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# An Application of Rouché's Theorem to Delimit the Zeros of a Certain Class of Robustly Stable Polynomials

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**Abstract:** An important problem related to the study of the robust stability of a linear system that presents variation in terms of an uncertain parameter consists of understanding the variation in the roots of a system's characteristic polynomial in terms of the uncertain parameter. In this contribution, we propose an algorithm to provide sufficient conditions on the uncertain parameter in such a way that a robustly stable family of polynomials has all of its zeros inside a specific subset of its stability region. Our method is based on the Rouché's theorem and uses robustly stable polynomials constructed by using basic properties of orthogonal polynomials.

**Keywords:** robust stability; Schur polynomials; orthogonal polynomials; Rouché's theorem

**MSC:** 33D45; 93C05



**Citation:** Martínez, N.; Garza, L.E.; Romero, G. An Application of Rouché's Theorem to Delimit the Zeros of a Certain Class of Robustly Stable Polynomials. *Mathematics* **2023**, *11*, 4244. https://doi.org/10.3390/math11204244

Academic Editors: Fangzheng Gao and Zhongcai Zhang

Received: 5 September 2023
Revised: 4 October 2023
Accepted: 9 October 2023
Published: 11 October 2023



## 1. Introduction

Hurwitz and Schur polynomials characterize the stability of time-invariant continuous and discrete linear systems, respectively [1]. More precisely, a continuous-time system is asymptotically stable if and only if all the roots (or zeros) of its characteristic polynomial have strictly negative real part. Such polynomials are known in the literature as Hurwitz polynomials [1–3]. A discrete-time system, on the other hand, is asymptotically stable if and only if its characteristic polynomial is a Schur polynomial, i.e., its zeros have magnitudes less than one [1,4]. Both classes of polynomials are also called *stable* polynomials. In general, a stable polynomial has its zeros within a given subset of the complex plane called the *stability region*. Their study is motivated by the important role that they play in the development and design of control systems [5,6]. There are several criteria to verify the stability of a polynomial by looking at its coefficients. This is useful given the fact that there are no general formulas to compute the zeros of polynomials of degree greater than or equal to five (see [1]).

In this contribution, we deal only with Hurwitz and Schur stability. These types of stability are related by the Möbius transformation $z = \frac{x+1}{x-1}$, which defines a conformal mapping between the open unit disk and the open left half-plane and can be utilized to determine the Schur stability by using a Hurwitz criterion, as follows.

**Theorem 1** ([1]). *Let $S(z)$ and $f(x)$ be polynomials of degree $n$ with real coefficients such that*

$$(x-1)^n S\left(\frac{x+1}{x-1}\right) = f(x). \tag{1}$$

*Then, $f(x)$ is a Hurwitz polynomial if and only if $S(z)$ is a Schur polynomial.*

On the other hand, a useful tool to study localization properties of the zeros of a pair of holomorphic functions is the well-known Rouché's Theorem [7], stated next.

**Theorem 2** ([7]). *Let $f$ and $g$ be holomorphic functions in an open set containing a circle C and its interior. If $|f(z)| > |g(z)|$ in C, then $f$ and $f+g$ have the same number of zeros inside C.*

Rouché's Theorem has found applications in several fields, such as its use to determine the probability generating function of the stationary queue length distribution [8], the design of minimum-phase, finite impulse response filters [9], and the design of disturbing rejection PID controllers for uncertain systems [10], among many others.

We will use an application of Rouché's Theorem (see [1]) for the particular case when $f$ and $g$ are polynomials.

**Proposition 1** ([1]). *If there exist $R \in \mathbb{R}^+$ and $0 \le k \le n$ integer such that*

$$|a_k|R^k > |a_0| + \cdots + |a_{k-1}|R^{k-1} + |a_{k+1}|R^{k+1} + \cdots + |a_n|R^n, \tag{2}$$

*then the polynomial*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \tag{3}$$

*has exactly $k$ roots of magnitude less than $R$.*

Notice that the above result can be used to determine Schur stability when $R = 1$ and $k = n$.

In applications, it is usually necessary to assume some degree of uncertainty in order to take into account variations in the system. Such uncertainty is typically represented by one or more parameters that are expected to vary within certain range. As a consequence, the design and implementation of control systems must take into account these variations. This is known in the literature as robust control [1,11].

**Definition 1** ([11]). *A polynomial of the form*

$$P(z,\boldsymbol{\lambda}) = \sum_{k=0}^{n} a_k(\boldsymbol{\lambda}) z^k, \quad n, d \in \mathbb{N},$$

*i.e., its coefficients depend on the entries of a vector of uncertain parameters $\boldsymbol{\lambda} \in Q \subset \mathbb{R}^d$, is called an uncertain polynomial.*

*The set $\mathcal{P} : \{P(z,\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in Q\}$ is called a family of uncertain polynomials. In particular, if $P(z,\boldsymbol{\lambda})$ is a Hurwitz (respectively Schur) polynomial for every value of $\boldsymbol{\lambda}$, we say $\mathcal{P}$ is a Hurwitz (respectively Schur) robustly stable family.*

For instance, if $a_k(\boldsymbol{\lambda}) \in [a_k^-, a_k^+]$ then we have

$$P(z,\boldsymbol{\lambda}) = [a_n^-, a_n^+]z^n + [a_{n-1}^-, a_{n-1}^+]z^{n-1} + \ldots + [a_0^-, a_0^+],$$

and $P$ is said to have *interval* uncertainty. Notice that if $[a_n^-, a_n^+]$ does not include the origin, then we have degree invariance. Kharitonov's theorem [12] states that this family is robustly stable if and only if four polynomials, obtained by taking their coefficients as the values $a_k^-$ and $a_k+$ in an appropriate way, are stable. Other studied uncertain structures include *affine* uncertainty [1,13], *multilinear* uncertainty [14,15], and *polynomial* uncertainty [1,11,16], among others. In the latter, the coefficients of the uncertain polynomial are expressed in terms of powers of the $\lambda_i$.

On the other hand, there are many applications where it is required to determine the location of the roots of a given polynomial. One of them is the problem of the design of control systems, which consists of proposing a controller such that when added to given closed-loop system, it has a desired performance. This problem is equivalent to locating the roots of the associated characteristic polynomial in a certain subset of the stability region [6,11]. Indeed, the location of the roots is directly related to the performance of the system. It is important to notice that some families of robustly stable polynomials (defined in terms of orthogonal polynomials) have been proposed in the literature [17–19], and the behavior of the roots in terms of the uncertain parameter has been studied [19].

Here, we consider several robustly stable families of polynomials with polynomial uncertainty that are defined by using some sequences of orthogonal polynomials. By using Proposition 1, we will be able to obtain sufficient conditions on the uncertain parameters such that zeros of such polynomials belong to a certain subset of the region of stability. There exist other methods that consider a similar problem. For instance, in [20] the authors obtained conditions based in the $H_\infty$ norm in order to obtain a desired performance. However, the approach described in this contribution presents a simpler and effective methodology to obtain a desired performance from an automatic control system. Moreover, there are many applications of the aforementioned techniques in real-life systems. Some examples can be found in [1,8,16]. The manuscript is organized as follows. Basic background of orthogonal polynomials on the unit circle, as well as their relation with Schur polynomials, is given in Section 2. Section 3 contains our main results. Namely, we present a method to establish conditions on the uncertain parameter such that the roots of a certain family of polynomials are located inside an annulus contained on the unit disk. Then, we use the relation (1) to extend our method for Hurwitz polynomials, providing conditions on the uncertain parameter so that the corresponding zeros are located inside a certain rectangle in the left half-plane. We conclude our contribution stating some open problems in Section 4.

## 2. Robustly Stable Polynomials from Orthogonal Polynomials

Orthogonal polynomials on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (OPUC) are defined by the relation

$$\int_{\mathbb{T}} \phi_n(z)\, \overline{\phi_m(z)}\, d\sigma(z) = k_n\, \delta_{n,m}, \quad k_n > 0, \qquad \forall\, n, m \geqslant 0, \tag{4}$$

where $\sigma$ is a measure that is positive on the unit circle and $\delta_{n,m}$ is Kronecker's delta. Typically, we have $d\sigma(z) = \omega(z)dz$ where the orthogonality function $\omega(z)$ is also positive. The sequence $\{\phi_n\}_{n\geq 0}$ is unique except for multiplications by constants, and we will assume without loss of generality that it is a monic sequence. For a general treatment of this theory, we refer the reader to [21,22]. A key property is that the roots of each $\phi_n$ belong to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In other words, $\phi_n$ has the Schur property. On the other hand, the Geronimus–Wendroff theorem (see [21]) states that every Schur polynomial is an element of a sequence of polynomials that is orthogonal with respect to some measure $\sigma$ supported on $\mathbb{T}$. In this sense, OPUC and Schur polynomials are equivalent.

Another important property of $\{\phi_n\}_{n\geq 0}$ is the Szegő relation

$$\phi_{n+1}(z) = z\phi_n(z) + \phi_{n+1}(0)\phi_n^*(z), \tag{5}$$

where $\phi_n^*(z) = z^n\overline{\phi_n}(z^{-1})$ is the so-called *reversed (or reciprocal)* polynomial. The complex numbers $\{\phi_n(0)\}_{n\geq 1}$ are called *Verblunsky coefficients* (they are also called reflection or Schur parameters) and satisfy $|\phi_n(0)| < 1$ for $n \geq 1$. Moreover, it is known that $\phi_n(z)$ has no zeros in $\{z : |z| \leq |\phi_n(0)|\}$ for $n \geq 2$ [21]. Finally, the Verblunsky's theorem states that given any sequence $\{\alpha_n\}_{n\geq 1}$ of complex numbers in $\mathbb{D}$, the sequence defined by (5) with $\phi_n(0) = \alpha_n$ for $n \geq 1$, starting from $\phi_0(z) = 1$, satisfies the orthogonality relation (4) [21].

Orthogonal polynomials on the unit circle have been recently used in [19] to build families of robustly stable Schur polynomials by using two different approaches:

1. By including an uncertain parameter $\lambda$ in the orthogonality weight $w(z, \lambda)$ without affecting its positivity. Thus, the associated family $\{\phi_n(z, \lambda)\}_{n\geq 0}$ will be orthogonal (and therefore Schur) for all values of $\lambda$.
2. By using Verblunsky's theorem along with the Szegő relation (5). Here, the idea is to consider $\lambda$-dependent Verblunsky coefficients $\alpha_n(\lambda)$ satisfying $\alpha_n(\lambda) \in \mathbb{D}$ for every

value of $\lambda$. As in the previous case, the constructed sequence $\{\phi_n(z,\lambda)\}_{n\geq 0}$ will be Schur for all values of $\lambda$.

Let us consider some examples that will be used in the following section. Notice that the first two correspond to the case 1 above, whereas the last one corresponds to the case 2. Furthermore, we are only considering the situation where there is only one uncertain parameter $\lambda$, i.e., $d = 1$ in Definition 1. More details can be found in [19].

- SIMP (singular inserted mass point) orthogonal polynomials are defined for $\lambda \in (0,1)$ by [21]
$$\phi_n(z) = z^n + v_n(z^{n-1} + z^{n-1} + \ldots + 1), \tag{6}$$
where $v_n$ is the $n$-th Verblunsky coefficient given by the expression
$$v_n = -\frac{\lambda}{1+(n-1)\lambda}, \quad n > 0.$$
These polynomials are orthogonal with respect to weight $\omega(z,\lambda) = (1-\lambda) + \lambda\delta_0$, where $\delta_0$ is the Dirac's delta distribution at the origin.
- Rogers–Szegő orthogonal polynomials are defined by [21]
$$\phi_n(z) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_\lambda \lambda^{(n-j)/2} z^j, \quad \lambda \in (0,1), \tag{7}$$
where $\begin{bmatrix} n \\ j \end{bmatrix}_\lambda$ are the so-called $\lambda$-*binomials* coefficients given by
$$\begin{bmatrix} n \\ j \end{bmatrix}_\lambda = \frac{(n)_\lambda}{(j)_\lambda (n-j)_\lambda} = \frac{(1-\lambda^n)\ldots(1-\lambda^{n-j+1})}{(1-\lambda)\ldots(1-\lambda^j)}, \tag{8}$$
with
$$(n)_\lambda = (1-\lambda)(1-\lambda^2)\ldots(1-\lambda^n), \quad (0)_\lambda \equiv 1. \tag{9}$$
Using the parametrization $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, this sequence is orthogonal with respect to the weight
$$\omega(\theta,\lambda) = \frac{2\pi}{\sqrt{2\pi \log(\frac{1}{\lambda})}} \sum_{j=-\infty}^{\infty} \lambda^{-(\theta-2\pi j)^2/2}.$$
- For $\lambda \in (-1,1)$, define the polynomials $\{\phi_n(z,\lambda)\}_{n\geq 0}$ with the Szegő relation
$$\phi_{n+1}(z,\lambda) = z\phi_n(z,\lambda) + \lambda\phi_n^*(z,\lambda), \quad \lambda \in (-1,1), \tag{10}$$
with the initial conditions $\phi_0(z,\lambda) = 1$. That is, taking $\alpha_n(\lambda) = \lambda \in (-1,1)$ in the approach 2 above.

We point out that a similar procedure was used to construct robustly stable families of Hurwitz polynomials by using orthogonal polynomials on the real line [17,18], because there is a close connection between both theories [23,24]. General information about orthogonal polynomials on the real line can be found in [22,25].

## 3. Delimiting the Zeros of Robustly Stable Polynomials

### *3.1. Schur's Robust Stability*

We aim to delimit the region where the zeros of uncertain polynomials are located using Proposition 1. For a polynomial of degree $n$ as in (3), setting $k = n$ and $R = R_n$ in (2), we obtain a sufficient condition so that the $n$ zeros of the polynomial lie inside the disk of radius $R_n$ centered at the origin. Similarly, setting $k = 0$ and $R = R_0$ we have a condition for the $n$ zeros of $p(z)$ to be outside the circle of radius $R_0$ centered at the origin. As a consequence, we propose the following algorithm to obtain sufficient conditions in $\lambda$ to guarantee that the roots of a Schur polynomial that is robustly stable belong to the annulus $R_0 < |z| < R_n$.

It is important to note that by using Algorithm 1 we can control the location of the roots of a *known* family of robustly stable polynomials defined in terms of a parameter $\lambda$. That is, the algorithm can be used once we have such a family of polynomials. In the previous section, we present several methods that have been used in the literature to construct robustly stable families of Schur polynomials. Thus, we present some examples of the results obtained from the algorithm when we use the polynomials $\phi_n(z, \lambda)$ introduced in the previous section.

**Algorithm 1:** Enclosing the zeros of $\phi_n(z, \lambda)$ in $R_0 < |z| < R_n$.

**Input:** A robustly stable polynomial $\phi_n(z, \lambda)$ and $R_0$, $R_1$ with $0 < R_0 < R_n < 1$.
**Output:** Sufficient conditions in $\lambda$ such that $\phi_n(z_0, \lambda) = 0 \Rightarrow z_0 \in R_0 < |z| < R_n$.

initialization;
Set $p(z) = \phi_n(z, \lambda)$ in (3) and $k = 0$, $R = R_0$ in (2). Solve inequality (2) for $\lambda$;
Set $p(z) = \phi_n(z, \lambda)$ in (3) and $k = n$, $R = R_n$ in (2). Solve inequality (2) for $\lambda$;
Determine the interval $(\lambda_{min}, \lambda_{max})$ such that $\lambda$ satisfies both inequalities;
end

**Example 1.** *Consider the orthogonal polynomial $\phi_3(z, \lambda)$ of the SIMP family introduced in the previous section. By Verblunsky's theorem, $\phi_3(z, \lambda)$ is Schur for $\lambda \in (0, 1)$. Suppose we want to enclose the zeros of this polynomial inside the annulus $R_0 < |z| < R_3$ with $R_0 = 0.2$ and $R_3 = 0.6$. Substituting* (6) *into* (2) *and setting $k = 0$ and $k = n = 3$, we find*

$$\lambda > \left(-\frac{R_0^{-3} - 1}{1 - R_0} + 2R_0^{-3} - 2\right)^{-1}, \tag{11}$$

*and*

$$\lambda < \left(-\frac{R_3^{-3} - 1}{1 - R_3} - 2\right)^{-1}, \tag{12}$$

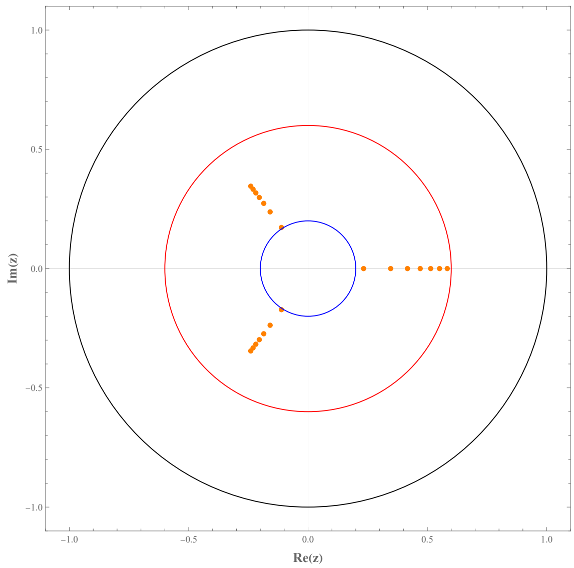
*respectively. Substituting the values $R_0 = 0.2$ and $R_3 = 0.6$ in these inequalities we find that if $\lambda \in (0.0107527, 0.141361)$, then the zeros of the polynomial will be located in the desired annulus. We can proceed in a similar way to obtain conditions on $\lambda$ in order for the polynomial $\phi_5(t, \lambda)$ to have its zeros in the annulus $R_0 = 0.1 < |z| < R_5 = 0.4$. In such a case, we obtain $\lambda \in (0.0000112504, 0.00636563)$. Figure 1 shows the motion the zeros with respect to $\lambda$ in both cases.*

We point out that the procedure above provides us with a sufficient condition in order for the zeros of a given polynomial to be located within some annulus, but such a condition is not necessary. Furthermore, Proposition 1 can be used only if the inequality (2) is satisfied for some $R$. Examples 2 and 3 illustrate these situations.

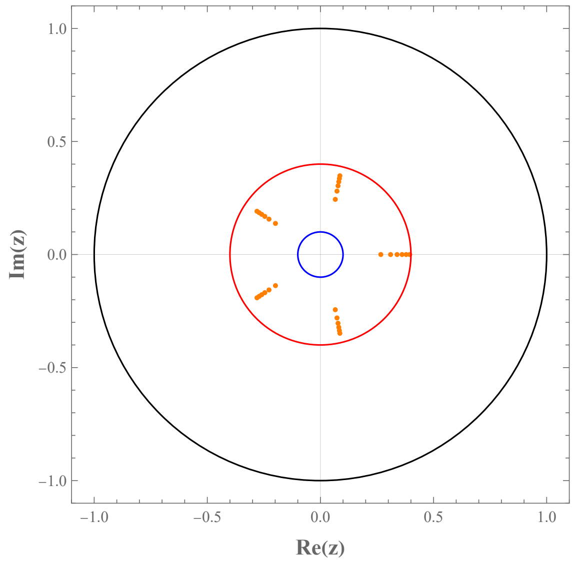
**Example 2.** *Consider $\phi_4(z, \lambda)$ of the Rogers–Szegő family. Suppose we want to keep its zeros inside the annulus $R_0 = 0.5 < |z| < R_n = 0.9$. Notice that Proposition 1 applies if the coefficients of the polynomial* (3) *are such that the polynomial $h_k(x) = |a_0| + \cdots + |a_{k-1}|x^{k-1} - |a_k|x^k + |a_{k+1}|x^{k+1} + \cdots + |a_n|x^n$ takes a negative value for some $x > 0$. Because we require $\lambda \in (0, 1)$ and $R_0 = 0.5$, we consider the function*

$$\begin{aligned} h_4(0.5, \lambda) \quad &= \quad -\lambda^2 + \frac{1}{2}(\lambda^{3/2} + \lambda^{5/2} + \lambda^{7/2} + \lambda^{9/2}) + \frac{1}{4}(\lambda + \lambda^2 + 2\lambda^3 + \lambda^4 + \lambda^5) \\ &+ \quad \frac{1}{8}(\lambda^{1/2} + \lambda^{3/2} + \lambda^{5/2} + \lambda^{7/2}) + \frac{1}{16}. \end{aligned}$$

*Thus, in order to guarantee that our polynomial does not contain zeros in $|z| < 0.5$, we must show that $h_4(0.5, \lambda) < 0$ for some value of $\lambda \in (0, 1)$. However, $h_4(0.5, 0) = 1/16$ and it is not difficult to see that $\frac{dh_4(0.5,\lambda)}{d\lambda} > 0$ for every $\lambda \in (0, 1)$. Therefore, we cannot apply Proposition 1 in this* *case. On the other hand, it is possible to find values of $\lambda$ such that all the zeros are inside $|z| < 0.9$. Indeed, that will be the case if $0 \le \lambda < 0.175065$ (see Figure 2).*




(**a**)




(**b**)

**Figure 1.** (**a**) Zeros of $\phi_3(z,\lambda)$ for $\lambda \in (0.01, 0.14)$ in steps of 0.02, $|z| = R_0 = 0.2$ in blue and $|z| = R_3 = 0.6$ in red. (**b**) Zeros of $\phi_5(z,\lambda)$ for $\lambda \in (0.001, 0.006)$ in steps of 0.001, $|z| = R_0 = 0.1$ in blue and $|z| = R_5 = 0.4$ in red. In both cases, the zeros move away as $\lambda$ increases.

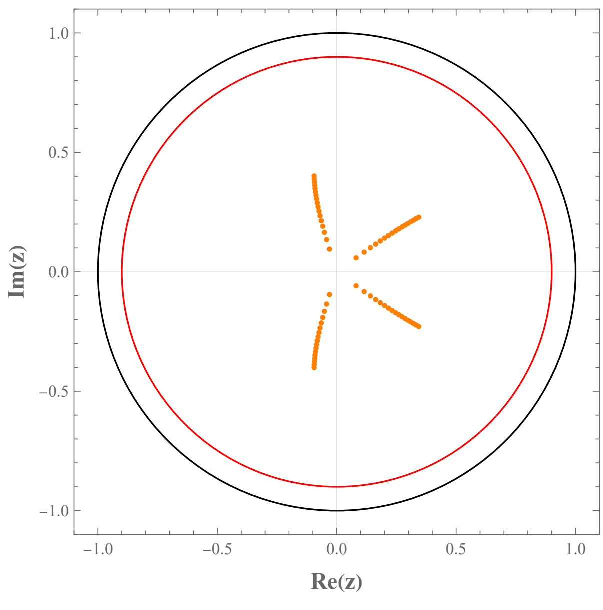



**Figure 2.** Zeros of the Rogers–Szegő polynomial $\phi_4(z,\lambda)$ for $\lambda \in (0, 0.17)$ in steps of 0.01, $|z| = R_4 = 0.9$ in red.

*Nevertheless, it is well known that the roots of the n-th degree Rogers–Szegő polynomial lie on the circle $|z| = \lambda^{1/2}$ [21]. As a consequence, if we want to enclose the zeros in an annular region $R_0 < |z| < R_n$, it is sufficient to take $R_0^2 < \lambda < R_n^2$.*

**Example 3.** *Let $\phi_3(z,\lambda) = \lambda + (\lambda + \lambda^2 + \lambda^3)z + (\lambda + 2\lambda^2)z^2 + z^3$ be defined from the recurrence relation (10). Assume that we want to determine the values of $\lambda$ such that the zeros of $\phi_3(z,\lambda)$*

*are located in* $0.5 < |z| < 1$*. According to Proposition 1, this will be the case if there exist values of* $\lambda \in (-1,1)$ *such that*

$$\lambda > (\lambda + \lambda^2 + \lambda^3)(0.7) + (\lambda + 2\lambda^2)(0.7)^2 + (0.7)^3.$$

*However, it is not difficult to see that* $f(\lambda) = (\lambda + \lambda^2 + \lambda^3)(0.7) + (\lambda + 2\lambda^2)(0.7)^2 + (0.7)^3 - \lambda$ *is positive for every* $\lambda \in (-1,1)$*, and thus Proposition 1 does not give us information about the zeros of* $\phi_3(z,\lambda)$ *in this situation. On the other hand, it is well known that* $\phi_3(z)$ *vanishes outside the disk* $|z| < |\phi_3(0)|$ *[21]. As a consequence, if* $|\lambda| > \frac{1}{2}$ *then the zeros of* $\phi_3(z,\lambda)$ *are located outside the disk* $|z| < \frac{1}{2}$ *(see Figure 3).*

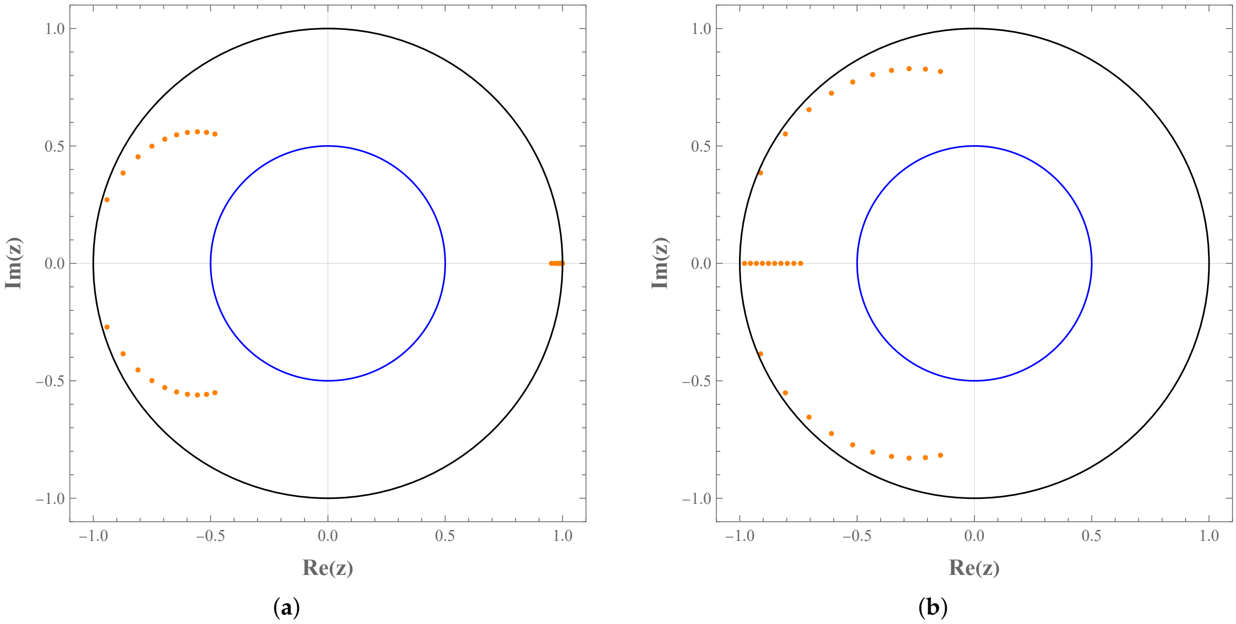



**Figure 3.** (**a**) Zeros of $\phi_3(z,\lambda)$ for $\lambda \in (-0.96,-0.51)$ in steps of 0.05. (**b**) Zeros of $\phi_3(z,\lambda)$ for $\lambda \in (0.51, 0.96)$ in steps of 0.05. In both cases, the zeros move away as $|\lambda|$ increases.

### *3.2. Hurwitz's Robust Stability*

In this section, we apply the Möbius transformation (1) to rewrite Proposition 1 in terms of Hurwitz polynomials. This will allow us to bound the region where the zeros of a Hurwitz polynomial are located, just as we did in the previous section for Schur polynomials.

From Theorem 1 it is easily deduced that $f_n(x)$ is a Hurwitz polynomial if and only if the polynomial $R(z)$ given by

$$(z-1)^n f_n\left(\frac{z+1}{z-1}\right) = R(z), \tag{13}$$

is a Schur polynomial. Notice that if $f_n(x) = \sum_{k=0}^{n} a_k x^k$ then

$$R(z) = (z-1)^n f_n\left(\frac{z+1}{z-1}\right) = \sum_{k=0}^{n} a_k (z+1)^k (z-1)^{n-k} = \sum_{k=0}^{n} b_k z^k, \tag{14}$$

and thus the coefficients $b_k$ of $R(z)$ can be easily computed in terms of the coefficients of $f_n(x)$.

We can now rewrite Proposition 1 to bound the zeros of a Hurwitz polynomial.

**Theorem 3.** *Let $f_n(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n$. If there exist $0 < R < 1$ and an integer $k$ with $0 \leq k \leq n$ such that*

$$|b_k|R^k > |b_0| + \cdots + |b_{k-1}|R^{k-1} + |b_{k+1}|R^{k+1} + \cdots + |b_n|R^n, \tag{15}$$

*where the $b_i$ are the coefficients of the polynomial $R(z)$ in (14), then $f_n(x)$ has exactly $k$ roots inside the disk $\{z \in \mathbb{C} : |z - \frac{R^2+1}{R^2-1}| < \frac{2R}{1-R^2}\}$.*

**Proof.** Proposition 1 guarantees that there are $k$ zeros of $R(z)$ in $|z| < R$. Given that $0 < R < 1$ and the transformation $z = \frac{x+1}{x-1}$ is its own inverse, we can apply this transformation to the set $\{z \in \mathbb{C} : |z| < R\}$ to find the region where $k$ zeros of $f_n$ are located, thus obtaining that these zeros are in the interior of $\{z \in \mathbb{C} : |z - \frac{R^2+1}{R^2-1}| < \frac{2R}{1-R^2}\}$. □

**Corollary 1.** *Let $f_n(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n$. If there exist $0 < R < 1$ such that*

$$|b_n|R^n > |b_0| + |b_1|R + \cdots + |b_{n-1}|R^{n-1}, \tag{16}$$

*where $b_i$, $1 \leq i \leq n$, are the coefficients of the polynomial $R(z)$ in (14), then $f_n(x)$ has exactly $k$ zeros inside the disk $\{z \in \mathbb{C} : |z - \frac{R^2+1}{R^2-1}| < \frac{2R}{1-R^2}\}$. Therefore, if $x_0$ is a zero of $f_n(x)$ then*

$$\frac{R+1}{R-1} < Re(x_0) < \frac{R-1}{R+1}, \quad \text{and} \quad -\frac{2R}{1-R^2} < Im(x_0) < \frac{2R}{1-R^2}. \tag{17}$$

As we did with the Schur polynomials, this result can be used to enclose the zeros of a Hurwitz polynomial in a disk of radius $R_n$ by setting $k = n$ in (15), or to guarantee that the zeros are outside of a disk of radius $R_0$ by setting $k = 0$. In this way, by applying Algorithm 1 we obtain inequalities that we can use to determine the range of values of the uncertain parameter so that the zeros stay within a certain region. Moreover, the corollary allows us to bound the real and imaginary parts of the zeros. The number of inequalities that must be satisfied will depend on the region where we want to enclose the zeros. Some observations are in order. First, the Möbius transformation maps disks centered at the origin of radius less than one to disks contained in the complex left half-plane centered at some point of the real axis. Second, the image of an annulus centered at the origin under the Möbius transformations is another annulus in the left half-plane, with a center on the real line. As a consequence, to delimit the region containing the zeros of $R(z)$ it is not necessary to verify several inequalities, but only the one involving the smallest radius.

We illustrate the use of the proposed algorithm in this situation, as well as some of its limitations, in the following examples. As in the Schur case, we need to use a known family of robustly stable Hurwitz polynomials. As a consequence, we use some robustly stable polynomials considered in [18], constructed by using recurrence relations satisfied by some sequences of orthogonal polynomials on the real line.

**Example 4.** *Let us consider the Hurwitz robustly stable polynomial $f(x,\lambda) = 170\lambda^6 + (119\lambda^5 + 80\lambda^4)x + (10\lambda^2 + 56\lambda^3 + 17\lambda^4)x^2 + (7\lambda + 8\lambda^2)x^3 + x^4$, constructed as shown in [18]. Suppose that we want to know the values of $\lambda$ for which the zeros of the polynomial $f$ have real part in the interval $(-3, -3/7)$. For the left bound, (17) implies $R = 1/2$ and for the right bound we obtain $R = 2/5$. Thus, it is required to solve (16) for $\lambda$ in both cases and determine the values of $\lambda$ satisfying both cases. However, the method is based on the fact that the polynomial $R(z)$ is Schur, and all of its zeros are located inside the disks centered at the origin with radii $R = 1/2$ and $R = 2/5$, respectively. By mapping these two disks to the complex left half-plane with the Möbius* *transformation we will also have two disks, one contained in the other, and this means that it suffices for $\lambda$ to satisfy* (16) *for the smaller radius $R = 2/5$. According to* (14)*, we obtain*

$$\begin{aligned} R(z) \;\; = \;\; & 1 - 7\lambda + 2\lambda^2 + 56\lambda^3 - 63\lambda^4 - 119\lambda^5 + 170\lambda^6 \\ & +(4 - 14\lambda - 16\lambda^2 + 160\lambda^4 + 238\lambda^5 + -680\lambda^6)z \\ & +(6 - 20\lambda^2 - 112\lambda^3 - 34\lambda^4 + 1020\lambda^6)z^2 \\ & +(4 + 14\lambda + 16\lambda^2 - 160\lambda^4 - 238\lambda^5 - 680\lambda^6)z^3 \\ & +(1 + 7\lambda + 18\lambda^2 + 56\lambda^3 + 97\lambda^4 + 119\lambda^5 + 170\lambda^6)z^4 \end{aligned}$$

*and by solving Equation* (16) *we obtain $\lambda \in (0.419, 7/15)$ (see Figure 4).*

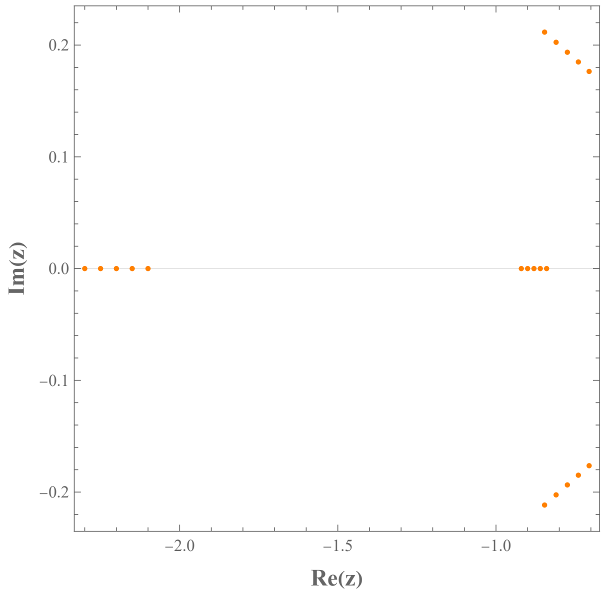



**Figure 4.** Zeros of $f(x)$ for $\lambda \in (0.42, 0.46)$ in steps of 0.01. The zeros move away from the abscissa $x = -\frac{3}{7}$ as the value of $\lambda$ increases.

Now, let us discuss some limitations of the proposed method. Let $x_0$ be any zero of $f(x, \lambda)$, and assume we require

$$a_i < Re(x_0) < a_s. \tag{18}$$

Thus, solving the equations

$$R_i = \frac{a_i + 1}{a_i - 1}, \quad \text{and} \quad R_s = \frac{a_s + 1}{1 - a_s}$$

and choosing $R = min\{R_i, R_s\}$, we can proceed to use (16) and obtain conditions in $\lambda$ so that the real parts of the zeros of $f(x, \lambda)$ satisfy (18). However, given that $R$ must satisfy $0 < R < 1$, it is clear that in order to apply this method we require $a_i \in (-\infty, -1)$ and $a_s \in (-1, 0)$.

On the other hand, if we want

$$o_i < Im(x_0) < o_s, \tag{19}$$

the symmetry of the involved disks with respect to the real axis implies that this method can only be applied to obtain values of $\lambda$ such that

$$-o < Im(x_i) < o, \quad o = min\{|o_i|, |o_s|\}. \tag{20}$$

Thus, (17) implies

$$R_o = \frac{-1+\sqrt{1+o^2}}{o},$$

and taking $R = R_o$, we can use (16) to obtain the corresponding values for $\lambda$. Notice that in this case $R_o$ can take any positive real value.

Finally, if we want to find values of $\lambda$ such that (18) and (20) are satisfied simultaneously, i.e., if the zeros must be located inside some rectangle in the left half-plane, then we must take $R = min\{R_i, R_s, R_o\}$.

**Example 5.** *Consider the polynomial* $f_3(x,\lambda) = x^3 + (5\lambda^3 + 12\lambda^2)x^2 + (60\lambda^5 + 61\lambda^4)x + 305\lambda^7$ *and assume that we require that its zeros have an imaginary part inside* $(-2, \frac{4}{3})$. *According to the previous observations, we can use the proposed method for the symmetrized interval* $(-\frac{4}{3}, \frac{4}{3})$. *Then, we obtain* $R = R_l = \frac{-1+\sqrt{1+(\frac{4}{3})^2}}{\frac{4}{3}} = \frac{1}{2}$, *which yields* $\lambda \in (0.407556, 0.463672)$ *(see Figure 5a).*

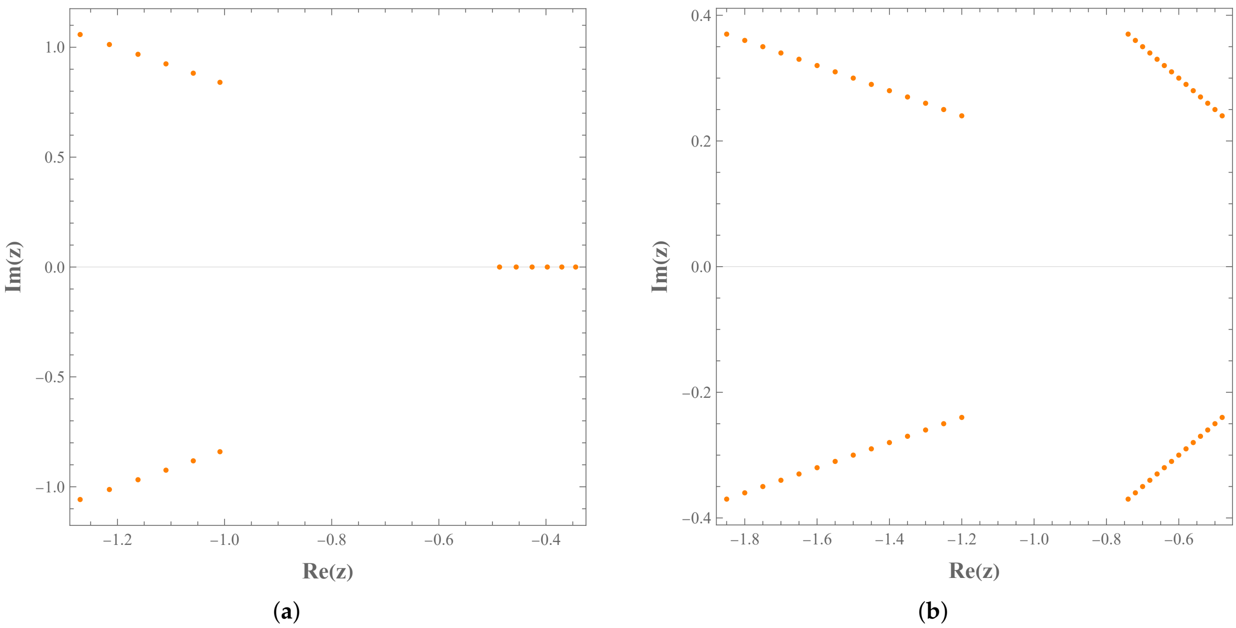


**Figure 5.** (**a**) Zeros of $f_3(x,\lambda)$ for $\lambda \in (0.41, 0.46)$ in steps of 0.01. (**b**) Zeros of $f_4(x,\lambda)$ for $\lambda \in (0.24, 0.37)$ in steps of 0.1. In both cases, the zeros move away from the imaginary axis when the value of $\lambda$ increases.

**Example 6.** *Finally, consider the polynomial* $f_4(x,\lambda) = x^4 + 14\lambda x^3 + 71\lambda^2 x^2 + 154\lambda^3 x + 130\lambda^4$, *and suppose we want to obtain conditions on* $\lambda$ *such that the corresponding zeros are located inside the rectangle defined by* $-5 < Re(z) < -\frac{1}{5}$ *and* $-\frac{15}{8} < Im(z) < \frac{15}{8}$. *Proceeding as above, we find* $R = min\{R_i, R_s, R_o\} = 0.6$ *and solving* (16) *we obtain* $\lambda \in (0.236017, 0.372254)$ *(see Figure 5b)* .

## 4. Discussion and Further Remarks

The problem of obtaining bounds for the zeros of a given polynomial has been widely considered in the literature because of the role that polynomials (and their zeros) play in many applications. In particular, the problem of determining conditions under which the zeros of a given polynomial lie within some annulus centered at the origin has been addressed, for instance, in [26,27], where the author considered a general context, and in [28] within the framework of Schur stability. In both cases, the given conditions depend on the polynomial's coefficients. The algorithm we propose in this contribution differs from these approaches due to the fact that we do not consider a single polynomial but a family of polynomials that are known to be robustly stable in terms of some parameter. More precisely, we have presented an algorithm to obtain sufficient conditions on $\lambda$ in such a way that the location of the zeros of robustly stable polynomials with uncertain parameter $\lambda$ are restricted to a certain subset of the stability region. A somewhat similar contribution is [29], where the authors considered stable polynomials with perturbations on the coefficients and obtained upper bounds for such perturbations in order that stability is preserved. Their approach was based in the principal argument and the Nyquist criterion. Although robust stability is considered, the problem considered there is different than the situation studied here. To the best of our knowledge, the setting that we consider in this contribution has not been considered elsewhere, and thus it is not possible to compare our results with those obtained with other methods, at least at the present time.

Our method can be applied to both Schur and Hurwitz stability and it requires solving some inequalities obtained from the Rouché's theorem. Here, robustly stable polynomials are constructed by using basic properties of orthogonal polynomials. Although we have only considered the case with one parameter $\lambda$, our approach can be used for an arbitrary number of parameters, and the only difficulty would be in solving inequalities with a larger number of parameters. Moreover, we use some known families of robustly stable polynomials to illustrate the algorithm, but the methods presented in [17–19] provide great flexibility to generate Schur and Hurwitz robustly stable polynomials in different ways, and this approach can be used with all of them. Finally, the proposed method for Hurwitz polynomials is based in the Möbius transformation relating the unit disk with the left half-plane, and this approach introduces some restrictions on the regions where the zeros are to be located, as explained in the discussion after Example 4. Finally, we consider that this method can be applied to develop a procedure for the tuning of a (robust) PID controller, by using the pole placement technique [11]. That is, to determine all possible values of the three parameters of the PID controller, in terms of an uncertain parameter, in such a way that the control system has a prescribed response. Another interesting problem is the development of a similar algorithm for Hurwitz polynomials that does not require the use of the Möbius transformation. Both problems will be addressed in a future contribution.

**Author Contributions:** Conceptualization, N.M. and G.R.; Methodology, L.G.; Validation, N.M., L.G. and G.R.; Formal analysis, N.M. and L.G.; Investigation, L.G. and G.R.; Writing—original draft, N.M.; Writing—review and editing, L.G. and G.R.; Supervision, L.G. and G.R.; Project administration, G.R.; Funding acquisition, G.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** The work of the first author was supported by Universidad de Colima, Convocatoria de Fortalecimiento de la Investigación Científica 2023. The work of the third author has been supported by Universidad Autónoma de Tamaulipas through the project: Fortalecimiento del Grupo de Investigación en Electrónica de la Unidad Académica Multidisciplinaria Reynosa Rodhe, PEI2018-UAT-2022.

**Conflicts of Interest:** The authors declare no conflict of interest.

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