mathematics

Article

A Relaxed Inertial Tseng's Extragradient Method for Solving Split Variational Inequalities with Multiple Output Sets

Timilehin Opeyemi Alakoya * and **Oluwatosin Temitope Mewomo**

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban 4041, South Africa
* Correspondence: timimaths@gmail.com or alakoyat1@ukzn.ac.za

Abstract: Recently, the split inverse problem has received great research attention due to its several applications in diverse fields. In this paper, we study a new class of split inverse problems called the split variational inequality problem with multiple output sets. We propose a new Tseng extragradient method, which uses self-adaptive step sizes for approximating the solution to the problem when the cost operators are pseudomonotone and non-Lipschitz in the framework of Hilbert spaces. We point out that while the cost operators are non-Lipschitz, our proposed method does not involve any linesearch procedure for its implementation. Instead, we employ a more efficient self-adaptive step size technique with known parameters. In addition, we employ the relaxation method and the inertial technique to improve the convergence properties of the algorithm. Moreover, under some mild conditions on the control parameters and without the knowledge of the operators' norm, we prove that the sequence generated by our proposed method converges strongly to a minimum-norm solution to the problem. Finally, we apply our result to study certain classes of optimization problems, and we present several numerical experiments to demonstrate the applicability of our proposed method. Several of the existing results in the literature in this direction could be viewed as special cases of our results in this study.

Keywords: split inverse problems; non-Lipschitz operators; pseudomonotone operators; Tseng's extragradient method; relaxation and inertial techniques

MSC: 65K15; 47J25; 65J15; 90C33

check for updates

Citation: Alakoya, T.O.; Mewomo, O.T. A Relaxed Inertial Tseng's Extragradient Method for Solving Split Variational Inequalities with Multiple Output Sets. *Mathematics* **2023**, *11*, 386. https://doi.org/10.3390/math11020386

Academic Editor: Christopher Goodrich

Received: 12 December 2022
Revised: 27 December 2022
Accepted: 5 January 2023
Published: 11 January 2023

1. Introduction

Let H be a real Hilbert space endowed with inner product $\langle\cdot,\cdot\rangle$ and induced norm $||\cdot||$. Let C be a nonempty, closed and convex subset of H, and let $A : H \to H$ be an operator. Recall that the variational inequality problem (VIP) is formulated as finding an element $p \in C$ such that

$$\langle x - p, Ap\rangle \geq 0, \quad \forall\, x \in C. \tag{1}$$

The solution set of the VIP (1) is denoted by $VI(C, A)$. Fichera [1] and Stampacchia [2] were the first to introduce and initiate a study independently on variational inequality theory. The variational inequality model is known to provide a general and useful framework for solving several problems in engineering, optimal control, data sciences, mathematical programming, economics, etc. (see [3–8] and the references therein). In recent times, the VIP has received great research attention owing to its several applications in diverse fields, such as economics, operations research, optimization theory, structural analysis, sciences and engineering (see [9–14] and the references therein). Several methods have been proposed and analyzed by authors for solving the VIP (see [15–19] and references therein).

One of the well-known and highly efficient methods is the Tseng extragradient method [20] (which is also known as the forward–backward–forward algorithm). The method is a two-step projection iterative method, which only requires single computation of the projection onto the feasible set per iteration. Several authors have modified and improved on the Tseng extragradient method to approximate the solution of the VIP (1) (for instance, see [19,21–23] and the references therein).

Another active area of research interest in recent years is the *split inverse problem* (SIP). The SIP finds applications in various fields, such as in medical image reconstruction, intensity-modulated radiation therapy, signal processing, phase retrieval, data compression, etc. (for instance, see [24–27]). The SIP model is presented as follows:

$$\text{Find } \hat{x} \in H_1 \quad \text{that solves IP}_1 \tag{2}$$

such that

$$\hat{y} := T\hat{x} \in H_2 \quad \text{solves IP}_2, \tag{3}$$

where H_1 and H_2 are real Hilbert spaces, IP_1 denotes an inverse problem formulated in H_1, and IP_2 denotes an inverse problem formulated in H_2, and $T : H_1 \to H_2$ is a bounded linear operator.

The first instance of the SIP, called the *split feasibility problem* (SFP), was introduced in 1994 by Censor and Elfving [26] for modeling inverse problems that arise from medical image reconstruction. The SFP has numerous areas of applications, for instance, in signal processing, biomedical engineering, control theory, approximation theory, geophysics, communications, etc. [25,27,28]. The SFP is formulated as follows:

$$\text{Find } \hat{x} \in C \text{ such that } \hat{y} = T\hat{x} \in Q, \tag{4}$$

where C and Q are nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, and $T : H_1 \to H_2$ is a bounded linear operator.

A well-known method for solving the SFP is the CQ method proposed by Byrne [29]. The CQ method has been improved and extended by several researchers. Moreover, many authors have proposed and analyzed several other iterative methods for approximating the solution of SFP (4) both in the framework of Hilbert and Banach spaces (for instance, see [25,27,28,30,31]).

Censor et al. [32] introduced an important generalization of the SFP called the *split variational inequality problem* (SVIP). The SVIP is defined as follows:

$$\text{Find } \hat{x} \in C \text{ that solves } \langle A_1\hat{x}, x - \hat{x}\rangle \geq 0, \quad \forall x \in C \tag{5}$$

such that

$$\hat{y} = T\hat{x} \in H_2 \text{ solves } \langle A_2\hat{y}, y - \hat{y}\rangle \geq 0, \quad \forall y \in Q, \tag{6}$$

where $A_1 : H_1 \to H_1, A_2 : H_2 \to H_2$ are single-valued operators. Many authors have proposed and analyzed several iterative techniques for solving the SVIP (e.g., see [33–36]).

Very recently, Reich and Tuyen [37] introduced and studied a new split inverse problem called the *split feasibility problem with multiple output sets* (SFPMOS) in the framework of Hilbert spaces. Let C and Q_i be nonempty, closed and convex subsets of Hilbert spaces H and $H_i, i = 1, 2, \ldots, N$, respectively. Let $T_i : H \to H_i,\ i = 1, 2, \ldots, N$ be bounded linear operators. The SFPMOS is formulated as follows: find an element $u^\dagger \in H$ such that

$$u^\dagger \in \Gamma := C \cap (\cap_{i=1}^N T_i^{-1}(Q_i)) \neq \emptyset. \tag{7}$$

Reich and Tuyen [38] proposed and analyzed two iterative methods for solving the SFPMOS (7) in the framework of Hilbert spaces. The proposed algorithms are presented as follows:

$$x_{n+1} = P_C\Big[x_n - \gamma_n \sum_{i=1}^N T_i^*(I - P_{Q_i})T_i x_n\Big], \tag{8}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_C \Big[x_n - \gamma_n \sum_{i=1}^{N} T_i^* (I - P_{Q_i}) T_i x_n \Big], \tag{9}$$

where $f : C \to C$ is a strict contraction, $\{\gamma_n\} \subset (0, +\infty)$ and $\{\alpha_n\} \subset (0, 1)$. The authors obtained weak and strong convergence results for Algorithm (8) and Algorithm (9), respectively.

Motivated by the importance and several applications of the split inverse problems, in this paper, we examine a new class of split inverse problems called the split variational inequality problem with multiple output sets. Let $H, H_i, i = 1, 2, \ldots, N$, be real Hilbert spaces and let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces H and $H_i, i = 1, 2, \ldots, N$, respectively. Let $T_i : H \to H_i, i = 1, 2, \ldots, N$, be bounded linear operators and let $A : H \to H, A_i : H_i \to H_i, i = 1, 2, \ldots, N$, be mappings. The *split variational inequality problem with multiple output sets* (SVIPMOS) is formulated as finding a point $x^* \in C$ such that

$$x^* \in \Omega := VI(C, A) \cap (\cap_{i=1}^{N} T_i^{-1} VI(C_i, A_i)) \neq \emptyset. \tag{10}$$

Observe that the SVIPMOS (10) is a more general problem than the SFPMOS (7).

In recent times, developing algorithms with high rates of convergence for solving optimization problems has become of great interest to researchers. There are two important techniques that are generally employed by researchers to improve the rate of convergence of iterative methods. These techniques include the *inertial technique* and the *relaxation technique*. The inertial technique first introduced by Polyak [39] originates from an implicit time discretization method (the heavy ball method) of second-order dynamical systems. The main feature of the inertial-algorithm is that the method uses the previous two iterates to generate the next iterate. We note that this small change can significantly improve the speed of convergence of an iterative method (for instance, see [21,23,40–45]). The relaxation method is another well-known technique employed by authors to improve the rate of convergence of iterative methods (see, e.g., [46–48]). The influence of these two techniques on the convergence properties of iterative methods was investigated in [46].

In this study, we introduce and analyze the convergence of a relaxed inertial Tseng extragradient method for solving the SVIPMOS (10) in the framework of Hilbert spaces when the cost operators are pseudomonotone and non-Lipschitz. Our proposed algorithm has the following key features:

- The proposed method does not require the Lipschitz continuity condition often imposed by the cost operator in the literature when solving variational inequality problems. In addition, while the cost operators are non-Lipschitz, the design of our algorithm does not involve any linesearch procedure, which could be time-consuming and too expensive to implement.
- Our proposed method does not require knowledge of the operators' norm for its implementation. Rather, we employ a very efficient self-adaptive step size technique with known parameters. Moreover, some of the control parameters are relaxed to enlarge the range of values of the step sizes of the algorithm.
- Our algorithm combines the relaxation method and the inertial techniques to improve its convergence properties.
- The sequence generated by our proposed method converges strongly to a minimum-norm solution to the SVIPMOS (10). Finding the minimum-norm solution to a problem is very important and useful in several practical problems.

Finally, we apply our result to study certain classes of optimization problems, and we carry out several numerical experiments to illustrate the applicability of our proposed method.

This paper is organized as follows: In Section 2, we present some definitions and lemmas needed to analyze the convergence of the proposed algorithm, while in Section 3, we present the proposed method. In Section 4, we discuss the convergence of the proposed method, and in Section 5, we apply our result to study certain classes of optimization problems. In Section 6, we present several numerical experiments with graphical illustrations. Finally, in Section 7, we give a concluding remark.

2. Preliminaries

Definition 1 ([21,22])**.** *An operator $A : H \to H$ is said to be*

(i) *α-strongly monotone, if there exists $\alpha > 0$ such that*

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|x - y\|^2, \ \forall\, x, y \in H;$$

(ii) *monotone, if*

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall\, x, y \in H;$$

(iii) *pseudomonotone, if*

$$\langle Ay, x - y \rangle \geq 0 \implies \langle Ax, x - y \rangle \geq 0, \forall x, y \in H,$$

(iv) *L-Lipschitz continuous, if there exists a constant $L > 0$ such that*

$$||Ax - Ay|| \leq L||x - y||, \quad \forall\, x, y \in H;$$

(v) *uniformly continuous, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that*

$$\|Ax - Ay\| < \epsilon \quad \text{whenever} \quad \|x - y\| < \delta, \quad \forall x, y \in H;$$

(vi) *sequentially weakly continuous, if for each sequence $\{x_n\}$, we have $x_n \rightharpoonup x \in H$ implies that $Ax_n \rightharpoonup Ax \in H$.*

Remark 1. *It is known that the following implications hold: $(i) \implies (ii) \implies (iii)$ but the converses are not generally true. We also note that uniform continuity is a weaker notion than Lipschitz continuity.*

It is well-known that if D is a convex subset of H, then $A : D \to H$ is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a constant $K < +\infty$ such that

$$\|Ax - Ay\| \leq K\|x - y\| + \epsilon \quad \forall x, y \in D. \tag{11}$$

Lemma 1 ([49])**.** *Suppose $\{a_n\}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\{b_n\}$ is a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n \quad \text{for all } n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 ([50])**.** *Suppose $\{\lambda_n\}$ and $\{\theta_n\}$ are two nonnegative real sequences such that*

$$\lambda_{n+1} \leq \lambda_n + \phi_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \phi_n < +\infty$, then $\lim_{n\to\infty} \lambda_n$ exists.

Lemma 3 ([51])**.** *Let H be a real Hilbert space. Then, the following results hold for all $x, y \in H$ and $\delta \in (0, 1)$:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle;$

(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2;$

(iii) $||\delta x+(1-\delta)y||^2=\delta||x||^2+(1-\delta)||y||^2-\delta(1-\delta)||x-y||^2.$

Lemma 4 ([52]). *Consider the VIP (1) with C being a nonempty, closed, convex subset of a real Hilbert space H and $A: C \to H$ being pseudomonotone and continuous. Then p is a solution of VIP (1) if and only if*

$$\langle Ax, x-p\rangle \geq 0, \qquad \forall x\in C$$

3. Main Results

In this section, we present our proposed iterative method for solving the SVIPMOS (10). We establish our convergence result for the proposed method under the following conditions:

Let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces $H, H_i, i = 1,2,\dots,N$, respectively, and let $T_i : H \to H_i, i = 1,2,\dots,N$ be bounded linear operators with adjoints T_i^*. Let $A : H \to H, A_i : H_i \to H_i, i = 1,2,\dots,N$, be uniformly continuous pseudomonotone operators satisfying the following property:

$$\text{whenever } \{T_ix_n\}\subset C_i,\ T_ix_n \rightharpoonup T_iz,\ \text{ then } \|A_iT_iz\| \leq \liminf_{n\to\infty}\|A_iT_ix_n\|,\ i=0,1,2\dots,N, C_0=C, A_0=A, T_0=I^H. \quad (12)$$

Moreover, we assume that the solution set $\Omega \neq \emptyset$ and the control parameters satisfy the following conditions:

Assumption B:

(A1) $\{\alpha_n\}\subset(0,1), \lim_{n\to\infty}\alpha_n=0, \sum_{n=1}^{\infty}\alpha_n=+\infty, \lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n}=0, \{\xi_n\}\subset[a,b]\subset(0,1), \theta>0;$

(A2) $0<c_i<c_i'<1, 0<\phi_i<\phi_i'<1, \lim_{n\to\infty}c_{n,i}=\lim_{n\to\infty}\phi_{n,i}=0, \lambda_{1,i}>0,\ \forall\, i=0,1,2,\dots,N;$

(A3) $\{\rho_{n,i}\}\subset\mathbb{R}_+, \sum_{n=1}^{\infty}\rho_{n,i}<+\infty, 0<a_i\leq\delta_{n,i}\leq b_i<1, \sum_{i=0}^{N}\delta_{n,i}=1$ for each $n\geq 1$.

Now, the Algorithm 1 is presented as follows:

Algorithm 1. A Relaxed Inertial Tseng's Extragradient Method for Solving SVIPMOS (10).

Step 0. Select initial points $x_0, x_1 \in H$. Let $C_0 = C$, $T_0 = I^H$, $A_0 = A$ and set $n = 1$.
Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n=\begin{cases}\min\left\{\theta,\ \frac{\epsilon_n}{\|x_n-x_{n-1}\|}\right\}, & \text{if } x_n\neq x_{n-1},\\ \theta, & \text{otherwise.}\end{cases} \quad (13)$$

Step 2. Compute

$$w_n=(1-\alpha_n)(x_n+\theta_n(x_n-x_{n-1})).$$

Step 3. Compute

$$y_{n,i}=P_{C_i}(T_iw_n-\lambda_{n,i}A_iT_iw_n).$$

Step 4. Compute

$$u_{n,i}=y_{n,i}-\lambda_{n,i}(A_iy_{n,i}-A_iT_iw_n),$$

$$\lambda_{n+1,i}=\begin{cases}\min\{\frac{(c_{n,i}+c_i)\|T_iw_n-y_{n,i}\|}{\|A_iT_iw_n-A_iy_{n,i}\|},\ \lambda_{n,i}+\rho_{n,i}\}, & \text{if } \quad A_iT_iw_n-A_iy_{n,i}\neq 0,\\ \lambda_{n,i}+\rho_{n,i}, & \text{otherwise.}\end{cases}$$

Step 5. Compute

$$v_n=\sum_{i=0}^{N}\delta_{n,i}\big(w_n+\eta_{n,i}T_i^*(u_{n,i}-T_iw_n)\big),$$

where

$$\eta_{n,i}=\begin{cases}\frac{(\phi_{n,i}+\phi_i)\|T_iw_n-u_{n,i}\|^2}{\|T_i^*(T_iw_n-u_{n,i})\|^2}, & \text{if } \quad \|T_i^*(T_iw_n-u_{n,i})\|\neq 0,\\ 0, & \text{otherwise.}\end{cases} \quad (14)$$

Step 6. Compute

$$x_{n+1}=\xi_nw_n+(1-\xi_n)v_n.$$

Set $n := n + 1$ and return to **Step 1**.

Remark 2. *Observe that by conditions (C1) and (C2) together with (13), we have that*

$$\lim_{n\to\infty} \theta_n ||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0.$$

Remark 3. *We also note that while the cost operators $A_i, i = 0, 1, 2, \ldots, N$ are non-Lipschitz, our method does not require any linesearch procedure, which could be computationally very expensive to implement. Rather, we employ self-adaptive step size techniques that only require simple computations of known parameters per iteration. Moreover, some of the parameters are relaxed to accommodate larger intervals for the step sizes.*

Remark 4. *We remark that condition (12) is a weaker assumption than the sequentially weakly continuity condition. We present the following example satisfying condition (12), which also illustrates that the condition is a weaker assumption than the sequentially weakly continuity condition.*

Let $A : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be an operator defined by

$$Ax = x\|x\|, \quad \forall x \in \ell_2(\mathbb{R}).$$

Suppose $\{z_n\} \subset \ell_2(\mathbb{R})$ such that $z_n \rightharpoonup z$. Then, by the weakly lower semi-continuity of the norm we obtain

$$\|z\| \leq \liminf_{n\to+\infty} \|z_n\|.$$

Thus, we have

$$\|Az\| = \|z\|^2 \leq (\liminf_{n\to+\infty} \|z_n\|)^2 \leq \liminf_{n\to+\infty} \|z_n\|^2 = \liminf_{n\to+\infty} \|Az_n\|.$$

Therefore, A satisfies condition (12).

On the other hand, to establish that A is not sequentially weakly continuous, choose $z_n = e_n + e_1$, where $\{e_n\}$ is a standard basis of $\ell_2(\mathbb{R})$, that is, $e_n = (0, 0, \ldots, 1, \ldots)$ with 1 at the n-th position. It is clear that $z_n \rightharpoonup e_1$ and $Az_n = A(e_n + e_1) = (e_n + e_1)\|e_n + e_1\| \rightharpoonup \sqrt{2}e_1$, but $Ae_1 = e_1\|e_1\| = e_1$. Consequently, A is not sequentially weakly continuous. Therefore, condition (12) is strictly weaker than the sequentially weakly continuity condition.

4. Convergence Analysis

First, we prove some lemmas needed for our strong convergence theorem.

Lemma 5. *Let $\{\lambda_{n,i}\}$ be the sequence generated by Algorithm 1 such that Assumption B holds. Then $\{\lambda_{n,i}\}$ is well-defined for each $i = 0, 1, 2, \ldots, N$ and $\lim_{n\to\infty} \lambda_{n,i} = \lambda_{1,i} \in [\min\{\frac{c_i}{M_i}, \lambda_{1,i}\}, \lambda_{1,i} + \Phi_i]$, where $\Phi_i = \sum_{n=1}^{\infty} \rho_{n,i}$.*

Proof. Observe that since A_i is uniformly continuous for each $i = 0, 1, 2, \ldots, N$, it follows from (11) that for any given $\epsilon_i > 0$, there exists $K_i < +\infty$ such that $\|A_i T_i w_n - A_i y_{n,i}\| \leq K_i \|T_i w_n - y_{n,i}\| + \epsilon_i$. Thus, for the case $A_i T_i w_n - A_i y_{n,i} \neq 0$ for all $n \geq 1$, we obtain

$$\frac{(c_{n,i} + c_i)\|T_i w_n - y_{n,i}\|}{\|A_i T_i w_n - A_i y_{n,i}\|} \geq \frac{(c_{n,i} + c_i)\|T_i w_n - y_{n,i}\|}{K_i \|T_i w_n - y_{n,i}\| + \epsilon_i} = \frac{(c_{n,i} + c_i)\|T_i w_n - y_{n,i}\|}{(K_i + \zeta_i)\|T_i w_n - y_{n,i}\|} = \frac{(c_{n,i} + c_i)}{M_i} \geq \frac{c_i}{M_i},$$

where $\epsilon_i = \zeta_i \|T_i w_n - y_{n,i}\|$ for some $\zeta_i \in (0, 1)$ and $M_i = K_i + \zeta_i$. Therefore, by the definition of $\lambda_{n+1,i}$, the sequence $\{\lambda_{n,i}\}$ has lower bound $\min\{\frac{c_i}{M_i}, \lambda_{1,i}\}$ and has upper bound $\lambda_{1,i} + \Phi_i$. By Lemma 2, the limit $\lim_{n\to\infty} \lambda_{n,i}$ exists and is denoted by $\lambda_i = \lim_{n\to\infty} \lambda_{n,i}$. Clearly, $\lambda_i \in \left[\min\{\frac{c_i}{M_i}, \lambda_{1,i}\}, \lambda_{1,i} + \Phi_i\right]$ for each $i = 0, 1, 2 \ldots, N$. □

Lemma 6. *If $\|T_i^*(T_i w_n - u_{n,i})\| \neq 0$, then the sequence $\{\eta_{n,i}\}$ defined by (14) has a positive lower bounded for each $i = 0, 1, 2, \ldots, N$.*

Proof. If $\|T_i^*(T_iw_n - u_{n,i})\| \neq 0$, it follows that for each $i = 0, 1, 2, \dots, N$

$$\eta_{n,i} = \frac{(\phi_{n,i} + \phi_i)\|T_iw_n - u_{n,i}\|^2}{\|T_i^*(T_iw_n - u_{n,i})\|^2}.$$

Since T_i is a bounded linear operator and $\lim_{n\to\infty} \phi_{n,i} = 0$ for each $i = 0, 1, 2, \dots, N$, we have

$$\frac{(\phi_{n,i} + \phi_i)\|T_iw_n - u_{n,i}\|^2}{\|T_i^*(T_iw_n - u_{n,i})\|^2} \geq \frac{(\phi_{n,i} + \phi_i)\|T_iw_n - u_{n,i}\|^2}{\|T_i\|^2\|T_iw_n - u_{n,i}\|\|^2} \geq \frac{\phi_i}{\|T_i\|^2},$$

which implies that $\frac{\phi_i}{\|T_i\|^2}$ is a lower bound of $\{\eta_{n,i}\}$ for each $i = 0, 1, 2, \dots, N$. □

Lemma 7. *Suppose Assumption B of Algorithm 1 holds. Then, there exists a positive integer N such that*

$$\phi_i + \phi_{n,i} \in (0,1), \quad \textit{and} \quad \frac{\lambda_{n,i}(c_{n,i} + ci)}{\lambda_{n+1,i}} \in (0,1), \quad \forall n \geq N.$$

Proof. Since $0 < \phi_i < \phi_i' < 1$ and $\lim_{n\to\infty} \phi_{n,i} = 0$ for each $i = 0, 1, 2, \dots, N$, there exists a positive integer $N_{1,i}$ such that

$$0 < \phi_i + \phi_{n,i} \leq \phi_i' < 1, \ \ \forall n \geq N_{1,i}.$$

Similarly, since $0 < c_i < c_i' < 1$, $\lim_{n\to\infty} c_{n,i} = 0$ and $\lim_{n\to\infty} \lambda_{n,i} = \lambda_i$ for each $i = 0, 1, 2, \dots, N$, we have

$$\lim_{n\to\infty} \left(1 - \frac{\lambda_{n,i}(c_{n,i} + c_i)}{\lambda_{n+1,i}}\right) = 1 - c_i > 1 - c_i' > 0.$$

Thus, for each $i = 0, 1, 2, \dots, N$, there exists a positive integer $N_{2,i}$ such that

$$1 - \frac{\lambda_{n,i}(c_{n,i} + c_i)}{\lambda_{n+1,i}} > 0, \ \ \forall n \geq N_{2,i}.$$

Now, setting $N = \max\{N_{1,i},\ N_{2,i} : i = 0, 1, 2, \dots, N\}$, we have the required result. □

Lemma 8. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 under Assumption B. Then the following inequality holds for all $p \in \Omega$:*

$$\|u_{n,i} - T_ip\|^2 \leq \|T_iw_n - T_ip\|^2 - \left(1 - \frac{\lambda_{n,i}^2}{\lambda_{n+1,i}^2}(c_{n,i} + c_i)^2\right)\|T_iw_n - y_{n,i}\|^2.$$

Proof. From the definition of $\lambda_{n+1,i}$, we have

$$\|A_iT_iw_n - A_iy_{n,i}\| \leq \frac{(c_{n,i} + c_i)}{\lambda_{n+1,i}}\|T_iw_n - y_{n,i}\|, \quad \forall n \in \mathbb{N},\ i = 0, 1, \dots, N. \tag{15}$$

Observe that (15) holds both for $A_iT_iw_n - A_iy_{n,i} = 0$ and $A_iT_iw_n - A_iy_{n,i} \neq 0$. Let $p \in \Omega$. Then, it follows that $T_ip \in VI(C_i, A_i),\ i = 0, 1, 2, \dots, N$. Using the definition of $u_{n,i}$ and applying Lemma 3, we have

$$\begin{aligned}\|u_{n,i}-T_ip\|^2 &= \|y_{n,i}-\lambda_{n,i}(A_iy_{n,i}-A_iT_iw_n)-T_ip\|^2\\ &= \|y_{n,i}-T_ip\|^2+\lambda_{n,i}^2\|A_iy_{n,i}-A_iT_iw_n\|^2-2\lambda_{n,i}\langle y_{n,i}-T_ip, A_iy_{n,i}-A_iT_iw_n\rangle\\ &= \|T_iw_n-T_ip\|^2+\|y_{n,i}-T_iw_n\|^2+2\langle y_{n,i}-T_iw_n, T_iw_n-T_ip\rangle+\lambda_{n,i}^2\|A_iy_{n,i}-A_iT_iw_n\|^2\\ &\quad -2\lambda_{n,i}\langle y_{n,i}-T_ip, A_iy_{n,i}-A_iT_iw_n\rangle\\ &= \|T_iw_n-T_ip\|^2+\|y_{n,i}-T_iw_n\|^2-2\langle y_{n,i}-T_iw_n, y_{n,i}-T_iw_n\rangle+2\langle y_{n,i}-T_iw_n, y_{n,i}-T_ip\rangle\\ &\quad +\lambda_{n,i}^2\|A_iy_{n,i}-A_iT_iw_n\|^2-2\lambda_{n,i}\langle y_{n,i}-T_ip, A_iy_{n,i}-A_iT_iw_n\rangle\\ &= \|T_iw_n-T_ip\|^2-\|y_{n,i}-T_iw_n\|^2+2\langle y_{n,i}-T_iw_n, y_{n,i}-T_ip\rangle+\lambda_{n,i}^2\|A_iy_{n,i}-A_iT_iw_n\|^2\\ &\quad -2\lambda_{n,i}\langle y_{n,i}-T_ip, A_iy_{n,i}-A_iT_iw_n\rangle. \end{aligned} \tag{16}$$

Since $y_{n,i}=P_{C_i}(T_iw_n-\lambda_{n,i}A_iT_iw_n)$ and $T_ip\in VI(C_i,A_i),\ i=0,1,2,\ldots,N$, by the property of the projection map we have

$$\langle y_{n,i}-T_iw_n+\lambda_{n,i}A_iT_iw_n, y_{n,i}-T_ip\rangle\le 0,$$

which is equivalent to

$$\langle y_{n,i}-T_iw_n, y_{n,i}-T_ip\rangle\le -\lambda_{n,i}\langle A_iT_iw_n, y_{n,i}-T_ip\rangle. \tag{17}$$

Furthermore, since $y_{n,i}\in C_i,\ i=0,1,2,\ldots,N$, we have

$$\langle A_iT_ip, y_{n,i}-T_ip\rangle\ge 0,$$

By the pseudomonotonicity of A_i, it follows that $\langle A_iy_{n,i}, y_{n,i}-T_ip\rangle\ge 0$. Since $\lambda_{n,i}>0,\ i=0,1,2,\ldots,N$, we obtain

$$\lambda_{n,i}\langle A_iy_{n,i}, y_{n,i}-T_ip\rangle\ge 0. \tag{18}$$

Next, by applying (15), (17) and (18) in (16), we obtain

$$\begin{aligned}\|u_{n,i}-T_ip\|^2 &\le \|T_iw_n-T_ip\|^2-\|y_{n,i}-T_iw_n\|^2-2\lambda_{n,i}\langle A_iT_iw_n, y_{n,i}-T_ip\rangle+(c_{n,i}+c_i)^2\frac{\lambda_{n,i}^2}{\lambda_{n+1,i}^2}\|T_iw_n-y_{n,i}\|^2\\ &\quad -2\lambda_{n,i}\langle y_{n,i}-T_ip, A_iy_{n,i}-A_iT_iw_n\rangle\\ &= \|T_iw_n-T_ip\|^2-\left(1-\frac{\lambda_{n,i}^2}{\lambda_{n+1,i}^2}(c_{n,i}+c_i)^2\right)\|T_iw_n-y_{n,i}\|^2-2\lambda_{n,i}\langle y_{n,i}-T_ip, A_iy_{n,i}\rangle\\ &\le \|T_iw_n-T_ip\|^2-\left(1-\frac{\lambda_{n,i}^2}{\lambda_{n+1,i}^2}(c_{n,i}+c_i)^2\right)\|T_iw_n-y_{n,i}\|^2, \end{aligned} \tag{19}$$

which is the required inequality. □

Lemma 9. *Suppose $\{x_n\}$ is a sequence generated by Algorithm 1 such that Assumption B holds. Then $\{x_n\}$ is bounded.*

Proof. Let $p\in\Omega$. By the definition of w_n and applying the triangular inequality, we have

$$\begin{aligned}\|w_n-p\| &= \|(1-\alpha_n)(x_n+\theta_n(x_n-x_{n-1}))-p\|\\ &= \|(1-\alpha_n)(x_n-p)+(1-\alpha_n)\theta_n(x_n-x_{n-1})-\alpha_np\|\\ &\le (1-\alpha_n)\|x_n-p\|+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|+\alpha_n\|p\|\\ &= (1-\alpha_n)\|x_n-p\|+\alpha_n\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|p\|\Big]. \end{aligned}$$

By Remark (2), we obtain

$$\lim_{n\to\infty}\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|\right] = \|p\|.$$

Thus, there exists $M_1 > 0$ such that $(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\| \le M_1$ for all $n \in \mathbb{N}$. It follows that

$$\|w_n - p\| \le (1-\alpha_n)\|x_n - p\| + \alpha_n M_1. \tag{20}$$

By Lemma 7, there exists a positive integer N such that $1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i) > 0, \ \ \forall n \ge N, \ i = 0,1,2,\dots,N$. Consequently, it follows from (19) that for all $n \ge N$ and $i = 0,1,2,\dots,N$

$$\le \|u_{n,i} - T_i p\|^2 \le \|T_i w_n - T_i p\|^2. \tag{21}$$

Next, since the function $\|\cdot\|^2$ is convex, we have

$$\begin{aligned}\|v_n - p\|^2 &= \|\sum_{i=0}^{N}\delta_{n,i}(w_n + \eta_{n,i}T_i^*(u_{n,i} - T_i w_n)) - p\|^2 \\ &\le \sum_{i=0}^{N}\delta_{n,i}\|w_n + \eta_{n,i}T_i^*(u_{n,i} - T_i w_n) - p\|^2.\end{aligned} \tag{22}$$

By Lemma 7, there exists a positive integer N such that $0 < \phi_{n,i} + \phi_i < 1, \ \ i = 0,1,2,\dots,N$ for all $n \ge N$. From (22) and by applying Lemma 3 and (21), we obtain

$$\begin{aligned}\|w_n + \eta_{n,i}T_i^*(u_{n,i} - T_i w_n) - p\|^2 &= \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(u_{n,i} - T_i w_n)\|^2 + 2\eta_{n,i}\langle w_n - p, T_i^*(u_{n,i} - T_i w_n)\rangle \\ &= \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(u_{n,i} - T_i w_n)\|^2 + 2\eta_{n,i}\langle T_i w_n - T_i p, u_{n,i} - T_i w_n\rangle \\ &= \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(u_{n,i} - T_i w_n)\|^2 + \eta_{n,i}[\|u_{n,i} - T_i p\|^2 - \|T_i w_n - T_i p\|^2 \\ &\quad - \|u_{n,i} - T_i w_n\|^2] \\ &\le \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(u_{n,i} - T_i w_n)\|^2 - \eta_{n,i}\|u_{n,i} - T_i w_n\|^2 \\ &= \|w_n - p\|^2 - \eta_{n,i}[\|u_{n,i} - T_i w_n\|^2 - \eta_{n,i}\|T_i^*(u_{n,i} - T_i w_n)\|^2].\end{aligned} \tag{23}$$

If $\|T_i^*(u_{n,i} - T_i w_n)\| \ne 0$, then by the definition of $\eta_{n,i}$, we have

$$\|u_{n,i} - T_i w_n\|^2 - \eta_{n,i}\|T_i^*(u_{n,i} - T_i w_n)\|^2 = [1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - u_{n,i}\|^2 \ge 0. \tag{24}$$

Now, applying (24) in (23) and substituting in (22), we have

$$\begin{aligned}\|v_n - p\|^2 &\le \|w_n - p\|^2 - \sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - u_{n,i}\|^2 \\ &\le \|w_n - p\|^2.\end{aligned} \tag{25}$$

Observe that if $\|T_i^*(u_{n,i} - T_i w_n)\| = 0$, (25) still holds from (23).

Next, using the definition of x_{n+1}, and applying (20) and (25), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\xi_n w_n + (1-\xi_n)v_n - p\| \\ &\le \xi_n\|w_n-p\| + (1-\xi_n)\|v_n-p\| \\ &\le \xi_n\|w_n-p\| + (1-\xi_n)\|w_n-p\| \\ &= \|w_n-p\| \\ &\le (1-\alpha_n)\|x_n-p\| + \alpha_n M_1 \\ &\le \max\{\|x_n-p\|, M_1\} \\ &\vdots \\ &\le \max\{\|x_N-p\|, M_1\},\end{aligned}$$

which implies that $\{x_n\}$ is bounded. Hence, $\{w_n\}$, $\{y_{n,i}\}$, $\{u_{n,i}\}$ and $\{v_n\}$ are all bounded. □

Lemma 10. *Let $\{w_n\}$ and $\{v_n\}$ be two sequences generated by Algorithm 1 with subsequences $\{w_{n_k}\}$ and $\{v_{n_k}\}$, respectively, such that $\lim_{k\to\infty}\|w_{n_k}-v_{n_k}\| = 0$. Suppose $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof. From (25), we have

$$\|v_{n_k}-p\|^2 \le \|w_{n_k}-p\|^2 - \sum_{i=0}^{N} \delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_i w_{n_k} - u_{n_k,i}\|^2. \tag{26}$$

From the last inequality, we obtain

$$\begin{aligned}\sum_{i=0}^{N} \delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_i w_{n_k} - u_{n_k,i}\|^2 &\le \|w_{n_k}-p\|^2 - \|v_{n_k}-p\|^2 \\ &\le \|w_{n_k}-v_{n_k}\|^2 + 2\|w_{n_k}-v_{n_k}\|\|v_{n_k}-p\| \end{aligned} \tag{27}$$

Since by the hypothesis of the lemma $\lim_{k\to\infty}\|w_{n_k}-v_{n_k}\| = 0$, it follows from (27) that

$$\sum_{i=0}^{N} \delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_i w_{n_k} - u_{n_k,i}\|^2 \to 0, \quad k\to\infty,$$

which implies that

$$\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_i w_{n_k} - u_{n_k,i}\|^2 \to 0, \quad k\to\infty, \ \ \forall i = 0,1,2,\ldots,N.$$

By the definition of $\eta_{n,i}$, we have

$$\delta_{n_k,i}(\phi_{n_k,i}+\phi_i)[1-(\phi_{n_k,i}+\phi_i)]\frac{\|T_i w_{n_k} - u_{n_k,i}\|^4}{\|T_i^*(T_i w_{n_k} - u_{n_k,i})\|^2} \to 0, \quad k\to\infty, \ \ \forall i = 0,1,2,\ldots,N.$$

From this, we obtain

$$\frac{\|T_i w_{n_k} - u_{n_k,i}\|^2}{\|T_i^*(T_i w_{n_k} - u_{n_k,i})\|} \to 0, \quad k\to\infty, \ \ \forall i = 0,1,2,\ldots,N,$$

Since $\{\|T_i^*(T_i w_{n_k} - u_{n_k,i})\|\}$ is bounded, it follows that

$$\|T_i w_{n_k} - u_{n_k,i}\| \to 0, \quad k\to\infty, \ \ \forall i = 0,1,2,\ldots,N. \tag{28}$$

Hence, we have

$$\|T_i^*(T_iw_{n_k} - u_{n_k,i})\| \le \|T_i^*\|\|(T_iw_{n_k} - u_{n_k,i})\| = \|T_i\|\|(T_iw_{n_k} - u_{n_k,i})\| \to 0, \quad k \to \infty, \ \forall i = 0,1,2,\ldots,N. \tag{29}$$

From (19), we obtain

$$\begin{aligned}\Big(1 - \frac{\lambda^2_{n_k,i}}{\lambda^2_{n_k+1,i}}(c_{n_k,i} + c_i)^2\Big)\|T_iw_{n_k} - y_{n_k,i}\|^2 &\le \|T_iw_{n_k} - T_ip\|^2 - \|u_{n_k,i} - T_ip\|^2 \\ &\le \|T_iw_{n_k} - u_{n_k,i}\|\big(\|T_iw_{n_k} - T_ip\| + \|u_{n_k,i} - T_ip\|\big).\end{aligned} \tag{30}$$

By applying (28), it follows from (30) that

$$\Big(1 - \frac{\lambda^2_{n_k,i}}{\lambda^2_{n_k+1,i}}(c_{n_k,i} + c_i)^2\Big)\|T_iw_{n_k} - y_{n_k,i}\|^2 \to 0, \quad k \to \infty, \ i = 0,1,\ldots,N.$$

Consequently, we have

$$\|T_iw_{n_k} - y_{n_k,i}\| \to 0, \quad k \to \infty, \ i = 0,1,\ldots,N. \tag{31}$$

Since $y_{n,i} = P_{C_i}(T_iw_n - \lambda_{n,i}A_iT_iw_n)$, by the property of the projection map, we obtain

$$\langle T_iw_{n_k} - \lambda_{n_k,i}A_iT_iw_{n_k} - y_{n_k,i}, T_ix - y_{n_k,i}\rangle \le 0, \quad \forall\, T_ix \in C_i, \quad i = 0,1,2,\ldots,N,$$

which implies that

$$\frac{1}{\lambda_{n_k,i}}\langle T_iw_{n_k} - y_{n_k,i}, T_ix - y_{n_k,i}\rangle \le \langle A_iT_iw_{n_k}, T_ix - y_{n_k,i}\rangle, \quad \forall\, T_ix \in C_i, \quad i = 0,1,2,\ldots,N.$$

From the last inequality, it follows that

$$\frac{1}{\lambda_{n_k,i}}\langle T_iw_{n_k} - y_{n_k,i}, T_ix - y_{n_k,i}\rangle + \langle A_iT_iw_{n_k}, y_{n_k,i} - T_iw_{n_k}\rangle \le \langle A_iT_iw_{n_k}, T_ix - T_iw_{n_k}\rangle, \ \forall\, T_ix \in C_i, \ i = 0,1,2,\ldots,N. \tag{32}$$

By applying (31) and the fact that $\lim_{k\to\infty} \lambda_{n_k,i} = \lambda_i > 0$, from (32) we obtain

$$\liminf_{k\to\infty}\langle A_iT_iw_{n_k}, T_ix - T_iw_{n_k}\rangle \ge 0, \quad \forall\, T_ix \in C_i, \ i = 0,1,2,\ldots,N. \tag{33}$$

Observe that

$$\langle A_iy_{n_k,i}, T_ix - y_{n_k,i}\rangle = \langle A_iy_{n_k,i} - A_iT_iw_{n_k}, T_ix - T_iw_{n_k}\rangle + \langle A_iT_iw_{n_k}, T_ix - T_iw_{n_k}\rangle + \langle A_iy_{n_k,i}, T_iw_{n_k} - y_{n_k,i}\rangle. \tag{34}$$

By the continuity of A_i, from (31) we obtain

$$\|A_iT_iw_{n_k} - A_iy_{n_k,i}\| \to 0, \quad k \to \infty, \ \forall i = 0,1,2,\ldots,N. \tag{35}$$

Using (31) and (35), it follows from (33) and (34) that

$$\liminf_{k\to\infty}\langle A_iy_{n_k,i}, T_ix - y_{n_k,i}\rangle \ge 0, \quad \forall\, T_ix \in C_i, \ i = 0,1,2,\ldots,N. \tag{36}$$

Next, let $\{\vartheta_{k,i}\}$ be a decreasing sequence of positive numbers such that $\vartheta_{k,i} \to 0$ as $k \to \infty$, $i = 0,1,2,\ldots,N$. For each k, let N_k denote the smallest positive integer such that

$$\langle A_iy_{n_j,i}, T_ix - y_{n_j,i}\rangle + \vartheta_{k,i} \ge 0, \quad \forall\, j \ge N_k, \ T_ix \in C_i, \ i = 0,1,2,\ldots,N, \tag{37}$$

where the existence of N_k follows from (36). Since $\{\vartheta_{k,i}\}$ is decreasing, then $\{N_k\}$ is increasing. Moreover, since $\{y_{N_k,i}\} \subset C_i$ for each k, we can suppose $A_iy_{N_k,i} \ne 0$ (otherwise, $y_{N_k,i} \in VI(C_i, A_i)$, $i = 0,1,2\ldots,N$) and let

$$z_{N_k,i} = \frac{A_i y_{N_k,i}}{\|A_i y_{N_k,i}\|^2}$$

Then, $\langle A_i y_{N_k,i}, z_{N_k,i}\rangle = 1$ for each $k,\ i = 0,1,2,\dots,N$. From (37), we have

$$\langle A_i y_{N_k,i}, T_i x + \vartheta_{k,i} z_{N_k,i} - y_{N_k,i}\rangle \geq 0, \quad \forall\, T_i x \in C_i,\ i = 0,1,2,\dots,N.$$

It follows from the pseudomonotonicity of A_i that

$$\langle A_i(T_i x + \vartheta_{k,i} z_{N_k,i}), T_i x + \vartheta_{k,i} z_{N_k,i} - y_{N_k,i}\rangle \geq 0, \quad \forall\, T_i x \in C_i, i = 0,1,2,\dots,N,$$

which is equivalent to

$$\langle A_i T_i x, T_i x - y_{N_k,i}\rangle \geq \langle A_i T_i x - A_i(T_i x + \vartheta_{k,i} z_{N_k,i}), T_i x + \vartheta_{k,i} z_{N_k,i} - y_{N_k,i}\rangle - \vartheta_{k,i}\langle A_i T_i x, z_{N_k,i}\rangle, \forall T_i x \in C_i, i = 0,1,\dots,N. \tag{38}$$

In order to complete the proof, we need to establish that $\lim_{k\to\infty} \vartheta_{k,i} z_{N_k,i} = 0$. Since $w_{n_k} \rightharpoonup z$ and T_i is a bounded linear operator for each $i = 0,1,2,\dots,N$, we have $T_i w_{n_k} \rightharpoonup T_i z,\ \ \forall i = 0,1,2,\dots,N$. Thus, from (31), we obtain $y_{n_k,i} \rightharpoonup T_i z,\ \ \forall\, i = 0,1,2,\dots,N$. Since $\{y_{n_k,i}\} \subset C_i,\ i = 0,1,2,\dots,N$, we have $T_i z \in C_i$. If $A_i T_i z = 0,\ \forall\, i = 0,1,2,\dots,N$, then $T_i z \in VI(C_i, A_i)\ \forall\, i = 0,1,2,\dots,N$, which implies that $z \in \Omega$. On the contrary, we suppose $A_i T_i z \neq 0,\ \forall\, i = 0,1,2,\dots,N$. Since A_i satisfies condition (12), we have for all $i = 0,1,2,\dots,N$

$$0 < \|A_i T_i z\| \leq \liminf_{k\to\infty} \|A_i y_{n_k,i}\|.$$

Applying the facts that $\{y_{N_k,i}\} \subset \{y_{n_k,i}\}$ and $\vartheta_{k,i} \to 0$ as $k \to \infty,\ i = 0,1,2\dots,N$, we have

$$0 \leq \limsup_{k\to\infty} \|\vartheta_{k,i} z_{N_k,i}\| = \limsup_{k\to\infty}\left(\frac{\vartheta_{k,i}}{\|A_i y_{n_k,i}\|}\right) \leq \frac{\limsup_{k\to\infty} \vartheta_{k,i}}{\liminf_{k\to\infty}\|A_i y_{n_k,i}\|} = 0,$$

which implies that $\limsup_{k\to\infty} \vartheta_{k,i} z_{N_k,i} = 0$. Applying the facts that A_i is continuous, $\{y_{N_k,i}\}$ and $\{z_{N_k,i}\}$ are bounded and $\lim_{k\to\infty} \vartheta_{k,i} z_{N_k,i} = 0$, from (38) we get

$$\liminf_{k\to\infty}\langle A_i T_i x, T_i x - y_{N_k,i}\rangle \geq 0, \quad \forall\, T_i x \in C_i,\ i = 0,1,2,\dots,N.$$

From the last inequality, we have

$$\langle A_i T_i x, T_i x - T_i z\rangle = \lim_{k\to\infty}\langle A_i T_i x, T_i x - y_{N_k,i}\rangle = \liminf_{k\to\infty}\langle A_i T_i x, T_i x - y_{N_k,i}\rangle \geq 0,\ \forall\, T_i x \in C_i,\ i = 0,1,2,\dots,N.$$

By Lemma 4, we obtain

$$T_i z \in VI(C_i, A_i),\ i = 0,1,2,\dots,N,$$

which implies that

$$z \in T_i^{-1}\big(VI(C_i, A_i)\big),\ i = 0,1,2,\dots,N,$$

Consequently, we have $z \in \bigcap_{i=0}^{N} T_i^{-1}\big(VI(C_i, A_i)\big)$, which implies that $z \in \Omega$ as desired. □

Lemma 11. *Suppose $\{x_n\}$ is a sequence generated by Algorithm 1 under Assumption B. Then, the following inequality holds for all $p \in \Omega$:*

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n d_n - (1-\xi_n)\sum_{i=0}^{N} \delta_{n,i}\eta_{n,i}[1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - u_{n,i}\|^2 - \xi_n(1-\xi_n)\|w_n - v_n\|^2.$$

Proof. Let $p \in \Omega$. By applying Lemma 3 together with the definition of w_n, we obtain

$$\begin{aligned}
\|w_n - p\|^2 &= \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n p\|^2 \\
&\le \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1})\|^2 + 2\alpha_n\langle -p,\ w_n - p\rangle \\
&\le (1-\alpha_n)^2\|x_n - p\|^2 + 2(1-\alpha_n)\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + (1-\alpha_n)^2\theta_n^2\|x_n - x_{n-1}\|^2 \\
&+ 2\alpha_n\langle -p,\ w_n - x_{n+1}\rangle + 2\alpha_n\langle -p,\ x_{n+1} - p\rangle \\
&\le (1-\alpha_n)\|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|p\|\|w_n - x_{n+1}\| \\
&+ 2\alpha_n\langle p,\ p - x_{n+1}\rangle. \qquad (39)
\end{aligned}$$

Now, using the definition of x_{n+1}, (25), (39) and applying Lemma 3, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\xi_n w_n + (1-\xi_n)v_n - p\|^2 \\
&= \xi_n\|w_n - p\|^2 + (1-\xi_n)\|v_n - p\|^2 - \xi_n(1-\xi_n)\|w_n - v_n\|^2 \\
&\le \xi_n\|w_n - p\|^2 + (1-\xi_n)\Big[\|w_n - p\|^2 - \sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\phi_{n,i}+\phi_i)]\|T_i w_n - u_{n,i}\|^2\Big] \\
&- \xi_n(1-\xi_n)\|w_n - v_n\|^2 \\
&= \|w_n - p\|^2 - (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\phi_{n,i}+\phi_i)]\|T_i w_n - u_{n,i}\|^2 - \xi_n(1-\xi_n)\|w_n - v_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|p\|\|w_n - x_{n+1}\| \\
&+ 2\alpha_n\langle p,\ p - x_{n+1}\rangle - (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\phi_{n,i}+\phi_i)]\|T_i w_n - u_{n,i}\|^2 - \xi_n(1-\xi_n)\|w_n - v_n\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\Big[2\|x_n - p\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&+ 2\|p\|\|w_n - x_{n+1}\| + 2\langle p,\ p - x_{n+1}\rangle\Big] - (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\phi_{n,i}+\phi_i)]\|T_i w_n - u_{n,i}\|^2 \\
&- \xi_n(1-\xi_n)\|w_n - v_n\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n d_n - (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\phi_{n,i}+\phi_i)]\|T_i w_n - u_{n,i}\|^2 - \xi_n(1-\xi_n)\|w_n - v_n\|^2,
\end{aligned}$$

where $d_n = 2\|x_n - p\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|p\|\|w_n - x_{n+1}\| + 2\langle p,\ p - x_{n+1}\rangle$, which is the required inequality. □

Theorem 1. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 under Assumption B. Then, $\{x_n\}$ converges strongly to $\hat{x} \in \Omega$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Omega\}$.*

Proof. Let $\|\hat{x}\| = \min\{\|p\| : p \in \Omega\}$, that is, $\hat{x} = P_\Omega(0)$. Then, from Lemma 11, we obtain

$$\|x_{n+1} - \hat{x}\|^2 \le (1-\alpha_n)\|x_n - \hat{x}\|^2 + \alpha_n\hat{d}_n, \qquad (40)$$

where $\hat{d}_n = 2\|x_n - \hat{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|\hat{x}\|\|w_n - x_{n+1}\| + 2\langle\hat{x},\ \hat{x} - x_{n+1}\rangle$.

Next, we claim that the sequence $\{\|x_n - \hat{x}\|\}$ converges to zero. To do this, in view of Lemma 1 it suffices to show that $\limsup_{k\to\infty}\hat{d}_{n_k} \le 0$ for every subsequence $\{\|x_{n_k} - \hat{x}\|\}$ of $\{\|x_n - \hat{x}\|\}$ satisfying

$$\liminf_{k\to\infty}\big(\|x_{n_k+1} - \hat{x}\| - \|x_{n_k} - \hat{x}\|\big) \ge 0. \qquad (41)$$

Suppose that $\{\|x_{n_k} - \hat{x}\|\}$ is a subsequence of $\{\|x_n - \hat{x}\|\}$ such that (41) holds. Again, from Lemma 11, we obtain

$$(1-\xi_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_i w_{n_k} - u_{n_k,i}\|^2 + \xi_{n_k}(1-\xi_{n_k})\|w_{n_k} - v_{n_k}\|^2$$
$$\leq (1-\alpha_{n_k})\|x_{n_k} - \hat{x}\|^2 - \|x_{n_k+1} - \hat{x}\|^2 + \alpha_{n_k}\hat{d}_{n_k}.$$

By (41), Remark 2 and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we obtain

$$(1-\xi_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_i w_{n_k} - u_{n_k,i}\|^2 + \xi_{n_k}(1-\xi_{n_k})\|w_{n_k} - v_{n_k}\|^2 \to 0, \quad k\to\infty.$$

Consequently, we obtain

$$\lim_{k\to\infty}\|w_{n_k} - v_{n_k}\| = 0; \quad \lim_{k\to\infty}\|T_i w_{n_k} - u_{n_k,i}\| = 0, \quad \forall i = 0,1,2,\ldots,N. \tag{42}$$

From the definition of w_n and by Remark 2, we have

$$\begin{aligned}\|w_{n_k} - x_{n_k}\| &= \|(1-\alpha_{n_k})(x_{n_k} + \theta_{n_k}(x_{n_k} - x_{n_k-1})) - x_{n_k}\| \\ &= \|(1-\alpha_{n_k})(x_{n_k} - x_{n_k}) + (1-\alpha_{n_k})\theta_{n_k}(x_{n_k} - x_{n_k-1}) - \alpha_{n_k}x_{n_k}\| \\ &\leq (1-\alpha_{n_k})\|x_{n_k} - x_{n_k}\| + (1-\alpha_{n_k})\theta_{n_k}\|x_{n_k} - x_{n_k-1}\| + \alpha_{n_k}\|x_{n_k}\| \to 0, \quad k\to\infty.\end{aligned} \tag{43}$$

Using (42) and (43), we obtain

$$\|v_{n_k} - x_{n_k}\| \to 0, \quad k\to\infty. \tag{44}$$

From the definition of x_{n+1} and by applying (43) and (44), we obtain

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \|\xi_{n_k}w_{n_k} + (1-\xi_{n_k})v_{n_k} - x_{n_k}\| \\ &\leq \xi_{n_k}\|w_{n_k} - x_{n_k}\| + (1-\xi_{n_k})\|v_{n_k} - x_{n_k}\| \to 0, \quad k\to\infty.\end{aligned} \tag{45}$$

Next, by combining (43) and (45), we obtain

$$\|w_{n_k} - x_{n_k+1}\| \to 0, \quad k\to\infty. \tag{46}$$

Since $\{x_n\}$ is bounded, $w_\omega(x_n) \neq \emptyset$. We choose an element $x^* \in w_\omega(x_n)$ arbitrarily. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. From (42), it follows that $w_{n_k} \rightharpoonup x^*$. Now, by invoking Lemma 10 and applying (42), we obtain $x^* \in \Omega$. Since $x^* \in w_\omega(x_n)$ was selected arbitrarily, it follows that $w_\omega(x_n) \subset \Omega$.

Next, by the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup q$ and

$$\limsup_{k\to\infty}\langle\hat{x}, \hat{x} - x_{n_k}\rangle = \lim_{j\to\infty}\langle\hat{x}, \hat{x} - x_{n_{k_j}}\rangle.$$

Since $\hat{x} = P_\Omega(0)$, it follows from the property of the metric projection map that

$$\limsup_{k\to\infty}\langle\hat{x}, \hat{x} - x_{n_k}\rangle = \lim_{j\to\infty}\langle\hat{x}, \hat{x} - x_{n_{k_j}}\rangle = \langle\hat{x}, \hat{x} - q\rangle \leq 0, \tag{47}$$

Thus, from (45) and (47), we obtain

$$\limsup_{k\to\infty}\langle\hat{x}, \hat{x} - x_{n_{k+1}}\rangle \leq 0. \tag{48}$$

Next, by Remark 2, (46) and (48) we have $\limsup_{k\to\infty} \hat{d}_{n_k} \leq 0$. Therefore, by invoking Lemma 1, it follows from (40) that $\{\|x_n - \hat{x}\|\}$ converges to zero as required. □

5. Applications

In this section, we apply our result to study related optimization problems.

5.1. Generalized Split Variational Inequality Problem

First, we apply our result to study and approximate the solution of the generalized split variational inequality problem (see [37]). Let D_i be nonempty, closed and convex subsets of real Hilbert spaces $H_i, i = 1, 2, \ldots, N$, and let $S_i : H_i \to H_{i+1}, i = 1, 2, \ldots, N-1$, be bounded linear operators, such that $S_i \neq 0$. Let $B_i : H_i \to H_i, i = 1, 2, \ldots, N$, be single-valued operators. The *generalized split variational inequality problem* (GSVIP) is formulated as finding a point $x^* \in D_1$ such that

$$x^* \in \Gamma := VI(D_1, B_1) \cap S_1^{-1}(VI(D_2, B_2)) \cap \ldots S_1^{-1}(S_2^{-1} \ldots (S_{N-1}^{-1}(VI(D_N, B_N)))) \neq \emptyset; \tag{49}$$

that is, $x^* \in D_1$ such that

$$x^* \in VI(D_1, B_1), S_1 x^* \in VI(D_2, B_2), \ldots, S_{N-1}(S_{N-2} \ldots S_1 x^*) \in VI(D_N, B_N).$$

We note that by setting $C = D_1, C_i = D_{i+1}, A = B_1, A_i = B_{i+1}, 1 \leq i \leq N-1, T_1 = S_1, T_2 = S_2 S_1, \ldots,$ and $T_{N-1} = S_{N-1} S_{N-2} \ldots S_1$, then the SVIPMOS (10) becomes the GSVIP (49). Consequently, we obtain the following strong convergence theorem for finding the solution of GSVIP (49) in Hilbert spaces when the cost operators are pseudomonotone and uniformly continuous.

Theorem 2. *Let D_i be nonempty, closed and convex subsets of real Hilbert spaces $H_i, i = 1, 2, \ldots, N$, and suppose $S_i : H_i \to H_{i+1}, i = 1, 2, \ldots, N-1$, are bounded linear operators with adjoints S_i^* such that $S_i \neq 0$. Let $B_i : H_i \to H_i$, $1, 2, \ldots, N$ be uniformly continuous pseudomonotone operators that satisfy condition (12), and suppose Assumption B of Theorem 1 holds and the solution set $\Gamma \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following Algorithm 2 converges in norm to $\hat{x} \in \Gamma$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Gamma\}$.*

Algorithm 2. A Relaxed Inertial Tseng's Extragradient Method for Solving GSVIP (49).

Step 0. Select initial points $x_0, x_1 \in H_1$. Let $S_0 = I^{H_1}$, $\hat{S}_{i-1} = S_{1-1}S_{i-2}\ldots S_0$, $\hat{S}^*_{i-1} = S^*_0 S^*_1 \ldots S^*_{i-1}$, $i = 1, 2, \ldots, N$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = (1-\alpha_n)(x_n + \theta_n(x_n - x_{n-1})).$$

Step 3. Compute

$$y_{n,i} = P_{D_i}(\hat{S}_{i-1}w_n - \lambda_{n,i}B_i\hat{S}_{i-1}w_n).$$

Step 4. Compute

$$u_{n,i} = y_{n,i} - \lambda_{n,i}(B_i y_{n,i} - B_i\hat{S}_{i-1}w_n),$$

$$\lambda_{n+1,i} = \begin{cases} \min\{\frac{(c_{n,i}+c_i)\|\hat{S}_{i-1}w_n - y_{n,i}\|}{\|B_i\hat{S}_{i-1}w_n - B_i y_{n,i}\|}, \ \lambda_{n,i} + \rho_{n,i}\}, & \text{if } \ B_i\hat{S}_{i-1}w_n - B_i y_{n,i} \neq 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

Step 5. Compute

$$v_n = \sum_{i=1}^{N} \delta_{n,i}(w_n + \eta_{n,i}\hat{S}^*_{i-1}(u_{n,i} - \hat{S}_{i-1}w_n)),$$

where

$$\eta_{n,i} = \begin{cases} \frac{(\phi_{n,i}+\phi_i)\|\hat{S}_{i-1}w_n - u_{n,i}\|^2}{\|\hat{S}^*_{i-1}(\hat{S}_{i-1}w_n - u_{n,i})\|^2}, & \text{if } \ \|\hat{S}^*_{i-1}(\hat{S}_{i-1}w_n - u_{n,i})\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 6. Compute

$$x_{n+1} = \xi_n w_n + (1-\xi_n)v_n.$$

Set $n := n+1$ and return to **Step 1**.

5.2. Split Convex Minimization Problem with Multiple Output Sets

Let C be a nonempty, closed and convex subset of a real Hilbert space H. The convex minimization problem is defined as finding a point $x^* \in C$, such that

$$g(x^*) = \min_{x \in C} g(x), \tag{50}$$

where g is a real-valued convex function. The solution set of Problem (50) is denoted by $\arg\min g$.

Let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces $H, H_i, i = 1, 2, \ldots, N$, respectively, and let $T_i : H \to H_i, i = 1, 2, \ldots, N$, be bounded linear operators with adjoints T^*_i. Let $g : H \to \mathbb{R}, g_i : H_i \to \mathbb{R}$ be convex and differentiable functions. In this subsection, we apply our result to find the solution of the following *split convex minimization problem with multiple output sets* (SCMPMOS): Find $x^* \in C$ such that

$$x^* \in \Psi := \arg\min g \cap \left(\cap_{i=1}^{N} T_i^{-1}\left(\arg\min g_i\right)\right) \neq \emptyset. \tag{51}$$

The following lemma is required to establish our next result.

Lemma 12 ([53])**.** *Suppose C is a nonempty, closed and convex subset of a real Banach space E, and let g be a convex function of E into $\mathbb{R}$. If g is Fréchet differentiable, then x is a solution of Problem (50) if and only if $x \in VI(C, \nabla g)$, where ∇g is the gradient of g.*

Applying Theorem 1 and Lemma 12, we obtain the following strong convergence theorem for finding the solution of the SCMPMOS (51) in the framework of Hilbert spaces.

Theorem 3. *Let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces $H, H_i, i = 1, 2, \dots, N$, respectively, and suppose $T_i : H \to H_i, i = 1, 2, \dots, N$, are bounded linear operators with adjoints T_i^*. Let $g : H \to \mathbb{R}, g_i : H_i \to \mathbb{R}$ be fréchet differentiable convex functions such that $\nabla g, \nabla g_i$ are uniformly continuous. Suppose that Assumption B of Theorem 1 holds and the solution set $\Psi \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following Algorithm 3 converges strongly to $\hat{x} \in \Psi$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Psi\}$.*

Algorithm 3. A Relaxed Inertial Tseng's Extragradient Method for Solving SCMPMOS (51).

Step 0. Select initial points $x_0, x_1 \in H$. Let $C_0 = C,\ T_0 = I^H,\ \nabla g_0 = \nabla g$ and set $n = 1$.
Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = (1-\alpha_n)(x_n + \theta_n(x_n - x_{n-1})).$$

Step 3. Compute

$$y_{n,i} = P_{C_i}(T_i w_n - \lambda_{n,i} \nabla g_i T_i w_n).$$

Step 4. Compute

$$u_{n,i} = y_{n,i} - \lambda_{n,i}(\nabla g_i y_{n,i} - \nabla g_i T_i w_n),$$

$$\lambda_{n+1,i} = \begin{cases} \min\{\frac{(c_{n,i}+c_i)\|T_i w_n - y_{n,i}\|}{\|\nabla g_i T_i w_n - \nabla g_i y_{n,i}\|},\ \lambda_{n,i} + \rho_{n,i}\}, & \text{if } \nabla g_i T_i w_n - \nabla g_i y_{n,i} \neq 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

Step 5. Compute

$$v_n = \sum_{i=0}^{N} \delta_{n,i}(w_n + \eta_{n,i} T_i^*(u_{n,i} - T_i w_n)),$$

where

$$\eta_{n,i} = \begin{cases} \frac{(\phi_{n,i}+\phi_i)\|T_i w_n - u_{n,i}\|^2}{\|T_i^*(T_i w_n - u_{n,i})\|^2}, & \text{if } \|T_i^*(T_i w_n - u_{n,i})\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 6. Compute

$$x_{n+1} = \xi_n w_n + (1-\xi_n) v_n.$$

Set $n := n + 1$ and return to **Step 1**.

Proof. We know that since g_i , $i = 0, 1, 2, \dots, N$ are convex, then ∇g_i are monotone [53] and, hence, pseudomonotone. Therefore, the required result follows by applying Lemma 12 and taking $A_i = \nabla g_i$ in Theorem 1. □

6. Numerical Experiments

Here, we carry out some numerical experiments to demonstrate the applicability of our proposed method (Proposed Algorithm 1). For simplicity, in all the experiments, we consider the case when $N = 5$. All numerical computations were carried out using Matlab version R2021(b).

In all the computations, we choose $\alpha_n = \frac{1}{3n+2}, \epsilon_n = \frac{5}{(3n+2)^3}, \xi_n = \frac{n+1}{2n+1}, \theta = 1.50, \lambda_{1,i} = i + 1.25, c_i = 0.10, \phi_i = 0.20, \rho_{n,i} = \frac{50}{n^2}, \delta_{n,i} = \frac{1}{6}$.

Now, we consider the following numerical examples both in finite and infinite dimensional Hilbert spaces for the proposed algorithm.

Example 1. *For each $i = 0, 1, \ldots, 5$, we define the feasible set $C_i = \mathbb{R}^m$, $T_i x = \frac{3x}{i+3}$ and $A_i(x) = Mx$, where M is a square $m \times m$ matrix given by*

$$a_{j,k} = \begin{cases} -1, & \text{if} \quad k = m+1-j \quad \text{and} \quad k > j, \\ 1 & \text{if} \quad k = m+1-j \quad \text{and} \quad k \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

We note that M is a Hankel-type matrix with a nonzero reverse diagonal.

Example 2. *Let $H_i = \mathbb{R}^2$ and $C_i = [-2-i, 2+i]^2$, $i = 0, 1, \ldots, 5$. We define $T_i x = \frac{2x}{i+2}$, and the cost operator $A_i : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by*

$$A_i(x, y) = (i+1)(-xe^y, y), \quad (i = 0, 1, \ldots, 5).$$

Finally, we consider the last example in infinite dimensional Hilbert spaces.

Example 3. *Let $H_i = \ell_2 := \{x = (x_1, x_2, \ldots, x_i, \ldots) : \sum_{j=1}^{\infty} |x_j|^2 < +\infty\}$, $i = 0, 1, \ldots, 5$. Let $r_i, R_i \in \mathbb{R}^+$ be such that $\frac{R_i}{k_i+1} < \frac{r_i}{k_i} < r_i < R_i$ for some $k_i > 1$. The feasible sets are defined as follows for each $i = 0, 1, \ldots, 5$:*

$$C_i = \{x \in H_i : \|x\| \leq r_i\}.$$

The cost operators $A_i : H_i \to H_i$ are defined by

$$A_i(x) = (R_i - \|x\|)x.$$

Then A_i are pseudomonotone and uniformly continuous. We choose $R_i = 1.4 + i, r_i = 0.8 + i, k_i = 1.2 + i$, and we define $T_i x = \frac{4x}{i+4}$.

We test Examples 1–3 under the following experiments:

Experiment 1. *In this experiment, we check the behavior of our method by fixing the other parameters and varying $c_{n,i}$ in Example 1. We do this to check the effects of this parameter and the sensitivity of our method on it.*

We consider $c_{n,i} \in \{0, \frac{20}{n^{0.1}}, \frac{40}{n^{0.01}}, \frac{60}{n^{0.001}}, \frac{80}{n^{0.0001}}\}$ with $m = 20$, $m = 40$, $m = 60$ and $m = 80$.

Using $\|x_{n+1} - x_n\| < 10^{-3}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations for each m. The numerical results are reported in Figures 1–4 and Table 1.

Table 1. Numerical results for Experiment 1.

	$m = 20$		$m = 40$		$m = 60$		$m = 80$	
Proposed Algorithm 1	**Iter.**	**CPU Time**	**Iter.**	**CPU Time**	**Iter.**	**CPU Time**	**Iter.**	**CPU Time**
$c_{n,i} = 0$	128	0.0889	156	0.1235	174	0.2028	189	0.2412
$c_{n,i} = \frac{20}{n^{0.1}}$	128	0.0652	156	0.1241	174	0.2664	189	0.2930
$c_{n,i} = \frac{40}{n^{0.01}}$	128	0.0719	156	0.1495	174	0.3013	189	0.3220
$c_{n,i} = \frac{60}{n^{0.001}}$	128	0.0695	156	0.1549	174	0.2959	189	0.3342
$c_{n,i} = \frac{80}{n^{0.0001}}$	128	0.0701	156	0.1678	174	0.2877	189	0.3129

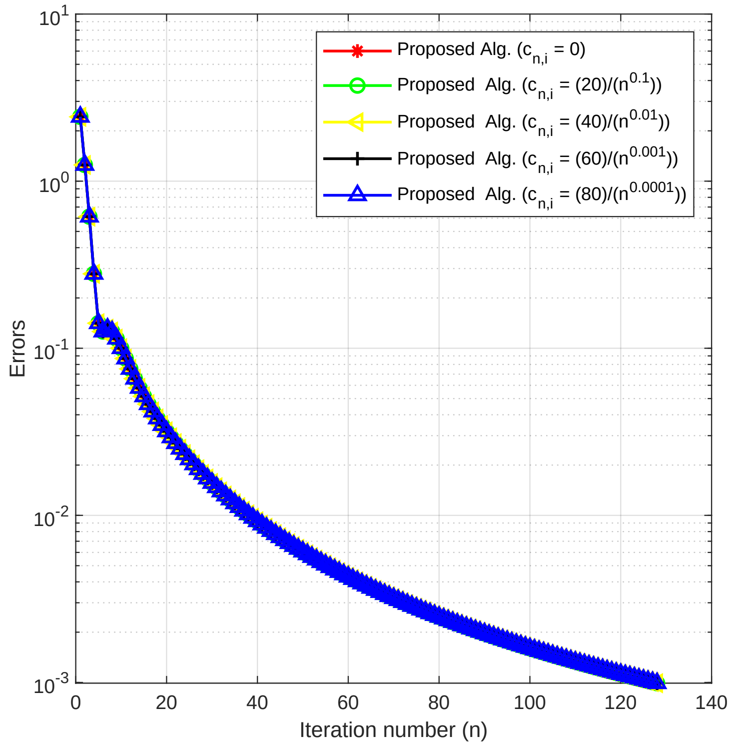

Figure 1. Experiment 1 : $m = 20$.

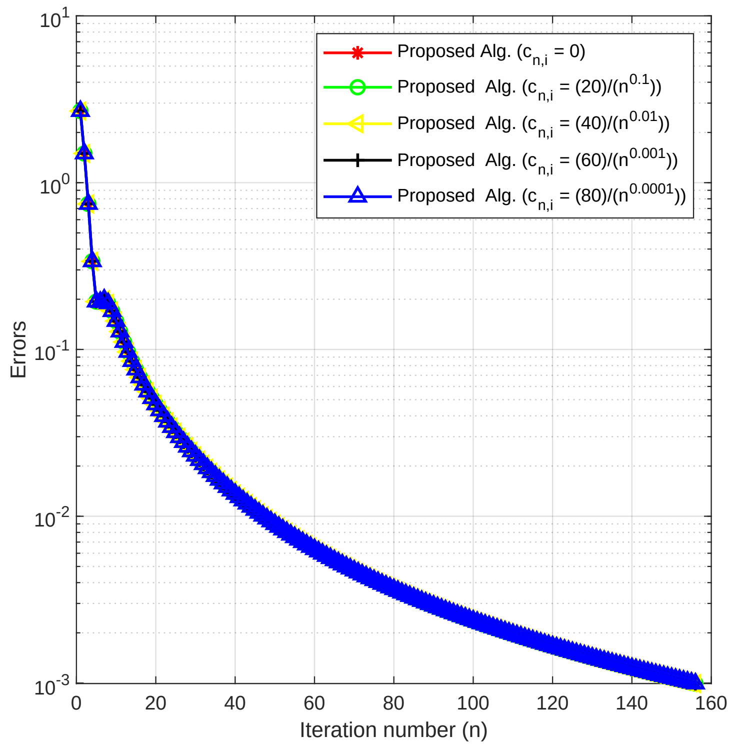

Figure 2. Experiment 1: $m = 40$.

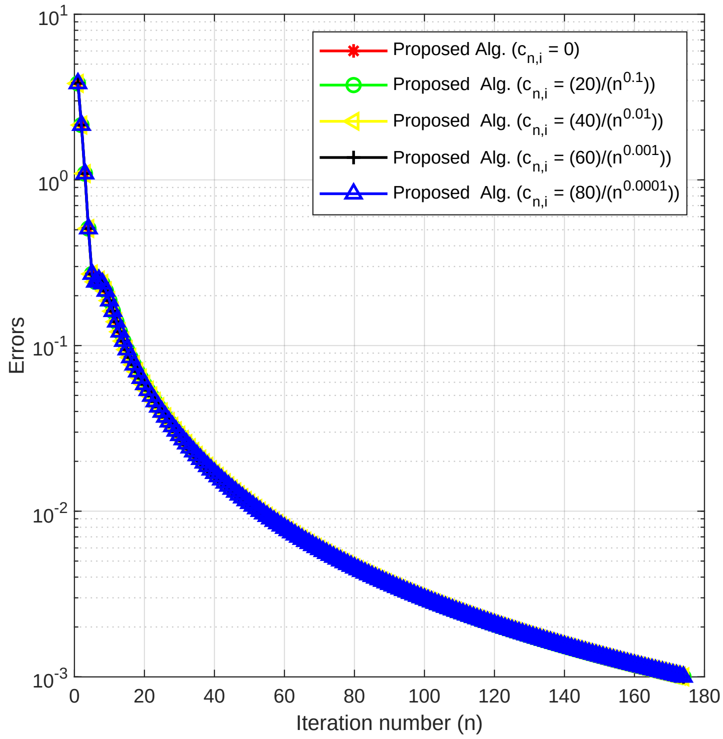

Figure 3. Experiment 1: $m = 60$.

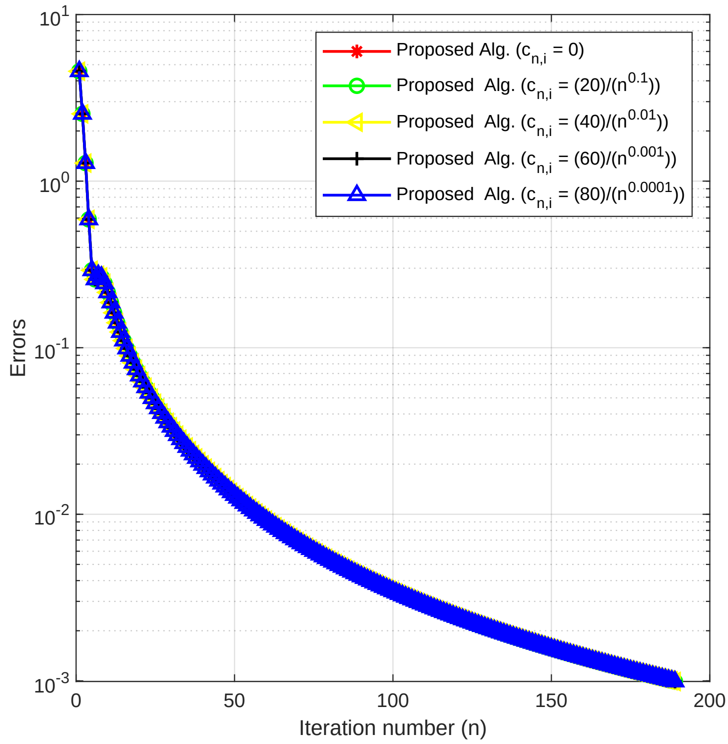

Figure 4. Experiment 1: $m = 80$.

Experiment 2. *In this experiment, we check the behavior of our method by fixing the other parameters and varying $c_{n,i}$ in Example 2. We do this to check the effects of this parameter and the sensitivity of our method to it.*

We consider $c_{n,i} \in \{0, \frac{20}{n^{0.1}}, \frac{40}{n^{0.01}}, \frac{60}{n^{0.001}}, \frac{80}{n^{0.0001}}\}$ with the following two cases of initial values x_0 and x_1 :

Case I: $x_0 = (2, 1)$; $x_1 = (0, 3)$;
Case II: $x_0 = (3, 2)$; $x_1 = (1, 1)$.

Using $\|x_{n+1} - x_n\| < 10^{-3}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations in each case. The numerical results are reported in Figures 5 and 6 and Table 2.*

Table 2. Numerical results for Experiment 2.

	Case I		Case II	
Proposed Algorithm 1	**Iter.**	**CPU Time**	**Iter.**	**CPU Time**
$c_{n,i} = 0$	248	0.0916	248	4.0980
$c_{n,i} = \frac{20}{n^{0.1}}$	248	0.0778	248	0.0816
$c_{n,i} = \frac{40}{n^{0.01}}$	248	0.0852	248	0.0818
$c_{n,i} = \frac{60}{n^{0.001}}$	248	0.0875	248	0.0753
$c_{n,i} = \frac{80}{n^{0.0001}}$	248	0.0817	248	0.0811

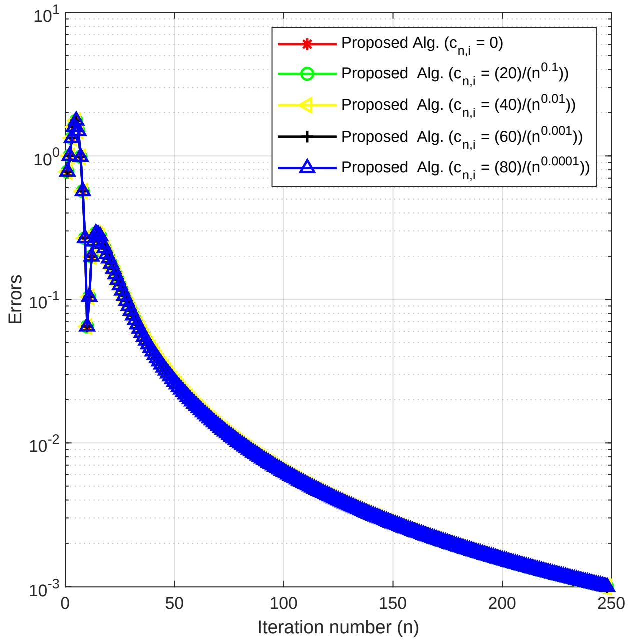

Figure 5. Experiment 2: Case 1.

Finally, we test Example 3 under the following experiment:

Experiment 3. *In this experiment, we check the behavior of our method by fixing the other parameters and varying* $c_{n,i}$ *in Example 3. We do this to check the effects of these parameters and the sensitivity of our method to it.*

We consider $c_{n,i} \in \{0, \frac{20}{n^{0.1}}, \frac{40}{n^{0.01}}, \frac{60}{n^{0.001}}, \frac{80}{n^{0.0001}}\}$ *with the following two cases of initial values* x_0 *and* x_1 :

Case I: $x_0 = (\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \cdots)$; $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$;
Case II: $x_0 = (\frac{3}{10}, \frac{3}{100}, \frac{3}{100}, \cdots)$; $x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots)$.

Using $\|x_{n+1} - x_n\| < 10^{-4}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations in each case. The numerical results are reported in Figures 7 and 8 and Table 3.*

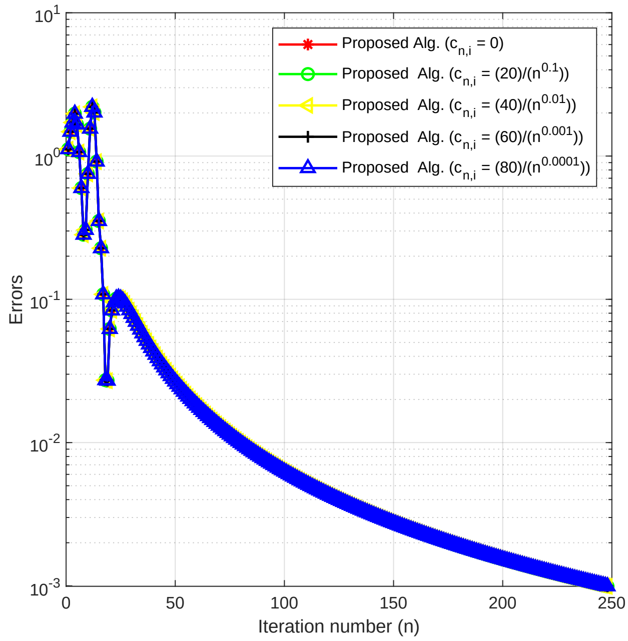

Figure 6. Experiment 2: Case 2.

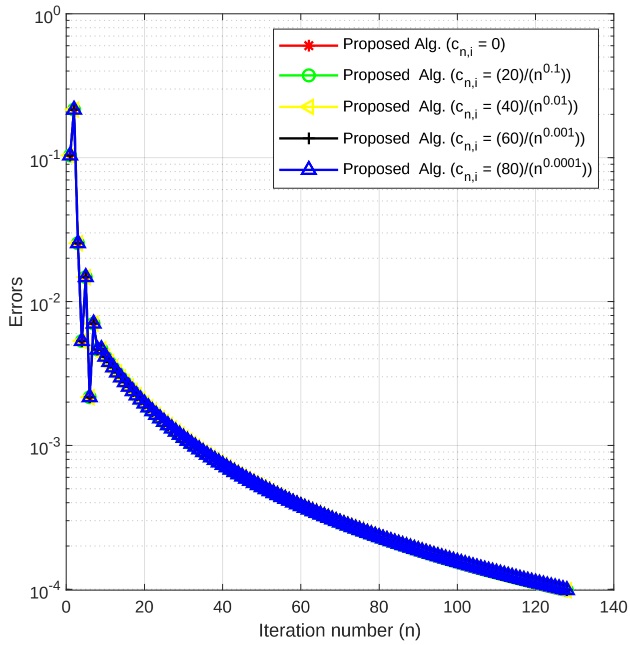

Figure 7. Experiment 3: Case 1.

Table 3. Numerical results for Experiment 3.

	Case I		Case II	
Proposed Algorithm 1	**Iter.**	**CPU Time**	**Iter.**	**CPU Time**
$c_{n,i} = 0$	128	0.0682	128	0.0620
$c_{n,i} = \frac{20}{n^{0.1}}$	128	0.0434	128	0.0422
$c_{n,i} = \frac{40}{n^{0.01}}$	128	0.0446	128	0.0474
$c_{n,i} = \frac{60}{n^{0.001}}$	128	0.0423	128	0.0414
$c_{n,i} = \frac{80}{n^{0.0001}}$	128	0.0416	128	0.0424

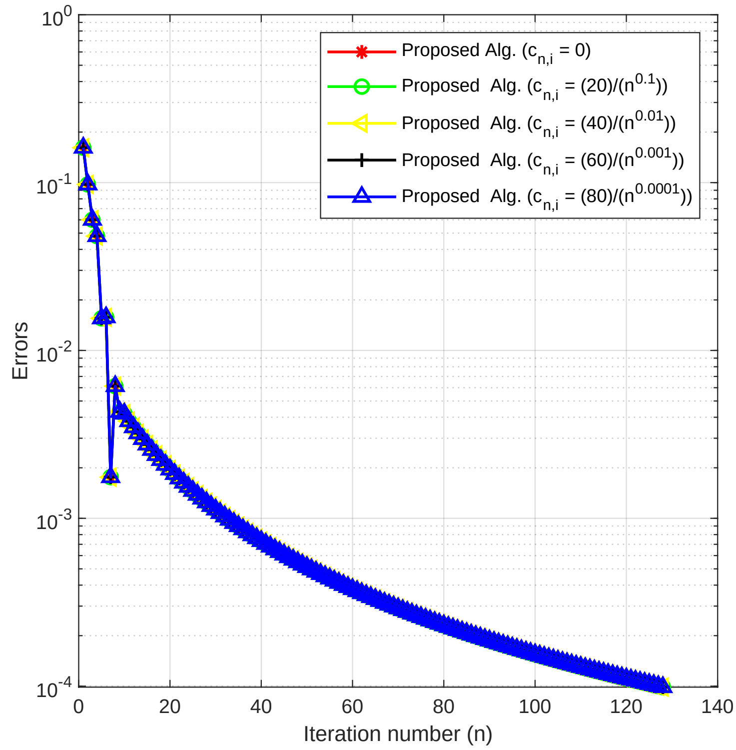

Figure 8. Experiment 3: Case 2.

Remark 5. *By using different initial values, cases of m and varying the key parameter in Experiments 1–3, we obtained the numerical results displayed in Tables 1–3 and Figures 1–8. In Figures 1–4, we considered different initial values and cases of m with varying values of the key parameter $c_{n,i}$ for Experiment 1 in $\mathbb{R}^m$. As observed from the figures, these varying choices do not have a significant effect on the behavior of the algorithm. Similarly, Figures 5 and 6 show that the behavior of our algorithm is consistent under varying initial starting points and different values of the key parameter $c_{n,i}$ for Experiment 2 in $\mathbb{R}^2$. Likewise, Figures 7 and 8 reveal that the behavior of the algorithm is not affected by varying starting points and values of $c_{n,i}$ for Experiment 3 in ℓ_2. From these results, we can conclude that our method is well-behaved since the choice of the key parameter and initial starting points do not affect the number of iterations or the CPU time in all the experiments.*

7. Conclusions

In this article, we studied a new class of split inverse problems called the split variational inequality problem with multiple output sets. We introduced a relaxed inertial Tseng extragradient method with self-adaptive step sizes for finding the solution to the problem when the cost operators are pseudomonotone and non-Lipschitz in the framework of Hilbert spaces. Moreover, we proved a strong convergence theorem for the proposed method under some mild conditions. Finally, we applied our result to study and approximate the solutions of certain classes of optimization problems, and we presented several numerical experiments to demonstrate the applicability of our proposed algorithm. The results of this study open up several opportunities for future research. As part of our future research, we would like to extend the results in this paper to a more general space, such as the reflexive Banach space. Furthermore, we would consider extending the results to a larger class of operators, such as the classes of quasimonotone and non-monotone operators. Moreover, in our future research, we would be interested in investigating the stochastic variant of our results in this study.

Author Contributions: Conceptualization, T.O.A.; Methodology, T.O.A.; Validation, O.T.M.; Formal analysis, T.O.A.; Investigation, T.O.A.; Resources, O.T.M.; Writing—original draft, T.O.A.; Writing —review & editing, O.T.M.; Visualization, O.T.M.; Supervision, O.T.M.; Project administration, O.T.M.; Funding acquisition, O.T.M. All authors have read and agreed to the published version of the manuscript.

Funding: The research of the first author is wholly supported by the University of KwaZulu-Natal, Durban, South Africa, Postdoctoral Fellowship. He is grateful for the funding and financial support. The second author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and conclusions arrived at are those of the authors and are not necessarily to be attributed to the NRF.

Acknowledgments: The authors thank the Reviewers and the Editor for the time spent in carefully going through the manuscript, and pointing out typos and areas of corrections, including constructive comments and suggestions, which have all helped to improve on the quality of the manuscript.

Conflicts of Interest: The authors declare that they have no competing interest.

References

1. Fichera, G. Sul problema elastostatico di Signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei VIII. Ser. Rend. Cl. Sci. Fis. Mat. Nat.* **1963**, *34*, 138–142.
2. Stampacchia, G. Formes bilineaires coercitives sur les ensembles convexes. *C. R. Acad. Sci. Paris* **1964**, *258*, 4413–4416.
3. Alakoya, T.O.; Uzor, V.A.; Mewomo, O.T. A new projection and contraction method for solving split monotone variational inclusion, pseudomonotone variational inequality, and common fixed point problems. *Comput. Appl. Math.* **2022**, *42*, 1–33. [CrossRef]
4. Ansari, Q.H.; Islam, M.; Yao, J.C. Nonsmooth variational inequalities on Hadamard manifolds. *Appl. Anal.* **2020**, *99*, 340–358. [CrossRef]
5. Cubiotti, P.; Yao, J.C. On the Cauchy problem for a class of differential inclusions with applications. *Appl. Anal.* **2020**, *99*, 2543–2554. [CrossRef]
6. Eskandari, Z.; Avazzadeh, Z.; Ghaziani, K.R.; Li, B. Dynamics and bifurcations of a discrete-time Lotka–Volterra model using nonstandard finite difference discretization method. *Math. Meth. Appl. Sci.* **2022**, 1–16. [CrossRef]
7. Li, B.; Liang, H.; He, Q. Multiple and generic bifurcation analysis of a discrete Hindmarsh-Rose model, Chaos. *Solitons Fractals* **2021**, *146*, 110856. [CrossRef]
8. Vuong, P.T.; Shehu, Y. Convergence of an extragradient-type method for variational inequality with applications to optimal control problems. *Numer. Algorithms* **2019**, *81*, 269–291. [CrossRef]
9. Aubin, J.; Ekeland, I. *Applied Nonlinear Analysis*; Wiley: New York, NY, USA, 1984.
10. Baiocchi, C.; Capelo, A. *Variational and Quasivariational Inequalities*; Applications to Free Boundary Problems; Wiley: New York, NY, USA, 1984.
11. Gibali, A.; Reich, S.; Zalas, R. Outer approximation methods for solving variational inequalities in Hilbert space. *Optimization* **2017**, *66*, 417–437. [CrossRef]
12. Kinderlehrer, D.; Stampacchia, G. An introduction to variational inequalities and their applications. In *Classics in Applied Mathematics*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 2000.
13. Li, B.; Liang, H.; Shi, L.; He, Q. Complex dynamics of Kopel model with nonsymmetric response between oligopolists. *Solitons Fractals* **2022**, *156*, 111860. [CrossRef]
14. Ogwo, G.N.; Alakoya, T.O.; Mewomo, O.T. Iterative algorithm with self-adaptive step size for approximating the common solution of variational inequality and fixed point problems. *Optimization* **2021**. [CrossRef]
15. Alakoya, T.O.; Jolaoso, L.O.; Mewomo, O.T. Modified inertial subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems. *Optimization* **2021**, *70*, 545–574. [CrossRef]
16. Ceng, L.C.; Coroian, I.; Qin, X.; Yao, J.C. A general viscosity implicit iterative algorithm for split variational inclusions with hierarchical variational inequality constraints. *Fixed Point Theory* **2019**, *20*, 469–482. [CrossRef]
17. Hai, T.N. Continuous-time ergodic algorithm for solving monotone variational inequalities, *J. Nonlinear Var. Anal.* **2021**, *5*, 391–401.
18. Khan, S.H.; Alakoya, T.O.; Mewomo, O.T. Relaxed projection methods with self-adaptive step size for solving variational inequality and fixed point problems for an infinite family of multivalued relatively nonexpansive mappings in Banach spaces *Math. Comput. Appl.* **2020**, *25*, 54. [CrossRef]
19. Mewomo, O.T.; Alakoya, T.O.; Yao, J.-C.; Akinyemi, L. Strong convergent inertial Tseng's extragradient method for solving non-Lipschitz quasimonotone variational Inequalities in Banach spaces. *J. Nonlinear Var. Anal.* **2023**, *7*, 145–172.
20. Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **2000**, *38*, 431–446. [CrossRef]

21. Alakoya, T.O.; Mewomo, O.T.; Shehu, Y. Strong convergence results for quasimonotone variational inequalities. *Math. Methods Oper. Res.* **2022**, *2022*, 47. [CrossRef]
22. Godwin, E.C.; Alakoya, T.O.; Mewomo, O.T.; Yao, J.-C. Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems. *Appl. Anal.* **2022**. [CrossRef]
23. Uzor, V.A.; Alakoya, T.O.; Mewomo, O.T. Strong convergence of a self-adaptive inertial Tseng's extragradient method for pseudomonotone variational inequalities and fixed point problems. *Open Math.* **2022**, *20*, 234–257. [CrossRef]
24. Alakoya, T.O.; Uzor, V.A.; Mewomo, O.T.; Yao, J.-C. On System of Monotone Variational Inclusion Problems with Fixed-Point Constraint. *J. Inequal. Appl.* **2022**, *2022*, 47. [CrossRef]
25. Censor, Y.; Borteld, T.; Martin, B.; Trofimov, A. A unified approach for inversion problems in intensity-modulated radiation therapy. *Phys. Med. Biol.* **2006**, *51*, 2353–2365. [CrossRef] [PubMed]
26. Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms* **1994**, *8*, 221–239. [CrossRef]
27. López, G.; Martín-Márquez, V.; Xu, H.K. Iterative algorithms for the multiple-sets split feasibility problem. In *Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems, Medical Physics Publishing, Madison*; Medical Physics Publishing: Madison, WI, USA, 2010; pp. 243–279.
28. Moudafi, A.; Thakur, B.S. Solving proximal split feasibility problems without prior knowledge of operator norms. *Optim. Lett.* **2014**, *8*, 2099–2110. [CrossRef]
29. Byrne, C. Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse Probl.* **2002**, *18*, 441–453. [CrossRef]
30. Censor, Y.; Motova, A.; Segal, A. Perturbed projections and subgradient projections for the multiple-sets split feasibility problem. *J. Math. Anal. Appl.* **2007**, *327*, 1244–1256. [CrossRef]
31. Godwin, E.C.; Izuchukwu, C.; Mewomo, O.T. Image restoration using a modified relaxed inertial method for generalized split feasibility problems. *Math. Methods Appl. Sci.* **2022**. [CrossRef]
32. Censor, Y.; Gibali, A.; Reich, S. Algorithms for the split variational inequality problem. *Numer. Algorithms* **2012**, *59*, 301–323. [CrossRef]
33. He, H.; Ling, C.; Xu, H.K. A relaxed projection method for split variational inequalities. *J. Optim. Theory Appl.* **2015**, *166*, 213–233. [CrossRef]
34. Kim, J.K.; Salahuddin, S.; Lim, W.H. General nonconvex split variational inequality problems. *Korean J. Math.* **2017**, *25*, 469–481.
35. Ogwo, G.N.; Izuchukwu, C.; Mewomo, O.T. Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity. *Numer. Algorithms* **2022**, *88*, 1419–1456. [CrossRef]
36. Tian, M.; Jiang, B.-N. Weak convergence theorem for a class of split variational inequality problems and applications in Hilbert space. *J. Ineq. Appl.* **2017**, *2017*, 123. [CrossRef]
37. Reich, S.; Tuyen, T.M. Iterative methods for solving the generalized split common null point problem in Hilbert spaces. *Optimization* **2020**, *69*, 1013–1038. [CrossRef]
38. Reich, S.; Tuyen, T.M. The split feasibility problem with multiple output sets in Hilbert spaces. *Optim. Lett.* **2020**, *14*, 2335–2353. [CrossRef]
39. Polyak, B.T. Some methods of speeding up the convergence of iteration methods. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **1964**, *4*, 1–17. [CrossRef]
40. Chang, S.-S.; Yao, J.-C.; Wang, L.; Liu, M.; Zhao, L. On the inertial forward-backward splitting technique for solving a system of inclusion problems in Hilbert spaces. *Optimization* **2021**, *70*, 2511–2525. [CrossRef]
41. Ogwo, G.N.; Alakoya, T.O.; Mewomo, O.T. Inertial Iterative Method With Self-Adaptive Step Size for Finite Family of Split Monotone Variational Inclusion and Fixed Point Problems in Banach Spaces. *Demonstr. Math.* **2022**, *55*, 193–216. [CrossRef]
42. Uzor, V.A.; Alakoya, T.O.; Mewomo, O.T. On Split Monotone Variational Inclusion Problem with Multiple Output Sets with fixed point constraints. *Comput. Methods Appl. Math.* **2022**. [CrossRef]
43. Wang, Z.-B.; Long, X.; Lei, Z.-Y.; Chen, Z.-Y. New self-adaptive methods with double inertial steps for solving splitting monotone variational inclusion problems with applications. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *114*, 106656. [CrossRef]
44. Yao, Y.; Iyiola, O.S.; Shehu, Y. Subgradient extragradient method with double inertial steps for variational inequalities. *J. Sci. Comput.* **2022**, *90*, 71. [CrossRef]
45. Godwin, E.C.; Alakoya, T.O.; Mewomo, O.T.; Yao, J.-C. Approximation of solutions of split minimization problem with multiple output sets and common fixed point problem in real Banach spaces. *J. Nonlinear Var. Anal.* **2022**, *6*, 333–358.
46. Iutzeler, F.; Hendrickx, J.M. A generic online acceleration scheme for optimization algorithms via relaxation and inertia. *Optim. Methods Softw.* **2019**, *34*, 383–405. [CrossRef]
47. Alvarez, F. Weak convergence of a relaxed-inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert Space. *SIAM J. Optim.* **2004**, *14*, 773–782. [CrossRef]
48. Attouch, H.; Cabot, A. Convergence of a relaxed inertial forward-backward algorithm for structured monotone inclusions. *Optimization* **2019**, *80*, 547–598. [CrossRef]
49. Saejung, S.; Yotkaew, P. Approximation of zeros of inverse strongly monotone operators in Banach spaces. *Nonlinear Anal.* **2012**, *75*, 742–750. [CrossRef]

50. Tan, K.K.; Xu, H.K. Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. *J. Math. Anal. Appl.* **1993**, *178*, 301–308. [CrossRef]
51. Chuang, C.S. Strong convergence theorems for the split variational inclusion problem in Hilbert spaces. *Fixed Point Theory Appl.* **2013**, *350*. [CrossRef]
52. Cottle, R.W.; Yao, J.C. Pseudomonotone complementary problems in Hilbert space. *J. Optim. Theory Appl.* **1992**, *75*, 281–295. [CrossRef]
53. Tian, M.; Jiang, B.-N. Inertial Haugazeau's hybrid subgradient extragradient algorithm for variational inequality problems in Banach spaces. *Optimization* **2021**, *70*, 987–1007. [CrossRef]