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Non-Parametric Test for Decreasing Uncertainty of Residual Life Distribution (DURL) †

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Abstract: In this paper, we propose a new statistic to test the monotonicity of uncertainty based on derivative criteria and the histogram method. We test the null hypothesis that residual entropy is constant against the fact that it decreases over time. Hence, by the fact that the exponential distribution is the distribution with a constant uncertainty, we establish the test exponential distribution against the decreasing uncertainty of residual life distribution. Consistency and asymptotic normality are proved. The critical values of the statistics are given by means of the Monte Carlo simulation method to decide on the test. Then, the power estimates of the new test are compared to those of the test based on the criteria of monotonicity of residual entropy. Finally, we show, with real survival data, that the distributions belong to a decreasing uncertainty residual life class. Moreover, by applying a test of goodness of fit, we confirm that the data follow parametric distributions belonging to a decreasing uncertainty of residual life class.

Keywords: reliability; information measures; uncertainty; residual entropy; lifetime distribution; non-parametric test; decreasing uncertainty of residual lifetime

MSC: 62E15; 62N05; 62G10

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1. Introduction

Several attempts have been made, in terms of effective procedures, at inherent uncertainty reasoning in medical diagnosis, expert systems, Artificial Intelligence and other engineering studies. "All of the time, agents are forced to make decisions based on incomplete information. Even when an agent senses the world to find out more information, it rarely finds out the exact state of the world. A robot does not know exactly where an object is. A doctor does not know exactly what is wrong with a patient. A teacher does not know exactly what a student understands. When intelligent agents must make decisions, they have to use whatever information they have" (Poole and Mackworth (2023) [1]). Probability theory is perhaps an adequate way to give a mathematical measure of an uncertain (or random) event from the point of view of the criteria enumerated by Waley (1996) [2], amongst others (interpretation, imprecision, calculus, consistency, assessment and computation), since it is naturally associated with mathematical statistics for inference (see also Poole and Mackworth (2023) [1], Zio and Pedroni (2013) [3]). Several objections have been made against probability models, whose role in dealing with uncertainty—particularly with clinical diagnosis—was the core of overheated debate that arose from misunderstanding questions. This led to new theories with different purposes of application: probability bounds, entropy, imprecise probability, fuzzy theory, possibility theory, belief functions, evidence theory, and so on, all of which tried to give more comprehensive insight into the notion of uncertainty.

In Colyvan (2008) [4], the author addressed this question through the famous Cox's theorem, which states that "Any measure of believe is isomorphic to a probability measure". His strategy was to first show that the claim that probability theory is the only coherent means of dealing with uncertainty is implausible. He did this by "considering different kinds of uncertainty and showing that probability theory seems ill-suited to the uncertainty arising in situations where the logical principle of excluded middle fails".

In this paper, we do not enter into this debate, but we refer to an idea that goes back to the mathematical theory of communication by Shannon (1948) [5], which defines uncertainty through the notion of entropy.

Recall that all probability distributions considered in classical uncertainty theories (probabilistic or not) are qualified as "parametric" in the sense that they depend on some parameters. For example, exponential distribution depends on only one parameter (the mean); normal distribution depends on two parameters (the mean and the variance). In any practical situation, we are satisfied with these parametric laws, which are simple and intuitive: they give a first answer to any given question. As a second step, we can question ourselves about how to improve the model from the "uncertainty" point of view. Indeed, we can have doubts about the chosen probability distribution (law), although it is given by the well-established statistical mathematic of decision or inference:

- First, experimentation with different statistical samples for a given variable can lead to the identification (estimation, in the statistical language) of different probability laws, without which that theory is false. This points, simply, to the fact that the identification is not unique, which is well known in theory and in practice.
- In practice, engineers are not always interested in the probability (parametric) distribution (or law) itself but simply in some "physical" property, such as the "aging" property of the component or the system. A more simple example is given by reliability or survival analysis studies (see Shaked and Shantikumar (1994) [6] and Lai and Xie (2006) [7]), in which practitioners are interested only in a stage of aging or rejuvenation: for example, "Increasing (Decreasing) Failure Rate", i.e., IFR (DFR). Therefore, we consider the class of all probability distributions with such a property IFR (DFR). Such a class is non-parametric rather than parametric, because it contains a collection of probability distribution functions (PDFs) that have in common a given aging property.
- Reliability studies have shown that it is possible to associate bounds (majoration/minotation) with a given class of aging distribution. Similar bounds can be obtained for any structured (coherent) system (Barlow and Proshan (1975) [8])—series, parallel, etc.—on the basis of the information about the class of the elements (stability or preservation properties).
- The properties of such classes are also of some interest for other stochastic models, such as queuing, insurance, networks, medicine or biological models (see Feng et al. (2017) [9], Feng et al. (2020) [10], Marshall and Olkin (1979) [11], Shaked and Shantikumar (1994) [6] and Senouci et al. (2012) [12] for more details).

Non-parametric probability distributions classes are always associated with the monotonicity and comparability properties of some stochastic order. In this paper, we consider the non-parametric class of DURL (IURL)—decreasing (increasing) uncertainty of residual life—which is based on the monotonicity of an uncertain order. These classes of probability distribution functions (PDFs), which were introduced by Ebrahimi (1996) [13], also provide several mathematical properties, particularly preservation results. Further results and references, particularly in connection to order statistics and record values, are provided in Asadi and Ebrahimi (2000) [14], Ebrahimi and Soofi (1994) [15], Ebrahimi and Pellery (1995) [16] and Ebrahimi and Kirmani (1996) [17]. Ebrahimi (1997) [18] provides some statistical hypothesis tests for DURL class based on the monotonicity of the residual entropy.

In this paper, we propose a new criterion based on a derivative criteria that tests such an uncertainty property. The paper is organized as follows: In the following Section 2, we present the notions of stochastic order and non-parametric aging probability distributions (NPDFs). The Less Uncertainty (LU) order and the DURL (IURL) non-parametric classes based on monotonicity of uncertainty are presented in Section 3. In Section 4, we propose the procedure of the test based on the histogram method and inspired by the work of Ebrahimi (1997) [18] but which is based on a derivative criteria rather than the monotonicity one. Section 5 is devoted to determining the mean and variance of the proposed statistic. The consistency and the asymptotic normality are discussed, respectively, in Sections 6 and 7. We determine in Section 8 the critical point values of the statistic by means of Monte Carlo simulation. Afterwards, we compare the power estimates of this test whith those of Ebrahimi (1997) [18] in Section 9. Finally, illustrative examples are provided in Section 10 followed by a conclusion in Section 11.

2. Stochastic Orders and Non-Parametric Aging Distribution

The problem arises in reliability theory when we have to compare reliability (or other stochastic) models; see Marshall and Olkin (1979) [11] and Shaked and Shantikumar (1994) [6]. This can be completed by using the notion of some partial stochastic ordering (it is a binary relation that is reflexive, transitive and anti-symmetric).

We consider a partial ordering defined on the set $\Im$ (or its suitable subsets) of all distribution functions of real-valued random variables. For example, if X and Y are two random variables with their distribution function F and G satisfying $F(x) \leq$ (resp.$\geq$)$G(x)$ for every x , then we say that X is stochastically larger (resp.smaller) than Y: we write $F \geq_{st} (resp.\ \leq_{st})G$ or $X \leq_{st} (resp.\ \geq_{st})Y$. It must be noted that even if X and Y have been defined in the same probability space, we can have anti-symmetry holding for F and G without it necessarily being the case that X and Y are the same random variables. However, stochastic ordering between two distributions, if it holds, is more informative than simply comparing their means or dispersions only. Thus, the proposed approach leads to a qualitative rather than purely quantitative estimation of the system under study. For example, such an approach can be used to design a better system. Since an agent can find two situations incomparable, one situation may be better in some stochastic sense but worse in another one.

In economic theory, this is known as the first-order stochastic dominance and is denoted by $F \geq_{FSD}$(resp. $\leq_{FSD}$)G. There is a growth in literature related to stochastic comparability (or dominance) and various stochastic orders have been introduced; most of them can be found in monographs by Marshall and Olkin (1979) [11] or Shaked and Shantikumar (1994) [6]. Note that we can use the terms "more variable", "riskier" and "more uncertain" synonymously, although the term "more variable" is related to specific variability orders.

For example, we say that X is smaller than Y in the increasing convex (concave) order ($X \leq_{icx} Y$) if $E(f(X)) \leq E(f(Y))$ for all increasing convex (concave) functions f: $R \rightarrow R$ where R is the space of real numbers. There is in the literature of reliability various variability orders:

- The convex (concave) order $\leq_{cx}$: the functions $f(.)$ are convex (concave);
- The convex transform order: $X \leq_c Y$ if $G^{-1}F(x)$ is convex in x on the support of F, where G^{-1} is the inverse of G;
- The star-shaped order $\leq_*$: $X \leq_* Y$ if $G^{-1}F(x)/x$ increases in x on the support of F.

Now, in reliability, we are interested in the lifetime of a component (or system of such components). Let X be a non-negative random variable considered as the failure time of such a component (or system) and denote by $F(t) = P(X \leq t), t \geq 0$, its probability distribution function, which is assumed to be absolutely continuous with the probability density function $f(t)$.

The function $\bar{F}(t) = 1 - F(t) = P(X > t)$ represents the survival (or reliability) function, i.e., the probability of survival of the system over the period $(0, t)$. We are interested also in the residual lifetime $X_t = X - t/X > t$ of a system which has survived until time t.

We denote by $F_t(x) = P(X_t < x/X > t) = \frac{P(t \le X < t+x)}{P(X>t)}$, $\bar{F}_t(x) = P(X_t > t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}$ the reliability of a system of age t and $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ the failure (or hazard) rate of the component at time t also called hazard or risk function. This is a local characteristic as the density function and $\lambda(t)dt$ represents the probability of no failure during the time interval $(t, t + dt)$ given no failure until time t.

Now, let us define some usual non-parametric PDFs. In the following, by increasing (decreasing), we mean non-decreasing (non-increasing).

Definition 1. *A non-negative random variable X is said to be IFR (Increasing Failure Rate) (resp. DFR (Decreasing Failure Rate)) if the failure rate $\lambda_F(t)$ is increasing (resp. decreasing) over $(0, t)$ [6,11].*

This definition assumes that $F(.)$ is absolutely continuous. Otherwise, the definition remains valid when considering the relation between comparability (or monotonicity) and reliability theory. A non-negative random variable X is IFR (DFR) if and only if $X_t \ge_{st} (\le_{st}) X_{t'}$ for all $t \le t'$. So, the stochastic order $\le_{st}$ characterizes the IFR (DFR) (non-parametric) probability distribution.

The order $\le_{icx}$ can characterize another usual aging notion in reliability, i.e., DMRL (IMRL): Decreasing (Increasing) Mean Residual Life.

Definition 2. *The random variable X has DMRL (IMRL) distribution if $m(t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}$ is decreasing (increasing) in $t \in [0, t]$ [6,11].*

Now, the random variable X is DMRL (IMRL) if, and only if, $X_t \ge_{icx} (\le_{icx}) X_{t'}$ whenever $t \le t'$.

The convex $\le_c$ and the star-shaped $\le_*$ orders can be used to characterize IFR and IFRA in the following sense.

Definition 3. *A non-negative random variable X is said to be IFRA (Increasing Failure Rate in Average) if the average of its cumulative failure rate over $(0, t)$ is increasing in $t \ge 0$ [6,11].*

Proposition 1. *Let Exp denote any exponential distributed random variable (no matter what its mean). Let X be a non-negative random variable. Then, X is IFR (resp. IFRA) if and only if $X \le_c$(resp $\le_*$)Exp [6,11].*

There are some relations between these non-parametric distributions (classification). If F is IFR, then it is also IFRA and DMRL, in the sense that for the class IFR $\subset$ IFRA and IFR $\subset$ DMRL, the inclusion is strict. However, there are no relations between IFRA and DMRL. The same relations hold for the dual classes DFR, DFRA and IMRL [6,11].

In the following section, we will consider a non-parametric probability distribution that is not based on aging properties but rather on differential entropy in the sense of Shannon.

3. Uncertainty Order and NPDFs DURL (IURL)

It is well known that we can define an uncertainty measure for the probability distribution function F (with density f) of the non-negative random variable X via differential entropy in the sense of Shannon (Shannon (1948) [5]):

$$H(f) = -\int_0^\infty f(x)log(f(x))dx$$
$$= -E(log(f(X)))$$

commonly referred to Shannon information measure, where $log(x)$ denotes the natural logarithm. The entropy is interpreted as the expected uncertainty contained in the density $f(x)$ about the predictability of an outcome of the random variable X, that is the quantity of information contained in the probability distribution of the random variable X. It measures the concentration of probabilities: low entropy distributions are more concentrated and hence more informative than higher ones. In this sense, the entropy can be used for qualitative studies. In the fields of reliability, survival analysis or insurance, we have additional information about the current age of the system S (or component) under study and we must reassess the uncertainty of the remaining lifetime of S.

Ebrahimi and Pellery (1995) [16] propose a similar notion of dynamic measure of entropy by considering the age of the component at time $t, t > 0$.

More precisely, the residual entropy gives the uncertainty contained in the conditional density of the residual lifetime $X_t = X - t/X > t$ on the predictability of the residual time of the component, which is denoted by $H(f;t)$, where:

$$H(f;t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)} log \frac{f(x)}{\overline{F}(t)} dx \tag{1}$$

After executing some algebraic manipulations and integration by parts, we can represent the residual entropy under the following form [9]:

$$H(f;t) = 1 - E[log\lambda_F(X)/X > t]$$
$$= log\overline{F}(t) - \frac{1}{\overline{F}(t)} \int_t^\infty f(x)logf(x)dx$$
$$= 1 - \frac{1}{\overline{F}(t)} \int_t^\infty f(x)log\lambda_F(x)dx \tag{2}$$

which can be seen as a dynamic measure of uncertainty about S associated to its lifetime distribution. In other words, $H(f;t)$ measures the expected uncertainty contained in conditional density of the residual lifetime X_t of a system (or component) of age t, i.e., given $X > t$. In this sense, $H(f;t)$ measures the concentration of conditional probability distributions. Note finally that the dynamic entropy of a new component (of age 0) $H(f;0) = H(f)$ is the ordinary Shannon's entropy and that the function $H(f;t)$ uniquely determines the reliability function $\overline{F}$, or equivalently, the PDF F.

We can now define an uncertainty ordering. The non-negative random variable X has less uncertainty than Y; we note $X \leq_{LU} Y$, if $H(f;t) \leq H(g;t), t \geq 0$ [17]. If X and Y are the lifetimes of two systems S and S' and if $X \leq_{LU} Y$, then the expected uncertainty contained in the conditional density of X_t, about the predictability of the residual lifetime of the first system S, is less than the expected uncertainty contained in the conditional density of Y_t about the remaining lifetime of the second system S' (see DiCrescenzo (2002) [19] for another notion of order related to past entropy). Note that the usual stochastic orderings used in the literature [6,11] can be interpreted in terms of aging properties, and in general, there is no relation between these orderings and the above-defined uncertainty order. So, intuitively speaking, the better system is the system which lives longer, and there is less uncertainty about its residual lifetime. This motivates the introduction of several definitions of preference based on aging and on uncertainty. This aspect is not considered here.

Note that Ebrahimi and Soofi (1994) [15] view X as less uncertain than Y, written $X \leq_{ENT} Y$, if $E[-logf(X)] \leq E[-logf(Y)]$. But the order $\leq_{LU}$ is stronger than the order $\leq_{ENT}$.

On the basis of the measure of uncertainty $H(f;t)$, Ebrahim (1996) [13] defines two non-parametric classes of life distributions.

Definition 4. *A survival function $\overline{F}$ has decreasing (resp. increasing) uncertainty of residual life DURL (resp. IURL) if $H(f;t)$ is decreasing (resp. increasing) in t [13].*

If a component has a survival PDF belonging to the class DURL, then as the component ages, the conditional probability density function becomes more informative. So, if a component of age X has a DURL distribution (resp. IURL), its residual life can be predicted with more (resp. less) precision [13].

Remark 1. *The exponential distribution is the only continuous distribution which is both DURL and IURL [13]. In fact, the exponential distribution has a constant uncertainty. Hence, the derivative of the uncertainty is null.*

Example 1. *Let X be a random variable having Weibull distribution with survival function $\overline{F}(t) = e^{-\lambda t^{\alpha}}, t > 0$. Then, X is DURL (IURL) for $\alpha > 1$ (resp. $0 < \alpha < 1$) [13].*

Remark 2. *The relation between aging non-parametric PDFs [6,11] and uncertainty ones [13] is that:*

1. *If F is $IFR(DFR)$, then it is $DURL(IURL)$. So that $IFR \subset DURL$ and $DFR \subset IURL$.*
2. *If F is $DMRL$, then it is $DURL$. So that $DMRL \subset DURL$.*
3. *IF F is $IURL$, then it is $IMRL$. So that $IURL \subset IMRL$.*

But there is no connection with IFRA (DFRA).

We can point out an interesting mathematical property of such DURL non-parametric probability distribution (Ebrahimi (1996) [13]) :

Lemma 1. *A survival function $\bar{F}$ is DURL (IURL), if $L(f;t) = -H'(f;t) \geq 0 (-H'(f;t) \leq 0)$. That is, if we have negative (positive) local reduction of uncertainty, then $\bar{F}$ is DURL (IURL) [13].*

Lemma 2. *A survival function $\bar{F}$ is $DURL$ if and only if $H(f;x) - H(f;y) \geq 0$ for all $y \geq x$ [13].*

As DURL (IURL) is a non-parametric class of probability distributions, then if we want to test H_0: $F \in$ DURL, we cannot compute the statistical criteria under H_0. So, it is more convenient to test H_0: $F \in$ Exp, against H_1: $F \in$ DURL, where Exp is the class of all exponential distributions. Ebrahimi (1997) [18] has initiated this test on the basis of the monotonicity property of uncertainty (Lemma 2). We note that under exponentiality, the residual entropy is constant (Ebrahimi (1996) [13]). So, in this paper, we propose a new criterion based on Lemma 1, which tests this uncertainty property (see also Baratpour and Habibi Rad (2012, 2016) [20,21]).

In the following, we explain the procedure of the test H_0: exponentiality against H_1: DURL, using the histogram method to estimate the density and the probability function [18] and derivative criteria.

We obtain the mean and variance of the proposed new statistic. The consistency and the asymptotic normality are discussed. We determine the critical point values of the statistic by means of Monte Carlo simulation. Then, we compare power estimates of this test with those of Ebrahimi (1997) [18]. Finally, results of the illustrative examples are discussed.

4. A Test Statistic Based on the Histogram Method and Derivative Criteria

Recall that Exp denotes the class of all exponential distributions (no matter what its parameter) and DURL denotes the class of all NPDFs which have the uncertain property of Definition 4, i.e., decreasing residual entropy.

In order to discriminate between the hypothesis:

$$H_0 : F \in Exp \tag{3}$$

against

$$H_1 : F \in DURL \ (and\ not\ exponential\ distribution) \tag{4}$$

We use the derivative property of Lemma 1, contrary to the test of Ebrahimi (1997) [18], which uses the monotonicity of the dynamic entropy (Lemma 2).

The derivative of the uncertainty measure $H(f;t)$ can be written as [13]:

$$H'(f;t) = [H(f;t) - 1 + log\lambda_F(t)]\lambda_F(t). \tag{5}$$

Under the hypothesis H_0, the exponential has a constant residual entropy; then, the function $H'(f;t) = 0$ and $\lambda_F(t) \neq 0, \forall t > 0$.

Let the function

$$g(t) = H(f;t) - 1 + log\lambda_F(t). \tag{6}$$

Then, $g(t) = 0$ under the hypothesis H_0.

Consider now the expectation:

$$S = E[g(t)]. \tag{7}$$

We have

$$S = \int_0^\infty g(t)f(t)dt =$$

$$= \int_0^\infty [H(f;t) - 1 + log\lambda_F(t)]f(t)dt =$$

$$= \int_0^\infty f(t)logf(t)dt + \int_0^\infty \frac{f(t)}{\bar{F}(t)}\left(\int_t^\infty f(y)logf(y)dy\right)dt - 1 \tag{8}$$

We use the empirical probability density function to estimate $f(t)$ using a histogram method [18] based on a partition of the order statistics. Let $t_1, t_2, \ldots, t_n$ be $n(n > 2)$ observations from F. Let $t(1), t(2) \ldots, t(n)$ be the corresponding order statistics. We choose k as an integer $(0 < k < n)$ such that:

$$\lambda_i = \left[\frac{(n-1)i}{k}\right] - a_{i-1} + 1, i = 1, \ldots, k,$$

where $a_0 = 1$ and $[.]$ indicates the nearest integer function.

We consider a partition of $[t(1), t(2) \ldots, t(n)]$ into k subintervals:

$$I_i = [t(a_{i-1}), t(a_i)[, i = 1, 2, \ldots, k$$

where

$$a_i = 1 + \sum_{j=1}^{i} \lambda_j, i = 1, 2, \ldots, k.$$

then,

$$a_i = \left[\frac{(n-1)i}{k}\right] + 1, i = 1, 2, \ldots, k.$$

The empirical density function is (see [22,23]):

$$f_n(x) = \begin{cases} \frac{\lambda_i}{(n-1)\left(t(a_i)-t(a_{i-1})\right)}, & \text{if} \quad x \in I_i, 1 \le i \le k, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

The empirical cumulative function is:

$$F_n(x) = \begin{cases} 0, & \text{if} \quad x < t(1), \\ \frac{a_{i-1}-1}{n-1} + \frac{\lambda_i}{n-1}\frac{x-t(a_{i-1})}{t(a_i)-t(a_{i-1})}, & \text{if} \quad x \in I_i, 1 \le i \le k, \\ 1, & \text{if} \quad x \ge t(n). \end{cases} \tag{10}$$

Let $d_i(k) = t(a_i) - t(a_{i-1}), i = 1, 2, \ldots, k$. Some algebras give:

$$\begin{aligned} \int_0^\infty f(t) log f(t) dt &= \sum_{i=1}^{k} \int_{I_i} \frac{\lambda_i}{(n-1)d_i(k)} log\left(\frac{\lambda_i}{(n-1)d_i(k)}\right) dx \\ &= \sum_{i=1}^{k} \frac{\lambda_i}{(n-1)} log\left(\frac{\lambda_i}{(n-1)d_i(k)}\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \int_t^\infty f(y) log f(y) dy &= \frac{\lambda_{i*}}{(n-1)d_{i*}(k)} log\left(\frac{\lambda_{i*}}{(n-1)d_{i*}(k)}\right) y\Big|_t^{t(a_{i*})} \\ &+ \sum_{i>i*} \frac{\lambda_i}{(n-1)d_i(k)} log\left(\frac{\lambda_i}{(n-1)d_i(k)}\right) y\Big|_{t(a_{i-1})}^{t(a_i)} \\ &= \frac{\lambda_{i*}}{(n-1)d_{i*}(k)} log\left(\frac{\lambda_{i*}}{(n-1)d_{i*}(k)}\right) [t(a_{i*}) - t] \\ &+ \sum_{i>i*} \frac{\lambda_i}{(n-1)} log\left(\frac{\lambda_i}{(n-1)d_i(k)}\right) \end{aligned} \tag{12}$$

where $i*$ is the i-th index such that the interval $[t(a_{i-1}), t(a_i)]$ contains t.

Now, if we replace (12) in the second term of expression (8), we obtain

$$\begin{aligned} &\int_0^\infty \frac{f(t)}{\overline{F}(t)} \left(\int_t^\infty f(y) log f(y) dy \right) dt = \\ &\sum_{i*=1}^{k} \int_{I_{i*}} b_{i*}(n) \left[\frac{\lambda_{i*}}{(n-1)d_{i*}(k)} log\left(\frac{\lambda_{i*}}{(n-1)d_{i*}(k)}\right) [t(a_{i*}) - t] \right] \\ &+ \sum_{i*=1}^{k-1} \int_{I_{i*}} b_{i*}(n) \sum_{i>i*} \frac{\lambda_i}{n-1} log\left(\frac{\lambda_i}{(n-1)d_i(k)}\right) dt \end{aligned} \tag{13}$$

where

$$b_{i*}(n) = \frac{\lambda_{i*}}{(n - a_{i*-1})d_{i*}(k) + \lambda_{i*} t(a_{i*-1}) - \lambda_{i*} t}. \tag{14}$$

so,

$$S_n(k) = -\sum_{i=1}^{k-1} \frac{1}{n-1} C_n(i) log\left(\frac{(n-1)d_i(k)}{\lambda_i}\right) - 1, \tag{15}$$

$$C_n(i) = 2\lambda_i + \big(\lambda_i - a_{i-1} + n + \sum_{j>i}\lambda_j\big)log\left(\frac{n-a_{i-1}}{n-a_i}\right) + \frac{\lambda_k}{k-1}. \quad (16)$$

Note that under H_0, we have $g(t) = 0$ and under H_1, $g(t) < 0$ according to Lemma 1. It is the same logic for the statistic $S_n(k)$. So, $S_n(k)$ is null under H_0 and negative under H_1. Then, small values of $S_n(k)$ favor H_1 or equivalently, large values of $W_n(k) = -S_n(k)$ favor H_1.

So, we reject H_0 in favor of H_1 at significance level α if $W_n(k) \geq C_{k,n}(\alpha)$. Note that $C_{k,n}(\alpha)$ is the critical point value determined by the $(1-\alpha)$-quantile of the distribution of $W_n(k)$ under exponentiality.

5. Mean and Variance of the Statistic $W_n(k)$

We now derive the mean and variance of $S_n(k)$ by using the linear property of mathematical expectation and some results cited in [18]. Let:

$$B_n(i,k) = E\big(t(a_i) - t(a_{i-1})\big) =$$

$$= \sum_{j=a_{i-1}}^{a_i-1} (\Gamma'(1) - log(n-j))\Pi_{m=a_{i-1},m\neq j}^{a_i-1}\frac{n-m}{j-m} \quad (17)$$

and

$$D_n(i,k) = Var\big(t(a_i) - t(a_{i-1})\big) =$$

$$= \sum_{j=a_{i-1}}^{a_i-1} (\Gamma''(1) + 2\Gamma'(1) - log(n-j)$$

$$+log^2(n-j)) \prod_{m=a_{i-1}}^{a_i-1}\frac{n-m}{j-m} - B_n^2(i,k), \quad (18)$$

where $\Gamma'(1)$ and $\Gamma''(1)$ are, respectively, the first and second derivatives of the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx \quad (19)$$

evaluated at $\alpha = 1$ [18].

Then, we have :

$$E\big(W_n(k)\big) = \sum_{i=1}^{k-1}\frac{C_n(i)}{n-1}B_n(i,k) + \sum_{i=1}^{k-1}\frac{C_n(i)}{n-1}log(\frac{n-1}{\lambda_i}) + 1 \quad (20)$$

and

$$Var\big(W_n(k)\big) = (\frac{1}{n-1})^2\sum_{i=1}^{k-1}C_n(i)^2D_n(i,k). \quad (21)$$

Inspired by the proofs cited in [18], we have established proofs in Sections 6 and 7 relating to the asymptotic properties of the new test.

6. Consistency of the Test $W_n(k)$

In this section, we prove the convergence of the test statistic $W_n(k)$.

Theorem 1. *The statistic $W_n(k)$ converges to S as $n \to \infty, k \to \infty$, and $\frac{k}{n} \to 0$.*

Proof. Let

$$U_n = \sum_{i=1}^{k-1} \left(log\left(\frac{n-1}{\lambda_i} \left(F\big(t(a_i)\big) - F\big(t(a_{i-1})\big) \right) \right) \right)^{\frac{C_n(i)}{n-1}} \quad (22)$$

and

$$V_n = -\sum_{i=1}^{k-1} log\left(\frac{F\big(t(a_i)\big) - F\big(t(a_{i-1})\big)}{t(a_i) - t(a_{i-1})} \right)^{\frac{C_n(i)}{n-1}}. \quad (23)$$

So, we have

$$W_n(k) = U_n(k) + V_n(k)$$

where $F(t(a_i))$ are order statistics from a uniformly distributed population. Next, we have

$$\begin{aligned} E(U_n) &= \sum_{i=1}^{k-1} \frac{C_n(i)}{n-1} E\left(log\big(F\big(t(a_i)\big) - F\big(t(a_{i-1})\big)\big) \right) \\ &\quad + \frac{1}{n-1} \sum_{i=1}^{k-1} C_n(i) log(\frac{n-1}{\lambda_i}) \\ &= \sum_{i=1}^{k-1} \frac{C_n(i)}{n-1} \Psi\left(\left[\frac{(n-1)i}{k} \right] + 1 \right) + \frac{1}{n-1} \sum_{i=1}^{k-1} C_n(i) log(\frac{n-1}{\lambda_i}) \\ &\quad - \sum_{i=1}^{k-1} \frac{C_n(i)}{n-1} \Psi(n+1) \\ &= \sum_{i=1}^{k-1} \frac{C_n(i)}{n-1} \Big(\Psi(a_i) - \Psi(n+1) \Big) + \frac{1}{n-1} \sum_{i=1}^{k-1} C_n(i) log(\frac{n-1}{\lambda_i}) \end{aligned} \quad (24)$$

where Ψ is the digamma function defined for each natural integer n by

$$\Psi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \gamma.$$

where $\gamma \simeq 0.557$ is the approximation of the Euler–Mascheroni constant.

It follows that $E(U_n) \to 0$ as $n \to \infty$. So, $U_n(k) \to 0$ in probability as $n \to \infty$.

Now, let G be the primitive of the fraction:

$$\begin{aligned} \frac{S(t)}{-log(f(t))} &= -f(t) - \frac{f(t)}{\bar{F}(t)\big(log(f(t))\big)} \int_t^{\infty} f(y) log(f(y)) dy \\ &\quad + \frac{f(t)}{log(f(t))} \end{aligned} \quad (25)$$

We express the statistic $V_n(k)$ as a Stieltjes sum of the function $-log(f(x))$ with respect to the measure G_n, where G_n is the empirical estimation of G.

Then,

$$\begin{aligned} &V_n(k) = \\ &-\sum_{i=1}^{k-1} log\left(\frac{F\big(t(a_i)\big) - F\big(t(a_{i-1})\big)}{t(a_i) - t(a_{i-1})} \right) \Big(G_n\big(t(a_i)\big) - G_n\big(t(a_{i-1})\big) \Big) \end{aligned} \quad (26)$$

Recall that the function $t(.)$ and the sequence a_i are chosen such as $\forall I_i, t(a_i) - t(a_{i-1}) \to 0$ when $n \to \infty$. So, from (26), we deduce that if $G_n(x) \to G_x$ almost surely uniformly over x [24], then $V_n(k) \to S$ almost surely as $n \to \infty$. Under this restriction, the test is consistent. □

7. Normality of the Statistic $W_n(k)$

In this section, we prove the normality of the statistic $W_n(k)$ as $n \to \infty$.

Theorem 2. *Under the null hypothesis H_0, the normalized statistic $\frac{k}{2}[W_n(k) - 1]$ converges to a standard normal distribution as $n \to \infty$, $k \to \infty$ and $\frac{k}{n} \to 0$.*

Proof. We can write $W_n(k)$ as follows,

$$W_n(k) = \sum_{i=1}^{k-1} \frac{C_n(i)}{n-1} log\Big(\sum_{j=a_{i-1}}^{a_i - 1} \big(t(j+1) - t(j)\big)\Big) + \frac{1}{n-1} \sum_{i=1}^{k-1} C_n(i) log(\frac{n-1}{\lambda_i}) + 1. \quad (27)$$

As proposed by David and Nagaraja (2003) [25] and using the fact that:

$$\frac{k}{2(n-1)} \sum_{i=1}^{k-1} C_n(i) log(\frac{n-1}{\lambda_i}) \to 0$$

when $n \to \infty, k \to \infty$ and $\frac{k}{n} \to 0$, the statistic $W_k(n)$ is asymptotically equivalent to:

$$\frac{1}{n-1} \sum_{i=1}^{k-1} C_n(i) log\big(U_{n-a_{i-1}}(n - a_{i-1} + 1)\big) \quad (28)$$

where $U_m(i)$ is the order statistic of a uniform distribution of size m. Then, the statistic (28) is statistically equivalent to:

$$I_n(k) = \frac{1}{n-1} \sum_{i=1}^{k-1} C_n(i) Z_i, \quad (29)$$

where $Z_i, i = 1, 2, \dots, k-1$ are independent and identically distributed random variables.

Now, let the normalized random variable:

$$\frac{k}{2} I_n(k) = \sum_{i=1}^{k-1} \frac{kC_n(i)}{2(n-1)} Z_i \quad (30)$$

and for $n \to \infty, k \to \infty$ and $\frac{k}{n} \to 0$, we have $\frac{kC_n(i)}{2(n-1)} \to 1$.

By the central limit theorem, we conclude that $\frac{k}{2} I_n(k)$ is asymptotically normal, and $\frac{k}{2}[W_n(k) - 1]$ is also asymptotically normal. □

8. Critical Values of the Test $W_n(k)$

In this section, by means of Monte Carlo simulation, we determine critical values $C_{k,n}(\alpha)$ of the statistic $W_n(k)$. For different confidence levels of $1 - \alpha$: $0.900, 0.950, 0.975$ and 0.990 and different values of sample size n, simulations are provided. A total of 5000 samples of exponential distribution with mean 1 are generated. Note that we compute for each sample the corresponding values of the empirical statistic distribution $W_n(k)$ for large spacings of the observations and $-W_n(k)$ for little spacings. The established algorithm gives the critical values for each α and n, which are summarized in Tables 1 and 2.

Table 1. Critical values of $W_n(k)$.

α	**0.100**	**0.050**	**0.025**	**0.010**
$n = 3$	4.8006	5.5334	6.3069	7.0340
$n = 4$	3.9738	4.5321	5.0834	5.5868
$n = 5$	6.8937	7.8342	8.6038	9.5242
$n = 6$	5.7476	6.4144	7.1087	7.6660
$n = 7$	6.2065	7.0206	7.6390	8.3710
$n = 8$	6.6750	7.6958	8.3703	8.9823
$n = 9$	7.2282	7.9532	8.4997	9.1910
$n = 10$	7.4526	8.2148	8.8700	9.5785
$n = 11$	6.8219	7.5294	8.1805	8.7718
$n = 12$	6.9761	7.7543	8.3426	8.9653
$n = 13$	7.1769	7.8341	8.5809	9.2141
$n = 14$	7.3460	8.0698	8.7684	9.5144
$n = 15$	6.9458	7.6614	8.2688	8.7816
$n = 19$	7.6312	8.3843	8.9691	9.5865
$n = 20$	7.8589	8.5677	9.1077	9.7696
$n = 25$	7.8795	8.5813	9.1813	8.8255
$n = 30$	7.7884	8.4882	9.0688	9.7825
$n = 40$	8.4485	9.1831	9.7124	10.3109
$n = 43$	8.4544	9.2073	9.7297	10.3426
$n = 50$	8.5087	9.2004	9.8074	10.3617
$n = 60$	8.4737	9.1195	9.6570	10.3008
$n = 70$	8.9838	9.5802	10.1954	10.7712
$n = 80$	9.0556	9.7066	10.2412	10.8923
$n = 90$	8.9335	9.5870	10.0512	10.6993
$n = 100$	8.9363	9.5745	10.1214	10.7857

Table 2. Critical values of $-W_n(k)$.

α	**0.100**	**0.050**	**0.025**	**0.010**
$n = 3$	2.3564	3.9297	5.3591	7.7634
$n = 4$	1.3378	2.4378	3.3115	4.4102
$n = 5$	2.3816	4.2727	6.1451	8.5942
$n = 6$	0.8225	1.9677	3.2572	4.7316
$n = 7$	1.0633	2.3907	3.5309	5.2719
$n = 8$	1.6147	3.0626	4.6911	6.3835
$n = 9$	−0.4653	0.7871	1.8714	3.1158
$n = 10$	−0.0337	1.2530	2.6525	4.1937
$n = 11$	−0.4963	0.7362	1.8106	2.8472
$n = 12$	−0.4329	0.8787	1.9971	3.1916
$n = 13$	−0.3341	0.8287	1.9519	3.0975
$n = 14$	−0.1302	1.1104	2.1776	3.9166
$n = 15$	−0.4373	0.7019	1.8144	2.8823
$n = 19$	−1.6532	0.7107	0.3514	1.5840
$n = 20$	−1.6547	−0.5765	0.4295	1.5230
$n = 25$	−1.6166	−0.5338	0.4630	1.6354
$n = 30$	−1.4796	−0.4397	0.6585	1.8028
$n = 40$	−2.5892	−1.5634	−0.7244	0.4818
$n = 43$	−2.6190	−1.7370	−0.8146	0.3497
$n = 50$	−2.5228	−1.4527	−0.5132	0.6715
$n = 60$	−2.7075	−1.8218	−0.7903	0.4079
$n = 70$	−3.5011	−2.6071	−1.8078	−0.6376
$n = 80$	−3.4030	−2.5228	−1.7164	−0.5773
$n = 90$	−3.4025	−2.4769	−1.5283	−0.3414
$n = 100$	−3.4964	−2.5482	−1.6047	−0.4008

9. Power Estimates

Under the alternative hypothesis, the distribution of the statistic $W_n(k)$ is complicated. However, we can obtain power estimates by using Monte Carlo simulation method. We have provided a large number of experiments under Weibull $(1, \theta)$

$$\bar{F}(x) = exp(-x^{\theta}), \theta > 0,$$

and Gamma$(\theta, 1)$

$$G(x) = \frac{1}{\Gamma(\theta)} \int_x^{\infty} y^{\theta-1} e^{-y} dy$$

alternatives.

This choice is motivated by the fact that when $\theta = 1$, both Weibull and Gamma distributions become an exponential distribution corresponding to the null hypothesis. In this case, the power estimates are very weak. So, we can compare our results with those of Ebrahimi (1997) [18]. In Tables 3 and 4, we give simulated powers for sample size n. Simulated results are based on 10,000 iterations. We notice that the proposed test based on the derivative criteria performs relatively well when the parameter θ of the distributions reaches 20.

Table 3. Power estimates for Weibull $(\theta, 1)$ of $W_n(k)$.

n	20	10	5
$\theta = 20$	0.9990	0.9862	0.8921
$\theta = 10$	0.9288	0.7831	0.5686
$\theta = 5$	0.5568	0.3750	0.2591
$\theta = 3$	0.2506	0.1784	0.1321
$\theta = 1$	0.0555	0.0522	0.0526

Table 4. Power estimates for Gamma $(\theta, 1)$ of $-W_n(k)$.

n	20	10	5
$\theta = 20$	0.9081	0.8410	0.7513
$\theta = 10$	0.7917	0.6940	0.6000
$\theta = 5$	0.5680	0.4743	0.3966
$\theta = 3$	0.3625	0.2926	0.2397
$\theta = 1$	0.0501	0.0511	0.0496

10. Illustrative Examples

Example 2. *As a first illustration, we consider survival times (in days) for 50 patients' head and neck cancer cited in Lai and Xie (2006) [7] and Efron (1988) [26]. We have computed the empirical value of the statistic $W_n(k)$ which is equal to $W_n(k) = 26.4194$. When comparing the result with the critical values given in Table 1, we deduce that the statistic exceeds all percentiles.*

We can conclude that this sample is provided from a DURL distribution. So, the remaining lifetimes of a patient from this population can be predicted with more precision.

From Figure 1, we notice that the more $t (t > 0)$ increases, the more the curve decreases. Thus, the empirical residual entropy from the dataset cited in [7,26] is DURL. It confirms the established test based on the derivative criteria. However, we notice that there is a part of the curve parallel to the abscissa axis. This is because during the time interval $[600, 1100]$, there are no data available regarding survival times, and therefore, the measure of information does not change.

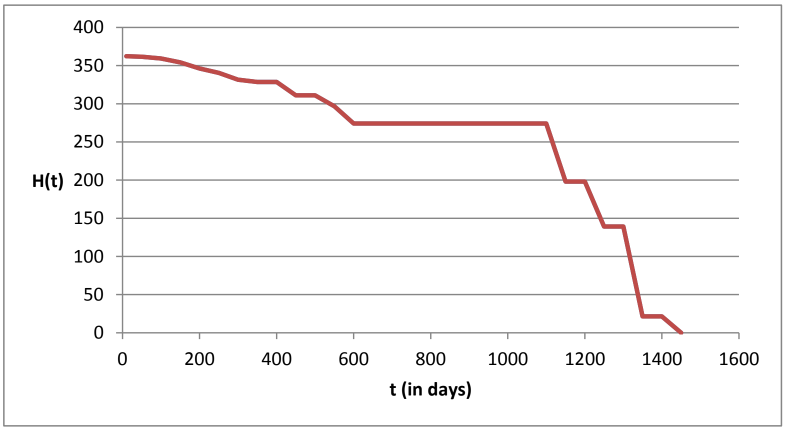

Figure 1. Curve of residual entropy of survival times (in days) for patients' head and neck cancer.

By applying the Chi-square goodness-of-fit test to the data cited in [7,26], we can test whether these data belong to Weibull distribution against another one. The null hypothesis is accepted at a 0.05 level of significance, where the estimated shape and scale parameters given by Matlab are, respectively, as follows: 365.7673 and 1.0223. This decision is further supported by the Weibull probability plot given by Matlab in Figure 2, where the data points (in cross line, representing the empirical data) appear along the reference line (in dashed line) representing the theoretical distribution.

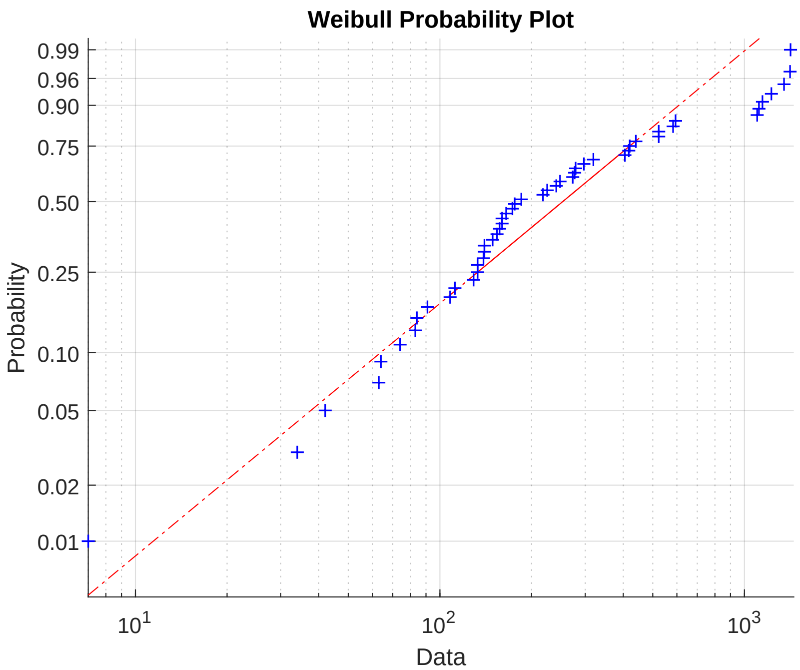

Figure 2. Weibull probability plot of Chi-2 goodness-of-fit test of survival times (in days) for patients' head and neck cancer.

Example 3. *As a second illustration, we consider times to failure of 18 electronics devices from Lai and Xie (2006) [7] and Wang (2000) [27]. The compilation of the algorithm gives the estimation of the statistic* $W_n(k) = 24.2029$*. This value exceeds all percentiles given in Table 1 when* $n = 18$*. We conclude also that this real sample is provided from DURL distribution.*

In Figure 3, we once again observe that the curve of the empirical residual entropy specific to the dataset cited in [7] decreases as $t(t > 0)$ increases. This confirms the result of the test based on the derivative criteria.

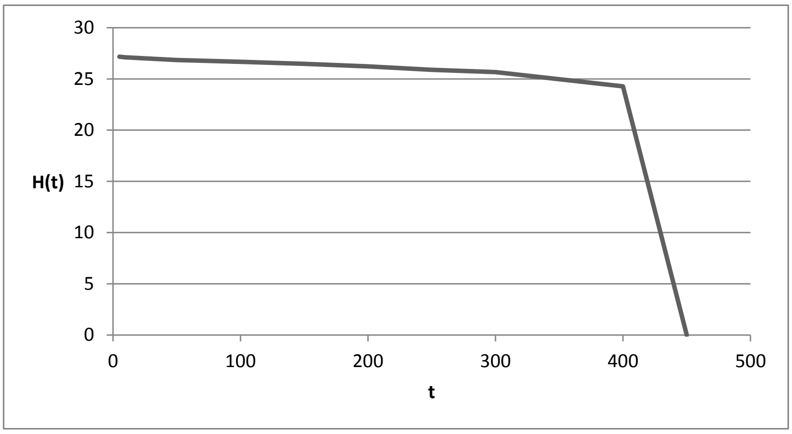

Figure 3. Curve of residual entropy of times to failure of 18 electronics devices.

The same conclusion leads by applying the Chi-square goodness-of-fit test to the data cited in [7,27]. Once again, we accept that the distribution is a Weibull at the 0.05 level of significance; see Figure 4, and the estimated shape and scale parameters given by Matlab are, respectively, 179.656 and 1.1458.

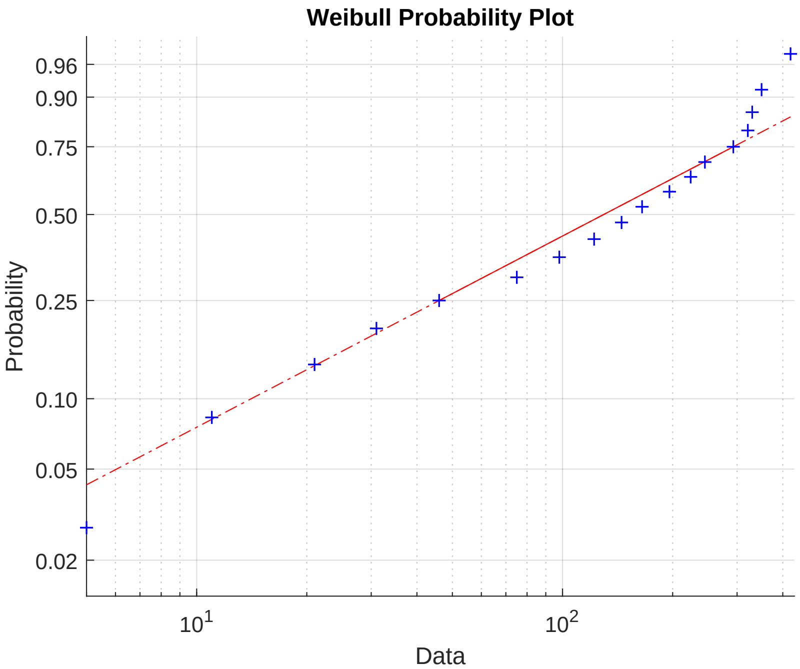

Figure 4. Weibull probability plot of Chi-2 goodness-of-fit-test of times to failure of 18 electronics devices.

Notice that the results above show that the shape parameters exceed 1, so the Weibull distributions are also DURL (Example 1). This is due to the fact that the DURL class does not only contain non-parametric distributions, but it also includes parametric ones.

Another example in medicine has been provided by Benaoudia and Aissani (2022) [28], on different real data by Bryson and Siddiqui, as cited in Lai and Xie (2006) [7]. It leads to the same results.

11. Conclusions

In this paper, we have proposed a new statistical test for detecting the monotonicity of uncertainty based on derivative criteria different from that of Ebrahimi (1997) [18]. The choice of the exponential distribution in the null hypothesis is based on the fact that this distribution gives a constant residual entropy. The derivative of this last one is null. Hence, we test whether the residual entropy is constant against the decreasing one. The consistency and the asymptotic normality properties have been discussed. Power estimates have been calculated and compared to those of the test of Ebrahimi (1997) [18]. Illustrations based on real survival data have been provided, and the application of the Chi-square goodness-of-fit test confirms the datasets distributions belong to parametric distributions that are DURL at the same time. A second new test has been studied using the kernel estimation method, which provides the same conclusions. This is the subject of a further paper.

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