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# Some Companions of Fejér Type Inequalities Using GA-Convex Functions

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**Abstract:** In this paper, we present some new and novel mappings defined over $[0,1]$ with the help of $GA$-convex functions. As a consequence, we obtain companions of Fejér-type inequalities for $GA$-convex functions with the help of these mappings, which provide refinements of some known results. The properties of these mappings are discussed as well.

**Keywords:** Hermite–Hadamard inequality; convex function; GA-convex function; fejér inequality

**MSC:** 26D15; 26D20; 26D07

## 1. Introduction

The following double inequality has significant literary importance for convex functions since a number of inequalities for means and quadrature rules can be obtained from it, and is recognized to as the Hermite–Hadamard inequality [1,2]:

Let $\nu : I \longrightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}, \lambda, \kappa \in I$ with $\lambda < \kappa$, be a convex function. Then,

$$\nu\left(\frac{\lambda+\kappa}{2}\right) \leq \frac{1}{\kappa-\lambda}\int_{\lambda}^{\kappa}\nu(\tau)d\tau \leq \frac{\nu(\lambda)+\nu(\kappa)}{2}, \tag{1}$$

the inequality holds in the reverse direction if $\nu$ is concave.

Fejér [3], established the following double inequality as a weighted generalization of (1):

$$\nu\left(\frac{\lambda+\kappa}{2}\right)\int_{\lambda}^{\kappa}\varphi(\tau)d\tau \leq \int_{\lambda}^{\kappa}\nu(\tau)\varphi(\tau)d\tau \leq \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_{\lambda}^{\kappa}\varphi(\tau)d\tau, \tag{2}$$

where $\nu : I \longrightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}, \lambda, \kappa \in I$ with $\lambda < \kappa$ is any convex function and $\varphi : [\lambda, \kappa] \to \mathbb{R}$ is non-negative integrable and symmetric about $\tau = \frac{\lambda+\kappa}{2}$.

The following mappings on $[0,1]$ are of interest:

$$G(\alpha) = \frac{1}{2}\left[\nu\left(\alpha\lambda + (1-\alpha)\frac{\lambda+\kappa}{2}\right) + \nu\left(\alpha\kappa + (1-\alpha)\frac{\lambda+\kappa}{2}\right)\right],$$

$$H(\alpha) = \frac{1}{\kappa-\lambda}\int_{\lambda}^{\kappa}\nu\left(\alpha\tau + (1-\alpha)\frac{\lambda+\kappa}{2}\right)d\tau,$$

$$H_{\varphi}(\alpha) = \int_{\lambda}^{\kappa}\nu\left(\alpha\tau + (1-\alpha)\frac{\lambda+\kappa}{2}\right)\varphi(\tau)d\tau,$$

$$L(\alpha) = \frac{1}{2(\kappa-\lambda)}\int_{\lambda}^{\kappa}[\nu(\alpha\lambda + (1-\alpha)\tau) + \nu(\alpha\kappa + (1-\alpha)\tau)]d\tau,$$

$$I(\alpha) = \frac{1}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\alpha\frac{\lambda+\tau}{2} + (1-\alpha)\frac{\lambda+\kappa}{2}\right) + \nu\left(\alpha\frac{\kappa+\tau}{2} + (1-\alpha)\frac{\lambda+\kappa}{2}\right)\right]\varphi(\tau)d\tau$$

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and

$$S_\varphi(\alpha) = \frac{1}{4}\int_\lambda^\kappa \left[\nu\left(\alpha\lambda + (1-\alpha)\frac{\lambda+\tau}{2}\right) + \nu\left(\alpha\lambda + (1-\alpha)\frac{\tau+\kappa}{2}\right)\right.$$
$$\left. + \nu\left(\alpha\kappa + (1-\alpha)\frac{\lambda+\tau}{2}\right) + \nu\left(\alpha\kappa + (1-\alpha)\frac{\tau+\kappa}{2}\right)\right]\varphi(\tau)d\tau,$$

where $\nu : [\lambda, \kappa] \to \mathbb{R}$ is a convex function and $\varphi : [\lambda, \kappa] \to \mathbb{R}$ is non-negative integrable and symmetric about $\tau = \frac{\lambda+\kappa}{2}$.

There are several modifications and generalizations of these inequalities that can be found in [4–16]. In [4], Ardic et al. proved some Ostrowski-type inequalities using $GG$-convex and $GA$-convex functions. Ardic et al. also obtained some new inequalities for $GG$-convexity and $GA$-convex functions in [5]. In the articles [6–9], Dragomir established some Fejér-, Hermite–Hadamard- and Jensen-type inequalities. Dragomir et al. also obtained different Hermite–Hadamard-type and refinement inequalities for Lipschitzian and convex mappings in [10–12]. There are number of articles that contain Hermite–Hadamard- and Fejér-type inequalities for convex, $GA$- and $GA$-$s$-convex functions; see, for instance, [17–45]. These kinds of inequalities also have a broad set of applications (see [31,32,34,38,46] and the references therein). The important results that characterize the properties of the above mappings and inequalities are discussed by a number of mathematicians.

Dragomir [7] established a result using the mapping $H$, which refines the first inequality of (1). Dragomir obtained another refinement inequality of (1) in [12] related to the mappings $H$, $G$ and $L$.

Tseng et al. [35] proved the following result and Yang and Tseng [40] used it to prove Fejér-type inequalities, which refines the first inequality of (2) with the help of the mapping $H_\varphi$.

**Lemma 1** ([35])**.** *Let $\nu : [\lambda, \kappa] \to \mathbb{R}$ be a convex function and let $\lambda \le \beta_1 \le \tau_1 \le \tau_2 \le \beta_2 \le \kappa$ with $\tau_1 + \tau_2 = \beta_1 + \beta_2$. Then,*

$$\nu(\tau_1) + \nu(\tau_2) \le \nu(\beta_1) + \nu(\beta_2).$$

In [35], Tseng et al. established results related to Fejér-type inequalities (2) using the mapping $I$ to provide refinement inequalities of (2).

Further Fejér-type inequalities have also been proven by Tseng et al. in [36].

**Theorem 1** ([36])**.** *Let $\nu$, $\varphi$ ,$I$ be defined as above. Then, the following Fejér-type inequalities hold:*

(i) *The following inequality holds:*

$$\nu\left(\frac{\lambda+\kappa}{2}\right)\int_\lambda^\kappa \varphi(\tau)d\tau \le 2\left[\int_{\frac{3\lambda+\kappa}{4}}^{\frac{\lambda+\kappa}{2}} \nu(\tau)\varphi(4\tau-2\lambda-\kappa)d\tau + \int_{\frac{\lambda+\kappa}{2}}^{\frac{\lambda+3\kappa}{4}} \nu(\tau)\varphi(4\tau-\lambda-2\kappa)d\tau\right]$$
$$\le \int_0^1 I(\alpha)d\alpha \le \frac{1}{2}\left[\nu\left(\frac{\lambda+\kappa}{2}\right)\int_\lambda^\kappa \varphi(\tau)d\tau\right.$$
$$\left. + \frac{1}{2}\int_\lambda^\kappa\left[\nu\left(\frac{\lambda+\tau}{2}\right) + \nu\left(\frac{\tau+\kappa}{2}\right)\right]\varphi(\tau)d\tau\right]. \tag{3}$$

(ii) *If $\nu$ is differentiable on $[\lambda, \kappa]$ and $\varphi$ is bounded on $[\lambda, \kappa]$, then for all $\alpha \in [0, 1]$, the following inequalities hold:*

$$0 \le \frac{1}{2}\int_\lambda^\kappa\left[\nu\left(\frac{\lambda+\tau}{2}\right) + \nu\left(\frac{\tau+\kappa}{2}\right)\right]\varphi(\tau)d\tau - I(\alpha)$$
$$\le (1-\alpha)\left[(\kappa-\lambda)\left[\frac{\nu(\lambda)+\nu(\kappa)}{2}\right] - \int_\lambda^\kappa \nu(\tau)d\tau\right]\|\varphi\|_\infty, \tag{4}$$

*where* $\|\varphi\|_\infty = \sup_{\tau\in[\lambda,\kappa]} \varphi(\tau)$.

(iii) *If* $\nu$ *is differentiable on* $[\lambda,\kappa]$*, then, for all* $\alpha \in [0,1]$*, we have the inequality*

$$0 \le \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa \varphi(\tau)d\tau - I(\alpha) \le \frac{(\kappa-\lambda)\Big(\nu'(\kappa)-\nu'(\lambda)\Big)}{4}\int_\lambda^\kappa \varphi(\tau)d\tau. \quad (5)$$

In the following theorems, we shall point out some inequalities from [36] for the mappings $G$, $H$, $I$, $S_\varphi$ considered above:

**Theorem 2** ([36])**.** *Let* $\nu$*,* $\varphi$*,* $G$*,*$I$ *be defined as above. Then, we have the following Fejér-type inequalities:*

(i) *The following inequality holds for all* $\alpha \in [0,1]$*:*

$$I(\alpha) \le G(\alpha)\int_\lambda^\kappa \varphi(\tau)d\tau. \quad (6)$$

(ii) *If* $\nu$ *is differentiable on* $[\lambda,\kappa]$ *and* $\varphi$ *is bounded on* $[\lambda,\kappa]$*, then, for all* $\alpha \in [0,1]$*, we have the inequality:*

$$0 \le I(\alpha) - \nu\left(\frac{\lambda+\kappa}{2}\right)\int_\lambda^\kappa \varphi(\tau)d\tau \le (\kappa-\lambda)[G(\alpha)-H(\alpha)]\|\varphi\|_\infty, \quad (7)$$

*where* $\|\varphi\|_\infty = \sup_{\tau\in[\lambda,\kappa]} \varphi(\tau)$.

**Theorem 3** ([36])**.** *Let* $\nu$*,* $\varphi$*,* $G$*,* $I$*,* $S_\varphi$ *be defined as above. Then, we have the following results:*

(i) $S_\varphi$ *is convex on* $[0,1]$.

(ii) *The following inequalities hold for all* $\alpha \in [0,1]$*:*

$$G(\alpha)\int_\lambda^\kappa \varphi(\tau)d\tau \le S_\varphi(\alpha) \le \frac{(1-\alpha)}{2}\int_\lambda^\kappa \left[\nu\left(\frac{\lambda+\tau}{2}\right)+\nu\left(\frac{\tau+\kappa}{2}\right)\right]\varphi(\tau)d\tau$$

$$+\alpha\cdot\frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa \varphi(\tau)d\tau \le \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa \varphi(\tau)d\tau, \quad (8)$$

$$I(1-\alpha) \le S_\varphi(\alpha) \quad (9)$$

*and*

$$\frac{I(\alpha)+I(1-\alpha)}{2} \le S_\varphi(\alpha). \quad (10)$$

(iii) *The following bound is true:*

$$\sup_{\alpha\in[0,1]} S_\varphi(\alpha) = \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa \varphi(\tau)d\tau. \quad (11)$$

One of the significant generalizations of the convex functions is geometrically, arithmetically convex functions, also known a $GA$-convex functions, stated below:

**Definition 1** ([7])**.** *Suppose* $I \subseteq (0,\infty)$ *is an interval of positive real numbers. A function* $\nu : I \to \mathbb{R}$ *is considered to be GA-convex, if*

$$\nu\Big(\tau^\alpha\beta^{1-\alpha}\Big) \le \alpha\nu(\tau) + (1-\alpha)\nu(\beta) \quad (12)$$

*for all* $\tau,\beta \in I$ *and* $\alpha \in [0,1]$*. A function* $\nu : I \to \mathbb{R}$ *is concave if the inequality in (12) reversed.*

We state some important facts which relate $GA$-convex and convex functions and use them to prove the main results.

**Theorem 4** ([7])**.** *If $[\lambda,\kappa] \subset (0,\infty)$ and the function $\mathcal{G} : [\ln\lambda, \ln\kappa] \to \mathbb{R}$ is convex (concave) on $[\ln\lambda, \ln\kappa]$, then the function $\nu : [\lambda,\kappa] \to \mathbb{R}, \nu(\alpha) = \mathcal{G}(\ln\alpha)$ is GA-convex (concave) on $[\lambda,\kappa]$.*

**Remark 1.** *It is obvious from Theorem 4 that if $\nu : [\lambda,\kappa] \to \mathbb{R}$ is GA-convex on $[\lambda,\kappa] \subset (0,\infty)$, then $\nu \circ \exp$ is convex on $[\ln\lambda, \ln\kappa]$. It follows that $\nu \circ \exp$ has finite lateral derivatives on $(\ln\lambda, \ln\kappa)$ and by gradient inequality for convex functions we have*

$$\nu \circ \exp(\tau) - \nu \circ \exp(\beta)(\tau - \beta) \geq \varphi(\exp\beta)\exp(\beta), \tag{13}$$

*where $\varphi(\exp\beta) \in \left[\nu'_-(\exp\beta), \nu'_+(\exp\beta)\right]$ for any $\tau, \beta \in (\ln\lambda, \ln\kappa)$.*

The following inequality of Hermite–Hadamard-type for $GA$-convex functions holds (see [27] for an extension for GA $h$-convex functions):

**Theorem 5** ([27])**.** *Let $\nu : I \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function and $\lambda,\kappa \in I$ with $\lambda < \kappa$. If $\nu \in L([\lambda,\kappa])$, then the following inequalities hold:*

$$\nu\left(\sqrt{\lambda\kappa}\right) \leq \frac{1}{\ln\kappa - \ln\lambda}\int_\lambda^\kappa \frac{\nu(\tau)}{\tau}d\tau \leq \frac{\nu(\lambda)+\nu(\kappa)}{2}. \tag{14}$$

The notion of geometrically symmetric functions was introduced in [23].

**Definition 2** ([23])**.** *A function $\varphi : [\lambda,\kappa] \subseteq (0,\infty) \to \mathbb{R}$ is geometrically symmetric with respect to $\sqrt{\lambda\kappa}$ if*

$$\varphi(\tau) = \varphi\left(\frac{\lambda\kappa}{\tau}\right)$$

*holds for all $\tau \in [\lambda,\kappa]$.*

Fejér-type inequalities using $GA$-convex functions and the notion of geometric symmetric functions were presented in the work of Latif et al. [23].

**Theorem 6** ([23])**.** *Let $\nu : I \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function and $\lambda,\kappa \in I$ with $\lambda < \kappa$. If $\nu \in L([\lambda,\kappa])$ and $\varphi : [\lambda,\kappa] \subseteq (0,\infty) \to \mathbb{R}$ is non-negative, integrable and geometrically symmetric with respect to $\sqrt{\lambda\kappa}$; then,*

$$\nu\left(\sqrt{\lambda\kappa}\right)\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau \leq \int_\lambda^\kappa \frac{\nu(\tau)\varphi(\tau)}{\tau}d\tau \leq \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau. \tag{15}$$

Suppose that $\nu : I \subseteq (0,\infty) \to \mathbb{R}$ is $GA$-convex on $I$ and $\lambda,\kappa \in I$. Let $\mathcal{H}, \mathcal{F}, \mathcal{V}, \mathcal{I}_\varphi : [0,1] \to \mathbb{R}$ be defined by

$$\mathcal{H}(\alpha) = \frac{1}{\ln\kappa - \ln\lambda}\int_\lambda^\kappa \frac{1}{\tau}\nu\left(\tau^\alpha\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)d\tau,$$

$$\mathcal{F}(\alpha) = \frac{1}{\ln\kappa - \ln\lambda}\int_\lambda^\kappa\int_\lambda^\kappa \frac{1}{\tau\beta}\nu\left(\tau^\alpha\beta^{1-\alpha}\right)d\tau d\beta,$$

$$\mathcal{V}(\alpha) = \frac{1}{2(\ln\kappa - \ln\lambda)}\int_\lambda^\kappa \frac{1}{\tau}\left[\nu\left(\kappa^{\frac{1+\alpha}{2}}\tau^{\frac{1-\alpha}{2}}\right) + \nu\left(\lambda^{\frac{1+\alpha}{2}}\tau^{\frac{1-\alpha}{2}}\right)\right]d\tau$$

and

$$\mathcal{I}_\varphi(\alpha) = \frac{1}{2}\int_\lambda^\kappa\left[\nu\left(\left(\sqrt{\lambda\tau}\right)^\alpha\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right) + \nu\left(\left(\sqrt{\tau\kappa}\right)^\alpha\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau,$$

where $\varphi : [\lambda, \kappa] \subseteq (0, \infty) \to \mathbb{R}$ is non-negative, integrable and geometrically symmetric with respect to $\sqrt{\lambda\kappa}$.

Latif et al. [19] obtained the following refinements for the inequalities (14):

**Theorem 7** ([19])**.** *A function $\nu : I \subseteq (0, \infty) \to \mathbb{R}$, as above. Then,*

(i) *$\mathcal{H}$ is GA-convex on $[0, 1]$.*
(ii) *We have*

$$\inf_{\alpha\in[0,1]} \mathcal{H}(\alpha) = \mathcal{H}(0) = \nu\left(\sqrt{\lambda\kappa}\right) \tag{16}$$

*and*

$$\sup_{\alpha\in[0,1]} \mathcal{H}(\alpha) = \mathcal{H}(1) = \frac{1}{\ln\kappa - \ln\lambda}\int_\lambda^\kappa \frac{\nu(\tau)}{\tau}d\tau. \tag{17}$$

(iii) *$\mathcal{H}$ increases monotonically on $[0, 1]$.*

The following theorem holds:

**Theorem 8** ([19])**.** *Let $\nu : [\lambda, \kappa] \subseteq (0, \infty) \to \mathbb{R}$ be as above. Then,*

(i) *$\mathcal{F}(\alpha + \frac{1}{2}) = \mathcal{F}(\frac{1}{2} - \alpha)$ for all $\alpha$ in $[0, \frac{1}{2}]$.*
(ii) *$\mathcal{F}$ is GA-convex on $[0, 1]$.*
(iii) *We have*

$$\sup_{\alpha\in[0,1]} \mathcal{F}(\alpha) = \mathcal{F}(0) = \mathcal{F}(1) = \frac{1}{(\ln\kappa - \ln\lambda)^2}\int_\lambda^\kappa \frac{1}{\tau}\nu(\tau)d\tau \tag{18}$$

*and*

$$\inf_{\alpha\in[0,1]} \mathcal{F}(\alpha) = \mathcal{F}\left(\frac{1}{2}\right) = \frac{1}{\ln\kappa - \ln\lambda}\int_\lambda^\kappa\int_\lambda^\kappa \frac{1}{\tau\beta}\nu\left(\sqrt{\tau\beta}\right)d\tau d\beta. \tag{19}$$

(iv) *The following inequality is valid:*

$$\nu\left(\sqrt{\tau\beta}\right) \leq \mathcal{F}\left(\frac{1}{2}\right). \tag{20}$$

(v) *$\mathcal{F}$ decreases monotonically on $[0, \frac{1}{2}]$ and increases monotonically on $\left[\frac{1}{2}, 1\right]$.*
(vi) *We have the inequality $\mathcal{H}(\alpha) \leq \mathcal{F}(\alpha)$ for all $\alpha \in [0, 1]$.*

**Theorem 9** ([19])**.** *Let $\mathcal{V} : [0, 1] \to \mathbb{R}$ and $\nu : [\lambda, \kappa] \subset (0, \infty) \to \mathbb{R}$ be as defined above. Then,*

(i) *$\mathcal{V}$ is GA-convex on $(0, 1]$.*
(ii) *The following hold:*

$$\inf_{\alpha\in[0,1]} \mathcal{V}(\alpha) = \mathcal{V}(0) = \frac{1}{\ln\kappa - \ln\lambda}\int_\lambda^\kappa \frac{\nu(\tau)}{\tau}d\tau \tag{21}$$

*and*

$$\sup_{\alpha\in[0,1]} \mathcal{V}(\alpha) = \mathcal{V}(1) = \frac{\nu(\lambda) + \nu(\kappa)}{2}. \tag{22}$$

(iii) *$\mathcal{V}$ increases monotonically on $[0, 1]$.*

**Theorem 10** ([25])**.** *Let $\nu, \varphi, \mathcal{I}_\varphi$ be defined as above. Then, $\mathcal{I}_\varphi$ is GA-convex, increasing on $[0, 1]$ , and for all $\alpha \in [0, 1]$, we have the following Fejér-type inequality*

$$\nu\left(\sqrt{\lambda\kappa}\right)\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau \leq \mathcal{I}_\varphi(0) \leq \mathcal{I}_\varphi(\alpha) \leq \mathcal{I}_\varphi(1)$$

$$= \frac{1}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau. \tag{23}$$

Motivated by the studies conducted in [10–13,15,16,30–36,40–44], we define some new mappings in connection to the inequalities (14) and (15) to prove to prove new Féjer type inequalities, which are variants of the inequalities given in Theorems 1–3 for GA-convex functions using novel techniques and using variant of Lemma 2 for GA-convex functions.

## 2. Main Results

Let us define some mappings on $[0,1]$ related to (15) and prove some refinement inequalities.

$$\mathcal{G}(\alpha) = \frac{1}{2}\left[\nu\left(\lambda^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right],$$

$$\mathcal{H}(\alpha) = \frac{1}{\ln\kappa-\ln\lambda}\int_{\lambda}^{\kappa}\frac{1}{\tau}\nu\left(\tau^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)d\tau,$$

$$\mathcal{L}(\alpha) = \frac{1}{2(\ln\kappa-\ln\lambda)}\int_{\lambda}^{\kappa}\frac{1}{\tau}\left[\nu\left(\lambda^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\tau^{1-\alpha}\right)\right]d\tau$$

and

$$\mathcal{S}_{\varphi}(\alpha) = \frac{1}{4}\int_{\lambda}^{\kappa}\left[\nu\left(\lambda^{\alpha}\left(\sqrt{\lambda\tau}\right)^{1-\alpha}\right)+\nu\left(\lambda^{\alpha}\left(\sqrt{\kappa\tau}\right)^{1-\alpha}\right)\right.$$
$$\left.+\nu\left(\kappa^{\alpha}\left(\sqrt{\lambda\tau}\right)^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\sqrt{\kappa\tau}\right)^{1-\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau,$$

where $\nu:[\lambda,\kappa]\to\mathbb{R}$ is a $GA$-convex function and $\varphi:[\lambda,\kappa]\to\mathbb{R}$ is non-negative integrable and symmetric about $\tau=\sqrt{\lambda\kappa}$.

The following result is very important to establish the results of this section.

**Lemma 2** ([25]). *Let $\nu:[\lambda,\kappa]\to\mathbb{R}$ be a $GA$-convex function and let $\lambda\le\beta_1\le\tau_1\le\tau_2\le\beta_2\le\kappa$ with $\tau_1\tau_2=\beta_1\beta_2$. Then,*

$$\nu(\tau_1)+\nu(\tau_2)\le\nu(\beta_1)+\nu(\beta_2). \tag{24}$$

Now, we present the first result, which is a variant of Theorem 1 for $GA$-convex functions.

**Theorem 11.** *Let $\nu$, $\varphi$ ,$\mathcal{I}_{\varphi}$ be defined as above. Then,*

(i) *The following inequality holds:*

$$\nu\left(\sqrt{\lambda\kappa}\right)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau\le 2\left[\int_{\lambda^{\frac{3}{4}}\kappa^{\frac{1}{4}}}^{\sqrt{\lambda\kappa}}\nu(\tau)\varphi\left(\frac{\tau^4}{\lambda^2\kappa}\right)\frac{d\tau}{\tau}+\int_{\sqrt{\lambda\kappa}}^{\lambda^{\frac{1}{4}}\kappa^{\frac{3}{4}}}\nu(\tau)\varphi\left(\frac{\tau^4}{\lambda\kappa^2}\right)\frac{d\tau}{\tau}\right]$$

$$\le\int_0^1\mathcal{I}_{\varphi}(\alpha)d\alpha\le\frac{1}{2}\left[\nu\left(\sqrt{\lambda\kappa}\right)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau+\frac{1}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau\right] \tag{25}$$

(ii) *If $\nu$ is differentiable on $I^{\circ}$ (interior of $I$) with $[\lambda,\kappa]\subseteq I^{\circ}$ and $\varphi$ bounded on $[\lambda,\kappa]$. Then, for all $\alpha\in[0,1]$, the inequality inequalities hold:*

$$0\le\frac{1}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau-\mathcal{I}_{\varphi}(\alpha)$$
$$\le(1-\alpha)\left[(\ln\kappa-\ln\lambda)\frac{\nu(\lambda)+\nu(\kappa)}{2}-\int_{\lambda}^{\kappa}\frac{\nu(\tau)}{\tau}d\tau\right]\|\varphi\|_{\infty}, \tag{26}$$

*where* $\|\varphi\|_{\infty}=\sup_{\tau\in[\lambda,\kappa]}\varphi(\tau)$.

(iii) *If $\nu$ is differentiable on $I^{\circ}$ (interior of $I$) with $[\lambda,\kappa]\subseteq I^{\circ}$, then the inequalities*

$$0\le \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau-\mathcal{I}_{\varphi}(\alpha)\le \frac{(\ln\kappa-\ln\lambda)\left(\kappa\nu'(\kappa)-\lambda\nu'(\lambda)\right)}{4}\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau \tag{27}$$

*hold for all $\alpha\in[0,1]$.*

**Proof.** (i) By applying the techniques of integration and the assumptions on $\varphi$, we obtain

$$\nu\left(\sqrt{\lambda\kappa}\right)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau=4\int_{\lambda}^{\sqrt{\lambda\kappa}}\int_{0}^{\frac{1}{2}}\nu\left(\sqrt{\lambda\kappa}\right)\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\alpha d\tau, \tag{28}$$

$$2\left[\int_{\lambda^{\frac{3}{4}}\kappa^{\frac{1}{4}}}^{\sqrt{\lambda\kappa}}\nu(\tau)\varphi\left(\frac{\tau^4}{\lambda^2\kappa}\right)\frac{d\tau}{\tau}+\int_{\sqrt{\lambda\kappa}}^{\lambda^{\frac{1}{4}}\kappa^{\frac{3}{4}}}\nu(\tau)\varphi\left(\frac{\tau^4}{\lambda\kappa^2}\right)\frac{d\tau}{\tau}\right]$$
$$=2\int_{\lambda^{\frac{3}{4}}\kappa^{\frac{1}{4}}}^{\sqrt{\lambda\kappa}}\left[\nu(\tau)+\nu\left(\frac{\lambda\kappa}{\tau}\right)\right]\varphi\left(\frac{\tau^4}{\lambda^2\kappa}\right)\frac{d\tau}{\tau}$$
$$=2\int_{\lambda}^{\sqrt{\lambda\kappa}}\int_{0}^{\frac{1}{2}}\left[\nu\left(\tau^{\frac{1}{2}}\lambda^{\frac{1}{4}}\kappa^{\frac{1}{4}}\right)+\nu\left(\tau^{-\frac{1}{2}}\lambda^{\frac{3}{4}}\kappa^{\frac{3}{4}}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\alpha d\tau, \tag{29}$$

$$\int_0^1\mathcal{I}_{\varphi}(\alpha)d\alpha=\frac{1}{2}\int_{\lambda}^{\kappa}\int_0^1\left[\nu\left(\left(\sqrt{\lambda\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right.$$
$$\left.+\nu\left(\left(\sqrt{\lambda\tau}\right)^{1-\alpha}\left(\sqrt{\lambda\kappa}\right)^{\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\alpha d\tau+\frac{1}{2}\int_{\lambda}^{\kappa}\int_0^1\left[\nu\left(\left(\sqrt{\tau\kappa}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right.$$
$$\left.+\nu\left(\left(\sqrt{\tau\kappa}\right)^{1-\alpha}\left(\sqrt{\lambda\kappa}\right)^{\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\alpha d\tau=\frac{1}{2}\int_{\lambda}^{\kappa}\int_0^1\left[\nu\left(\left(\sqrt{\lambda\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right.$$
$$\left.+\nu\left(\left(\sqrt{\lambda\tau}\right)^{1-\alpha}\left(\sqrt{\lambda\kappa}\right)^{\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\alpha d\tau+\frac{1}{2}\int_{\lambda}^{\kappa}\int_0^1\left[\nu\left(\left(\sqrt{\frac{\lambda\kappa^2}{\tau}}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right.$$
$$\left.+\nu\left(\left(\sqrt{\frac{\lambda\kappa^2}{\tau}}\right)^{1-\alpha}\left(\sqrt{\lambda\kappa}\right)^{\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\alpha d\tau$$
$$=\int_{\lambda}^{\sqrt{\lambda\kappa}}\int_0^{\frac{1}{2}}\left[\nu\left(\left(\sqrt{\lambda\kappa}\right)^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\tau^{\alpha}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\alpha d\tau$$
$$+\int_{\lambda}^{\sqrt{\lambda\kappa}}\int_0^{\frac{1}{2}}\left[\nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\tau^{\alpha}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\alpha d\tau \tag{30}$$

and

$$\frac{1}{2}\left[\nu\left(\sqrt{\lambda\kappa}\right)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau+\int_{\lambda}^{\kappa}\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau\right]$$
$$=\frac{1}{2}\int_{\lambda}^{\kappa}\int_0^{\frac{1}{2}}\left[2\nu\left(\sqrt{\lambda\kappa}\right)+\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\frac{\lambda\kappa^2}{\tau}}\right)\right]\frac{\varphi(\tau)}{\tau}d\alpha d\tau$$
$$=\int_{\lambda}^{\sqrt{\lambda\kappa}}\int_0^{\frac{1}{2}}\left[\nu(\tau)+\nu\left(\sqrt{\lambda\kappa}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\alpha d\tau+\int_{\lambda}^{\sqrt{\lambda\kappa}}\int_0^{\frac{1}{2}}\left[\nu\left(\sqrt{\lambda\kappa}\right)+\nu\left(\frac{\lambda\kappa}{\tau}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\alpha d\tau. \tag{31}$$

According to Lemma 2, the following inequalities hold for all $\alpha\in\left[0,\frac{1}{2}\right]$ and $\tau\in\left[\lambda,\sqrt{\lambda\kappa}\right]$:

The inequality

$$4\nu\left(\sqrt{\lambda\kappa}\right)\le 2\left[\nu\left(\tau^{\frac{1}{2}}\lambda^{\frac{1}{4}}\kappa^{\frac{1}{4}}\right)+\nu\left(\tau^{-\frac{1}{2}}\lambda^{\frac{3}{4}}\kappa^{\frac{3}{4}}\right)\right] \tag{32}$$

when $\tau_1 = \tau_2 = \sqrt{\lambda\kappa}$, $\beta_1 = \tau^{\frac{1}{2}}\lambda^{\frac{1}{4}}\kappa^{\frac{1}{4}}$ and $\beta_2 = \tau^{-\frac{1}{2}}\lambda^{\frac{3}{4}}\kappa^{\frac{3}{4}}$ in Lemma 2.
The inequality

$$2\nu\left(\tau^{\frac{1}{2}}\lambda^{\frac{1}{4}}\kappa^{\frac{1}{4}}\right) \leq \nu\left(\left(\sqrt{\lambda\kappa}\right)^{\alpha}\tau^{1-\alpha}\right) + \nu\left(\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\tau^{\alpha}\right) \tag{33}$$

when $\tau_1 = \tau_2 = \tau^{\frac{1}{2}}\lambda^{\frac{1}{4}}\kappa^{\frac{1}{4}}$, $\beta_1 = \left(\sqrt{\lambda\kappa}\right)^{\alpha}\tau^{1-\alpha}$ and $\beta_2 = \left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\tau^{\alpha}$ in Lemma 2.
The inequality

$$2\nu\left(\tau^{-\frac{1}{2}}\lambda^{\frac{3}{4}}\kappa^{\frac{3}{4}}\right) \leq \nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\tau^{1-\alpha}\right) + \nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\tau^{\alpha}\right) \tag{34}$$

when $\tau_1 = \tau_2 = \tau^{-\frac{1}{2}}\lambda^{\frac{3}{4}}\kappa^{\frac{3}{4}}$, $\beta_1 = \left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\tau^{1-\alpha}$ and $\beta_2 = \left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\tau^{\alpha}$ in Lemma 2.
The inequality

$$\nu\left(\left(\sqrt{\lambda\kappa}\right)^{\alpha}\tau^{1-\alpha}\right) + \nu\left(\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\tau^{\alpha}\right) \leq \nu(\tau) + \nu\left(\sqrt{\lambda\kappa}\right) \tag{35}$$

when $\tau_1 = \left(\sqrt{\lambda\kappa}\right)^{\alpha}\tau^{1-\alpha}$, $\tau_2 = \left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\tau^{\alpha}$, $\beta_1 = \tau$ and $\beta_2 = \sqrt{\lambda\kappa}$ in Lemma 2.
Finally, the inequality

$$\nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\tau^{1-\alpha}\right) + \nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\tau^{\alpha}\right) \leq \nu\left(\sqrt{\lambda\kappa}\right) + \nu\left(\frac{\lambda\kappa}{\tau}\right) \tag{36}$$

when $\tau_1 = \left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\tau^{1-\alpha}$, $\tau_2 = \left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\tau^{\alpha}$, $\beta_1 = \sqrt{\lambda\kappa}$ and $\beta_2 = \frac{\lambda\kappa}{\tau}$ in Lemma 2.
Multiplying the inequalities (32)–(36) by $\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}$ and integrating with respect to $\alpha$ over $\left[0, \frac{1}{2}\right]$ and with respect to $\tau$ over $\left[\lambda, \sqrt{\lambda\kappa}\right]$ and using the identities (28)–(31), we obtain (25).

(ii) $\nu : [\lambda, \kappa] \to \mathbb{R}$ is $GA$-convex on $[\lambda, \kappa]$, hence the mapping $\sigma : [\ln\lambda, \ln\kappa] \to \mathbb{R}$ defined by $\sigma(\tau) = \nu \circ \exp(\tau)$ is convex on $[\ln\lambda, \ln\kappa]$.
By integration by parts, we find that following identity holds:

$$\int_{\ln\lambda}^{\frac{\ln\lambda+\ln\kappa}{2}} \left(\frac{\ln\lambda + \ln\kappa}{2} - \tau\right)\left[\sigma'(\ln\lambda + \ln\kappa - \tau) - \sigma'(\tau)\right]d\tau$$

$$= \left(\frac{\ln\kappa - \ln\lambda}{2}\right)[\sigma(\ln\lambda) + \sigma(\ln\kappa)] - \int_{\ln\lambda}^{\frac{\ln\lambda+\ln\kappa}{2}} [\sigma(\ln\lambda + \ln\kappa - \tau) + \sigma(\tau)]d\tau. \tag{37}$$

The equality (37) is equivalent to the following equality:

$$\int_{\lambda}^{\sqrt{\lambda\kappa}} \frac{1}{\tau}\left(\frac{\ln\lambda + \ln\kappa}{2} - \ln\tau\right)\left[\left(\frac{\lambda\kappa}{\tau}\right)\nu'\left(\frac{\lambda\kappa}{\tau}\right) - \tau\nu'(\tau)\right]$$

$$= (\ln\kappa - \ln\lambda)\left[\frac{\nu(\lambda) + \nu(\kappa)}{2}\right] - \int_{\lambda}^{\kappa} \frac{\nu(\tau)}{\tau}d\tau.d\tau \tag{38}$$

Using substitution rules of integration and the hypothesis on $\varphi$, we have the following identities:

$$\frac{1}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\sqrt{\lambda\tau}\right) + \nu(\sqrt{\tau\kappa})\right]\frac{\varphi(\tau)}{\tau}d\tau = \frac{1}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\sqrt{\lambda\tau}\right) + \nu\left(\frac{\lambda\kappa}{\tau}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau$$

$$= \int_{\lambda}^{\sqrt{\lambda\kappa}} \left[ \nu\left(\sqrt{\lambda\tau}\right) + \nu\left(\sqrt{\frac{\lambda\kappa^2}{\tau}}\right) \right] \frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau} d\tau \tag{39}$$

and

$$\mathcal{I}_{\varphi}(\alpha) = \frac{1}{2} \int_{\lambda}^{\kappa} \left[ \nu\left( \left(\sqrt{\lambda\tau}\right)^{\alpha} \left(\sqrt{\lambda\kappa}\right)^{1-\alpha} \right) + \nu\left( \left(\sqrt{\frac{\lambda\kappa^2}{\tau}}\right)^{\alpha} \left(\sqrt{\lambda\kappa}\right)^{1-\alpha} \right) \right] \frac{\varphi(\tau)}{\tau} d\tau$$

$$= \int_{\lambda}^{\sqrt{\lambda\kappa}} \left[ \nu\left( \tau^{\alpha} \left(\sqrt{\lambda\kappa}\right)^{1-\alpha} \right) + \nu\left( \left(\frac{\lambda\kappa}{\tau}\right)^{\alpha} \left(\sqrt{\lambda\kappa}\right)^{1-\alpha} \right) \right] \frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau} d\tau \tag{40}$$

hold for all $\alpha \in [0,1]$.

Using the convexity of $\sigma(\tau) = \nu \circ \exp(\tau)$ on $[\ln\lambda, \ln\kappa]$ and the hypothesis on $\varphi$, we obtain that

$$\left[ \sigma(\tau) - \sigma\left( \alpha\tau + (1-\alpha)\frac{\ln\lambda + \ln\kappa}{2} \right) \right] \varphi \circ \exp(2\tau - \ln\lambda)$$

$$+ \left[ \sigma(\ln\lambda + \ln\kappa - \tau) - \sigma\left( \alpha(\ln\lambda + \ln\kappa - \tau) + (1-\alpha)\frac{\ln\lambda + \ln\kappa}{2} \right) \right] \varphi \circ \exp(2\tau - \ln\lambda)$$

$$\leq (1-\alpha)\left( \tau - \frac{\ln\lambda + \ln\kappa}{2} \right) \sigma'(\tau) \varphi \circ \exp(2\tau - \ln\lambda)$$

$$+ (1-\alpha)\left( \frac{\ln\lambda + \ln\kappa}{2} - \tau \right) \sigma'(\ln\lambda + \ln\kappa - \tau) \varphi \circ \exp(2\tau - \ln\lambda)$$

$$= (1-\alpha)\left( \frac{\ln\lambda + \ln\kappa}{2} - \tau \right) \left[ \sigma'(\ln\lambda + \ln\kappa - \tau) - \sigma'(\tau) \right] \varphi \circ \exp(2\tau - \ln\lambda)$$

$$\leq (1-\alpha)\left( \frac{\ln\lambda + \ln\kappa}{2} - \tau \right) \left[ \sigma'(\ln\lambda + \ln\kappa - \tau) - \sigma'(\tau) \right] \|\varphi\|_{\infty} \tag{41}$$

holds for all $\alpha \in [0,1]$ and $\tau \in \left[\ln\lambda, \frac{\ln\lambda + \ln\kappa}{2}\right]$.

From (41), we obtain

$$\left[ \nu(\tau) - \nu\left( \tau^{\alpha} \left(\sqrt{\lambda\kappa}\right)^{1-\alpha} \right) \right] \varphi\left(\frac{\tau^2}{\lambda}\right) + \left[ \nu\left(\frac{\lambda\kappa}{\tau}\right) - \nu\left( \left(\frac{\lambda\kappa}{\tau}\right)^{\alpha} \left(\sqrt{\lambda\kappa}\right)^{1-\alpha} \right) \right] \varphi\left(\frac{\tau^2}{\lambda}\right)$$

$$\leq (1-\alpha)\frac{1}{\tau}\left( \frac{\ln\lambda + \ln\kappa}{2} - \tau \right) \left[ \nu'\left(\frac{\lambda\kappa}{\tau}\right) - \nu'(\tau) \right] \|\varphi\|_{\infty} \tag{42}$$

holds for all $\alpha \in [0,1]$ and $\tau \in \left[\lambda, \sqrt{\lambda\kappa}\right]$. Integrating the inequality (42) over $\tau$ on $\left[\lambda, \sqrt{\lambda\kappa}\right]$ and using (38), (39), (40) and (23), we obtain (26).

(iii) We use the fact that $\nu : [\lambda, \kappa] \to \mathbb{R}$ is $GA$-convex on $[\lambda, \kappa]$; hence, $\sigma : [\ln\lambda, \ln\kappa] \to \mathbb{R}$ defined by $\sigma(\tau) = \nu \circ \exp(\tau)$ is convex on $[\ln\lambda, \ln\kappa]$. Thus,

$$\frac{\sigma(\ln\lambda) - \sigma\left(\frac{\ln\lambda + \ln\kappa}{2}\right)}{2} \leq \frac{\ln\lambda - \ln\kappa}{4} \sigma'(\ln\lambda) \tag{43}$$

and

$$\frac{\sigma(\ln\kappa) - \sigma\left(\frac{\ln\lambda + \ln\kappa}{2}\right)}{2} \leq \frac{\ln\kappa - \ln\lambda}{4} \sigma'(\ln\kappa) \tag{44}$$

Adding the above inequalities,

$$\frac{\sigma(\ln\lambda) + \sigma(\ln\kappa)}{2} - \sigma\left(\frac{\ln\lambda + \ln\kappa}{2}\right) \leq \frac{(\ln\kappa - \ln\lambda)\left(\sigma'(\ln\kappa) - \sigma'(\ln\lambda)\right)}{4}. \tag{45}$$

The inequality (45) becomes

$$\frac{\nu(\lambda)+\nu(\kappa)}{2}-\nu\left(\sqrt{\lambda\kappa}\right)\leq\frac{(\ln\kappa-\ln\lambda)\left(\kappa\nu'(\kappa)-\lambda\nu'(\lambda)\right)}{4}. \tag{46}$$

Multiplying (46) both sides by $\frac{\varphi(\tau)}{\tau^2}$ and integrating over $[\lambda,\kappa]$, we obtain

$$\frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau-\nu\left(\sqrt{\lambda\kappa}\right)\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau$$
$$\leq\frac{(\ln\kappa-\ln\lambda)\left(\kappa\nu'(\kappa)-\lambda\nu'(\lambda)\right)}{4}\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau. \tag{47}$$

From (23) and (47), we obtain (27). □

**Example 1.** *Let $\mathcal{G}(\tau)=\exp(\tau)$, $\tau\in[\ln 2,\ln 3]$; then, according to Theorem 4, $\nu(\tau)=\tau$ is a GA-convex function on $[2,3]$. Moreover, the mapping $\varphi(\tau)=\left(\tau-\frac{6}{\tau}\right)^2$ is symmetric with respect to $\sqrt{6}$ over the interval $[2,3]$. Now,*

$$\nu\left(\sqrt{\lambda\kappa}\right)\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau=\sqrt{6}\int_2^3\frac{\left(\tau-\frac{6}{\tau}\right)^2}{\tau}d\tau=5-12\ln\left(\frac{3}{2}\right)=0.134419, \tag{48}$$

$$2\left[\int_{\lambda^{\frac34}\kappa^{\frac14}}^{\sqrt{\lambda\kappa}}\nu(\tau)\varphi\left(\frac{\tau^4}{\lambda^2\kappa}\right)\frac{d\tau}{\tau}+\int_{\sqrt{\lambda\kappa}}^{\lambda^{\frac14}\kappa^{\frac34}}\nu(\tau)\varphi\left(\frac{\tau^4}{\lambda\kappa^2}\right)\frac{d\tau}{\tau}\right]$$
$$=2\left[\int_{2^{\frac34}3^{\frac14}}^{\sqrt{6}}\left(\frac{\tau^4}{12}-\frac{72}{\tau^4}\right)^2d\tau+\int_{\sqrt{6}}^{2^{\frac14}3^{\frac34}}\left(\frac{\tau^4}{18}-\frac{108}{\tau^4}\right)^2d\tau\right]$$
$$=2\left[\frac{809}{21}\left(\frac{2}{3}\right)^{\frac34}-\frac{18\sqrt{6}}{5}+\frac{1}{63}\left(-729+2^{\frac14}3^{\frac34}+809\sqrt{6}\right)\right]=0.329935,$$

$$\int_0^1\mathcal{I}_\varphi(\alpha)d\alpha=\frac{1}{2}\int_0^1\left[\int_\lambda^\kappa\left[\nu\left(\left(\sqrt{\lambda\tau}\right)^\alpha\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\left(\sqrt{\tau\kappa}\right)^\alpha\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau\right]d\alpha$$
$$=\frac{1}{2}\int_0^1\left[\int_2^3\left[\left(\sqrt{2\tau}\right)^\alpha\left(\sqrt{6}\right)^{1-\alpha}+\left(\sqrt{3\tau}\right)^\alpha\left(\sqrt{6}\right)^{1-\alpha}\right]\frac{\left(\tau-\frac{6}{\tau}\right)^2}{\tau}d\tau\right]d\alpha=0.330161$$

*and*

$$\frac{1}{2}\left[\nu\left(\sqrt{\lambda\kappa}\right)\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau+\frac{1}{2}\int_\lambda^\kappa\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau\right]$$
$$=\frac{1}{2}\left[\sqrt{6}\int_2^3\frac{\left(\tau-\frac{6}{\tau}\right)^2}{\tau}d\tau+\frac{1}{2}\int_2^3\left[\sqrt{2\tau}+\sqrt{3\tau}\right]\frac{\left(\tau-\frac{6}{\tau}\right)^2}{\tau}d\tau\right]$$
$$=\frac{1}{2}\left[\frac{1}{2}\left(-\frac{62}{5}+16\sqrt{\frac{2}{3}}\right)+\sqrt{6}\left(5-12\ln\left(\frac{3}{2}\right)\right)\right]=0.330615.$$

*The above calculations validate the inequality (25).*

*Let $\alpha=\frac{1}{2}$ and consider now*

$$\frac{1}{2}\int_\lambda^\kappa\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau-\mathcal{I}_\varphi(\alpha)=\frac{1}{2}\int_2^3\left[\sqrt{2\tau}+\sqrt{3\tau}\right]\frac{\left(\tau-\frac{6}{\tau}\right)^2}{\tau}d\tau-\mathcal{I}_\varphi\left(\frac{1}{2}\right)$$
$$=\frac{1}{2}\left(-\frac{62}{5}+16\sqrt{\frac{2}{3}}\right)-\frac{2}{63}\left(80\sqrt{6}-\sqrt{2903285\sqrt{6}-7077132}\right)=0.00203793$$

*and*

$$(1-\alpha)\left[(\ln\kappa-\ln\lambda)\frac{\nu(\lambda)+\nu(\kappa)}{2}-\int_\lambda^\kappa\frac{\nu(\tau)}{\tau}d\tau\right]\|\varphi\|_\infty=\frac{1}{2}\left[\frac{5}{2}(\ln 3-\ln 2)-1\right]=0.00683139.$$

*The last two calculations prove that (26) is valid.*

*Lastly, for $\alpha=\frac{1}{2}$, we observe that*

$$\frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau-\mathcal{I}_\varphi(\alpha)=\frac{5}{2}\int_2^3\frac{\left(\tau-\frac{6}{\tau}\right)^2}{\tau}d\tau-\mathcal{I}_\varphi\left(\frac{1}{2}\right)$$
$$=\frac{5}{2}\left(5-12\ln\left(\frac{3}{2}\right)\right)-\frac{2}{63}\left(80\sqrt{6}-\sqrt{2903285\sqrt{6}-7077132}\right)=0.00611204$$

*and*

$$\frac{(\ln\kappa-\ln\lambda)\left(\kappa\nu'(\kappa)-\lambda\nu'(\lambda)\right)}{4}\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau=\frac{(\ln 3-\ln 2)}{4}\int_2^3\frac{\left(\tau-\frac{6}{\tau}\right)^2}{\tau}d\tau$$
$$=\frac{1}{4}\left(5-12\ln\left(\frac{3}{2}\right)\right)(\ln 3-\ln 2)=0.0136255.$$

*Hence, the last two calculations show that the inequality (27) is true as well.*

**Corollary 1.** *Let $\varphi(\tau)=\frac{1}{\ln\kappa-\ln\lambda}$ ($\tau\in[\lambda,\kappa]$) in Theorem 11. Then,*

(i) *The following inequality holds:*

$$\nu\left(\sqrt{\lambda\kappa}\right)\le\frac{2}{\ln\kappa-\ln\lambda}\left[\int_{\lambda^{\frac{3}{4}}\kappa^{\frac{1}{4}}}^{\sqrt{\lambda\kappa}}\frac{\nu(\tau)}{\tau}d\tau+\int_{\sqrt{\lambda\kappa}}^{\lambda^{\frac{1}{4}}\kappa^{\frac{3}{4}}}\frac{\nu(\tau)}{\tau}d\tau\right]\le\int_0^1\mathcal{H}(\alpha)d\alpha$$
$$\le\frac{1}{2}\left[\nu\left(\sqrt{\lambda\kappa}\right)+\frac{1}{2(\ln\kappa-\ln\lambda)}\int_\lambda^\kappa\frac{1}{\tau}\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]d\tau\right].\tag{49}$$

(ii) *If $\nu$ is differentiable on $I^\circ$ (interior of $I$) with $[\lambda,\kappa]\subseteq I^\circ$ and $\varphi$ bounded on $[\lambda,\kappa]$, then for all $\alpha\in[0,1]$, the inequality inequalities hold:*

$$0\le\frac{1}{2(\ln\kappa-\ln\lambda)}\int_\lambda^\kappa\frac{1}{\tau}\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]d\tau-\mathcal{H}(\alpha)$$
$$\le\frac{1-\alpha}{\ln\kappa-\ln\lambda}\left[(\ln\kappa-\ln\lambda)\frac{\nu(\lambda)+\nu(\kappa)}{2}-\int_\lambda^\kappa\frac{\nu(\tau)}{\tau}d\tau\right].\tag{50}$$

(iii) *If $\nu$ is differentiable on $I^\circ$ (Interior of $I$) with $[\lambda,\kappa]\subseteq I^\circ$, then the inequalities*

$$0\le\frac{\nu(\lambda)+\nu(\kappa)}{2}-\mathcal{H}(\alpha)\le\frac{(\ln\kappa-\ln\lambda)\left(\kappa\nu'(\kappa)-\lambda\nu'(\lambda)\right)}{4}\tag{51}$$

*hold for all $\alpha\in[0,1]$.*

**Proof.** If $\varphi(\tau)=\frac{1}{\ln\kappa-\ln\lambda}$ ($\tau\in[\lambda,\kappa]$), then

$$\begin{aligned}\mathcal{I}_{\varphi}(\alpha) &= \frac{1}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\left(\sqrt{\lambda\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\left(\sqrt{\tau\kappa}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau \\ &= \frac{1}{2(\ln\kappa-\ln\lambda)}\int_{\lambda}^{\kappa}\frac{1}{\tau}\left[\nu\left(\left(\sqrt{\lambda\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\left(\sqrt{\tau\kappa}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]d\tau \\ &= \frac{2}{2(\ln\kappa-\ln\lambda)}\int_{\lambda}^{\kappa}\frac{1}{\tau}\nu\left(\left(\sqrt{\lambda\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)d\tau \\ &= \frac{2}{2(\ln\kappa-\ln\lambda)}\int_{\lambda}^{\kappa}\frac{1}{\tau}\nu\left(\left(\sqrt{\lambda\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)d\tau = \mathcal{H}(\alpha)\end{aligned}$$

for all $\alpha \in [0,1]$ and therefore the proof is completed. □

**Remark 2.** *The inequalities (49) provide a refinement of Theorem 7.*

Now, we provide a generalization of Theorem 2 using $GA$-convex mappings.

**Theorem 12.** *Let $\nu$, $\varphi$, $\mathcal{G}$, $\mathcal{I}_{\varphi}$, $\mathcal{H}$ be defined as above. Then, we have the following Fejér-type inequalities:*

(i) *The following inequality holds for all $\alpha \in [0,1]$:*

$$\mathcal{I}_{\varphi}(\alpha) \leq \mathcal{G}(\alpha)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau. \quad (52)$$

(ii) *If $\nu$ is differentiable on $I^{\circ}$ (interior of $I$) with $[\lambda,\kappa] \subseteq I^{\circ}$ and $\varphi$ bounded on $[\lambda,\kappa]$, then for all $\alpha \in [0,1]$, the inequality inequalities hold:*

$$0 \leq \mathcal{I}_{\varphi}(\alpha) - \nu\left(\sqrt{\lambda\kappa}\right)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau \leq (\ln\kappa-\ln\lambda)[\mathcal{G}(\alpha)-\mathcal{H}(\alpha)]\|\varphi\|_{\infty}, \quad (53)$$

*where* $\|\varphi\|_{\infty} = \sup_{\tau\in[\lambda,\kappa]}\varphi(\tau)$.

**Proof.** (i) Using the suitable substitution and assumptions on $\varphi$, we obtain the following identity:

$$\mathcal{G}(\alpha)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau = \int_{\lambda}^{\sqrt{\lambda\kappa}}\left[\nu\left(\lambda^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\tau. \quad (54)$$

By using Lemma 2, we observed that the following inequality holds for all $\alpha \in [0,1]$ and $\tau \in \left[\lambda, \sqrt{\lambda\kappa}\right]$:

$$\nu\left(\tau^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right) \leq \nu\left(\lambda^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right) \quad (55)$$

when we take $\tau_1 = \tau^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}$, $\tau_2 = \left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}$, $\beta_1 = \lambda^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}$ and $\beta_2 = \kappa^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}$ in Lemma 2.

Multiplying the inequalities (55) by $\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}$, integrating with respect to $\tau$ over $\left[\lambda, \sqrt{\lambda\kappa}\right]$ and using the identities (40) and (54), we obtain (52).

(ii) Using an integration by parts, we have that the following identity holds on $[0,1]$:

$$\alpha\int_{\ln\lambda}^{\frac{\ln\lambda+\ln\kappa}{2}}\left[\left(\tau-\frac{\ln\lambda+\ln\kappa}{2}\right)\sigma'\left(\alpha\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right.$$

$$+\left(\frac{\ln\lambda+\ln\kappa}{2}-\tau\right)\sigma'\left(\alpha(\ln\lambda+\ln\kappa-\tau)+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\Bigg]d\tau$$
$$=\alpha\int_{\ln\lambda}^{\ln\kappa}\left(\tau-\frac{\ln\lambda+\ln\kappa}{2}\right)\sigma'\left(\alpha\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)d\tau$$
$$=\left(\frac{\ln\kappa-\ln\lambda}{2}\right)\left[\sigma\left(\alpha\ln\kappa+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)+\sigma\left(\alpha\ln\lambda+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right]$$
$$-\int_{\ln\lambda}^{\ln\kappa}\sigma\left(\alpha\ln\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)d\tau. \tag{56}$$

Using the convexity of $\sigma$ and the hypothesis of $\varphi$, the inequality holds for all $\alpha\in[0,1]$ and $\tau\in\left[\ln\lambda,\frac{\ln\lambda+\ln\kappa}{2}\right]$:

$$\left[\sigma\left(\alpha\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)-\sigma\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right]\varphi\circ\exp(2\tau-\ln\lambda)$$
$$+\left[\sigma\left(\alpha(\ln\lambda+\ln\kappa-\tau)+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)-\sigma\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right]\varphi\circ\exp(2\tau-\ln\lambda)$$
$$\le\alpha\left(\tau-\frac{\ln\lambda+\ln\kappa}{2}\right)\sigma'\left(\alpha\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\varphi\circ\exp(2\tau-\ln\lambda)$$
$$+\alpha\left(\frac{\ln\lambda+\ln\kappa}{2}-\tau\right)\sigma'\left(\alpha(\ln\lambda+\ln\kappa-\tau)+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\varphi\circ\exp(2\tau-\ln\lambda)$$
$$=\alpha\left(\frac{\ln\lambda+\ln\kappa}{2}-\tau\right)\left[\sigma'\left(\alpha(\ln\lambda+\ln\kappa-\tau)+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right.$$
$$\left.-\sigma'\left(\alpha\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right]\varphi\circ\exp(2\tau-\ln\lambda)$$
$$\le\alpha\left(\frac{\ln\lambda+\ln\kappa}{2}-\tau\right)\left[\sigma'\left(\alpha(\ln\lambda+\ln\kappa-\tau)+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right.$$
$$\left.-\sigma'\left(\alpha\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right]\|\varphi\|_{\infty}. \tag{57}$$

Integrating (57) and using (56), we obtain

$$\int_{\ln\lambda}^{\frac{\ln\lambda+\ln\kappa}{2}}\left[\sigma\left(\alpha\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)-\sigma\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right]\varphi\circ\exp(2\tau-\ln\lambda)d\tau$$
$$+\int_{\ln\lambda}^{\frac{\ln\lambda+\ln\kappa}{2}}\left[\sigma\left(\alpha(\ln\lambda+\ln\kappa-\tau)+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)-\sigma\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right]$$
$$\times\varphi\circ\exp(2\tau-\ln\lambda)d\tau\le\left(\frac{\ln\kappa-\ln\lambda}{2}\right)\left[\sigma\left(\alpha\ln\kappa+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right.$$
$$\left.+\sigma\left(\alpha\ln\lambda+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)\right]-\int_{\ln\lambda}^{\ln\kappa}\sigma\left(\alpha\ln\tau+(1-\alpha)\left(\frac{\ln\lambda+\ln\kappa}{2}\right)\right)d\tau\Bigg]\|\varphi\|_{\infty}. \tag{58}$$

Inequality (58) can be re-written as

$$\int_{\lambda}^{\sqrt{\lambda\kappa}}\left[\nu\left(\tau^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\tau-2\int_{\lambda}^{\sqrt{\lambda\kappa}}\nu\left(\sqrt{\lambda\kappa}\right)\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\tau$$
$$\le(\ln\kappa-\ln\lambda)\left\{\frac{1}{2}\left[\nu\left(\lambda^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]\right.$$
$$\left.-\frac{1}{\ln\kappa-\ln\lambda}\int_{\lambda}^{\kappa}\frac{1}{\tau}\nu\left(\tau^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)d\tau\right\}\|\varphi\|_{\infty}=(\ln\kappa-\ln\lambda)[\mathcal{G}(\alpha)-\mathcal{H}(\alpha)]\|\varphi\|_{\infty}. \tag{59}$$

Using (23) and (40), we obtain (53) from (59). □

**Example 2.** *Let $\mathcal{G}(\tau) = \exp(\tau)$, $\tau \in [\ln 2, \ln 3]$, then according to Theorem 4, $\nu(\tau) = \tau$ is GA-convex function on $[2,3]$. Moreover, the mapping $\varphi(\tau) = \left(\tau - \frac{6}{\tau}\right)^2$ is symmetric with respect to $\sqrt{6}$ over the interval $[2,3]$. Now, for $\alpha = \frac{1}{2}$, we obtain*

$$\begin{aligned}
\mathcal{I}_\varphi(\alpha) = \mathcal{I}_\varphi\left(\frac{1}{2}\right) &= \frac{1}{2}\int_\lambda^\kappa \left[\nu\left(\left(\sqrt{\lambda\tau}\right)^{\frac{1}{2}}\left(\sqrt{\lambda\kappa}\right)^{\frac{1}{2}}\right) + \nu\left(\left(\sqrt{\tau\kappa}\right)^{\frac{1}{2}}\left(\sqrt{\lambda\kappa}\right)^{\frac{1}{2}}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau \\
&= \frac{1}{2}\int_2^3 \left[\nu\left(\left(\sqrt{2\tau}\right)^{\frac{1}{2}}\left(\sqrt{6}\right)^{\frac{1}{2}}\right) + \nu\left(\left(\sqrt{3\tau}\right)^{\frac{1}{2}}\left(\sqrt{6}\right)^{\frac{1}{2}}\right)\right]\frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau \\
&= \frac{2}{63}\left(80\sqrt{6} - \sqrt{2903285\sqrt{6} - 7077132}\right) = 0.329935
\end{aligned}$$

*and*

$$\begin{aligned}
&\mathcal{G}(\alpha)\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau = \mathcal{G}\left(\frac{1}{2}\right)\int_2^3 \frac{\varphi(\tau)}{\tau}d\tau = \frac{1}{2}\left[\sqrt{2}\left(\sqrt{6}\right)^{\frac{1}{2}} + \sqrt{3}\left(\sqrt{6}\right)^{\frac{1}{2}}\right]\int_2^3 \frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau \\
&= \frac{\sqrt[4]{3}\left(\sqrt{2} + \sqrt{3}\right)\left(5 - 12\ln\left(\frac{3}{2}\right)\right)}{2^{\frac{3}{4}}} = 0.33095.
\end{aligned}$$

*Thus, the validity of the inequality (52) in Theorem 12 is established.*

*We now prove the validity of the inequality (53) in Theorem 12. Let $\alpha = \frac{1}{2}$; then,*

$$\begin{aligned}
&\mathcal{I}_\varphi(\alpha) - \nu\left(\sqrt{\lambda\kappa}\right)\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau = \mathcal{I}_\varphi\left(\frac{1}{2}\right) - \sqrt{6}\int_2^3 \frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau \\
&= \frac{2}{63}\left(80\sqrt{6} - \sqrt{2903285\sqrt{6} - 7077132}\right) - \sqrt{6}\left(5 - 12\ln\left(\frac{3}{2}\right)\right) = 0.000677484.
\end{aligned}$$

*and*

$$\begin{aligned}
(\ln\kappa - \ln\lambda)[\mathcal{G}(\alpha) - \mathcal{H}(\alpha)]\|\varphi\|_\infty &= (\ln 3 - \ln 2)\left[\mathcal{G}\left(\frac{1}{2}\right) - \mathcal{H}\left(\frac{1}{2}\right)\right] \\
&= \frac{(\ln 3 - \ln 2)}{2}\left[\sqrt{2}\left(\sqrt{6}\right)^{\frac{1}{2}} + \sqrt{3}\left(\sqrt{6}\right)^{\frac{1}{2}}\right] - \left(\sqrt{6}\right)^{\frac{1}{2}}\int_2^3 \tau^{-\frac{1}{2}}d\tau \\
&= \frac{1}{2}\left(2^{\frac{3}{4}}\sqrt[4]{3} + \sqrt[4]{2} \times 3^{\frac{3}{4}}\right)(\ln 3 - \ln 2) - \sqrt[4]{6}\left(2\sqrt{3} - 2\sqrt{2}\right) = 0.00340519.
\end{aligned}$$

*Hence, the validity of the inequality (53) in Theorem 12 is proved.*

**Corollary 2.** *Let $\varphi(\tau) = \frac{1}{\ln\kappa - \ln\lambda}(\tau \in [\lambda, \kappa])$ in Theorem 12. Then, the inequalities (52) and (53) reduce to the inequalities:*

$$\mathcal{H}(\alpha) \le \mathcal{G}(\alpha)\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau \tag{60}$$

*and*

$$0 \le \mathcal{H}(\alpha) - \nu\left(\sqrt{\lambda\kappa}\right)\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau \le (\ln\kappa - \ln\lambda)[\mathcal{G}(\alpha) - \mathcal{H}(\alpha)]\|\varphi\|_\infty, \tag{61}$$

*where $\|\varphi\|_\infty = \sup_{\tau \in [\lambda,\kappa]} \varphi(\tau)$.*

**Proof.** Suppose that $\varphi(\tau) = \frac{1}{\ln\kappa - \ln\lambda}$ $(\tau \in [\lambda, \kappa])$ in Theorem 12; then, it has been observed that $\mathcal{I}_\varphi(\alpha) = \mathcal{H}(\alpha)$ $(\alpha \in [0,1])$. Hence, the inequalities (60) and (61) are achieved. □

We can now prove the variant of Theorem 3 for GA-convex functions.

**Theorem 13.** *Let $\nu$, $\varphi$, $\mathcal{G}$, $\mathcal{I}_\varphi$ and $\mathcal{S}_\varphi$ be defined as above. Then, the following Fejér-type inequalities hold:*

(i) *$\mathcal{S}_\varphi$ is convex on $[0,1]$.*

(ii) *The following inequalities hold for all $\alpha \in [0,1]$:*

$$\mathcal{G}(\alpha)\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau \leq \mathcal{S}_{\varphi}(\alpha) \leq \frac{(1-\alpha)}{2}\int_{\lambda}^{\kappa}\left[\nu\left(\sqrt{\lambda\tau}\right)+\nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau$$
$$+\alpha\cdot\frac{\nu(\lambda)+\nu(\kappa)}{2}\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau \leq \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau, \tag{62}$$

$$\mathcal{I}_{\varphi}(1-\alpha) \leq \mathcal{S}_{\varphi}(\alpha) \tag{63}$$

*and*

$$\frac{\mathcal{I}_{\varphi}(\alpha)+\mathcal{I}_{\varphi}(1-\alpha)}{2} \leq \mathcal{S}_{\varphi}(\alpha). \tag{64}$$

(iii) *The following equality holds:*

$$\sup_{\alpha\in[\lambda,\kappa]}\mathcal{S}_{\varphi}(\alpha) = \frac{\nu(\lambda)+\nu(\kappa)}{2}\int_{\lambda}^{\kappa}\frac{\varphi(\tau)}{\tau}d\tau. \tag{65}$$

**Proof.** (i) It suffices to prove that the mapping $\mathcal{S}_{\varphi} : [0,1] \to \mathbb{R}$ is $GA$-convex on $(0,1]$ if and only the mapping $\mathcal{S}'_{\varphi} : [0,1] \to \mathbb{R}$ defined by

$$\mathcal{S}'_{\varphi}(\alpha) = \frac{1}{4}\int_{\ln\lambda}^{\ln\kappa}\Bigg[\sigma\circ\left(\alpha\ln\lambda+(1-\alpha)\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left(\alpha\ln\lambda+(1-\alpha)\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)+\sigma\circ\left(\alpha\ln\kappa+(1-\alpha)\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left(\alpha\ln\kappa+(1-\alpha)\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)\Bigg]\frac{\varphi(\tau)}{\tau}d\tau,$$

is convex for a convex mapping $\sigma : [\ln\lambda, \ln\kappa] \to \mathbb{R}$. Let $\alpha_1, \alpha_2 \in [0,1]$, $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$. Then,

$$\mathcal{S}'_{\varphi}(\alpha\alpha_1+\beta\alpha_2) = \frac{1}{4}\int_{\ln\lambda}^{\ln\kappa}\Bigg[\sigma\circ\left((\alpha\alpha_1+\beta\alpha_2)\ln\lambda+(1-(\alpha\alpha_1+\beta\alpha_2))\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left((\alpha\alpha_1+\beta\alpha_2)\ln\lambda+(1-(\alpha\alpha_1+\beta\alpha_2))\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)$$
$$+\sigma\circ\left((\alpha\alpha_1+\beta\alpha_2)\ln\kappa+(1-(\alpha\alpha_1+\beta\alpha_2))\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left((\alpha\alpha_1+\beta\alpha_2)\ln\kappa+(1-(\alpha\alpha\alpha_1+\beta\alpha_2))\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)\Bigg]\varphi\circ\exp(\tau)d\tau$$
$$\leq \alpha\Bigg\{\frac{1}{2}\int_{\ln\lambda}^{\ln\kappa}\Bigg[\sigma\circ\left(\alpha_1\ln\lambda+(1-\alpha_1)\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left(\alpha_1\ln\lambda+(1-\alpha_1)\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)+\sigma\circ\left(\alpha_1\ln\kappa+(1-\alpha_1)\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left(\alpha_1\ln\kappa+(1-\alpha_1)\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)\Bigg]\varphi\circ\exp(\tau)d\tau\Bigg\}$$
$$+\beta\Bigg\{\frac{1}{2}\int_{\ln\lambda}^{\ln\kappa}\Bigg[\sigma\circ\left(\alpha_2\ln\lambda+(1-\alpha_2)\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left(\alpha_2\ln\lambda+(1-\alpha_2)\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)+\sigma\circ\left(\alpha_2\ln\kappa+(1-\alpha_2)\left(\frac{\ln\lambda+\ln\tau}{2}\right)\right)$$
$$+\sigma\circ\left(\alpha_2\ln\kappa+(1-\alpha_2)\left(\frac{\ln\tau+\ln\kappa}{2}\right)\right)\Bigg]\varphi\circ\exp(\tau)d\tau\Bigg\}$$
$$= \alpha\mathcal{S}'_{\varphi}(\alpha_1)+\beta\mathcal{S}'_{\varphi}(\alpha_2)$$

This proves the harmonic convexity of $\mathcal{S}_{\varphi} : [0,1] \to \mathbb{R}$ on $(0,1]$.

(ii) Using substitution techniques of integration and under the hypothesis of $\varphi$, we have the following identities:

$$\mathcal{S}_{\varphi}(\alpha)=\frac{1}{4}\int_{\lambda}^{\sqrt{\lambda\kappa}}\left[\nu\left(\lambda^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\lambda^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\right)\right.$$
$$\left.+\nu\left(\kappa^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\tau. \tag{66}$$

By Lemma 2, the following inequalities hold for all $\alpha\in[0,1]$ and $\tau\in\left[\lambda,\sqrt{\lambda\kappa}\right]$:
The inequality

$$2\nu\left(\lambda^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\leq\nu\left(\lambda^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\lambda^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\right) \tag{67}$$

holds for the choices $\tau_1=\tau_2=\lambda^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}$,$\beta_1=\lambda^{\alpha}\tau^{1-\alpha}$ and $\beta_2=\lambda^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}$ in Lemma 2.
The inequality

$$2\nu\left(\kappa^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\leq\nu\left(\kappa^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\right) \tag{68}$$

holds for the choices $\tau_1=\tau_2=\kappa^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}$,$\beta_1=\kappa^{\alpha}\tau^{1-\alpha}$ and $\beta_2=\kappa^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}$ in Lemma 2.
Multiplying the inequalities (67), (68) by $\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}$, integrating them over $\tau$ on $\left[\lambda,\sqrt{\lambda\kappa}\right]$, adding the resulting inequalities and using identities (54) and (66), we derive the first inequality of (62).
Using the $GA$-convexity of $\nu$ and Theorem 7, the last part of (62) holds.
By the $GA$-convexity of $\nu$ and the identity (66), we obtain

$$\mathcal{I}_{\varphi}(1-\alpha)=\int_{\lambda}^{\sqrt{\lambda\kappa}}\left[\nu\left(\tau^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)+\nu\left(\left(\frac{\lambda\kappa}{\tau}\right)^{\alpha}\left(\sqrt{\lambda\kappa}\right)^{1-\alpha}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\tau$$
$$=\frac{1}{2}\int_{\lambda}^{\sqrt{\lambda\kappa}}\left[\nu\left(\left(\sqrt{\lambda^{\alpha}\tau^{1-\alpha}}\right)\left(\sqrt{\kappa^{\alpha}\tau^{1-\alpha}}\right)\right)\right.$$
$$\left.+\nu\left(\left(\sqrt{\lambda^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}}\right)\left(\sqrt{\kappa^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}}\right)\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\tau$$
$$\leq\frac{1}{2}\int_{\lambda}^{\sqrt{\lambda\kappa}}\left[\nu\left(\lambda^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\lambda^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\right)\right.$$
$$\left.+\nu\left(\kappa^{\alpha}\tau^{1-\alpha}\right)+\nu\left(\kappa^{\alpha}\left(\frac{\lambda\kappa}{\tau}\right)^{1-\alpha}\right)\right]\frac{\varphi\left(\frac{\tau^2}{\lambda}\right)}{\tau}d\tau=\mathcal{S}_{\varphi}(\alpha). \tag{69}$$

From (52) , (62) and (63) , we obtain (64).

(iii) Using (62) , we obtain (65).

□

**Example 3.** *Let $\mathcal{G}(\tau) = \exp(\tau)$, $\tau \in [\ln 2, \ln 3]$. Then, according to Theorem 4, $\nu(\tau) = \tau$ is GA-convex function on $[2,3]$. Moreover, the mapping $\varphi(\tau) = \left(\tau - \frac{6}{\tau}\right)^2$ is symmetric with respect to $\sqrt{6}$ over the interval $[2,3]$. Now, for $\alpha = \frac{1}{2}$, we obtain*

$$\mathcal{G}(\alpha)\int_\lambda^\kappa \frac{\varphi(\tau)}{\tau}d\tau = \mathcal{G}\left(\frac{1}{2}\right)\int_2^3 \frac{\varphi(\tau)}{\tau}d\tau = \frac{1}{2}\left[\sqrt{2}\left(\sqrt{6}\right)^{\frac{1}{2}} + \sqrt{3}\left(\sqrt{6}\right)^{\frac{1}{2}}\right]\int_2^3 \frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau$$
$$= \frac{\sqrt[4]{3}\left(\sqrt{2}+\sqrt{3}\right)\left(5 - 12\ln\left(\frac{3}{2}\right)\right)}{2^{\frac{3}{4}}} = 0.33095,$$

$$\mathcal{S}_\varphi\left(\frac{1}{2}\right) = \frac{1}{4}\int_2^3\left[2^{\frac{1}{2}}\left(\sqrt{2\tau}\right)^{\frac{1}{2}} + 2^{\frac{1}{2}}\left(\sqrt{3\tau}\right)^{\frac{1}{2}} + 3^{\frac{1}{2}}\left(\sqrt{2\tau}\right)^{\frac{1}{2}} + 3^{\frac{1}{2}}\left(\sqrt{3\tau}\right)^{\frac{1}{2}}\right]\frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau$$
$$= \frac{1}{63}\left(80\sqrt{6} + 80\sqrt{5\sqrt{6}+12} - 569\right) = 0.331631,$$

$$\frac{(1-\alpha)}{2}\int_\lambda^\kappa\left[\nu\left(\sqrt{\lambda\tau}\right) + \nu\left(\sqrt{\tau\kappa}\right)\right]\frac{\varphi(\tau)}{\tau}d\tau + \alpha\cdot\frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau$$
$$= \frac{1}{4}\int_2^3\left[\sqrt{2\tau}+\sqrt{3\tau}\right]\frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau + \frac{5}{4}\int_2^3\frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau$$
$$= \frac{1}{4}\left(16\sqrt{\frac{2}{3}} - \frac{62}{5}\right) + \frac{5}{4}\left(5 - 12\ln\left(\frac{3}{2}\right)\right) = 0.33401$$

*and*

$$\frac{\nu(\lambda)+\nu(\kappa)}{2}\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau}d\tau = \frac{5}{2}\int_2^3\frac{\left(\tau - \frac{6}{\tau}\right)^2}{\tau}d\tau = \frac{5}{2}\left(5 - 12\ln\left(\frac{3}{2}\right)\right) = 0.336047. \tag{70}$$

*Thus, the validity of the inequality (62) in Theorem 13 is established.*

*The inequalities are very easy to verify with the functions given above and we omit the details for the readers.*

**Corollary 3.** *Let $\varphi(\tau) = \frac{1}{\ln\kappa - \ln\lambda}$, $\tau \in [\lambda,\kappa]$ in Theorem 13. Then, we observe that*

(i) *$\mathcal{L}$ is convex on $[0,1]$.*
(ii) *The following inequalities hold for all $\alpha \in [0,1]$:*

$$\mathcal{G}(\alpha)\int_\lambda^\kappa\frac{\varphi(\tau)}{\tau^2}d\tau \le \mathcal{L}(\alpha) \le (1-\alpha)\cdot\frac{1}{\ln\kappa - \ln\lambda}\int_\lambda^\kappa\frac{1}{\tau}\left[\nu\left(\sqrt{\lambda\tau}\right) + \nu\left(\sqrt{\tau\kappa}\right)\right]d\tau$$
$$+\alpha\cdot\frac{\nu(\lambda)+\nu(\kappa)}{2} \le \frac{\nu(\lambda)+\nu(\kappa)}{2}, \tag{71}$$
$$\mathcal{H}(1-\alpha) \le \mathcal{L}(\alpha) \tag{72}$$

*and*

$$\frac{\mathcal{H}(\alpha)+\mathcal{H}(1-\alpha)}{2} \le \mathcal{L}(\alpha). \tag{73}$$

(iii) *The following equality holds:*

$$\sup_{\alpha\in[\lambda,\kappa]}\mathcal{L}(\alpha) = \frac{\nu(\lambda)+\nu(\kappa)}{2}. \tag{74}$$

**Proof.** If we take $\varphi(\tau) = \frac{1}{\ln\kappa - \ln\lambda}$, $\tau \in [\lambda,\kappa]$, then it has been proved earlier in Corollary 1 that $\mathcal{I}_\varphi(\alpha) = \mathcal{H}(\alpha)$.

Now,

$$\begin{aligned}\mathcal{S}_{\varphi}(\alpha) = \frac{1}{2(\ln\kappa - \ln\lambda)} \int_{\lambda}^{\kappa} \Big[ & \nu\left(\lambda^{\alpha}\left(\sqrt{\lambda\tau}\right)^{1-\alpha}\right) + \nu\left(\lambda^{\alpha}\left(\sqrt{\kappa\tau}\right)^{1-\alpha}\right) \\ & + \nu\left(\kappa^{\alpha}\left(\sqrt{\lambda\tau}\right)^{1-\alpha}\right) + \nu\left(\kappa^{\alpha}\left(\sqrt{\kappa\tau}\right)^{1-\alpha}\right)\Big] \frac{\varphi(\tau)}{\tau} d\tau \\ = & \frac{1}{2(\ln\kappa - \ln\lambda)} \int_{\lambda}^{\sqrt{\lambda\kappa}} \frac{1}{\tau}\left[\nu\left(\lambda^{\alpha}\tau^{1-\alpha}\right) + \nu\left(\kappa^{\alpha}\tau^{1-\alpha}\right)\right] d\tau \\ & + \frac{1}{2(\ln\kappa - \ln\lambda)} \int_{\sqrt{\lambda\kappa}}^{\kappa} \frac{1}{\tau}\left[\nu\left(\lambda^{\alpha}\tau^{1-\alpha}\right) + \nu\left(\kappa^{\alpha}\tau^{1-\alpha}\right)\right] d\tau \\ = & \frac{1}{2(\ln\kappa - \ln\lambda)} \int_{\lambda}^{\kappa} \frac{1}{\tau}\left[\nu\left(\lambda^{\alpha}\tau^{1-\alpha}\right) + \nu\left(\kappa^{\alpha}\tau^{1-\alpha}\right)\right] d\tau = \mathcal{L}(\alpha),\end{aligned}$$

for all $\alpha \in [0,1]$. □

## 3. Conclusions

In this study, we have considered some mappings defined on $[0,1]$, which are related to the Hermite–Hadamard and Fejér-type inequalities proven for $GA$-convex functions. We discussed very important properties of these mappings and obtained novel Hermite–Hadamard and Fejér-type inequalities using $GA$-convex function and geometrically symmetric functions. As a consequent, the obtained Hermite–Hadamard and Fejér-type inequalities provide some refinement inequalities. The results presented in this study can be a source for young researchers to further explore the topic of mathematical inequalities, especially related to the topic of generalization of convexity in details.

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