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# Asymptotic Behavior and Oscillation of Third-Order Nonlinear Neutral Differential Equations with Mixed Nonlinearities

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**Abstract:** In this paper, we investigate the asymptotic properties of third-order nonlinear neutral differential equations with mixed nonlinearities using the comparison principle. Our results not only vastly improve upon but also broadly generalize many previously known ones. Examples demonstrating the applicability and efficacy of our results are provided.

**Keywords:** oscillation; asymptotic behavior; neutral differential equation; third-order

**MSC:** 34K11; 39A10; 39A99

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## 1. Introduction

We consider third-order nonlinear neutral differential equations with mixed nonlinearities of the following form:

$$\left[b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}\right]' + \sum_{\kappa=1}^{m} q_\kappa(s)\Phi_{\alpha_\kappa}(y(\tau_\kappa(s))) = 0, \tag{1}$$

where $s \in [s_0, \infty)$ with $s_0 \geq 0$ as a constant, $z(s) = y(s) + py(s - \tau_0)$, and $\Phi_\delta(\theta) = |\theta|^{\delta-1}\theta$, $\delta > 0$. Here, we assume the following:

(1) $\gamma_1, \gamma_2, \alpha_\kappa \in (0, \infty)$, $\kappa = 1, 2, \ldots, m$, $p \in [0, \infty)$, $p \neq 1$, and $\tau_0 \in (-\infty, \infty)$ are constants;
(2) $b_1, b_2, q_\kappa \in C([s_0, \infty), (0, \infty))$ such that

$$\int_{s_0}^{\infty} \frac{ds}{b_i^{1/\gamma_i}(s)} = \infty, \quad i = 1, 2, \tag{2}$$

(3) $\tau_\kappa \in C^1([s_0, \infty), (-\infty, \infty))$, satisfying $\lim_{s\to\infty} \tau_\kappa(s) = \infty$ for $\kappa = 1, 2, \ldots, m$.

Let $\tau(s) := \min\{\tau_1(s), \tau_2(s), ..., \tau_m(s)\}$. If there exists a function $y \in C([t_y, \infty), \mathbb{R})$, $t_y := \min\{s - \tau_0, \tau(s)\}$ such that $z(s)$, $b_1(s)\Phi_{\gamma_1}(z'(s))$, and $b_2(s)\Phi_{\gamma_2}\{(b_1(s)\Phi_{\gamma_1}(z'(s)))'\}$ are continuously differentiable for all $s \in [s_y, \infty)$ and satisfy Equation (1) for all $s \in [s_y, \infty)$ and $\sup\{|y(s)| : s \geq T\} > 0$ for all $T \in [s_y, \infty)$. If such a solution contains arbitrarily large zeros, it is said to be oscillatory; otherwise, it is said to be nonoscillatory. The theory of neutral differential equations has drawn increasing interest over the past three decades see, for example [1–6]. Since neutral equations are used to describe a variety of real-world phenomena, such as the motion of radiating electrons, population development, the spread of epidemics, and networks incorporating lossless transmission lines, studying these equations is crucial both for theory and for applications. For additional applications and general theory of these equations, the reader is directed to the monographs in [7–9]. It is noteworthy to observe that some third-order delay differential equations have both oscillatory and nonoscillatory solutions, or they have only an oscillatory solution. For example, in [10], the third-order delay differential equation

$$y'''(s) + 2y'(s) - y(s - \frac{3\pi}{2}) = 0,$$

has the oscillatory solution $y_1(s) = \sin s$ and a nonoscillatory solution $y_2(s) = \exp(\mu s)$, where $\mu > 0$ such that

$$\mu^3 + 2\mu - \exp\left(-\frac{3\pi}{2}\mu\right) = 0.$$

While the result is due to [11], all solutions to the third-order delay differential equation

$$y'''(s) + y(s - \sigma) = 0, \ \sigma > 0$$

are oscillatory if and only if $\sigma e > 3$. However, the associated ordinary differential equation

$$y'''(s) + y(s) = 0,$$

has the oscillatory solutions $y_1(s) = \exp(s/2)\sin\left(s\sqrt{3}/2\right)$ and $y_2(s) = \exp(s/2)\cos\left(s\sqrt{3}/2\right)$ and a nonoscillatory solution $y_3(s) = \exp(-s)$. There has been increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of different classes of differential equations. We refer the reader to [12–26]. Graef et al. [27] obtained sufficient conditions for oscillation for the third-order neutral differential equation

$$\left[b_2(s)\left\{\left(b_1(s)z'(s)\right)'\right\}\right]' + q(s)f(y(s-\sigma)) = 0,$$

where $0 \le p < 1, \sigma > 0, \ f(y) \in C(-\infty, \infty), f$ is nondecreasing, $yf(y) > 0$ for all $y \neq 0$, and $\int_{s_0}^{\infty} \frac{ds}{b_i(s)} = \infty, \ i = 1, 2$. Baculíková and Džurina [28] discussed the third-order delay differential equation

$$\left[b_2(s)\left(y''(s)\right)^{\gamma_2}\right]' + q(s)f(y(\sigma(s))) = 0,$$

where $\gamma_2$ is the quotient of the odd positive integers, $\sigma(s) \le s$, $f(y) \in C(-\infty, \infty)$, $yf(y) > 0$, $f'(y) \ge 0$ for all $y \neq 0$, $-f(-xy) \ge f(xy) \ge f(x)f(y)$ for $xy > 0$, and

$$\int_{s_0}^{\infty} \frac{ds}{b_2^{1/\gamma_2}(s)} < \infty.$$

Very recently, Li and Rogovchenko [24] studied the oscillation criteria for the third-order neutral functional differential equation

$$\left[b_2(s)\left(z''(s)\right)^{\gamma_2}\right]' + q(s)y^{\gamma_2}(\sigma(s)) = 0,$$

where $\gamma_2$ is the quotient of the odd positive integers and

$$\int_{s_0}^{\infty} \frac{ds}{b_2^{1/\gamma_2}(s)} = \infty.$$

This paper was inspired by recent works [24,29] which established new oscillation criteria that extend and generalize the result in [24] as well as some previously known results. For investigating the oscillation of Equation (1), common techniques include a reduction in order and comparing it with the oscillation of first-order delay differential equations for both delayed and advanced arguments.

## 2. Main Results

We begin this section with some preliminary lemmas, which will be used in the statement of the main results:

**Lemma 1** ([8] Lemma 1.5.1)**.** *Let $h, g : [s_0, \infty) \to (-\infty, \infty)$ such that $h(s) = g(s) + pg(s - c)$, $s \geq s_0 + \max\{0, c\}$, where $p, c \in (-\infty, \infty)$ and $p \neq 1$. Assume that $\limsup_{s\to\infty} h(s) = l \in (-\infty, \infty)$ exists. Then, the following statements hold:*

1. *If $\liminf_{s\to\infty} g(s) = a \in \mathbb{R}$, then $l = (1 + p)a$;*
2. *If $\limsup_{s\to\infty} g(s) = b \in \mathbb{R}$, then $l = (1 + p)b$.*

The next lemma improves upon [30] (Lemma 1) (see also [29,31,32]):

**Lemma 2** ([30] Lemma 1)**.** *Assume that*

$$\alpha_\kappa > \gamma := \gamma_1\gamma_2,\ \kappa = 1, 2, ..., l;\ \text{ and }\ \alpha_\kappa < \gamma := \gamma_1\gamma_2,\ \kappa = l + 1, l + 2, ..., m. \tag{3}$$

*Then, an m-tuple $(\eta_1, \eta_2, ..., \eta_m)$ exists with $\eta_\kappa > 0$ satisfying the conditions*

$$\sum_{\kappa=1}^{m} \alpha_\kappa \eta_\kappa = \gamma \quad \text{and} \quad \sum_{\kappa=1}^{m} \eta_\kappa = 1. \tag{4}$$

**Lemma 3** ([33], Lemma 2.1)**.** *Let Equation (2) hold. If $y(s)$ is an eventually positive solution of Equation (1), then either*

*$(H_1)$* $z'(s) < 0,\ \left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)' > 0,\ \left[b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}\right]' \leq 0$ *or*

*$(H_2)$* $z'(s) > 0,\ \left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)' > 0,\ \left[b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}\right]' \leq 0$

*eventually.*

**Lemma 4.** *Let $y(s)$ be an eventually positive solution to Equation (1) and the corresponding $y(s)$ satisfy condition $(H_1)$ of Lemma 3. If for a sufficiently large $T \in [s_0, \infty)$ we have*

$$\int_T^\infty \left[\frac{1}{b_1(w)} \int_w^\infty \left(\frac{1}{b_2(v)} \int_v^\infty q(u)du\right)^{1/\gamma_2} dv\right]^{1/\gamma_1} dw = \infty, \tag{5}$$

*where*

$$q(s) := \prod_{\kappa=1}^{m} \left[\frac{q_\kappa(s)}{\eta_\kappa}\right]^{\eta_\kappa}, \tag{6}$$

*with $\eta_\kappa$ defined as in Lemma 2, then every solution to Equation (1) tends toward zero eventually.*

**Proof.** Since $z(s) > 0$ and $z'(s) < 0$, then there exists a constant $l \geq 0$ such that $\lim_{s\to\infty} z(s) = l$. We claim $l = 0$. If not, then using Lemma 1, we see that $\lim_{s\to\infty} y(s) = \frac{l}{1+p} > 0$. Then, there exists $s_1 \in [s_0, \infty)$ such that for $s \geq s_1$, we have

$$y(\tau_\kappa(s))) > \frac{l}{2(1 + p)},\ \kappa = 1, ..., m.$$

However, we have

$$\begin{aligned}\sum_{\kappa=1}^{m} q_{\kappa}(s)\Phi_{\alpha_{\kappa}}(y(\tau_{\kappa}(s))) &\geq \sum_{\kappa=1}^{m} q_{\kappa}(s)\Phi_{\alpha_{\kappa}}\left(\frac{l}{2(1+p)}\right) \\ &= \left(\frac{l}{2(1+p)}\right)^{\gamma} \sum_{\kappa=1}^{m} q_{\kappa}(s)\left[\frac{l}{2(1+p)}\right]^{\alpha_{\kappa}-\gamma}. \end{aligned} \tag{7}$$

Through Lemma 2, there exists $\eta_1, ..., \eta_m$ with

$$\sum_{\kappa=1}^{m} \alpha_{\kappa}\eta_{\kappa} - \gamma \sum_{\kappa=1}^{m} \eta_{\kappa} = 0.$$

The arithmetic-geometric mean inequality (see [34] (p. 17)) leads to

$$\sum_{\kappa=1}^{m} \eta_{\kappa} v_{\kappa} \geq \prod_{\kappa=1}^{m} v_{\kappa}^{\eta_{\kappa}}, \quad \text{for any } v_{\kappa} \geq 0,\ \kappa = 1, \dots, m.$$

Then, we obtain

$$\begin{aligned}\sum_{\kappa=1}^{m} q_{\kappa}(s)\left[\frac{l}{2(1+p)}\right]^{\alpha_{\kappa}-\gamma} &= \sum_{\kappa=1}^{m} \eta_{\kappa}\frac{q_{\kappa}(s)}{\eta_{\kappa}}\left[\frac{l}{2(1+p)}\right]^{\alpha_{\kappa}-\gamma} \\ &\geq \prod_{\kappa=1}^{m}\left[\frac{q_{\kappa}(s)}{\eta_{\kappa}}\right]^{\eta_{\kappa}}\left[\frac{l}{2(1+p)}\right]^{\eta_{\kappa}(\alpha_{\kappa}-\gamma)} \\ &= \prod_{\kappa=1}^{m}\left[\frac{q_{\kappa}(s)}{\eta_{\kappa}}\right]^{\eta_{\kappa}} = q(s).\end{aligned}$$

This, together with Equation (7), shows that

$$\sum_{\kappa=1}^{m} q_{\kappa}(s)\Phi_{\alpha_{\kappa}}(y(\tau_{\kappa}(s))) \geq q(s)\left(\frac{l}{2(1+p)}\right)^{\gamma}. \tag{8}$$

By combining Equations (1) and (8), we obtain

$$\begin{aligned}\left[b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}\right]' &= -\sum_{\kappa=1}^{m} q_{\kappa}(s)\ \Phi_{\alpha_{\kappa}}(y(\tau_{\kappa}(s))) \\ &\leq -q(s)\left(\frac{l}{2(1+p)}\right)^{\gamma}.\end{aligned}$$

By integrating the latter inequality from $s$ to $v$ and letting $v \to \infty$, we obtain

$$b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\} \geq \left(\frac{l}{2(1+p)}\right)^{\gamma}\int_s^{\infty} q(u)du.$$

It follows that

$$\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)' \geq \left(\frac{l}{2(1+p)}\right)^{\gamma_1}\left(\frac{1}{b_2(s)}\int_s^{\infty} q(u)du\right)^{1/\gamma_2}. \tag{9}$$

Again, by integrating this inequality from $s$ to $\infty$, we see that

$$-z'(s) \geq \frac{l}{2(1+p)}\left[\frac{1}{b_1(s)}\int_s^{\infty}\left(\frac{1}{b_2(v)}\int_v^{\infty} q(u)du\right)^{1/\gamma_2} dv\right]^{1/\gamma_1}.$$

Finally, by integrating the last inequality from $s_1$ to $\infty$, we obtain

$$z(s_1) \geq \frac{l}{2(1+p)} \int_{s_1}^{\infty} \left[ \frac{1}{b_1(w)} \int_{w}^{\infty} \left( \frac{1}{b_2(v)} \int_{v}^{\infty} q(u) du \right)^{1/\gamma_2} dv \right]^{1/\gamma_1} dw,$$

which a contradiction to Equation (5). This shows that $\lim_{s\to\infty} z(s) = 0$ and hence $\lim_{s\to\infty} y(s) = 0$ due to $0 < y(s) \leq z(s)$. □

The following result deals with the delayed argument case, namely

$$\tau_0 \geq 0. \tag{10}$$

**Theorem 1.** *Let Equations (2), (5), and (10) hold. If $\tau(s) < s$, and the first-order delay differential equation*

$$x'(s) + Q_1(s)x(\tau(s)) = 0, \tag{11}$$

*where*

$$Q_1(s) := \left[ \frac{1}{1+p} \int_{s_2}^{\tau(s)} \left( \frac{1}{b_1(v)} \int_{s_1}^{v} \left( \frac{1}{b_2(u)} \right)^{1/\gamma_2} du \right)^{1/\gamma_1} dv \right]^{\gamma} q(s), \tag{12}$$

*with $q(s)$, defined as in Equation (6), is oscillatory for all large $s_1 \geq s_0$ and for some $s_2 \geq s_1$, then every solution to Equation (1) is either oscillatory or tends toward zero eventually.*

**Proof.** Assume that $y(s)$ is a nonoscillatory solution to Equation (1). Then, without loss of generality, assume $y(s) > 0$ for $s \in [s_0, \infty)$. It follows from Lemma 3 that there exists $s_1 \geq s_0$ such that either $(H_1)$ or $(H_2)$ holds for $s \geq s_1$. If $(H_1)$ is satisfied, then from Lemma 4, $y(s)$ tends toward zero eventually. Now, we assume that $(H_2)$ is satisfied. By virtue of

$$\left[ b_2(s)\Phi_{\gamma_2}\left\{ \left( b_1(s)\Phi_{\gamma_1}(z'(s)) \right)' \right\} \right]' \leq 0, \quad \text{for } s \geq s_1,$$

it then follows that

$$\begin{aligned} b_1(s)\Phi_{\gamma_1}(z'(s)) &= b_1(s)\Phi_{\gamma_1}(z'(s_1)) + \int_{s_1}^{s} \frac{\left[ b_2(u)\Phi_{\gamma_2}\left\{ \left( b_1(u)\Phi_{\gamma_1}(z'(u)) \right)' \right\} \right]^{1/\gamma_2}}{b_2^{1/\gamma_2}(u)} du \\ &\geq \left[ b_2(s)\Phi_{\gamma_2}\left\{ \left( b_1(s)\Phi_{\gamma_1}(z'(s)) \right)' \right\} \right]^{1/\gamma_2} \int_{s_1}^{t} \frac{du}{b_2^{1/\gamma_2}(u)}, \end{aligned}$$

and hence

$$z'(s) \geq \left[ b_2(s)\Phi_{\gamma_2}\left\{ \left( b_1(s)\Phi_{\gamma_1}(z'(s)) \right)' \right\} \right]^{1/\gamma} \left( \frac{1}{b_1(s)} \int_{s_1}^{s} \frac{du}{b_2^{1/\gamma_2}(u)} \right)^{1/\gamma_1}.$$

By integrating this inequality from $s_2$ to $s$, we obtain

$$\begin{aligned} z(s) &= z(s_2) + \int_{s_2}^{s} \left[ b_2(v)\Phi_{\gamma_2}\left\{ \left( b_1(v)\Phi_{\gamma_1}(z'(v)) \right)' \right\} \right]^{1/\gamma} \left( \frac{1}{b_1(v)} \int_{s_1}^{v} \frac{du}{b_2^{1/\gamma_2}(u)} \right)^{1/\gamma_1} dv \\ &\geq \left[ b_2(s)\Phi_{\gamma_2}\left\{ \left( b_1(s)\Phi_{\gamma_1}(z'(s)) \right)' \right\} \right]^{1/\gamma} \int_{s_2}^{s} \left( \frac{1}{b_1(v)} \int_{s_1}^{v} \frac{du}{b_2^{1/\gamma_2}(u)} \right)^{1/\gamma_1} dv. \end{aligned} \tag{13}$$

However, there is a positive constant $l_1$ such that $\lim_{s\to\infty} z'(s) = l_1$. Then, according to Lemma 1, we obtain $\lim_{s\to\infty} y'(s) = \frac{l_1}{1+p} > 0$, and hence $y'(s) > 0$. From Equation (10) and the fact that $y'(s) > 0$, we obtain

$$z(s) = y(s) + py(s-\tau_0) \le (1+p)y(s).$$

Consequently, we obtain

$$y(\tau(s)) \ge \frac{1}{1+p} z(\tau(s)). \tag{14}$$

In addition, we have

$$\begin{aligned} \sum_{\kappa=1}^{m} q_\kappa(s)\Phi_{\alpha_\kappa}(y(\tau_\kappa(s))) &\ge \sum_{\kappa=1}^{m} q_\kappa(s)\Phi_{\alpha_\kappa}(y(\tau(s))) \\ &= \Phi_\gamma(y(\tau(s))) \sum_{\kappa=1}^{m} q_\kappa(s)[y(\tau(s))]^{\alpha_\kappa-\gamma}. \end{aligned} \tag{15}$$

According to Lemma 2, there exists $\eta_1, ..., \eta_m$ with

$$\sum_{\kappa=1}^{m} \alpha_\kappa \eta_\kappa - \gamma \sum_{\kappa=1}^{m} \eta_\kappa = 0.$$

The arithmetic-geometric mean inequality (see [34] (p. 17)) gives us

$$\sum_{\kappa=1}^{m} \eta_\kappa v_\kappa \ge \prod_{\kappa=1}^{m} v_\kappa^{\eta_\kappa}, \quad \text{for any } v_\kappa \ge 0,\ \kappa = 1,\dots,m.$$

Therefore, we have

$$\begin{aligned} \sum_{\kappa=1}^{m} q_\kappa(s)[y(\tau(s))]^{\alpha_\kappa-\gamma} &= \sum_{\kappa=1}^{m} \eta_\kappa \frac{q_\kappa(s)}{\eta_\kappa}[y(\tau(s))]^{\alpha_\kappa-\gamma} \\ &\ge \prod_{\kappa=1}^{m} \left[\frac{q_\kappa(s)}{\eta_\kappa}\right]^{\eta_\kappa} [y(\tau(s))]^{\eta_\kappa(\alpha_\kappa-\gamma)} \\ &= \prod_{\kappa=1}^{m} \left[\frac{q_\kappa(s)}{\eta_\kappa}\right]^{\eta_\kappa} = q(s). \end{aligned}$$

This, together with Equation (15), shows that

$$\sum_{\kappa=1}^{m} q_\kappa(s)\Phi_{\alpha_\kappa}(y(\tau_\kappa(s))) \ge q(s)\Phi_\gamma(y(\tau(s))). \tag{16}$$

Now, we have

$$\begin{aligned} \left[b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}\right]' &\le -q(s)\Phi_\gamma(y(\tau(s))) \\ &\le -\frac{q(s)}{(1+p)^\gamma}\Phi_\gamma(z(\tau(s))). \end{aligned}$$

Using Equation (13), we obtain

$$x'(s) \le -\left[\frac{1}{1+p}\int_{s_2}^{\tau(s)} \left(\frac{1}{b_1(v)}\int_{s_1}^{v} \frac{du}{b_2^{1/\gamma_2}(u)}\right)^{1/\gamma_1} dv\right]^\gamma q(s)x(\tau(s)),$$

where $x(s) := b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}$. Due to [35] (Theorem 1), the corresponding delay differential equation also has a positive solution. The proof is completed by this contradiction. □

The next result is extracted from Theorem 1 and [23] (Theorem 2.1.1):

**Corollary 1.** *Assume that Equations (2), (5), and (10) hold. If* $\tau(s) < s$, *and*

$$\liminf_{s\to\infty}\int_{\tau(s)}^{s} Q_1(w)dw \geq \frac{1}{\mathrm{e}},$$

*where* $Q_1(w)$ *is defined as in (12) then every solution to Equation (1) is either oscillatory or tends toward zero eventually.*

The following results address the advanced argument case, namely

$$\tau_0 \leq 0. \tag{17}$$

**Theorem 2.** *Assume that Equations (2), (5), and (17) hold. If* $\tau(s) < s - \tau_0$, *and the first order delay differential equation*

$$x'(s) + Q_2(s)x(\tau(s)+\tau_0) = 0, \tag{18}$$

*where*

$$Q_2(s) := \left[\frac{1}{1+p}\int_{s_2}^{\tau(s)+\tau_0}\left(\frac{1}{b_1(v)}\int_{s_1}^{v}\left(\frac{1}{b_2(u)}\right)^{1/\gamma_2}du\right)^{1/\gamma_1}dv\right]^{\gamma} q(s), \tag{19}$$

*with* $q(s)$ *defined as in (6)is oscillatory, then every solution to Equation (1) is either oscillatory or tends toward zero eventually.*

**Proof.** Assume that $y(s)$ is a nonoscillatory solution to Equation (1). Then, without loss of generality, assume $y(s) > 0$ for $s \in [s_0,\infty)$. It follows from Lemma 3 that there exists $s_1 \geq s_0$ such that either $(H_1)$ or $(H_2)$ hold for $s \geq s_1$. If $(H_1)$ is satisfied, then from Lemma 4, $y(s)$ tends toward zero eventually. Now, we assume that $(H_2)$ is satisfied. With the same proof as in the proof for Theorem 1, we find that $y'(s) > 0$ on $[s_1,\infty)$, and Equations (13) and (16) hold. From Equation (17), we obtain

$$z(s) = y(s) + py(s-\tau_0) \leq (1+p)y(s-\tau_0),$$

which implies

$$y(s) \geq \frac{1}{1+p}z(s+\tau_0).$$

Consequently, we have

$$y(\tau(s)) \geq \frac{1}{1+p}z(\tau(s)+\tau_0). \tag{20}$$

Using Equation (16), we get

$$\begin{aligned}\left[b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}\right]' &\leq -q(s)\Phi_\gamma(y(\tau(s)))\\ &\leq -\frac{q(s)}{(1+p)^\gamma}\Phi_\gamma(z(\tau(s)+\tau_0)).\end{aligned}$$

From Equation (13), we have

$$x'(s) \leq -\left[\frac{1}{1+p}\int_{s_2}^{\tau(s)+\tau_0}\left(\frac{1}{b_1(v)}\int_{s_1}^{v}\left(\frac{1}{b_2(u)}\right)^{1/\gamma_2}du\right)^{1/\gamma_1}dv\right]^{\gamma} q(s)x(\tau(s)+\tau_0),$$

where $x(s) := b_2(s)\Phi_{\gamma_2}\left\{\left(b_1(s)\Phi_{\gamma_1}(z'(s))\right)'\right\}$. The associated delay differential equation also has a positive solution because of [35] (Theorem 1). The proof is completed by this contradiction. □

According to Theorem 2 and [23] (Theorem 2.1.1), we have the next result:

**Corollary 2.** *Assume that Equations (2), (5), and (17) hold. If $\tau(s) < s - \tau_0$ and*

$$\liminf_{s\to\infty}\int_{\tau(s)+\tau_0}^{s} Q_2(w)dw \geq \frac{1}{e},$$

*where $Q_2(w)$ is defined as in Equation (12), then every solution to Equation (1) is either oscillatory or tends toward zero eventually.*

The effectiveness and efficiency of our results are shown in the examples below:

**Example 1.** *Consider the third-order nonlinear neutral differential equation of the form*

$$\left[\Phi_{\gamma_2}\left\{\left(\frac{1}{s}\Phi_{\gamma_1}\left(y(s)+py(s-1)\right)'\right)'\right\}\right]' + e^2\Phi_{\alpha_1}(y(s-1)) + e^2\Phi_{\alpha_2}(y(s)) = 0, s \geq 1 \quad (21)$$

*where $p \neq 1$, $p \geq 0$, $\gamma_1 = \frac{1}{3}$, $\gamma_2 = 3$, $\eta_1 = \eta_2 = \frac{1}{2}$, $\alpha_1 = \frac{3}{2}$, and $\alpha_2 = \frac{1}{2}$. With appropriate software (e.g., Maple), we see that Equation (2) holds, where*

$$q(s) = \prod_{\kappa=1}^{m}\left[\frac{q_\kappa(s)}{\eta_\kappa}\right]^{\eta_\kappa} = 2e^2,$$

*and*

$$\int_1^\infty w^3\left[\int_w^\infty\left(\int_v^\infty 2e^2 du\right)^{\frac{1}{3}}dv\right]^3 dw = \infty.$$

*We also have*

$$\begin{aligned}\liminf_{s\to\infty}\int_{\tau(s)}^{s} Q_1(w)dw &= \frac{1}{1+p}\liminf_{s\to\infty}\left[\int_{s-1}^{s}\left\{\int_1^{w-1}\left(v\int_1^v du\right)^{\frac{1}{3}}dv\right\}2e^2dw\right]\\ &= (1+p)^{-1}\liminf_{s\to\infty}\left[\frac{10079\,e^2 s}{70} + 2/7\,e^2s^7 - 4\,e^2s^6 + \frac{121\,e^2s^5}{5}\right.\\ &\quad\left. - 82\,e^2s^4 + 168\,e^2s^3 - 208\,e^2s^2 - \frac{6011\,e^2}{140}\right] \geq \frac{1}{e}.\end{aligned}$$

*Then according to Corollary 1, every solution to Equation (21) is either oscillatory or tends toward zero eventually.*

**Example 2.** *Consider the third-order nonlinear neutral differential equation of the form*

$$\left[\frac{1}{s^2}\Phi_{\gamma_2}\left\{\left(\frac{1}{s}\Phi_{\gamma_1}\left(y(s)+py(s+1)\right)'\right)'\right\}\right]' + s\Phi_{\alpha_1}(y(s)) + s\Phi_{\alpha_2}(y(s+2)) = 0,\ s \geq 1 \quad (22)$$

*where* $p \neq 1,\ p \geq 0,\ \gamma_1 = \gamma_2 = 1,\ \eta_1 = \eta_2 = \frac{1}{2},\ \alpha_1 = \frac{3}{2}$, *and* $\alpha_2 = \frac{1}{2}$. *With appropriate software (e.g., Maple), we see that Equation* (2) *holds, where*

$$q(s) = \prod_{\kappa=1}^{m} \left[ \frac{q_\kappa(s)}{\eta_\kappa} \right]^{\eta_\kappa} = 2s,$$

*and*

$$\int_1^\infty w \left[ \int_w^\infty w^2 \left( \int_v^\infty 2u du \right) dv \right] dw = \infty.$$

*We also have*

$$\begin{aligned} \liminf_{s\to\infty} \int_{\tau(s)+\tau_0}^{s} Q_2(w) dw &= \frac{1}{1+p} \liminf_{s\to\infty} \Big[ \int_{s-1}^{s} \Big\{ \int_1^{w-1} \Big( v \int_1^v u^2 du \Big)^{\frac{1}{\frac{1}{3}}} dv \Big\} 2w dw \Big] \\ &= \frac{1}{1+p} \liminf_{s\to\infty} \left[ \frac{1753}{1260} + \frac{257\, s^2}{30} - \frac{65\, s^3}{9} + 11/3\, s^4 - \frac{16\, s^5}{15} \right. \\ &\quad \left. + 2/15\, s^6 - \frac{27\, s}{5} \right] \geq \frac{1}{e}. \end{aligned}$$

*Then according to Corollary* 2, *every solution to* (22) *is either oscillatory or tends toward zero eventually.*

## 3. Conclusions

In this study, we investigated the oscillation criteria for third-order nonlinear neutral differential equations with mixed nonlinearities. We discovered new oscillation criteria that enhanced numerous earlier efforts. Two examples were used to demonstrate the relevance and power of our results.

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