properties of the original quasi-metric space are preserved by the generated KM-fuzzy quasi-metric space.

Example 4. *See, e.g., [19]. Given a quasi-metric space (X, q), let $\mathcal{Q}_{01}^q : X \times X \times \mathbb{R}^+ \to [0, 1]$ be defined, for every $u, v \in X$, as $\mathcal{Q}_{01}^q(u, v, t) = 1$ if $q(u, v) < t$, with $t > 0$, and $\mathcal{Q}_{01}^q(u, v, t) = 0$ if $q(u, v) \geq t$, with $t \geq 0$. Then $(X, \mathcal{Q}_{01}^q, *)$ is a KM-fuzzy quasi-metric space for any continuous t-norm $*$, such that $\tau(\mathcal{Q}) = \tau(q)$.*

The next two properties, which will be used later, are straightforward.

(p1) $(X, \mathcal{Q}_{01}^q, *)$ is (doubly) Hausdorff if and only if (X, q) is (doubly) Hausdorff.
(p2) $(X, \mathcal{Q}_{01}^q, *)$ is left (resp. right) complete if and only if (X, q) is left (respectively, right) complete.

3. Fixed Point Results and Examples

Samet et al. proved, in [26] (Theorem 2.2), an outstanding fixed point theorem for complete metric spaces which was generalized and extended by other authors in several directions (see, e.g., [33,43–49] and the references therein). In particular, it was checked in [33] that such a theorem provides a characterization of metric completeness.

Recently, a KM-fuzzy metric version of Samet et al.'s theorem was established in [29]. Since that version is a basic piece in our study, we next give the concepts and definitions that are involved in its statement.

Let X be a (nonempty) set. A function $\alpha : X \times X \to \mathbb{R}^+$ is triangular ([44]) provided that for any $u, v, w \in X$, $\alpha(u, v) \geq 1$ and $\alpha(v, w) \geq 1$ imply $\alpha(u, w) \geq 1$.

Now let X be a (nonempty) set, $\alpha : X \times X \to \mathbb{R}^+$ be a function and $T : X \to X$ be a mapping. According to [26] we say that T is α-admissible if $\alpha(Tu, Tv) \geq 1$ whenever $\alpha(u, v) \geq 1$.

The idea of α-regularity, as given in condition (iii) of [26] (Theorem 2.2), was adapted to the KM-fuzzy metric framework in [29] (Definition 2). In a natural way, we generalize this concept to the KM-fuzzy quasi-metric setting as follows.

Definition 1. *Let $(X, \mathcal{Q}, *)$ be a KM-fuzzy quasi-metric space and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then:*

(a) *$(X, \mathcal{Q}, *)$ is called α-regular if for each sequence $(u_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $u_n \to u \in X$ for $\tau(\mathcal{Q})$, it follows that $\alpha(u_n, u) \geq 1$ for all $n \in \mathbb{N}$.*

(b) *$(X, \mathcal{Q}, *)$ is called α^--regular if for each sequence $(u_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $u_n \to u \in X$ for $\tau(\mathcal{Q})$, it follows that $\alpha(u, u_n) \geq 1$ for all $n \in \mathbb{N}$.*

Definition 2 ([27,28])**.** *A comparison function is a nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi^n(t) \to 0$ as $n \to +\infty$, for all $t \geq 0$.*

In the rest of the paper we denote by Φ the set of all comparison functions. Note that if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

In [29] (Definition 1) they introduced the notion of a fuzzy $\alpha - \phi$-contractive mapping in the realm of KM-fuzzy metric spaces. In a natural way, we generalize this concept to the KM-fuzzy quasi-metric setting as follows.

Definition 3. *Let $(X, \mathcal{Q}, *)$ be a KM-fuzzy quasi-metric space, $T : X \to X$ be a mapping, $\alpha : X \times X \to \mathbb{R}^+$ be a function and $\phi \in \Phi$. We say that T is fuzzy $\alpha - \phi$-contractive if for any $u, v \in X$ such that $\alpha(u, v) > 0$ and any $t > 0$, the following condition holds:*

$$Q(u, v, t) > 1 - t \Longrightarrow Q(Tu, Tv, \frac{\phi(t)}{\alpha(u, v)}) > 1 - \frac{\phi(t)}{\alpha(u, v)}.$$

With the above ingredients, it was proved, in [29], the next fuzzy metric version of Samet et al.'s fixed point theorem.

Theorem 1. *Let $(X, \mathcal{M}, *)$ be a complete KM-fuzzy metric space and $T : X \to X$ be a fuzzy $\alpha - \phi$-contractive mapping satisfying the following conditions:*

(i) *α is triangular and T is α-admissible;*
(ii) *there is $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$;*
(iii) *$(X, \mathcal{M}, *)$ is α-regular.*

Then T has a fixed point.

In [29], a variant of Theorem 1 was also obtained in the following terms.

Theorem 2. *Let $(X, \mathcal{M}, *)$ be a complete KM-fuzzy metric space and $T : X \to X$ be a fuzzy $\alpha - \phi$-contractive mapping that fulfills conditions (i) and (ii) of Theorem 1 and the following one:*
(iii$^-$) $(X, \mathcal{M}, *)$ is α^--regular.

Then T has a fixed point.

In our quasi-metric context, it seems reasonable to start by exploring the possibility of obtaining full generalizations of Theorems 1 and 2 to left complete and/or right complete

(Hausdorff) KM-fuzzy quasi-metric spaces. This attempt is motivated by the fact that both Theorems 1 and 2 provide characterizations of complete KM-fuzzy metric spaces, as was shown in [29].

In the rest of this section, we check the next facts which show that the situation in the quasi-metric setting presents interesting differences from the corresponding to the fuzzy metric context:

- Theorem 1 does not admit a full generalization for left complete or for right complete Hausdorff KM-fuzzy quasi-metric spaces. Specifically (see Example 5 below), we give an instance of a left complete and right complete Hausdorff KM-fuzzy quasi-metric space $(X, \mathcal{Q}, *)$, and a fuzzy $\alpha - \phi$-contractive mapping $T : X \to X$ without fixed points, satisfying the following conditions: (i) α is triangular and T is α-admissible; (ii) there is $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$; (iii) $(X, \mathcal{Q}, *)$ is α-regular.
- Theorem 2 admits a full generalization for left complete Hausdorff KM-fuzzy quasi-metric spaces. In fact, we prove a more general result involving an adaptation to our framework of conditions of Suzuki-type (see Theorem 3 below).
- Theorem 2 does not admit a full generalization to right complete Hausdorff KM-fuzzy quasi-metric spaces (see Example 8 below).

We would like to emphasize that this circumstance, far from being a drawback, opens the door to examining a suggestive modification of the Suzuki-type conditions and of α-regularity conditions that will be critical to obtaining some satisfactory results in our setting (see Theorems 4 and 6 below).

At the end of this section, we give an example (see Example 11) showing that in the obtained results, Hausdorffness of $(X, \mathcal{Q}, *)$ cannot be relaxed to T_1.

Example 5. *Let* $X = \{0\} \cup \{n \in \mathbb{N} : n \geq 2\}$, *and let* q *be the quasi-metric on* X *given by*

$$q(u,u) = 0 \text{ for all } u \in X,$$
$$q(0,n) = 1/n \text{ for all } n \in X \backslash \{0\},$$
$$q(n,0) = 1 \text{ for all } n \in X \backslash \{0\},$$

and

$$q(n,m) = 1/n + 1/m \text{ for all } n,m \in X \backslash \{0\} \text{ with } n \neq m.$$

It is clear that (X, q) *is a Hausdorff compact quasi-metric space (note that* $q(0, n) \to 0$ *as* $n \to +\infty$*), so it is left and right complete.*

Therefore, by properties **(p1)** *and* **(p2)**, *the KM-fuzzy quasi-metric space* $(X, \mathcal{Q}_{01}^q, *)$ *is also Hausdorff and left and right complete, for any continuous t-norm* $*$.

Let $T : X \to X$ *be defined as* $T0 = 2$ *and* $Tn = 2n$ *for all* $n \in X \backslash \{0\}$. *Evidently,* T *has no fixed points.*

Consider the function $\alpha : X \times X \to \mathbb{R}^+$ *defined as:*

$$\alpha(n,m) = 1 \text{ for all } n,m \in X \backslash \{0\},$$
$$\alpha(n,0) = 1 \text{ for all } n \in X \backslash \{0\},$$

and

$$\alpha(0,u) = 0 \text{ for all } u \in X.$$

It is obvious that α *is triangular and* T *is* α*-admissible with* $\alpha(n, Tn) = 1$ *for any* $n \in X \backslash \{0\}$.

Furthermore, $(X, \mathcal{Q}_{01}^q, *)$ *is* α*-regular. Indeed, let* $(u_n)_{n \in \mathbb{N}}$ *be a sequence in* X *such that* $\alpha(u_n, u_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *and* $u_n \to u$ *for* $\tau(\mathcal{Q})$. *If* $(u_n)_{n \in \mathbb{N}}$ *is eventually constant, then* $\alpha(u_n, u) = 1$ *for all* $n \in \mathbb{N}$. *Otherwise, we have* $u = 0$, *and* $u_n \neq 0$ *for all* $n \in \mathbb{N}$. *Hence,* $\alpha(u_n, u) = 1$ *for all* $n \in \mathbb{N}$.

Finally, we show that T *is a fuzzy* $\alpha - \phi$*-contractive mapping on* $(X, \mathcal{Q}_{01}^q, *)$ *for* α *as defined above and* $\phi \in \Phi$ *given by* $\phi(t) = t/2$ *for all* $t \geq 0$.

Let $u, v \in X$ *and* $t > 0$ *such that* $\mathcal{Q}_{01}^q(u, v, t) > 1 - t$. *Since* $\alpha(0, u) = 0$ *for all* $u \in X$, *it suffices to consider two cases.*

Case 1. $u, v \in X \backslash \{0\}$.
Set $u = n$ *and* $v = m$.

If $\mathcal{Q}_{01}^q(Tn, Tm, \phi(t)/\alpha(n,m)) = 1$, *we have finished.*

If $\mathcal{Q}_{01}^q(Tn, Tm, \phi(t)/\alpha(n,m)) = 0$, *we deduce that* $q(Tn, Tm) \geq t/2$, *i.e.,* $1/2n + 1/2m \geq t/2$, *so* $1/n + 1/m \geq t$, *which implies that* $q(n,m) \geq t$. *Hence,* $\mathcal{Q}_{01}^q(n, m, t) = 0$, *so* $t > 1$, *and thus* $1/n + 1/m > 1$. *We reach a contradiction because* $n, m \geq 2$.

Case 2. $u \in X \backslash \{0\}$ *and* $v = 0$.
Set $u = n$.

If $\mathcal{Q}_{01}^q(Tn, T0, \phi(t)/\alpha(n,0)) = 1$, *we have finished.*

If $\mathcal{Q}_{01}^q(Tn, T0, \phi(t)/\alpha(n,0)) = 0$, *we deduce that* $q(Tn, T0) \geq t/2$, *i.e.,* $1/2n + 1/4 \geq t/2$, *so* $1/n + 1/2 \geq t$. *Suppose that* $\mathcal{Q}_{01}^q(n, 0, t) = 0$. *Then* $t > 1$ *and also* $q(n,0) \geq t$, *a contradiction because* $q(n,0) = 1$. *Therefore,* $\mathcal{Q}_{01}^q(n, 0, t) = 1$, *so* $q(n,0) < t$, *and thus* $1 < t$, *which contradicts that* $1/n + 1/2 \geq t$.

Hence, in both Case 1 and Case 2 we obtain $\mathcal{Q}_{01}^q(Tu, Tv, \phi(t)/\alpha(u,v)) = 1$, *so* T *is a fuzzy* $\alpha - \phi$*-contractive mapping on* $(X, \mathcal{Q}_{01}^q, *)$.

In [32], we proposed and discussed a notion of Suzuki-type contraction for KM-fuzzy metric spaces, in the following terms.

Let $(X, \mathcal{M}, *)$ be a KM-fuzzy metric space and let $T : X \to X$ be a mapping. For each $t > 0$, set

$$S(t) = \{(u,v) \in X \times X : \min\{\mathcal{M}(u, Tu, t), \mathcal{M}(u, v, t)\} > 1 - t\}.$$

Then T is called a Suzuki fuzzy ϕ-contraction (on $(X, \mathcal{M}, *)$) if there is $\phi \in \Phi$ such that for each $u, v \in X$ and $t > 0$, the following condition holds:

$$(u,v) \in S(t) \Longrightarrow \mathcal{M}(Tu, Tv, \phi(t)) > 1 - \phi(t).$$

It was proved in [32] (Theorem 3.5) that every Suzuki fuzzy ϕ-contraction on a complete KM-fuzzy metric space has a unique fixed point.

Now let $(X, \mathcal{Q}, *)$ be a KM-fuzzy quasi-metric space and let $T : X \to X$ be a mapping. For each $t > 0$, set

$$S^+(t) = \{(u,v) \in X \times X : \min\{\mathcal{Q}(u, v, t), \mathcal{Q}(v, Tv, t)\} > 1 - t\}.$$

Thus, we propose the following.

Definition 4. *Let* $(X, \mathcal{Q}, *)$ *be a KM-fuzzy quasi-metric space,* $T : X \to X$ *be a mapping,* $\alpha : X \times X \to \mathbb{R}^+$ *be a function, and* $\phi \in \Phi$. *We say that* T *is a Suzuki*$^+$ *fuzzy* $\alpha - \phi$*-contractive mapping (on* $(X, \mathcal{Q}, *)$*) if for any* $u, v \in X$ *such that* $\alpha(u,v) > 0$ *and any* $t > 0$, *the following condition holds:*

$$(u,v) \in S^+(t) \Longrightarrow Q(Tu, Tv, \frac{\phi(t)}{\alpha(u,v)}) > 1 - \frac{\phi(t)}{\alpha(u,v)}.$$

Remark 2. *Note that for each* $u, v \in X$ *and* $t > 1$, *we have* $(u,v) \in S^+(t)$.

Remark 3. *If* $(X, \mathcal{M}, *)$ *is a KM-fuzzy metric space, we obtain* $S^+(t) = S(t)$ *for all* $t > 0$, *by the symmetry of* $\mathcal{M}$. *It was noticed in [32] (Remark 3.4) that* $S(t)$ *can be the empty set in some cases.*

Remark 4. *It is not difficult to find examples of Suzuki*$^+$ *fuzzy* $\alpha - \phi$*-contractive mappings on complete KM-fuzzy metric spaces (and, hence, on left and right complete Hausdorff KM-fuzzy quasi-metric spaces) which are not Suzuki fuzzy* ϕ*-contractions for any* $\phi \in \Phi$. *For instance, let* $(X, \mathcal{M}, *)$ *be the complete KM-fuzzy metric space such that* $X = \{0, 1\}$, $\mathcal{M}(x, y, 0) = 0$ *for* *all $x, y \in X$, $\mathcal{M}(x, x, t) = 1$ for all $x \in X$ and $t > 0$, $\mathcal{M}(0, 1, t) = \mathcal{M}(1, 0, t) = 0$ for all $t > 0$, and $*$ is any continuous t-norm. If $T : X \to X$ is defined as $T0 = 0$ and $T1 = 1$, and $\alpha : X \times X \to \mathbb{R}^+$ is defined as $\alpha(0, 0) = \alpha(1, 1) = 1$, and $\alpha(0, 1) = \alpha(1, 0) = 0$, then α is triangular, T is α-admissible, $\alpha(0, T0) = \alpha(1, T1) = 1$, and $(X, \mathcal{M}, *)$ is clearly both α-regular and α^--regular. Moreover, by the definition of T and α, it is obvious that T is a Suzuki$^+$ fuzzy $\alpha - \phi$-contractive mapping for any $\phi \in \Phi$ verifying that $\phi(t) > 0$ for all $t > 0$. However, T is not a Suzuki fuzzy ϕ-contraction for any $\phi \in \Phi$ because it does not have a unique fixed point [32] (Theorem 3.5).*

In order to simplify the proof of our next theorem, the following two auxiliary lemmas will be useful. The proof of the first one is obvious, while the proof of the second one is based upon a very effective idea from Radu [50].

Lemma 1. *Let X be a set, $T : X \to X$ be a mapping, and $\alpha : X \times X \to \mathbb{R}^+$ be a triangular function such that T is α-admissible.*

(a) If there is $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$, then $\alpha(T^n u_0, T^m u_0) \geq 1$ whenever $m > n$.

(b) If there is $u_0 \in X$ such that $\alpha(Tu_0, u_0) \geq 1$, then $\alpha(T^m u_0, T^n u_0) \geq 1$ whenever $m > n$.

Lemma 2. *Let $(X, \mathcal{Q}, *)$ be a KM-fuzzy quasi-metric space, $T : X \to X$ be a mapping, and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. If there exists $u', v' \in X$ verifying, for any $t > 0$, the following condition:*

$$(u', v') \in S^+(t) \Longrightarrow \mathcal{Q}(Tu', Tv', \phi(t)) > 1 - \phi(t), \tag{1}$$

then, for each $t_0 > 1$ and each $n \in \mathbb{N}$, we obtain

$$\mathcal{Q}(T^n u', T^n v', \phi^n(t_0)) > 1 - \phi^n(t_0).$$

Proof. Fix $t_0 > 1$. Since $(u', v') \in S^+(t_0)$ and $(v', Tv') \in S^+(t_0)$, it follows from condition (1) that

$$\mathcal{Q}(Tu', Tv', \phi(t_0)) > 1 - \phi(t_0), \text{ and } \mathcal{Q}(Tv', T^2v', \phi(t_0)) > 1 - \phi(t_0),$$

which implies that $(Tu', Tv') \in S^+(\phi(t_0))$.

Consequently, again by condition (1), $\mathcal{Q}(T^2u', T^2v', \phi^2(t)) > 1 - \phi^2(t)$.

Repeating this process, we deduce, by mathematical induction, that

$$\mathcal{Q}(T^n u', T^n v', \phi^n(t_0)) > 1 - \phi^n(t_0),$$

for all $n \in \mathbb{N}$. □

Theorem 3. *Let $(X, \mathcal{Q}, *)$ be a left complete Hausdorff KM-fuzzy quasi-metric space and $T : X \to X$ be a Suzuki$^+$ fuzzy $\alpha - \phi$-contractive mapping satisfying the following conditions:*

(i) α is triangular and T is α-admissible;

(ii) There exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$;

(iii$^-$) $(X, \mathcal{Q}, *)$ is α^--regular.

Then T has a fixed point.

Proof. By Lemma 1 (a), $\alpha(u_0, T^k u_0) \geq 1$ for all $k \in \mathbb{N}$.

Let $t > 0$ such that $(u_0, T^k u_0) \in S^+(t)$ for all $k \in \mathbb{N}$ (recall that the existence of such a t is guaranteed by Remark 2). Since T is Suzuki$^+$ fuzzy $\alpha - \phi$-contractive, we obtain

$$\mathcal{Q}(Tu_0, T^{k+1}u_0, \frac{\phi(t)}{\alpha(u_0, T^k u_{0)}}) > 1 - \frac{\phi(t)}{\alpha(u_0, T^k u_{0)}}, \tag{2}$$

for all $k \in \mathbb{N}$.

From the inequality (2) and the fact that $\phi(t) \geq \phi(t)/\alpha(u_0, T^k u_0)$, we obtain

$$\mathcal{Q}(Tu_0, T^{k+1}u_0, \phi(t)) > 1 - \phi(t),$$

for all $k \in \mathbb{N}$. Hence, we can apply Lemma 2 to deduce that

$$\mathcal{Q}(T^n u_0, T^{n+k}u_0, \phi^n(t_0)) > 1 - \phi^n(t_0),$$

for all $t_0 > 1$ and $n, k \in \mathbb{N}$.

Fix $t_0 > 1$. Given an arbitrary $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\phi^n(t_0) < \varepsilon$ for all $n \geq n_\varepsilon$. Therefore,

$$\mathcal{Q}(T^n u_0, T^{n+k}u_0, \varepsilon) > 1 - \varepsilon, \tag{3}$$

for all $n \geq n_\varepsilon$ and $k \in \mathbb{N}$.

Thus, $(T^n u_0)_{n\in\mathbb{N}}$ is a left Cauchy sequence in $(X, \mathcal{Q}, *)$. Let $v \in X$ such that $T^n u_0 \to v$ for $\tau(\mathcal{Q})$.

Fix $t > 0$. Then, there exists $n_t \in \mathbb{N}$ such that $\mathcal{Q}(v, T^n u_0, t) > 1 - t$ for all $n \geq n_t$, and also $\mathcal{Q}(T^n u_0, T^{n+1}u_0, t) > 1 - t$ for all $n \geq n_t$, as a consequence of the inequality (3). This implies that $(v, T^n u_0) \in S^+(t)$ for all $n \geq n_t$.

On the other hand, from the facts that $\alpha(u_0, Tu_0) \geq 1$ and T is α-admissible, it follows that $\alpha(T^n u_0, T^{n+1}u_0) \geq 1$ for all $n \in \mathbb{N}$. Thus, by condition (iii$^-$), $\alpha(v, T^n u_0) \geq 1$ for all $n \in \mathbb{N}$. Therefore,

$$\mathcal{Q}(Tv, T^{n+1}u_0, \frac{\phi(t)}{\alpha(v, T^n u_0)}) > 1 - \frac{\phi(t)}{\alpha(v, T^n u_0)},$$

for all $n \geq n_t$. Since $t > \phi(t)/\alpha(v, T^n u_0)$, we deduce that

$$\mathcal{Q}(Tv, T^{n+1}u_0, t) > 1 - t,$$

for all $n \geq n_t$.

Consequently, $T^n u_0 \to Tv$ for $\tau(\mathcal{Q})$. Since $(X, \mathcal{Q}, *)$ is Hausdorff, we conclude that $v = Tv$. □

Remark 5. *The example given in Remark 4 (see also Example 6 below) shows that the fixed point of the preceding theorem is not necessarily unique.*

In the sequel, every (Suzuki$^+$) fuzzy $\alpha - \phi$-contractive mapping on a KM-fuzzy quasi-metric space $(X, \mathcal{Q}, *)$ verifying conditions (i), (ii), and (iii$^-$) of Theorem 3 will be called a coherent (Suzuki$^+$) fuzzy $\alpha - \phi$-contractive mapping (on $(X, \mathcal{Q}, *)$).

The following two examples illustrate Theorem 3. It is interesting to point out that the coherent Suzuki$^+$ fuzzy $\alpha - \phi$-contractive mapping of Example 6 is not a fuzzy $\alpha - \varphi$-contractive mapping for any $\varphi \in \Phi$.

Example 6. *Let $X = [0,1]$ and let $\mathcal{Q} : X \times X \times \mathbb{R}^+ \to [0,1]$ be defined as:*

$\mathcal{Q}(u, v, 0) = 0$ *for all* $u, v \in X$,
$\mathcal{Q}(u, u, t) = 1$ *for all* $u \in X$ *and* $t > 0$,
$\mathcal{Q}(u, v, t) = 1/2$ *if* $u > v$ *and* $t > 0$,

and

$\mathcal{Q}(u, v, t) = 0$ *if* $u < v$ *and* $t > 0$.

It is easy to check that $(X, \mathcal{Q}, *_L)$ is a KM-fuzzy quasi-metric space, where by $*_L$ we denote the famous Łukasiewicz continuous t-norm, which is defined as $x *_L y = \max\{0, x + y - 1\}$ for all $x, y \in [0,1]$.

Moreover, $(X, \mathcal{Q}, *_L)$ is Hausdorff because for each $u \in X$, $B_{\mathcal{Q}}(u, 1/2, 1/2) = \{u\}$, so $\tau(\mathcal{Q})$ is the discrete topology on X. It is also left complete because the left Cauchy sequences are those that are eventually constant.

Let $T : X \to X$ be defined as $Tu = u^{1/2}$ for all $u \in X$, and let $\alpha : X \times X \to \mathbb{R}^+$ be defined as $\alpha(0,0) = 1$, $\alpha(u,v) = 1$ if $1 > u > v > 0$, and $\alpha(u,v) = 0$ otherwise.

Clearly α is triangular, and T is α-admissible with $\alpha(0, T0) = 1$.

Furthermore, $(X, \mathcal{Q}, *_L)$ is α^--regular because if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ such that $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and $u_n \to u$ for $\tau(\mathcal{Q})$, then $u_n = u = 0$ for all $n \in \mathbb{N}$ by the definition of α and the fact that $\tau(\mathcal{Q})$ is the discrete topology on X. Thus, $\alpha(u, u_n) = 1$ for all $n \in \mathbb{N}$.

Next we show that T is Suzuki$^+$ fuzzy $\alpha - \phi$-contractive for α as defined above and $\phi \in \Phi$ given by $\phi(t) = t/2$ for all $t \geq 0$.

Indeed, let $u, v \in X$ such that $\alpha(u,v) > 0$, and $(u,v) \in S^+(t)$, $t > 0$.

It suffices to only consider the case $1 > u > v > 0$. We have $\mathcal{Q}(u,v,t) > 1 - t$ and $\mathcal{Q}(v, Tv, t) > 1 - t$. Since $v < Tv$, we obtain $\mathcal{Q}(v, Tv, t) = 0$, so $t > 1$. Taking into account that $Tu > Tv$, we obtain

$$\mathcal{Q}(Tu, Tv, \frac{\phi(t)}{\alpha(u,v)}) = \frac{1}{2} > 1 - \frac{t}{2} = 1 - \frac{\phi(t)}{\alpha(u,v)}.$$

Hence, all conditions of Theorem 3 are fulfilled. In fact, T has two fixed points, 0 and 1.

Finally, observe that for every $t \in (1/2, 1)$ and $0 < v < u < 1$ one has $\mathcal{Q}(u,v,t) = 1/2 > 1 - t$ and $\alpha(u,v) = 1$. Suppose that there is $\varphi \in \Phi$ such that

$$\mathcal{Q}(Tu, Tv, \frac{\varphi(t)}{\alpha(u,v)}) > 1 - \frac{\varphi(t)}{\alpha(u,v)},$$

for all $t \in (1/2, 1)$.

Since $\mathcal{Q}(Tu, Tv, \varphi(t)/\alpha(u,v)) = 1/2$ and $\alpha(u,v) = 1$, we would have $\varphi(t) > 1/2$ for all $t \in (1/2, 1)$, which is not possible because $\varphi \in \Phi$. Consequently, T is not fuzzy $\alpha - \varphi$-contractive for any $\varphi \in \Phi$.

Example 7. *Let $X = \{1\} \cup \{n/(n+1) : n \in \mathbb{N}\}$ and let $\mathcal{Q} : X \times X \times \mathbb{R}^+ \to [0,1]$ be defined as:*

$\mathcal{Q}(u,v,0) = 0$ *for all $u, v \in X$,*

$\mathcal{Q}(u,u,t) = 1$ *for all $u \in X$ and $t > 0$,*

$\mathcal{Q}(u,1,t) = 1/2$ *for all $u \in X \backslash \{1\}$ and $t \in (0,1]$,*

$\mathcal{Q}(u,v,t) = \min\{u,v\}$ *for all $u, v \in X$ with $u \neq v$, $v \neq 1$, and $t \in (0,1]$,*

and

$\mathcal{Q}(u,v,t) = 1$ *for all $u, v \in X$ and $t > 1$.*

It is routine to check that $(X, \mathcal{Q}, \wedge)$ is a Hausdorff KM-fuzzy quasi-metric space. Furthermore, it is left complete (note that $(n/(n+1))_{n\in\mathbb{N}}$ is a left Cauchy sequence that tau($\mathcal{Q}$))-converges to 1).

Fix $k \in \mathbb{N} \backslash \{1\}$. Let $T : X \to X$ be defined as $T1 = 1$ and

$$T\frac{n}{n+1} = \frac{kn}{kn+1},$$

for all $n \in \mathbb{N}$.

Define $\alpha : X \times X \to \mathbb{R}^+$ as $\alpha(u,v) = 1$ if $u < v$, $\alpha(1,v) = 1$ for all $v \in X$, and $\alpha(u,v) = 0$ otherwise.

Then α is triangular and T is α-admissible with $\alpha(1, T1) = 1$.

Furthermore, $(X, \mathcal{Q}, \wedge)$ is α^--regular. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and $u_n \to u \in X$ for $\tau(\mathcal{Q})$. We can suppose, without loss of generality, that the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly increasing, so $u = 1$ and thus $\alpha(u, u_n) = 1$ for all $n \in \mathbb{N}$.

Next, we show that T is Suzuki$^+$ $\alpha - \phi$-contractive for α as defined above and $\phi \in \Phi$ given by $\phi(t) = 1$ if $t > 1$, and $\phi(t) = 2t/(k+1)$ if $t \in [0,1]$.

Indeed, let $u, v \in X$ such that $\alpha(u,v) > 0$ and $(u,v) \in S^+(t)$, $t > 0$.
We distinguish two cases.
Case 1. $u = 1$ and $v = n/(n+1)$, $n \in \mathbb{N}$.
If $t > 1$, we have

$$\mathcal{Q}(T1, Tv, \frac{\phi(t)}{\alpha(1,v)}) = \mathcal{Q}(1, \frac{kn}{kn+1}, 1) = \frac{kn}{kn+1} > 0 = 1 - \frac{\phi(t)}{\alpha(1,v)}.$$

If $t \in (0,1]$, we have

$$\mathcal{Q}(T1, Tv, \frac{\phi(t)}{\alpha(1,v)}) = \mathcal{Q}(1, \frac{kn}{kn+1}, \frac{2t}{k+1}) = \frac{kn}{kn+1} > 0 = 1 - \frac{\phi(t)}{\alpha(1,v)}.$$

Case 2. $1 < u < v$.
Set $u = n/(n+1)$ and $v = m/(m+1)$.
If $t > 1$, we have

$$\mathcal{Q}(Tu, Tv, \frac{\phi(t)}{\alpha(1,v)}) = Tu = \frac{kn}{kn+1} > 0 = 1 - \frac{\phi(t)}{\alpha(1,v)}.$$

If $t \in (0,1]$, we have $t > 1 - u$, because $(u,v) \in S^+(t)$ and thus $\mathcal{Q}(u,v,t) > 1 - t$. An easy computation shows that

$$\frac{2t}{k+1} > \frac{1}{kn+1},$$

and hence

$$\mathcal{Q}(Tu, Tv, \frac{\phi(t)}{\alpha(1,v)}) = Tu = \frac{kn}{kn+1} > 1 - \frac{2t}{k+1} = 1 - \frac{\phi(t)}{\alpha(1,v)}.$$

We showed that all conditions of Theorem 3 are fulfilled (note that we actually proved that T is fuzzy $\alpha - \phi$-contractive). In fact, T has a (unique) fixed point, $u = 1$.

Next, we present the example announced above showing that Theorem 2 does not admit a full generalization to right complete Hausdorff KM-fuzzy quasi-metric spaces.

Example 8. *Let $q : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ be defined as:*
$q(n,n) = 0$ *for all* $n \in \mathbb{N}$,
$q(n,m) = 1/n$ *if* $n < m$,
and
$q(n,m) = 1/2$ *if* $n > m$.

Then q is a quasi-metric on $\mathbb{N}$ such that $\tau(q)$ is the discrete topology on $\mathbb{N}$, so $(\mathbb{N}, q)$ is a Hausdorff quasi-metric space. Furthermore, it is right complete because the right Cauchy sequences are eventually constant.

*Hence, by properties **(p1)** and **(p2)**, $(\mathbb{N}, \mathcal{Q}_{01}^q, *)$ is a right complete Hausdorff KM-fuzzy quasi-metric space, for any continuous t-norm $*$. Note also that $(\mathbb{N}, q)$, and hence $(\mathbb{N}, \mathcal{Q}_{01}^q, *)$, is not left complete.*

Let $T : \mathbb{N} \to \mathbb{N}$ be defined as $Tn = 2n$ for all $n \in \mathbb{N}$, and $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ be defined as $\alpha(n,m) = 1$ if $n < m$, and $\alpha(n,m) = 0$ if $n \geq m$.

It is clear that α is triangular. In addition, T is α-admissible, with $\alpha(n, Tn) = 1$ for all $n \in \mathbb{N}$.

Furthermore $(\mathbb{N}, \mathcal{Q}_{01}^q, *)$ is α^--regular. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ satisfying $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then $u_n < u_{n+1}$ for all $n \in \mathbb{N}$, so $(u_n)_{n\in\mathbb{N}}$ is not $tau(\mathcal{Q})$)-convergent.

Finally, we show that T is fuzzy $\alpha - \phi$-contractive for α as defined above and $\phi \in \Phi$ given by $\phi(t) = t/2$ for all $t \geq 0$.

Indeed, let $n, m \in \mathbb{N}$ such that $\alpha(n,m) > 0$, and let $t > 0$ such that $\mathcal{Q}_{01}^q(n,m,t) > 1 - t$. Then $n < m$.

If $\mathcal{Q}_{01}^{q}(n,m,t) = 0$, it follows that $q(n,m) \geq t$ and $t > 1$. Thus, $1/n \geq t > 1$, a contradiction. Therefore, $\mathcal{Q}_{01}^{q}(n,m,t) = 1$, so $1/n < t$, and thus $1/2n < t/2$, which implies that $q(2n,2m) < t/2$. Hence,

$$\mathcal{Q}_{01}^{q}(Tn, Tm, \frac{\phi(t)}{\alpha(n,m)}) = \mathcal{Q}_{01}^{q}(2n, 2m, \frac{t}{2}) = 1 > 1 - \frac{\phi(t)}{\alpha(n,m)}.$$

In the light of the preceding example, we proceed to modify in a slight but suitable fashion the notions of α-regularity and of a Suzuki^{+} fuzzy $\alpha - \phi$-contractive mapping, in the following way.

Definition 5. *Let $(X, \mathcal{Q}, *)$ be a KM-fuzzy quasi-metric space and let $\alpha : X \times X \to \mathbb{R}^{+}$ be a function. $(X, \mathcal{Q}, *)$ is said to be α^{-1}-regular if for each sequence $(u_n)_{n\in\mathbb{N}}$ in X satisfying $\alpha(u_{n+1}, u_n) \geq 1$ for all $n \in \mathbb{N}$ and such that $u_n \to u \in X$ for $\tau(\mathcal{Q})$, so it follows that $\alpha(u, u_n) \geq 1$ for all $n \in \mathbb{N}$.*

Definition 6. *Let $(X, \mathcal{Q}, *)$ be a KM-fuzzy quasi-metric space, $T : X \to X$ be a mapping, and $\alpha : X \times X \to \mathbb{R}^{+}$ be a function and $\phi \in \Phi$. We say that T is a Suzuki^{-} fuzzy $\alpha - \phi$-contractive mapping (on $(X, \mathcal{Q}, *)$) if for any $u, v \in X$ such that $\alpha(u,v) > 0$ and any $t > 0$, the following condition holds:*

$$(u,v) \in S^{-}(t) \Longrightarrow Q(Tu, Tv, \frac{\phi(t)}{\alpha(u,v)}) > 1 - \frac{\phi(t)}{\alpha(u,v)},$$

where

$$S^{-}(t) = \{(u,v) \in X \times X : \min\{\mathcal{Q}(u,v,t), \mathcal{Q}(Tv,v,t)\} > 1 - t\}.$$

Remark 6. *Note that if in the statement of Lemma 2, we replace condition (1) with*

$$(u', v') \in S^{-}(t) \Longrightarrow \mathcal{Q}(Tu', Tv', \phi(t)) > 1 - \phi(t),$$

then Lemma 2 remains valid, i.e., we obtain

$$\mathcal{Q}(T^{n}u', T^{n}v', \phi^{n}(t_0)) > 1 - \phi^{n}(t_0),$$

for all $t_0 > 1$ and $n \in \mathbb{N}$.

Now we are in a position to prove the following result.

Theorem 4. *Let $(X, \mathcal{Q}, *)$ be a right complete Hausdorff KM-fuzzy quasi-metric space and $T : X \to X$ be a Suzuki^{-} fuzzy $\alpha - \phi$-contractive mapping satisfying the following conditions:*

(i) α is triangular and T is α-admissible;

(ii^{-}) there exists $u_0 \in X$ such that $\alpha(Tu_0, u_0) \geq 1$;

(iii^{-1}) $(X, \mathcal{Q}, *)$ is α^{-1}-regular.

Then T has a fixed point.

Proof. By Lemma 1 (b), $\alpha(T^{k}u_0, u_0) \geq 1$ for all $k \in \mathbb{N}$.

Let $t > 0$ such that $(T^{k}u_0, u_0) \in S^{-}(t)$ for all $k \in \mathbb{N}$ (this is guaranteed because for any $u, v \in X$ and $t > 1$ one has $(u,v) \in S^{-}(t)$).

Since T is Suzuki^{-} fuzzy $\alpha - \phi$-contractive, we obtain

$$\mathcal{Q}(T^{k+1}u_0, Tu_0, \frac{\phi(t)}{\alpha(T^{k}u_0, u_0)}) > 1 - \frac{\phi(t)}{\alpha(T^{k}u_0, u_0)}, \tag{4}$$

for all $k \in \mathbb{N}$.

From the inequality (4) and the fact that $\phi(t) \geq \phi(t)/\alpha(T^k u_0, u_0)$, it follows that

$$\mathcal{Q}(T^{k+1}u_0, Tu_0, \phi(t)) > 1 - \phi(t),$$

for all $k \in \mathbb{N}$. Thus, by Remark 6, we obtain

$$\mathcal{Q}(T^{n+k}u_0, T^n u_0, \phi^n(t_0)) > 1 - \phi^n(t_0),$$

for all $t_0 > 1$ and $n, k \in \mathbb{N}$.

Fix $t_0 > 1$. Given an arbitrary $\varepsilon \in (0,1)$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\phi^n(t_0) < \varepsilon$ for all $n \geq n_\varepsilon$. Consequently,

$$\mathcal{Q}(T^{n+k}u_0, T^n u_0, \varepsilon) > 1 - \varepsilon,$$

for all $n \geq n_\varepsilon$ and $k \in \mathbb{N}$.

We showed that $(T^n u_0)_{n\in\mathbb{N}}$ is a right Cauchy sequence in $(X, \mathcal{Q}, *)$. Let $v \in X$ such that $T^n u_0 \to v$ for $\tau(\mathcal{Q})$.

Fix $t > 0$. Exactly as in the proof of Theorem 3, there exists $n_t \in \mathbb{N}$ such that $(v, T^n u_0) \in S^-(t)$ for all $n \geq n_t$.

Since $\alpha(T^{n+1}u_0, T^n u_0) \geq 1$ for all $n \in \mathbb{N}$, we deduce from condition (iii^{-1}) that $\alpha(v, T^n u_0) \geq 1$ for all $n \in \mathbb{N}$. Therefore,

$$\mathcal{Q}(Tv, T^{n+1}u_0, \frac{\phi(t)}{\alpha(v, T^n u_0)}) > 1 - \frac{\phi(t)}{\alpha(v, T^n u_0)},$$

for all $n \geq n_t$. Since $t > \phi(t)/\alpha(v, T^n u_0)$, we infer that

$$\mathcal{Q}(Tv, T^{n+1}u_0, t) > 1 - t,$$

for all $n \geq n_t$.

Consequently, $T^n u_0 \to Tv$ for $\tau(\mathcal{Q})$. Since $(X, \mathcal{Q}, *)$ is Hausdorff, we obtain $v = Tv$. □

In the sequel, every (Suzuki$^-$) fuzzy $\alpha - \phi$-contractive mapping on a KM-fuzzy quasi-metric space $(X, \mathcal{Q}, *)$ verifying conditions (i), (ii$^-$), and (iii^{-1}) of Theorem 4 will be called a reverse coherent (Suzuki$^-$) fuzzy $\alpha - \phi$-contractive mapping (on $(X, \mathcal{Q}, *)$).

Next, we give two examples where Theorem 4 is applied. In Example 9, we cannot apply Theorem 3 and, in addition, the comparison function ϕ is not a (c)-comparison function, while in Example 10, the reverse coherent Suzuki$^-$ fuzzy $\alpha - \phi$-contractive mapping is not Suzuki$^+$ fuzzy $\alpha - \phi$-contractive for any $\varphi \in \Phi$ (notice that the last part of Example 6 shows that the coherent Suzuki$^+$ fuzzy $\alpha - \varphi$-contractive mapping of such example is not Suzuki$^-$ fuzzy $\alpha - \varphi$-contractive for any $\varphi \in \Phi$).

Example 9. *Let $(\mathbb{R}, q_S)$ be the right complete Hausdorff quasi-metric space of Example 3. Then, the KM-fuzzy quasi-metric space $(\mathbb{R}, \mathcal{Q}_{01}^{q_S}, *)$ of Example 4 is right complete and Hausdorff for any continuous t-norm $*$, by properties* ***(p1)*** *and* ***(p2)****.*

Let $T : \mathbb{R} \to \mathbb{R}$ be defined as $Tu = 0$ if $u \leq 0$, and $Tu = u/(1+u)$ if $u > 0$.

Now define $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ as $\alpha(u, v) = 1$ if $0 \leq u < v < 1$, and $\alpha(u, v) = 0$ otherwise.

It is routine to check that α is triangular, with T α-admissible and $\alpha(Tu, u) = 1$ for all $u \in (0,1)$.

Furthermore, $(\mathbb{R}, \mathcal{Q}_{01}^{q_S}, *)$ is α^{-1}-regular because if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ satisfying $\alpha(u_{n+1}, u_n) \geq 1$ for all $n \in \mathbb{N}$, and such that $u_n \to u$ for $\tau(\mathcal{Q}_{01}^{q_S})$, we infer that $(u_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $(0,1)$ with $u = \inf_{n\in\mathbb{N}} u_n$. Therefore, $0 \leq u < u_n$ for all $n \in \mathbb{N}$, so $\alpha(u, u_n) = 1$ for all $n \in \mathbb{N}$.

Finally, we show that T is a fuzzy $\alpha - \phi$-contractive mapping on $(\mathbb{R}, \mathcal{Q}_{01}^{q_S}, *)$, and, hence, a coherent (Suzuki$^-$) fuzzy $\alpha - \phi$-contractive mapping, for α as defined above and $\phi \in \Phi$ given by $\phi(t) = 1$ for all $t > 1$, and $\phi(t) = t/(t+1)$ for all $t \in [0,1)$ (it is well known that ϕ is not a *(c)-comparison function because the series $\sum_{n=1}^{\infty} \phi^n(1)$ is not convergent; in fact, $\sum_{n=1}^{\infty} \phi^n(t)$ does not converge for any $t > 0$).*

Let $u, v \in \mathbb{R}$ such that $\alpha(u,v) > 0$ and $\mathcal{Q}_{01}^{q_S}(u,v,t) > 1 - t$, $t > 0$.

If $\mathcal{Q}_{01}^{q_S}(u,v,t) = 0$, we deduce that $q_S(u,v) \geq t > 1$, so $v - u > 1$, which is not possible because $0 \leq u < v < 1$.

Hence, we have $\mathcal{Q}_{01}^{q_S}(u,v,t) = 1$. Thus, $q_S(u,v) = v - u < t$, and $q_S(Tu,Tv) = Tv - Tu < 1$, so $\mathcal{Q}_{01}^{q_S}(Tu,Tv,1) = 1$.

Suppose $t > 1$. Then $\phi(t) = 1$, and thus

$$\mathcal{Q}_{01}^{q_S}(Tu,Tv,\frac{\phi(t)}{\alpha(u,v)}) = \mathcal{Q}_{01}^{q_S}(Tu,Tv,1) = 1 > 0 = 1 - \frac{\phi(t)}{\alpha(u,v)}.$$

Now suppose $t \leq 1$. Then

$$\begin{aligned} q_S(Tu,Tv) &= Tv - Tu = \frac{v-u}{(1+u)(1+v)} < \frac{v-u}{1+v-u} \\ &= \frac{q_S(u,v)}{1+q_S(u,v)} < \frac{t}{1+t}, \end{aligned}$$

and, consequently,

$$\mathcal{Q}_{01}^{q_S}(Tu,Tv,\frac{\phi(t)}{\alpha(u,v)}) = \mathcal{Q}_{01}^{q_S}(Tu,Tv,\frac{t}{1+t}) = 1 > 1 - \frac{\phi(t)}{\alpha(u,v)}.$$

We verified that all conditions of Theorem 4 are fulfilled. In fact, T has a (unique) fixed point, $u = 0$.

*Note that we cannot apply Theorem 3 to this example because $(\mathbb{R}, q_S)$, and hence $(\mathbb{R}, \mathcal{Q}_{01}^{q_S}, *)$, is not left complete (see Example 3 and property **(p2)**).*

Example 10. *Let $\mathcal{Q} : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to [0,1]$ be defined as:*

$\mathcal{Q}(n,m,0) = 0$ *for all* $n, m \in \mathbb{N}$,

$\mathcal{Q}(n,m,t) = (n+t)/(m+t)$ *if* $n \leq m$ *and* $t > 0$,

and

$\mathcal{Q}(n,m,t) = 0$ *if* $n > m$ *and* $t > 0$.

It is easily verified that $(\mathbb{N}, \mathcal{Q}, \mathrm{Pr})$ is a KM-fuzzy quasi-metric space, where by Pr we denote the well-known product continuous t-norm, i.e., $x \,\mathrm{Pr}\, y = xy$ for all $x, y \in [0,1]$ (compare [51] (Example 2.5), [52] (Example 3(A))). It is also Hausdorff because $\tau(\mathcal{Q})$ is the discrete topology on $\mathbb{N}$. Indeed, for each $n \in \mathbb{N}$, we have

$B_{\mathcal{Q}}(n, 1/(n+2), 1/(n+2)) = \{n\}$.

Moreover, it is right complete because if $(u_n)_{n\in\mathbb{N}}$ is a right Cauchy sequence in $(\mathbb{N}, \mathcal{Q}, \mathrm{Pr})$ and we fix $t_0 \in (0,1)$, there is $n_0 \in \mathbb{N}$ such that $\mathcal{Q}(u_m, u_n, t_0) > 1 - t_0$ whenever $m \geq n \geq n_0$. Hence, $\mathcal{Q}(u_m, u_n, t_0) > 0$ whenever $m \geq n \geq n_0$, which implies, in particular, that $u_m \leq u_{n_0}$ for all $m \geq n_0$, so $(u_n)_{n\in\mathbb{N}}$ is eventually constant.

Let $T : \mathbb{N} \to \mathbb{N}$ be defined as $Tn = n^2 - 2n + 2$ for all $n \in \mathbb{N}$, and $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ be defined as $\alpha(1,1) = 1$, $\alpha(n,m) = 1$ if $1 < n < m$, and $\alpha(n,m) = 0$ otherwise.

Then, we have $T1 = 1$, $T2 = 2$, and $Tn > n$ for all $n > 2$.

Clearly, α is triangular with T α-admissible, and $\alpha(T1, 1) = 1$.

Furthermore $(\mathbb{N}, \mathcal{Q}, \mathrm{Pr})$ is α^{-1}-regular because if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ satisfying $\alpha(u_{n+1}, u_n) \geq 1$ for all $n \in \mathbb{N}$, and such that $u_n \to u$ for $\tau(\mathcal{Q})$, we infer that $u = u_n = 1$ for all $n \in \mathbb{N}$, so $\alpha(u, u_n) = 1$ for all $n \in \mathbb{N}$.

Now, we show that T is a Suzuki$^-$ fuzzy $\alpha - \phi$-contractive mapping on $(\mathbb{N}, \mathcal{Q}, \mathrm{Pr})$ for α defined as above and $\phi \in \Phi$ given by $\phi(t) = 1$ for all $t > 1$, and $\phi(t) = t/2$ for all $t \in [0,1)$.

Let $n, m \in \mathbb{N}$ such that $\alpha(n,m) > 0$ and $(n,m) \in S^{-}(t)$, $t > 0$. By the definition of α, we only consider the case $1 < n < m$.

From the fact that $(n,m) \in S^{-}(t)$, we deduce that $\mathcal{Q}(n,m,t) > 1-t$ and $\mathcal{Q}(Tm,m,t) > 1-t$. Since $m > 2$, we have $Tm > m$, so $\mathcal{Q}(Tm,m,t) = 0$, and thus $t > 1$.

On the other hand, from $n < m$ it follows that $Tn < Tm$, so

$$\mathcal{Q}(Tn, Tm, \frac{\phi(t)}{\alpha(n,m)}) = \mathcal{Q}(Tn, Tm, 1) = \frac{Tn+1}{Tm+1} > 0 = 1 - \frac{\phi(t)}{\alpha(n,m)}.$$

We verified that all conditions of Theorem 4 are fulfilled. In fact, T has two fixed points, 1 and 2.

Finally, we check that T is not a Suzuki^{+} fuzzy $\alpha - \varphi$-contractive mapping for any $\varphi \in \Phi$. Take $n = 2$ and $m = 3$. Then $\alpha(2,3) = 1 > 0$, and $T3 = 5$.

Set $a = (\sqrt{33} - 5)/2$. We show that

$$\mathcal{Q}(2,3,t) > 1-t \text{ and } \mathcal{Q}(3,T3,t) > 1-t,$$

for all $t \in (a,1)$.

We have

$$\mathcal{Q}(2,3,t) = (2+t)/(3+t) \text{ and } \mathcal{Q}(3,T3,t) = (3+t)/(5+t),$$

for all $t > 0$, so, in particular, $\mathcal{Q}(3,5,a) = (3+a)/(5+a) = 1-a$. Hence, for each $t \in (a,1)$, we obtain

$$\mathcal{Q}(3,5,t) > \frac{3+a}{5+a} = 1-a > 1-t.$$

As $(2+t)/(3+t) > (3+t)/(5+t)$, we infer that

$$\mathcal{Q}(2,3,t) > \mathcal{Q}(2,5,t) > 1-t,$$

for all $t \in (a,1)$.

Suppose that there exists $\varphi \in \Phi$ satisfying

$$\mathcal{Q}(T2, T3, \frac{\varphi(t)}{\alpha(2,3)}) > 1 - \frac{\varphi(t)}{\alpha(2,3)},$$

for all $t \in (a,1)$. This implies that

$$\frac{2+\varphi(t)}{5+\varphi(t)} > 1 - \varphi(t),$$

i.e., $(\varphi(t))^2 + 5\varphi(t) > 3$ for all $t \in (a,1)$.

As $(\sqrt{37} - 5)/2$ is the positive solution of the equation $(\varphi(t))^2 + 5\varphi(t) = 3$, and also $(\sqrt{37} - 5)/2 > a$, it follows that $\varphi(t) > a$ for all $t \in (a,1)$, which contradicts our assumption that $\varphi \in \Phi$.

We conclude this section with the promised example showing that Hausdorffness cannot be relaxed to T_1 in the statement of Theorems 3 and 4.

Example 11. *Let $(\mathbb{N}, q)$ be the Hausdorff quasi-metric space of Example 1. Then $(\mathbb{N}, q^{-1})$ is T_1 but not Hausdorff, so, by property **(p1)**, the KM-fuzzy quasi-metric $(\mathbb{N}, \mathcal{Q}_{01}^{q^{-1}}, *)$ is T_1 but not Hausdorff for any continuous t-norm $*$. In addition, $(\mathbb{N}, \mathcal{Q}_{01}^{q}, *)$ is both left complete and right complete by property **(p2)**.*

Define $T : X \to X$ as $Tn = 2n$ for all $n \in \mathbb{N}$, and $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^{+}$ as $\alpha(n,m) = 1$ for all $n, m \in \mathbb{N}$.

Obviously, α is triangular, T is α-admissible with $\alpha(n, Tn) = \alpha(Tn, n) = 1$ for all $n \in \mathbb{N}$, and $(\mathbb{N}, \mathcal{Q}_{01}^{q}, *)$ is both α^{-}-regular and α^{-1}-regular.

We show that T is fuzzy $\alpha - \phi$-contractive for α as defined above and ϕ given by $\phi(t) = t/2$ for all $t \geq 0$.

Let $n, m \in \mathbb{N}$ such that $\mathcal{Q}_{01}^{q^{-1}}(n, m, t) > 1 - t, t > 0$.

If $\mathcal{Q}_{01}^{q^{-1}}(n, m, t) = 0$, we deduce that $t > 1$ and $q^{-1}(n, m) \geq t$, so $1/m > 1$, which is not possible.

Hence, $\mathcal{Q}_{01}^{q^{-1}}(n, m, t) = 1$, and thus $q^{-1}(n, m) < t$, i.e., $1/m < t$. Therefore, $1/2m < t/2$, so $q^{-1}(2n, 2m) < t/2$, and, consequently,

$$\mathcal{Q}_{01}^{q^{-1}}(Tn, Tm, \frac{\phi(t)}{\alpha(n,m)}) = \mathcal{Q}_{01}^{q^{-1}}(2n, 2m, \frac{t}{2}) = 1 > 1 - \frac{\phi(t)}{\alpha(n,m)}.$$

We showed that T verifies all conditions of Theorems 3 and 4. Nevertheless, it has no fixed points.

4. Characterizations of Left Completeness and Right Completeness

In order to obtain our characterizations of left and right completeness, the following technical results will be useful.

Lemma 3. *Let $(u_k)_{k\in\mathbb{N}}$ be a left or right Cauchy sequence in a KM-fuzzy quasi-metric space $(X, \mathcal{Q}, *)$ and let $v \in X$. If for each $\varepsilon \in (0,1)$ there is a subsequence $(u_{k_\varepsilon(n)})_{n\in\mathbb{N}}$ of $(u_k)_{k\in\mathbb{N}}$ such that $\mathcal{Q}(v, u_{k_\varepsilon(n)}, \varepsilon) > 1 - \varepsilon$ for all $n \in \mathbb{N}$, then the sequence $(u_k)_{k\in\mathbb{N}}$ $\tau(\mathcal{Q})$-converges to v.*

Proof. Fix $\varepsilon \in (0,1)$. By continuity of $*$ there is $\delta \in (0, \varepsilon)$ such that $(1-\varepsilon) * (1-\delta) > 1 - 2\varepsilon$.

- If $(u_k)_{k\in\mathbb{N}}$ be left Cauchy there is $k_\delta \in \mathbb{N}$ such that

  $$\mathcal{Q}(u_i, u_j, \delta) > 1 - \delta,$$

  whenever $k_\delta \leq i \leq j$. Take $n_0 \in \mathbb{N}$ such that $k_\varepsilon(n_0) \geq k_\delta$. Then, for each $j \geq k_\varepsilon(n_0)$ we obtain

  $$\mathcal{Q}(v, u_j, 2\varepsilon) \geq \mathcal{Q}(v, u_{k_\varepsilon(n_0)}, \varepsilon) * \mathcal{Q}(u_{k_\varepsilon(n_0)}, u_j, \delta) \geq (1-\varepsilon) * (1-\delta) > 1 - 2\varepsilon,$$

  which implies that $u_k \to v$ for $\tau(\mathcal{Q})$.
- If $(u_k)_{k\in\mathbb{N}}$ be right Cauchy there is $k_\delta \in \mathbb{N}$ such that

  $$\mathcal{Q}(u_i, u_j, \delta) > 1 - \delta,$$

  whenever $k_\delta \leq j \leq i$. For each $j \geq k_\delta$ take $n_j \in \mathbb{N}$ such that $k_\varepsilon(n_j) \geq j$. Therefore, we obtain

  $$\mathcal{Q}(v, u_j, 2\varepsilon) \geq \mathcal{Q}(v, u_{k_\varepsilon(n_j)}, \varepsilon) * \mathcal{Q}(u_{k_\varepsilon(n_j)}, u_j, \delta) \geq (1-\varepsilon) * (1-\delta) > 1 - 2\varepsilon,$$

  for all $j \geq k_\delta$, which implies that $u_k \to v$ for $\tau(\mathcal{Q})$.

□

Corollary 1. *Let $(u_k)_{k\in\mathbb{N}}$ be a left or right Cauchy sequence in a KM-fuzzy quasi-metric space $(X, \mathcal{Q}, *)$. If $(u_k)_{k\in\mathbb{N}}$ has a subsequence that is $\tau(\mathcal{Q})$-convergent to some $v \in X$, then it is $\tau(\mathcal{Q})$-convergent to v.*

Lemma 4. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in a T_1 KM-fuzzy quasi-metric space $(X, \mathcal{Q}, *)$ and let $v \in X$. If $(u_k)_{k\in\mathbb{N}}$ is left or right Cauchy and it is not $\tau(\mathcal{Q})$-convergent to v, then there is $\varepsilon_v \in (0,1)$ such that*

$$\mathcal{Q}(v, u_k, \varepsilon_v) \leq 1 - \varepsilon_v,$$

whenever $v \neq u_k$.

Proof. Assume, without loss of generality, that $u_j \neq u_k$ whenever $j \neq k$, and $v \neq u_k$ for all $k \in \mathbb{N}$. By Lemma 3, there exist $\varepsilon \in (0,1)$ and $k_\varepsilon \in \mathbb{N}$ such that

$$\mathcal{Q}(v, u_k, \varepsilon) \leq 1 - \varepsilon,$$

for all $k \geq k_\varepsilon$.

Since $(X, \mathcal{Q}, *)$ is T_1, for each $k \in \{1, ..., k_\varepsilon - 1\}$ (assuming $k_\varepsilon > 1$), there is $\varepsilon_k \in (0,1)$ such that

$$\mathcal{Q}(v, u_k, \varepsilon_k) \leq 1 - \varepsilon_k.$$

Setting $\varepsilon_v = \min\{\varepsilon, \varepsilon_1, ..., \varepsilon_{k_\varepsilon - 1}\}$, we conclude that

$$\mathcal{Q}(v, u_k, \varepsilon_v) \leq 1 - \varepsilon_v,$$

for all $k \in \mathbb{N}$. □

Theorem 5. *For a Hausdorff KM-fuzzy quasi-metric space $(X, \mathcal{Q}, *)$ the following statements are equivalent:*

(1) $(X, \mathcal{Q}, *)$ is left complete.
(2) Every coherent Suzuki$^+$ fuzzy $\alpha - \phi$-contractive mapping on $(X, \mathcal{Q}, *)$ has a fixed point.
(3) Every coherent fuzzy $\alpha - \phi$-contractive mapping on $(X, \mathcal{Q}, *)$ has a fixed point.

Proof. (1) $\Rightarrow$ (2) It follows from Theorem 3.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Suppose that $(X, \mathcal{Q}, *)$ is not left complete. Then, there is a left Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $(X, \mathcal{Q}, *)$ such that $u_n \neq u_m$ whenever $n \neq m$, and that is not $tau(\mathcal{Q})$)-convergent.

By Lemma 4, we can find a strictly decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$ such that

$$\mathcal{Q}(u_n, u_k, \varepsilon_n) \leq 1 - \varepsilon_n,$$

for all $n \in \mathbb{N}$ and $k \neq n$.

As $(u_n)_{n\in\mathbb{N}}$ is left Cauchy, we can find a sequence $(l(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $l(1) > 1$, $l(n) > \max\{n, l(n-1)\}$ for all $n > 1$, and

$$\mathcal{Q}(u_j, u_k, \frac{\varepsilon_n}{2}) > 1 - \frac{\varepsilon_n}{2}, \tag{5}$$

whenever $l(n) \leq j \leq k$.

Now, we define a mapping $T : X \to X$ as $Tu_n = u_{l(n)}$ for all $n \in \mathbb{N}$, and $Tu = u_1$ for all $u \in X \backslash \{u_n : n \in \mathbb{N}\}$.

Note that T is free of fixed points because $l(n) > n$ for all $n \in \mathbb{N}$, and thus $u_n \neq u_{l(n)}$.

Let $\alpha : X \times X \to \mathbb{R}^+$ be defined as $\alpha(u, v) = 1$ if $u = u_n$ and $v = u_m$ with $n < m$, and $\alpha(u, v) = 0$ otherwise.

Obviously, α is triangular. Moreover, T is α-admissible because if $\alpha(u, v) = 1$, we have $u = u_n$, $v = u_m$, with $n < m$; so $l(n) < l(m)$, and thus $\alpha(Tu, Tv) = \alpha(u_{l(n)}, u_{l(m)}) = 1$. We also have $\alpha(u_n, Tu_n) = 1$ for all $n \in \mathbb{N}$.

On the other hand, $(X, \mathcal{Q}, *)$ is α^--regular because if $(v_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(v_{n+1}, v_n) \geq 1$ for all $n \in \mathbb{N}$, it follows that $(v_n)_{n\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$, which is left Cauchy but not $\tau(\mathcal{Q})$-convergent, so $(v_n)_{n\in\mathbb{N}}$ is not $\tau(\mathcal{Q})$-convergent by Corollary 1.

Finally, we prove that T is a fuzzy $\alpha - \phi$-contractive mapping for the function α defined as above, and $\phi \in \Phi$ given by $\phi(t) = t/2$ for all $t \geq 0$.

Indeed, let $u, v \in X$ such that $\alpha(u, v) > 0$. Then $u = u_n$ and $v = u_m$ with $n < m$ and $\alpha(u_n, u_m) = 1$.

Suppose that $\mathcal{Q}(u,v,t) > 1-t$, $t>0$. Then

$$\mathcal{Q}(u_n,u_m,t) > 1-t.$$

If $\varepsilon_n \geq t$, we obtain

$$1-\varepsilon_n \geq \mathcal{Q}(u_n,u_m,\varepsilon_n) \geq \mathcal{Q}(u_n,u_m,t) > 1-t > 1-\varepsilon_n,$$

a contradiction. Therefore, $t > \varepsilon_n$. From the inequality 6, it follows that

$$\mathcal{Q}(u_{l(n)},u_{l(m)},\frac{\varepsilon_n}{2}) > 1-\frac{\varepsilon_n}{2}.$$

Hence,

$$\mathcal{Q}(Tu,Tv,\frac{\phi(t)}{\alpha(u,v)}) = \mathcal{Q}(u_{l(n)},u_{l(m)},\frac{t}{2}) \geq \mathcal{Q}(u_{l(n)},u_{l(m)},\frac{\varepsilon}{2}) > 1-\frac{\varepsilon}{2} > 1-\frac{t}{2}.$$

Thus, we constructed a coherent fuzzy $\alpha-\phi$-contractive mapping on $(X,\mathcal{Q},*)$ without fixed point. This concludes the proof. □

Theorem 6. *For a doubly Hausdorff KM-fuzzy quasi-metric space $(X,\mathcal{Q},*)$, the following statements are equivalent:*

(1) $(X,\mathcal{Q},*)$ is right complete.

(2) Every reverse coherent Suzuki$^-$ fuzzy $\alpha-\phi$-contractive mapping on $(X,\mathcal{Q},*)$ has a fixed point.

(3) Every reverse coherent fuzzy $\alpha-\phi$-contractive mapping on $(X,\mathcal{Q},*)$ has a fixed point.

Proof. $(1)\Rightarrow(2)$ It follows from Theorem 4.

$(2)\Rightarrow(3)$ It is obvious.

$(3)\Rightarrow(1)$ Suppose that $(X,\mathcal{Q},*)$ is not right complete. Then, there is a right Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $(X,\mathcal{Q},*)$ that is not $tau(\mathcal{Q})$)-convergent and verifies $u_n \neq u_m$ whenever $n\neq m$.

As $(u_n)_{n\in\mathbb{N}}$ is right Cauchy, we can find a sequence $(l(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $l(1)>1$, $l(n) > \max\{n, l(n-1)\}$ for all $n>1$, and

$$\mathcal{Q}(u_k,u_j,\frac{\varepsilon_n}{2}) > 1-\frac{\varepsilon_n}{2},$$

whenever $k \geq j \geq l(n)$.

Now, we distinguish two cases.

Case 1. The sequence $(u_n)_{n\in\mathbb{N}}$ is not $\tau(\mathcal{Q}^{-1})$-convergent or it is $\tau(\mathcal{Q}^{-1})$-convergent to a point $v\in X\backslash\{u_n : n\in\mathbb{N}\}$.

Case 2. There is an $n_0\in\mathbb{N}$ such that the sequence $(u_n)_{n\in\mathbb{N}}$ is $tau(\mathcal{Q})$)-convergent to u_{n_0}.

In Case 1, by our assumption that $(X,\mathcal{Q}^{-1},*)$ is Hausdorff, it follows from Lemma 4 that there is the existence of a strictly decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$ such that

$$\mathcal{Q}(u_k,u_n,\varepsilon_n) \leq 1-\varepsilon_n, \tag{6}$$

for all $n\in\mathbb{N}$ and $k\neq n$.

Set $C=\{u_n : n\in\mathbb{N}\}$, and define a mapping $T: X\to X$ as $Tu_n = u_{l(n)}$ for all $n\in\mathbb{N}$, and $Tu = u_1$ for all $u\in X\backslash C$.

Note that T is free of fixed points because $l(n)>n$ for all $n\in\mathbb{N}$, and thus $u_n\neq u_{l(n)}$.

Let $\alpha : X\times X\to\mathbb{R}^+$ be defined as $\alpha(u,v)=1$ if $u=u_m$ and $v=u_n$ with $m>n$, and $\alpha(u,v)=0$ otherwise.

Clearly, α is triangular. Moreover, T is α-admissible because if $\alpha(u,v) = 1$, we have $u = u_m$, $v = u_n$, with $m > n$; so $l(m) > l(n)$, and thus $\alpha(Tu, Tv) = \alpha(u_{l(m)}, u_{l(n)}) = 1$. We also have $\alpha(u_n, Tu_n) = 1$ for all $n \in \mathbb{N}$.

Furthermore, $(X, \mathcal{Q}, *)$ is α^{-1}-regular because if $(v_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(v_{n+1}, v_n) \geq 1$ for all $n \in \mathbb{N}$, it follows that $(v_n)_{n\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$ which is right Cauchy but not $\tau(\mathcal{Q})$-convergent, so $(v_n)_{n\in\mathbb{N}}$ is not $\tau(\mathcal{Q})$-convergent by Corollary 1.

Next, we show that T is a fuzzy $\alpha - \phi$-contractive mapping for the function α defined as above, and $\phi \in \Phi$ given by $\phi(t) = t/2$ for all $t \geq 0$.

Indeed, let $u, v \in X$ such that $\alpha(u,v) > 0$. Then $u = u_m$ and $v = u_n$ with $m > n$ and $\alpha(u_m, u_n) = 1$.

Suppose that $\mathcal{Q}(u, v, t) > 1 - t$, $t > 0$. Then

$$\mathcal{Q}(u_m, u_n, t) > 1 - t.$$

Similarly to the proof of Theorem 5, if $\varepsilon_n \geq t$, we obtain, by the inequality (6),

$$1 - \varepsilon_n \geq \mathcal{Q}(u_m, u_n, \varepsilon_n) \geq \mathcal{Q}(u_m, u_n, t) > 1 - t \geq 1 - \varepsilon_n,$$

a contradiction. Therefore, $t > \varepsilon_n$. Taking into account that

$$\mathcal{Q}(u_{l(m)}, u_{l(n)}, \frac{\varepsilon_n}{2}) > 1 - \frac{\varepsilon_n}{2},$$

we deduce that

$$\mathcal{Q}(Tu, Tv, \frac{\phi(t)}{\alpha(u,v)}) = \mathcal{Q}(u_{l(m)}, u_{l(n)}, \frac{t}{2}) \geq \mathcal{Q}(u_{l(m)}, u_{l(n)}, \frac{\varepsilon}{2}) > 1 - \frac{\varepsilon}{2} > 1 - \frac{t}{2}.$$

In Case 2, we set $C = \{u_{n_0+k} : k \in \mathbb{N}\}$, and define a mapping $T : X \to X$ as $Tu_{n_0+k} = u_{l(n_0+k)}$ for all $k \in \mathbb{N}$, and $Tu = u_{n_0}$ for all $u \in X \backslash C$.

We also define $\alpha : X \times X \to \mathbb{R}^+$ as $\alpha(u,v) = 1$ if $u = u_{n_0+j}$ and $v = u_{n_0+k}$ with $j > k$, and $\alpha(u,v) = 0$ otherwise.

Exactly as in the argument developed in Case 1, we obtain that T is a reverse coherent fuzzy $\alpha - \phi$-contractive mapping (on $(X, \mathcal{Q}, *)$) without fixed points, for α defined as above, and $\phi \in \Phi$ given by $\phi(t) = t/2$ for all $t \geq 0$. This finishes the proof. □

As an immediate consequence of Theorems 5 and 6, we obtain the following improvement of [29] (Theorem 4).

Corollary 2. *For a KM-fuzzy metric space $(X, \mathcal{M}, *)$ the following statements are equivalent:*
(1) $(X, \mathcal{M}, *)$ is complete.
(2) Every coherent Suzuki$^+$ fuzzy $\alpha - \phi$-contractive mapping on $(X, \mathcal{M}, *)$ has a fixed point.
(3) Every coherent fuzzy $\alpha - \phi$-contractive mapping on $(X, \mathcal{M}, *)$ has a fixed point.
(4) Every reverse coherent Suzuki$^-$ fuzzy $\alpha - \phi$-contractive mapping on $(X, \mathcal{M}, *)$ has a fixed point.
(5) Every reverse coherent fuzzy $\alpha - \phi$-contractive mapping on $(X, \mathcal{M}, *)$ has a fixed point.

We finish the paper with an open question: Can "doubly Hausdorff" be replaced with "Hausdorff" in the statement of Theorem 6?

5. Conclusions

Involving conditions of Suzuki-type joint with contractions of $\alpha - \phi$-type in the sense of Samet, Vetro, and Vetro, we introduced the notions of coherent Suzuki$^+$ fuzzy $\alpha - \phi$-contractive mapping and of reverse coherent Suzuki$^-$ fuzzy $\alpha - \phi$-contractive mapping in the realm of KM-fuzzy quasi-metric spaces, which allowed us to obtain general fixed point theorems for left complete and right complete Hausdorff KM-fuzzy quasi-metric spaces. We present, among others, an example showing that Hausdorffness cannot be relaxed to T_1 in such theorems. As an application, we deduce characterizations of left complete Hausdorff KM-fuzzy quasi-metric spaces and of right complete doubly Hausdorff KM-fuzzy quasi-metric spaces, respectively. These characterizations generalize and improve a recent characterization of complete KM-fuzzy metric spaces given by Pedro Tirado and the author.

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