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Maps Preserving Zero $*$-Products on $\mathcal{B}(\mathcal{H})$

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Abstract: The conventional research topic in operator algebras involves exploring the structure of algebras and using homomorphic mappings to study the classification of algebras. In this study, a new invariant is developed based on the characteristics of the operator using the linear preserving method. The results show that the isomorphic mapping is used for preserving this invariant, which provides the classification information of operator algebra from a new perspective. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces with dimensions greater than two, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ be the set of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. For $A, B \in \mathcal{B}(\mathcal{H})$, the $*$, $*$-Lie, and $*$-Jordan products are defined by A^*B, $A^*B - B^*A$, and $A^*B + B^*A$, respectively. Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive unital surjective map. It is confirmed that if Φ preserves zero $*$, $*$-Lie, and $*$-Jordan products, then Φ is unitary or conjugate unitary isomorphisms.

Keywords: Hilbert space; $*$-product; unitary isomorphic

MSC: 47B49; 46L40

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1. Introduction

In recent years, numerous studies have studied preserver problems concerning the characterization of maps on operator algebras that yield certain functions, subsets, relations, and invariants. Therefore, certain intrinsic isomorphism invariants between operator algebras must be determined [1–9].

Accordingly, the maps preserving certain products, such as the Lie and Jordan products, can be considered. The commutative elements in operator algebras play a crucial role in the study of algebraic structure, and scholars are actively studying the maps that preserve the product commutable. Let R be a ring; for any $A, B \in R$, $AB - BA$ is Lie product. Notably, if the Lie product is zero, then the product of A and B is commutative. Therefore, studying the commutability is equivalent to studying a zero-Lie product. Studies by [10–23] have previously demonstrated the preservation of zero products.

In addition to the special relationship where the product is a zero element, studies have reported the relationship characteristics of other products, such as whether the product is a self-adjoint element, positive element, idempotent element, nilpotent element, or projection [24–30].

In [31], Cui and Li proved that the nonlinear bijective map that preserves the product $XY - YX^*$ on factor von Neumann algebras is a $*$-ring isomorphism. This result shows that some new product which is related to the $*$-operation and Lie (resp. Jordan) product can entirely determine the isomorphisms between factor von Neumann algebras. Thus, the study of invariants with the $*$-operation has become an active area of research [31–34].

In our study, the isomorphism invariants of operator algebras are determined from another new perspective. We define new invariants by combining the $*$-operation of the product, the Lie product, and the Jordan product of operators. For any $A, B \in \mathcal{B}(\mathcal{H})$, we call A^*B the $*$ product of A, B, $A^*B + B^*A$ the $*$-Jordan product of A, B, and $A^*B - B^*A$ the

$*$-Lie product of A, B. For more on the $*$-Jordan product and $*$-Lie product, refer to [32,33]. These products are collectively referred to as the $*$ products.

In this study, we find that there are very close relationships between the $*$ products and the zero product. Interestingly, $A^*B = 0$ reflects a number of characteristics of the kernel and domain of operators A and B. Similarly, $A^*B - B^*A = 0$ and $A^*B + B^*A = 0$, illustrating that A^*B are self-adjoint and anti-self-adjoint, respectively. Lastly, we investigate additive maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ preserving zero-$*$ products. The zero-$*$ products are highly correlated with the self-adjoint properties of the operator. At the same time, ref. [32] recently provided the form of a map that preserves commutation on the set of self-adjoint elements. Using their conclusion, we solved the following three problems, finding three new invariants of $\mathcal{B}(\mathcal{H})$.

First, we chose the zero-$*$ product as invariant and considered an additive unital surjective map Φ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ preserving the zero-$*$ product; that is, for any $A, B \in \mathcal{B}(\mathcal{H})$, we have

$$A^*B = 0 \Leftrightarrow \Phi(A)^*\Phi(B) = 0.$$

Therefore, we can prove that Φ is a unitary or conjugate unitary isomorphism, thereby indicating that the zero-$*$ product can act as an isomorphism invariant of $\mathcal{B}(\mathcal{H})$, keeping the algebraic structure intact.

Second, we chose the zero $*$-Lie product as invariant and considered an additive unital surjective map Φ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ preserving the zero $*$-Lie product; that is, for any $A, B \in \mathcal{B}(\mathcal{H})$, we have

$$A^*B - B^*A = 0 \Leftrightarrow \Phi(A)^*\Phi(B) - \Phi(B)^*\Phi(A) = 0.$$

Accordingly, we can prove that Φ is a unitary or conjugate unitary isomorphism, which indicates that the zero-$*$ Lie-product can act as an isomorphism invariant of $\mathcal{B}(\mathcal{H})$, keeping the algebraic structure intact.

Lastly, we chose the zero $*$-Jordan product as invariant and considered an additive unital surjective map Φ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ preserving the zero $*$-Jordan product; that is, for any $A, B \in \mathcal{B}(\mathcal{H})$, we have

$$A^*B + B^*A = 0 \Leftrightarrow \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A) = 0.$$

Therefore, we can prove that Φ is a unitary or conjugate unitary isomorphism, which indicates that the zero $*$-Jordan product can act as an isomorphism invariant of $\mathcal{B}(\mathcal{H})$, keeping the algebraic structure intact.

The completion of the above three problems shows that we have developed three new invariants which can provide new tools and perspectives for operator algebra classifications. The following sections present our main results.

Theorem 1. *Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces with dimensions greater than two, and let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive unital surjective map. If Φ satisfies*

$$A^*B = 0 \Leftrightarrow \Phi(A)^*\Phi(B) = 0,$$

for any $A, B \in \mathcal{B}(\mathcal{H})$, then there exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{B}(\mathcal{H})$.

Theorem 2. *Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces with dimensions greater than two and let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive unital surjective map. If Φ satisfies*

$$A^*B - B^*A = 0 \Leftrightarrow \Phi(A)^*\Phi(B) - \Phi(B)^*\Phi(A) = 0,$$

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Theorem 3. *Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces with dimensions greater than two and let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive unital surjective map. If Φ satisfies*

$$A^*B + B^*A = 0 \Leftrightarrow \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A) = 0,$$

for any $A, B \in \mathcal{B}(\mathcal{H})$, then there exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $\Phi(A) = UAU^*$ for all $A \in \mathcal{B}(\mathcal{H})$.

2. Preliminaries

In this section, we review several necessary preliminaries.

- $\mathbb{R}$ denotes the real number field.
- $\mathbb{C}$ denotes the complex number field.
- $\mathcal{H}, \mathcal{K}$ denote the complex Hilbert spaces with dimensions greater than two.
- $\mathcal{B}(\mathcal{H})$ denotes all bounded linear operators on $\mathcal{H}$.
- $\mathcal{B}(\mathcal{K})$ denotes all bounded linear operators on $\mathcal{K}$.
- $\mathcal{B}_s(\mathcal{H})$ denotes the set of all self-adjoint operators of $\mathcal{B}(\mathcal{H})$.
- $\mathcal{P}(\mathcal{H})$ denotes the set of projections of $\mathcal{B}(\mathcal{H})$.
- $\mathcal{P}_1(\mathcal{H})$ denotes the set of one-rank projections of $\mathcal{B}(\mathcal{H})$.
- $\mathcal{L}(\mathcal{P}(\mathcal{H}))$ denotes the linear manifold spanned by $\mathcal{P}(\mathcal{H})$.
- $\mathcal{M}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : A^* = -A\}$.
- $\mathcal{M}(\mathcal{K}) = \{A \in \mathcal{B}(\mathcal{K}) : A^* = -A\}$.
- $[x] = \{\lambda x : \lambda \in \mathbb{C}\}$ for any $x \in \mathcal{H}$.
- $[x]^\perp = \{y \in \mathcal{H} : \langle x, y\rangle = 0\}$ for any $x \in \mathcal{H}$.
- $[x, y]$ denotes the subspace generated by x and y.
- $\langle x, y\rangle$ denotes the inner product of x and y.
- $\ker A$ denotes the kernel space of A for any $A \in \mathcal{B}(\mathcal{H})$.

Definition 1 ([35])**.** *If $x, y \in \mathcal{H}$, then the one rank operator $x \otimes y$ is defined as $(x \otimes y)z = \langle z, y\rangle x$, $\forall z \in \mathcal{H}$. If $x \in \mathcal{H}$ is a unit vector, then $x \otimes x$ is one rank projection.*

Definition 2 ([35])**.** *If $A \in \mathcal{B}(\mathcal{H})$, then: (1) A is unitary if $A^* = A^{-1}$; (2) A is conjugate unitary if $A^* = -A^{-1}$.*

Theorem 4 ([36])**.** *Each operator in $\mathcal{B}(\mathcal{H})$ can be written as the sum of five idempotent operators.*

Theorem 5 ([34])**.** *Let $\mathcal{H}, \mathcal{K}$ be two Hilbert spaces of dimensions greater than two; then, $\Phi : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{K})$ preserves the orthogonality of projections bilaterally.*

(1) If $\mathcal{H}, \mathcal{K}$ are real, then there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$, such that $\Phi(P) = UPU^*$ for every projection $P \in \mathcal{P}(\mathcal{H})$.

(2) If $\mathcal{H}, \mathcal{K}$ are complex, then there exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $\Phi(P) = UPU^*$ for every projection of $P \in \mathcal{P}(\mathcal{H})$.

Theorem 6 ([34])**.** *Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces with dimensions greater than two and let $\Phi : \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{K})$ be an additive surjection. If Φ satisfies*

$$AB = BA \Leftrightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)$$

for any $A, B \in \mathcal{B}_s(\mathcal{H})$, then there exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$ with the additive injective $\tau : \mathbb{R} \to \mathbb{R}$ and real additive functional f on $\mathcal{L}(\mathcal{P}(\mathcal{H}))$ such that $\Phi(aP) = \tau(a)UPU^* + f(aP)I$ for all $P \in \mathcal{P}(\mathcal{H})$ and $a \in \mathbb{R}$.

3. Proof of Theorem 1

Proof. The proof of Theorem 1 is completed by the following claims.

Claim 1 Φ is bijective.

We need to prove that $A = 0$ when $\Phi(A) = 0$. As $\Phi(A) = 0$, $\Phi(A)^*\Phi(B) = 0$ for all $B \in \mathcal{B}(\mathcal{H})$, therefore, $A^*B = 0$ holds for all $B \in \mathcal{B}(\mathcal{H})$, that is, $A = 0$.

Claim 2 There exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $\Phi(P) = UPU^*$ for any $P \in \mathcal{P}(\mathcal{H})$.

For any projection P, we have

$$P^*(I - P) = (I - P)^*P = 0.$$

Therefore,

$$\Phi(P)^*\Phi(I - P) = \Phi(I - P)^*\Phi(P) = 0$$

which is equivalent to

$$\Phi(P)^* - \Phi(P)^*\Phi(P) = \Phi(P) - \Phi(P)^*\Phi(P) = 0.$$

Therefore,

$$\Phi(P) = \Phi(P)^*,\ \Phi(P) = \Phi(P)^2.$$

This demonstrates that Φ preserves the projection. As Φ is bijective, the same method can be used for Φ^{-1} to obtain two-sided Φ, which preserves the projection. For any $P, Q \in \mathcal{P}(\mathcal{H})$, if $PQ = 0$, then $\Phi(P)\Phi(Q) = 0$. In contrast, if $\Phi(P)\Phi(Q) = 0$, then there exists $PQ = 0$, indicating that Φ preserves the orthogonality of the projection on both sides. Theorem 5 demonstrates that for any projection $P \in \mathcal{P}(\mathcal{H})$ there exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $\Phi(P) = UPU^*$.

Let

$$\Psi(A) = U^*\Phi(A)U,\ \forall A \in \mathcal{B}(\mathcal{H}).$$

Then, $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ remains an additive bijection and satisfies

$$\forall A, B \in \mathcal{B}(\mathcal{H}),\ A^*B = 0 \Leftrightarrow \Psi(A)^*\Psi(B) = 0.$$

Notably,

$$\Psi(I) = I,\ \Psi(P) = P,\ \forall P \in \mathcal{P}(\mathcal{H}).$$

Therefore, we need to prove that $\Psi(A) = A, \forall A \in \mathcal{B}(\mathcal{H})$. Claims 3 and 4 will complete this proof.

Claim 3 For any one rank operator $x \otimes y$, we have $\Psi(x \otimes y) = x \otimes y$.

Let $x \otimes y \in \mathcal{B}(\mathcal{H})$ be an arbitrary rank operator. For any unit vector $z \in [x]^\perp$, we have

$$(x \otimes y)^*(z \otimes z) = (z \otimes z)^*(x \otimes y) = 0,$$

meaning that

$$\Psi(x \otimes y)^*\Psi(z \otimes z) = \Psi(z \otimes z)^*\Psi(x \otimes y) = 0.$$

From Claim 2, it is evident that

$$\Psi(x \otimes y)^*(z \otimes z) = (z \otimes z)\Psi(x \otimes y) = 0. \tag{1}$$

Therefore, $\Psi(x \otimes y)^*z = 0$, indicating that $\Psi(x \otimes y)^* \mid_{[x]^\perp} = 0$. Thus, there exists a vector $u_{x,y} \in \mathcal{H}$ such that $\Psi(x \otimes y)^* = u_{x,y} \otimes x$, that is, $\Psi(x \otimes y) = x \otimes u_{x,y}$.

We now prove that $u_{x,y} \in [x, y]$. If x and y are linearly dependent, then Equation (1) suggests that

$$0 = \Psi(x \otimes y)^*(z \otimes z)x = (z \otimes z)\Psi(x \otimes y)x, \forall z \in [x]^\perp.$$

This means that $\Psi(x \otimes y)x \in [x]$. Therefore, $u_{x,y} \in [x]$. We assume that $\Psi(x \otimes y) = \lambda_{x,y} x \otimes y$, where $\lambda_{x,y}$ is related to x and y. If x and y are linearly independent, then $x \otimes y = x \otimes y_1 + x \otimes y_2$, where x and y_1 are linearly dependent and $x \perp y_2$. Therefore, we only need to prove the case of $x \perp y$. Because $\dim \mathcal{H} > 2$, we can determine a nonzero unit vector $z \in [x, y]^\perp$. By applying the same method above, we obtain $\Psi(x \otimes y)x \in [x, y]$ and $\Psi(x \otimes y)y \in [x, y]$.

Evidently, $\Psi(x \otimes y)^*([x,y]) \subseteq [x,y]$, that is, $u_{x,y} \in [x,y]$. For any $x_1, x_2 \in \mathcal{H}$, if x_1 and x_2 are linearly independent, then there exists $u_{x_1,y}, u_{x_2,y}$, and $u_{x_1+x_2,y}$ such that

$$\Psi(x_1 \otimes y) = x_1 \otimes u_{x_1,y},$$

$$\Psi(x_2 \otimes y) = x_2 \otimes u_{x_2,y},$$

and

$$\Psi((x_1 + x_2) \otimes y) = (x_1 + x_2) \otimes u_{x_1+x_2,y}.$$

From the additivity of Ψ, we obtain

$$x_1 \otimes (u_{x_1+x_2,y} - u_{x_1,y}) + x_2 \otimes (u_{x_1+x_2,y} - u_{x_2,y}) = 0.$$

Thus, $u_{x_1,y} = u_{x_2,y}$. This means that $u_{x,y}$ does not depend on x; that is, $\Psi(x \otimes y) = x \otimes \lambda_y y$. If x_1 and x_2 are linearly dependent, then there exists $x_3 \in \mathcal{H}$ such that x_3 is linearly independent on both x_1 and x_2. Similarly, λ_y is not dependent on y. The above discussion indicates that $\Psi(x \otimes y) = \lambda x \otimes y$ and $\forall x, y \in \mathcal{H}$. As $\Psi(x \otimes x) = x \otimes x$ holds for all unit vectors x, we have $\Psi(x \otimes y) = x \otimes y, \forall x, y \in \mathcal{H}$.

Claim 4 For any $A \in \mathcal{B}(\mathcal{H})$, it is the case that $\Psi(A) = A$.

If $\dim \mathcal{H} < \infty$, then $\Psi(A) = A, \forall A \in \mathcal{B}(\mathcal{H})$.

If $\dim \mathcal{H} = \infty$, then any $A \in \mathcal{B}(\mathcal{H})$ can be written as the sum of five idempotent operators (Theorem 4). Therefore, we only need to prove that $\Psi(Q) = Q$ holds for every idempotent $Q \in \mathcal{B}(H)$. For any non-trivial idempotent element Q, if Q is a finite rank, then $\Psi(Q) = Q$. Else, there should be $\ker Q^* \neq \{0\}$. Let P be the projection on $\ker Q^*$; then, $Q^*P = 0 = P^*Q$ and $\Psi(P) = P$ indicate that $\Psi(Q)^*P = 0 = P\Psi(Q)$, that is,

$$\Psi(Q)^* = Q^*\Psi(Q)^*Q^* + (I-Q)^*\Psi(Q)^*Q^*.$$

Similarly, for idempotent $I - Q$, we have

$$\Psi(I-Q)^* = (I-Q)^*\Psi(I-Q)^*(I-Q)^* + Q^*\Psi(I-Q)^*(I-Q)^*.$$

Therefore,

$$\begin{aligned} I &= \Psi(I) = \Psi(Q)^* + \Psi(I-Q)^* \\ &= Q^*\Psi(Q)^*Q^* + (I-Q)^*\Psi(Q)^*Q^* \\ &\quad + (I-Q)^*\Psi(I-Q)^*(I-Q)^* + Q^*\Psi(I-Q)^*(I-Q)^* \end{aligned} . \tag{2}$$

Multiplying Equation (2) by $(I-Q)^*$ and Q^* from the left and right, respectively, we obtain $(I-Q)^*\Psi(Q)^*Q^* = 0$. By further multiplying Equation (2) by Q^* from the left and right, respectively, we obtain $Q^*\Psi(Q)^*Q^* = Q^*$. Therefore, $\Psi(Q)^* = Q^*$, that is, $\Psi(Q) = Q$. □

4. Proof of Theorem 2

Proof. The proof of Theorem 2 is completed by the following claims.

Claim 1 A is bijective.

If $\Phi(A) = 0$, then for any $B \in \mathcal{B}(\mathcal{H})$ we have

$$\Phi(A)^*\Phi(B) = \Phi(B)^*\Phi(A) = 0.$$

Therefore, $A^*B = B^*A$ holds for all $B \in \mathcal{B}(\mathcal{H})$. If $B = I$, we can obtain $A = A^*$, while $AB = BA$ holds for all $B \in \mathcal{B}_s(\mathcal{H})$. Therefore, there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I$. If $B \neq B^*$, then $A = 0$.

Claim 2 $\Phi(\mathcal{B}_s(\mathcal{H})) = \mathcal{B}_s(\mathcal{K})$.

For any $A \in \mathcal{B}_s(\mathcal{H})$, there exists $A^*I = I^*A$; therefore, $\Phi(A)^*\Phi(I) = \Phi(I)^*\Phi(A)$. As Φ is unital, we have $\Phi(A)^* = \Phi(A)$, which is $\Phi(A) \in \mathcal{B}_s(\mathcal{K})$. By applying the same method to Φ^{-1}, we obtain Φ, which preserves the self-adjoint nature of both sides.

The following conclusion can be obtained from Theorem 6 and Claims 1 and 2. There exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$, an additive monomorphism $\tau : \mathbb{R} \to \mathbb{R}$, and an additive mapping $f : \mathcal{L}(\mathcal{P}(\mathcal{H})) \to \mathbb{R}$ such that

$$\Phi(aP) = \tau(a)UPU^* + f(aP)I, \forall P \in \mathcal{P}(\mathcal{H}), a \in \mathbb{R}.$$

Let

$$d = \tau(1)^{-1}, \Psi(A) = dU^*\Phi(A)U, \forall A \in \mathcal{B}(\mathcal{H}).$$

Therefore, $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ remains an additive bijection and satisfies

$$A^*B = B^*A \Leftrightarrow \Psi(A)^*\Psi(B) = \Psi(B)^*\Psi(A), \forall A, B \in \mathcal{B}(\mathcal{H}).$$

Furthermore,

$$\Psi(I) = dI, d = \tau(1)^{-1} \in \mathbb{R}. \tag{3}$$

Therefore, it can be verified that

$$\Phi(\mathcal{B}_s(\mathcal{H})) = \mathcal{B}_s(\mathcal{H})$$

and

$$\Psi(P) = P + h(P)I,\ \forall P \in \mathcal{P}(\mathcal{H}),\ h(P) = df(P).$$

We then need to prove $\Psi(A) = A, \forall A \in \mathcal{B}(\mathcal{H})$. Claims 3–5 will complete the proof of $\Psi(A) = A$.

Claim 3 For any $P \in \mathcal{P}_1(\mathcal{H})$, $\Psi(P) = P$.

For any one rank projection $x \otimes x \in \mathcal{P}_1(\mathcal{H})$, we have $\Psi(x \otimes x) = x \otimes x + h(x \otimes x)I$. Suppose there is a unit vector $x \in \mathcal{H}$ such that $h(x \otimes x) \neq 0$. For any operator A, if $A^*x = \lambda x, \lambda \in \mathbb{R}$, then $A^*(x \otimes x) = (x \otimes x)A$; therefore,

$$\Psi(A)^*\Psi(x \otimes x) = \Psi(x \otimes x)^*\Psi(A). \tag{4}$$

Because $\mathcal{H} = [x] \oplus [x]^\perp$, if A is a self-adjoint operator, then $A = \begin{pmatrix} \lambda & 0 \\ 0 & A_{22} \end{pmatrix}$. Let

$$\Psi(A) = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix},\ S_{11}^* = S_{11},\ S_{22}^* = S_{22}.$$

According to Equation (4), we have $S_{12} = 0$, that is,

$$\Psi(A) = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix},\ S_{11}^* = S_{11},\ S_{22}^* = S_{22}. \tag{5}$$

If A is not a self-adjoint operator, we can set $A = \begin{pmatrix} \lambda & 0 \\ A_{21} & A_{22} \end{pmatrix}$. Let $\Psi(A) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$. From Equation (5), we obtain

$$\begin{pmatrix} (1 + h(x \otimes x))T_{11}^* & h(x \otimes x)T_{21}^* \\ (1 + h(x \otimes x))T_{12}^* & h(x \otimes x)T_{22}^* \end{pmatrix} = \begin{pmatrix} (1 + h(x \otimes x))T_{11} & (1 + h(x \otimes x))T_{12} \\ h(x \otimes x)T_{21} & h(x \otimes x)T_{22} \end{pmatrix}.$$

We claim that $h(x \otimes x) \neq -1$. If $h(x \otimes x) = -1$, then

$$\Psi(A) = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}, T_{22}^* = T_{22}.$$

By considering any unit vector $y \in [x]^{\perp}$, we obtain

$$\Psi(y \otimes y) = y \otimes y + h(y \otimes y)I.$$

If $h(y \otimes y) \neq 0$, then we let $B = y \otimes y = \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix}$. There is a non-zero operator A_{21} such that $B_{22}A_{21} = 0$. We let $A = \begin{pmatrix} 1 & 0 \\ A_{21} & 1 \end{pmatrix}$ such that $A^*B = B^*A$; therefore, $\Psi(A)^*\Psi(B) = \Psi(B)^*\Psi(A)$, i.e.,

$$\begin{pmatrix} h(y \otimes y)T_{11}^* & 0 \\ 1 + h(y \otimes y)T_{12}^* & T_{22}S_{22} \end{pmatrix} = \begin{pmatrix} h(y \otimes y)T_{11} & h(y \otimes y)T_{12} \\ 0 & S_{22}T_{22} \end{pmatrix}$$

where $S_{22} = y \otimes y + h(y \otimes y)(I - x \otimes x)$. Therefore, $T_{11} = T_{11}^*, T_{12} = 0$, and $\Psi(A)$ is self-adjoint, indicating that A is self-adjoint as well. This is a clear contradiction; therefore, $h(y \otimes y) = 0$.

Let

$$B = I - y \otimes y = \begin{pmatrix} 1 & 0 \\ 0 & B_{22} \end{pmatrix}, \; B_{22} = I - x \otimes x - y \otimes y.$$

There exists a non-zero operator A_{21} such that $B_{22}A_{21} = 0$; moreover, order $A = \begin{pmatrix} 1 & 0 \\ A_{21} & 1 \end{pmatrix}$. Notably, $A^*B = B^*A$; hence,

$$\Psi(A)^*\Psi(B) = \Psi(B)^*\Psi(A).$$

Therefore,

$$\begin{pmatrix} T_{11}^* & 0 \\ T_{12}^* & T_{22}S_{22} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & S_{22}T_{22} \end{pmatrix}$$

where $S_{22} = I - x \otimes x - y \otimes y$. Evidently, $T_{11} = T_{11}^*, T_{12} = 0$, and $\Psi(A)$ are self-adjoint. Consequently, A is self-adjoint, which is a contradiction. Hence, $h(x \otimes x) \neq -1$. From the above proof, the following conclusion can be drawn. If

$$A = \begin{pmatrix} \lambda & 0 \\ A_{21} & A_{22} \end{pmatrix}, \; \lambda \in \mathbb{R},$$

then

$$\Psi(A) = \begin{pmatrix} T_{11} & T_{12} \\ \frac{1+h}{h}T_{12}^* & T_{22} \end{pmatrix}, \; h = h(x \otimes x), \; T_{11} = T_{11}^*, \; T_{22} = T_{22}^*. \tag{6}$$

From the surjectivity of Ψ, $\Psi(P) = P + h(P)I$, and Equation (6), we can determine an operator

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \; \alpha, \beta \in \mathbb{R}$$

such that

$$\Psi(B) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{h}{1+h} \end{pmatrix}.$$

Notably, $\Psi(A)^*\Psi(B) = \Psi(B)^*\Psi(A)$. Thus, $A^*B = B^*A$, that is,

$$\begin{pmatrix} \alpha\lambda & \beta A_{21}^* \\ 0 & \beta A_{22} \end{pmatrix} = \begin{pmatrix} \alpha\lambda & 0 \\ \beta A_{21} & \beta A_{22} \end{pmatrix}.$$

Therefore, $\beta A_{21} = 0$. If $\beta \neq 0$, then $A_{21} = 0$. This is a clear contradiction; therefore, $h(x \otimes x) = 0$.

If $\beta = 0$, then $B = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$. Additionally, $\forall \gamma \in \mathbb{R}$ has $\Psi(\gamma x \otimes x) = g(\gamma)\Psi(x \otimes x)$, where $g : \mathbb{R} \to \mathbb{R}$ is bijection. In fact,

$$(\gamma x \otimes x)^* B = B^*(\gamma x \otimes x),$$

$$(\gamma x \otimes x)^* A = A^*(\gamma x \otimes x).$$

Therefore, we obtain

$$\Psi(\gamma x \otimes x)\Psi(B) = \Psi(B)^*\Psi(\gamma x \otimes x)$$

and

$$\Psi(\gamma x \otimes x)\Psi(A) = \Psi(A)^*\Psi(\gamma x \otimes x).$$

There exists $g(\gamma) \in \mathbb{R}$, such that

$$\Psi(\gamma x \otimes x) = g(\gamma)\Psi(x \otimes x).$$

However, as $\forall \delta \in \mathbb{R}$, the surjectivity of Ψ shows that there exists $C \in \mathcal{B}(\mathcal{H})$ such that $\Psi(C) = \delta\Psi(x \otimes x)$. Note that

$$\Psi(B)^*\Psi(C) = \Psi(C)^*\Psi(B),$$

$$\Psi(B + (I - x \otimes x))^*\Psi(C) = \Psi(C)^*\Psi(B + (I - x \otimes x))$$

and

$$\Psi(C)^*\Psi(A) = \Psi(A)^*\Psi(C).$$

Consequently, there exists $\gamma \in \mathbb{R}$ such that $C = \gamma x \otimes x$ and $\delta = g(\gamma)$.

If $h(x \otimes x) \neq 1$, then

$$A_3 = \begin{pmatrix} A_{11} & A_{12} \\ aA_{21} & A_{22} \end{pmatrix}, A_{11} = A_{11}^*, A_{22} = A_{22}^*, A_{12} \neq 0, a \in \mathbb{R} \setminus \{0\}.$$

Because

$$(ax \otimes x + (I - x \otimes x))^* A_3 = A_3^*(ax \otimes x + (I - x \otimes x)),$$

we obtain

$$\Psi(ax \otimes x + (I - x \otimes x))^*\Psi(A_3) = \Psi(A_3)^*\Psi(ax \otimes x + (I - x \otimes x)). \tag{7}$$

We know that

$$\Psi(ax \otimes x + (I - x \otimes x)) = \begin{pmatrix} g(a) + (g(a) - 1)h & 0 \\ 0 & (g(a) - 1)h + 1 \end{pmatrix}.$$

As $g : \mathbb{R} \to \mathbb{R}$ is bijective, there exists a nonzero real number a such that $g(a) + (g(a) - 1)h \neq 0$ and $(g(a) - 1)h + 1 \neq 0$. Therefore, from Equation (7) we obtain

$$\Psi(A_3) = \begin{pmatrix} S_{11} & S_{12} \\ bS_{12}^* & S_{22} \end{pmatrix}, S_{11} = S_{11}^*, S_{22} = S_{22}^*, b \in \mathbb{R}.$$

Similarly, there exists a nonzero real number a such that $g(a) = \frac{h-1}{h}$, i.e.,

$$\Psi(ax \otimes x + (I - x \otimes x)) = \begin{pmatrix} g(a) + (g(a) - 1)h & 0 \\ 0 & 0 \end{pmatrix}$$

where $g(a) + (g(a) - 1)h \neq 0$. Evidently, $S_{12} = 0$ and $\Psi(A_3)$ are self-adjoint. However, A_3 is not self-adjoint. Therefore, $h(x \otimes x) = 0$.

If $h(x \otimes x) = 1$, then

$$A_1 = \begin{pmatrix} \lambda & 0 \\ A_{12}^* & A_{22} \end{pmatrix}, A_2 = \begin{pmatrix} \lambda & A_{12} \\ 0 & A_{22} \end{pmatrix}, \lambda \in \mathbb{R}, A_{22} = A_{22}^*.$$

From Equation (5), we obtain

$$\Psi(A_1) = \begin{pmatrix} T_{11} & T_{12} \\ 2T_{12}^* & T_{22} \end{pmatrix}, T_{11} = T_{11}^*, T_{22} = T_{22}^*.$$

Meanwhile,

$$A_2^*(I - x \otimes x) = (I - x \otimes x)^* A_2,$$
$$\Psi(I - x \otimes x) = -x \otimes x$$

and

$$\Psi(A_2)^*(-x \otimes x) = (-x \otimes x)^* \Psi(A_2).$$

Evidently,

$$\Psi(A_2) = \begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}, S_{11} = S_{11}^*, S_{22} = S_{22}^*.$$

As $A_1 + A_2$ is self-adjoint, we know that $S_{21} = -T_{12}^*$ because of the additivity of Ψ. Therefore,

$$\Psi(A_1 + 2A_2)^* \Psi(I - x \otimes x) = \Psi(I - x \otimes x)^* \Psi(A_1 + 2A_2).$$

By contrast,

$$(A_1 + 2A_2)^*(I - x \otimes x) \neq (I - x \otimes x)^*(A_1 + 2A_2).$$

This is contradictory; hence, $h(x \otimes x) = 0$.

Claim 4 For any one rank operator $x \otimes y \in \mathcal{B}(\mathcal{H})$, there exists $\lambda_{x,y} \in i\mathbb{R}$ related to x, y such that $\Psi(x \otimes y) = x \otimes y + \lambda_{x,y} I$.

Let $x \otimes y \in \mathcal{B}(\mathcal{H})$ be an arbitrary rank-one operator. For any unit vector $z \in [x]^\perp$, we obtain

$$(x \otimes y)^*(z \otimes z) = (z \otimes z)^*(x \otimes y).$$

Therefore,

$$\Psi(x \otimes y)^* \Psi(z \otimes z) = \Psi(z \otimes z)^* \Psi(x \otimes y).$$

From Claim 2, it can be observed that

$$\Psi(x \otimes y)^*(z \otimes z) = (z \otimes z)\Psi(x \otimes y). \tag{8}$$

Therefore, there exists $\lambda_z \in \mathbb{C}$, such that $\Psi(x \otimes y)^* z = \lambda_z z$. This implies that there exists constant $\lambda_{x,y}$ such that

$$\Psi(x \otimes y)^* \mid_{[x]^\perp} = \lambda_{x,y} I \mid_{[x]^\perp}.$$

More specifically, there exists vector $u_{x,y} \in \mathcal{H}$ such that

$$\Psi(x \otimes y)^* = u_{x,y} \otimes x + \lambda_{x,y} I.$$

From the above results and Equation (3), it can be concluded that $\lambda_{x,y} z \otimes z = \overline{\lambda_{x,y}} z \otimes z$. Therefore, $\lambda_{x,y} \in \mathbb{R}$, that is,

$$\Psi(x \otimes y) = x \otimes u_{x,y} + \lambda_{x,y} I, \lambda_{x,y} \in \mathbb{R}.$$

Subsequently, we prove that $u_{x,y} \in [x, y]$. If x and y are linearly dependent, then, based on Equation (3), for any unit vector $z \in [x]^\perp$ there is

$$0 = \Psi(x \otimes y)^*(z \otimes z)x = (z \otimes z)\Psi(x \otimes y)x,$$

that is, $\Psi(x \otimes y)x \in [x]$. If x and y are linearly independent, then $x \otimes y = x \otimes y_1 + x \otimes y_2$, where x and y_1 are linearly dependent and $x \perp y_2$. Therefore, we need to prove the case of $x \perp y$. Because $\dim \mathcal{H} > 2$, we can determine a nonzero unit vector $z \in [x, y]^{\perp}$. By applying the above method, we obtain $\Psi(x \otimes y)x \in [x, y]$ and $\Psi(x \otimes y)y \in [x, y]$. It is evident that

$$\Psi(x \otimes y)^*([x, y]) \subseteq [x, y], u_{x,y} \in [x, y].$$

Here, $\forall x_1, x_2 \in \mathcal{H}$, and if x_1 and x_2 are linearly independent, then there exists $u_{x_1,y}, u_{x_2,y}, u_{x_1+x_2,y}$ such that

$$\Psi(x_1 \otimes y) = x_1 \otimes u_{x_1,y} + \lambda_{x_1,y} I,$$

$$\Psi(x_2 \otimes y) = x_2 \otimes u_{x_2,y} + \lambda_{x_2,y} I$$

and

$$\Psi((x_1 + x_2) \otimes y) = (x_1 + x_2) \otimes u_{x_1+x_2,y} + \lambda_{x_1+x_2,y} I.$$

From the additivity of Ψ, we obtain

$$x_1 \otimes (u_{x_1+x_2,y} - u_{x_1,y}) + x_2 \otimes (u_{x_1+x_2,y} - u_{x_2,y}) = (\lambda_{x_1+x_2,y} - \lambda_{x_1,y} - \lambda_{x_2,y}) I.$$

As $\dim \mathcal{H} > 2$, $u_{x_1,y} = u_{x_2,y}$, we know that $u_{x,y}$ does not depend on x, that is,

$$\Psi(x \otimes y) = x \otimes \lambda_y y + \lambda_{x,y} I.$$

If y_1 and y_2 are linearly dependent, then there exists $y_3 \in \mathcal{H}$ such that y_3 is linearly independent on both y_1 and y_2. Similarly, λ_y does not depend on y; therefore,

$$\Psi(x \otimes y) = \lambda x \otimes y + \lambda_{x,y} I.$$

As $\Psi(x \otimes x) = x \otimes x$ holds for all unit vectors x, we have $\lambda = 1$. For any $x, y \in \mathcal{H}$, there exists $\lambda_{x,y} \in \mathbb{R}$ such that $\Psi(x \otimes y) = x \otimes y + \lambda_{x,y} I$.

Claim 5 For any $A \in \mathcal{B}(\mathcal{H})$, $\Psi(A) = A$.

When $\dim \mathcal{H} < \infty$, there is $\Psi(A) = A + \lambda_A I$ for any $A \in \mathcal{B}(\mathcal{H})$, where $\lambda_A \in \mathbb{R}$ depends on A. If $\ker A^* \neq \{0\}$, then for any non-zero vector $x \in \ker A^*$ there exists a non-zero vector $y \in \mathcal{H}$ such that $x \perp y$. Therefore, we obtain

$$A^*(x \otimes y) = (x \otimes y)^* A = 0.$$

From Claim 4, it is evident that

$$(A + \lambda_A I)^*(x \otimes y + \lambda_{x,y} I) = (x \otimes y + \lambda_{x,y} I)^*(A + \lambda_A I).$$

The above equation can be reduced to

$$\lambda_{x,y}(A - A^*) = \lambda_A x \otimes y - \lambda_A y \otimes x.$$

If $\lambda_{x,y} = 0$, then $\lambda_A = 0$. This is equivalent to $\Psi(A) = A$.

If $\lambda_{x,y} \neq 0$, then

$$A - A^* = \frac{\lambda_A}{\lambda_{x,y}} x \otimes y - \frac{\lambda_A}{\lambda_{x,y}} y \otimes x.$$

Because $\dim \mathcal{H} \geq 3$, there exists a nonzero vector $z \in \mathcal{H}$ such that $z \perp x$ and $z \perp y$. Similarly, if $\lambda_{x,z} = 0$, then $\lambda_A = 0$. If $\lambda_{x,z} \neq 0$, then

$$A - A^* = \frac{\lambda_A}{\lambda_{x,z}} x \otimes z - \frac{\lambda_A}{\lambda_{x,z}} z \otimes x.$$

Therefore, we have

$$x \otimes (\frac{\lambda_A}{\lambda_{x,y}}y - \frac{\lambda_A}{\lambda_{x,z}}z) = (\frac{\lambda_A}{\lambda_{x,y}}y - \frac{\lambda_A}{\lambda_{x,z}}z) \otimes x.$$

Thus, $\lambda_A = 0$.

For any unit vector $x \in \mathcal{H}$, let $B = A - (x \otimes x)A$. If $\ker A^* = \{0\}$, then $\ker B \neq \{0\}$. Therefore,

$$\Psi(B) = B, \Psi((x \otimes x)A) = (x \otimes x)A,$$

that is,

$$\Psi(A) = \Phi(B) - \Psi((x \otimes x)A) = A.$$

If $\dim \mathcal{H} = \infty$, then every $A \in \mathcal{B}(\mathcal{H})$ can be written as the sum of five idempotents (Theorem 4). Accordingly, we only need to prove $\Psi(Q) = Q$ for every idempotent $Q \in \mathcal{B}(H)$.

First, it is known that $\Psi(P) = P$ holds for all $P \in \mathcal{P}(\mathcal{H})$. For any non-zero projection P, there exists a non-zero vector x such that $Px = 0$. Thus, there exists a non-zero vector y such that $x \perp y$. This indicates that $P^*(x \otimes y) = (x \otimes y)^*P$. Therefore, Claim 3 suggests that

$$(P + h(P)I)(x \otimes y) = (x \otimes y)^*(P + h(P)I).$$

The above equation can be simplified to $h(P)x \otimes y = h(P)y \otimes x$, which indicates that $h(P) = 0$. Note that for all $P \in \mathcal{P}(\mathcal{H})$, there exists

$$\Psi(P) = dU^*\Phi(P)U = P.$$

Therefore,

$$I = \Phi(I) = \Phi(P) + \Phi(I - P) = d^{-1}I.$$

that is, $d = 1$. Hence, $\Psi(I) = I$ can be obtained using Equation (3).

Second, we prove that $\Psi(Q) = Q$ holds for all idempotents $Q \in \mathcal{B}(\mathcal{H})$. For any idempotent Q, if Q is a finite rank operator, there exists $\Psi(Q) = Q$, else there should be $\ker Q^* \neq \{0\}$. Let P be a projection on $\ker Q^*$. Evidently, $Q^*P = P^*Q$ and $\Psi(P) = P$; therefore, $\Psi(Q)^*P = P\Psi(Q)$, that is,

$$\Psi(Q)^* = Q^*\Psi(Q)^*Q^* + Q^*\Psi(Q)^*(I - Q)^* + (I - Q)^*\Psi(Q)^*(I - Q)^*.$$

Similarly, for idempotent I-Q we obtain

$$\Psi(I - Q)^* = (I - Q)^*\Psi(I - Q)^*(I - Q)^* + (I - Q)^*\Psi(I - Q)^*Q^* + Q^*\Psi(I - Q)^*Q^*.$$

Therefore,

$$\begin{aligned} I &= \Psi(I) = \Psi(Q)^* + \Psi(I - Q)^* \\ &= Q^*\Psi(Q)^*Q^* + Q^*\Psi(Q)^*(I - Q)^* + (I - Q)^*\Psi(Q)^*(I - Q)^* \\ &\quad + (I - Q)^*\Psi(I - Q)^*(I - Q)^* + (I - Q)^*\Psi(I - Q)^*Q^* \\ &\quad + Q^*\Psi(I - Q)^*Q^* \end{aligned} . \tag{9}$$

By multiplying the equality in Equation (9) by Q^* and $(I - Q)^*$ from the left and right, respectively, we obtain $Q^*\Psi(Q)^*(I - Q)^* = 0$. Thus,

$$\Psi(Q)^* = Q^*\Psi(Q)^*Q^* + (I - Q)^*\Psi(Q)^*(I - Q)^*.$$

For any unit vector $x \in \ker Q^*$, $Q^*(x \otimes x) = (x \otimes x)^*Q$. Therefore, as seen from Claim 2,

$$\Psi(Q)^*(x \otimes x) = (x \otimes x)\Psi(Q).$$

This result implies that $\Psi(Q)^*x = \mu_x x$ for all unit vectors $x \in \ker Q^*$, where $\mu_x \in \mathbb{C}$. Accordingly,

$$\Psi(Q)^*|_{\ker Q^*} = \mu_Q I|_{\ker Q^*},$$

that is,

$$\Psi(Q)^* = Q^*\Psi(Q)^*Q^* + \mu_Q(I-Q)^*.$$

Similarly, for idempotent $I-Q$, there exists $\gamma_Q \in \mathbb{C}$ such that

$$\Psi(I-Q)^* = (I-Q)^*\Psi(I-Q)^*(I-Q)^* + \mu_{I-Q}Q^*.$$

Again, using Equation (9), we obtain

$$I = Q^*\Psi(Q)^*Q^* + \mu_Q(I-Q)^* + (I-Q)^*\Psi(I-Q)^*(I-Q)^* + \mu_{I-Q}Q^*. \tag{10}$$

Using Q^* to multiply Equation (10) from the left and right, respectively, we obtain

$$\Psi(Q)^* = (1-\mu_{I-Q}-\mu_Q)Q^* + \mu_Q I.$$

Therefore, we obtain

$$\Psi(Q) = \alpha_Q Q + \beta_Q I, \alpha_Q = 1 - \overline{\mu_{I-Q}} - \overline{\mu_Q}, \beta_Q = \overline{\mu_Q}.$$

If Q is a finite rank operator, then $\Psi(Q) = Q$. If Q is not a finite rank operator, then let Q_1 be a non-zero finite rank idempotent such that $QQ_1 = Q_1Q = Q_1$. Thus, $Q = Q_1 + Q_2$ and $Q_1Q_2 = Q_2Q_1 = 0$, where $Q_2 = Q - Q_1$ is an idempotent. Therefore,

$$\Psi(Q) = \Psi(Q_1+Q_2) = \alpha_Q Q + \beta_Q I = Q_1 + \beta_{Q_1} I + \alpha_{Q_2}Q_2 + \beta_{Q_2} I,$$

that is,

$$(\alpha_Q - 1)Q_1 + (\alpha_Q - \alpha_{Q_1})Q_2 = (\beta_{Q_1} + \beta_{Q_2} - \beta_Q)I.$$

This implies that $\alpha_Q = 1 = \alpha_{Q_2}$, that is, $\Psi(Q) = Q + \beta_Q I$. Considering any two non-zero vectors $x \in \ker Q^*$ and $y \in \mathcal{H}$ such that $x \perp y$, owing to

$$0 = Q^*(x \otimes y) = (x \otimes y)^*Q,$$

we obtain

$$(Q + \beta_Q I)^*(x \otimes y) = (x \otimes y)^*(Q + \beta_Q I),$$

that is, $\overline{\beta_Q} x \otimes y = \beta_Q y \otimes x$; thus, $\beta_Q = 0$. □

5. Proof of Theorem 3

Proof. The proof of Theorem 3 is completed by the following claims.

Claim 1 Φ is bijective.

If $\Phi(A) = 0$, then for any $B \in \mathcal{B}(\mathcal{H})$ there is

$$\Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A) = 0.$$

Therefore, $A^*B + B^*A = 0$ holds for all $B \in \mathcal{B}(\mathcal{H})$. If $B = I$, then we can obtain $A = -A^*$. This implies that $AB - BA = 0$ holds for all $B \in \mathcal{B}_s(\mathcal{H})$. Consequently, $A = \lambda i I, \lambda \in \mathbb{R}$. However, if $B \neq -B^*$, then $A = 0$.

Claim 2 $\Phi(\mathcal{M}(\mathcal{H})) = \mathcal{M}(\mathcal{K})$.

For any $A \in \mathcal{M}(\mathcal{H})$, we have $A^*I + I^*A = 0$. Therefore, $\Phi(A)^*\Phi(I) + \Phi(I)^*\Phi(A) = 0$. As Φ is unital, $\Phi(A) \in \mathcal{M}(\mathcal{K})$. Applying the same method to Φ^{-1}, we find that Φ preserves anti-self-adjoint elements on both sides.

Claim 3 $\Phi(\mathbb{R}I) = \mathbb{R}I$.

For any $B \in \mathcal{M}(\mathcal{H})$ and $\lambda \in \mathbb{R}$, we obtain $(\lambda I)^* B + B^*(\lambda I) = 0$. Therefore,

$$\Phi(\lambda I)^* \Phi(B) + \Phi(B)^* \Phi(\lambda I) = 0.$$

Accordingly, there exists $B \in \mathcal{B}(\mathcal{H})$ such that $\Phi(B) = iI$. Consequently, $\Phi(\lambda I)^* = \Phi(\lambda I)$. Evidently, $\Phi(\lambda I)\Phi(B) = \Phi(B)\Phi(\lambda I)$ holds for all $B \in \mathcal{M}(\mathcal{H})$; thus, $\Phi(\lambda I) \in \mathbb{R}I$. As indicated from Claim 1, there exists $A \in \mathcal{B}(\mathcal{H})$ for any $\lambda \in \mathbb{R}$ such that

$$\Phi(A) = \lambda I, \Phi(A)^* \Phi(iI) = \Phi(iI)\Phi(A).$$

Therefore, $A^*(iI) = (iI)A$, that is, $A^* = A$. For any $M \in \mathcal{M}(\mathcal{K})$, there exists $B \in \mathcal{M}(\mathcal{H})$ such that $\Phi(B) = M$ and $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)$. Thus, from Claim 1, we know that $AB = BA$, that is, $A \in \mathbb{R}I$.

Claim 4 $\Phi(iI) = \pm iI$.

Because

$$[(1+i)I]^*(1-i)I + [(1-i)I]^*(1+i)I = 0,$$

we have

$$\Phi((1+i)I)^* \Phi((1-i)I) + \Phi((1-i)I)^* \Phi((1+i)I) = 0.$$

Therefore, $\Phi(iI)^2 = -I$. If $\Phi(iI) \neq \pm iI$, then $\Phi(iI) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. For any self-adjoint operator A, we have $A^*(iI) + (iI)^* A = 0$; hence,

$$\Phi(A)^* \Phi(iI) + \Phi(iI)^* \Phi(A) = 0.$$

Letting

$$\Phi(A) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

we have

$$\Phi(A) = \begin{pmatrix} 0 & T_{12} \\ -T_{12}^* & 0 \end{pmatrix} \in \mathcal{M}(\mathcal{K}).$$

As this contradicts Claim 2, Claim 4 is proved.

Claim 5 There exists a unitary or conjugate unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $\Phi(P) = UPU^*, \forall P \in \mathcal{P}(\mathcal{H})$.

First, it can be easily proven that Φ preserves self-adjoint elements on both sides. For any self-adjoint element A,

$$(iI)^* A + A^*(iI) = 0,$$

therefore,

$$\Phi(iI)^* \Phi(A) + \Phi(A)^* \Phi(iI) = 0,$$

that is, $\Phi(A)^* = \Phi(A)$. Applying the same method to Φ^{-1}, we obtain Φ, preserving the self-adjoint nature on both sides.

Second, we prove that Φ preserves projections on both sides. For any projection $P \in \mathcal{P}(\mathcal{H})$,

$$P^*(I-P) + (I-P)^* P = 0.$$

Therefore,

$$\Phi(P)^* \Phi(I-P) + \Phi(I-P)^* \Phi(P) = 0.$$

Because Φ preserves self-adjoint elements bilaterally, we obtain

$$\Phi(P)^* = \Phi(P), \Phi(P)^2 = \Phi(P),$$

that is, Φ preserves the projection. The same method is applied to Φ^{-1} to obtain Φ bilateral preserving projection.

Because Φ is an additive map and preserves the projection bilaterally, it is easy to verify that Φ preserves orthogonality on both sides. In fact, for all $P, Q \in \mathcal{P}(\mathcal{H})$, if $PQ = QP = 0$, then $P + Q$ remains a projection. Therefore, $\Phi(P) + \Phi(Q)$ is a projection. Thus,

$$\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0,$$

that is, Φ preserves the orthogonality of the projection. Applying the same method to Φ^{-1}, we find that Φ preserves orthogonality on both sides. Therefore, Claim 5 is proved by Theorem 5.

Let

$$\Psi(A) = U^*\Phi(A)U, \forall A \in \mathcal{B}(\mathcal{H}). \tag{11}$$

Then, it is easy to verify that $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ remains an additive bijection and satisfies the requirement that for any $A, B \in \mathcal{B}(\mathcal{H})$, there is

$$A^*B + B^*A = 0 \Leftrightarrow \Psi(A)^*\Psi(B) + \Psi(B)^*\Psi(A) = 0.$$

Additionally, we can verify that $\Psi(I) = I$, $\Psi(P) = P$ holds for all $P \in \mathcal{P}(\mathcal{H})$. Consequently, we only need to prove that $\Psi(A) = A$ holds for all $A \in \mathcal{B}(\mathcal{H})$. Claims 6 and 7 complete this proof.

Claim 6 For any one rank operator $x \otimes y \in \mathcal{B}(\mathcal{H})$, there exists $\lambda_{x,y} \in i\mathbb{R}$ that is related to x, y such that $\Psi(x \otimes y) = x \otimes y + \lambda_{x,y} I$.

Let $x \otimes y \in \mathcal{B}(\mathcal{H})$ be an arbitrary one rank operator. For any unit vector $z \in [x]^\perp$, we have

$$(x \otimes y)^*(z \otimes z) + (z \otimes z)^*(x \otimes y) = 0.$$

Thus,

$$\Psi(x \otimes y)^*\Psi(z \otimes z) + \Psi(z \otimes z)^*\Psi(x \otimes y) = 0.$$

Evidently,

$$\Psi(x \otimes y)^*(z \otimes z) + (z \otimes z)\Psi(x \otimes y) = 0. \tag{12}$$

Therefore, there exists $\lambda_z \in \mathbb{C}$ such that $\Psi(x \otimes y)^* z = \lambda_z z$. This indicates that there exists constant $\lambda_{x,y}$ such that

$$\Psi(x \otimes y)^* \mid_{[x]^\perp} = \lambda_{x,y} I \mid_{[x]^\perp} .$$

More specifically, there exists a vector $u_{x,y} \in \mathcal{H}$ such that

$$\Psi(x \otimes y)^* = u_{x,y} \otimes x + \lambda_{x,y} I.$$

From the above result and Equation (12), we can observe that $\lambda_{x,y} z \otimes z = \overline{\lambda_{x,y}} z \otimes z$. Evidently, $\lambda_{x,y} \in \mathbb{R}$, that is,

$$\Psi(x \otimes y) = x \otimes u_{x,y} + \lambda_{x,y} I, \lambda_{x,y} \in \mathbb{R}.$$

Next, we prove that $u_{x,y} \in [x, y]$. If x and y are linearly dependent, then, as seen from Equation (12), for any $z \in [x]^\perp$, we have

$$0 = \Psi(x \otimes y)^*(z \otimes z)x = (z \otimes z)\Psi(x \otimes y)x,$$

that is, $\Psi(x \otimes y)x \in [x]$. If x and y are linearly independent, then $x \otimes y = x \otimes y_1 + x \otimes y_2$, where x and y_1 are linearly dependent and $x \perp y_2$. Therefore, we only need to prove the case of $x \perp y$. Owing to $\dim \mathcal{H} > 2$, we can determine a non-zero unit vector $z \in [x, y]^\perp$. Applying the same method as above, we obtain

$$\Psi(x \otimes y)x \in [x, y], \Psi(x \otimes y)y \in [x, y].$$

Thus,

$$\Psi(x \otimes y)^*([x, y]) \subseteq [x, y], u_{x,y} \in [x, y].$$

For any $x_1, x_2 \in \mathcal{H}$, if x_1 and x_2 are linearly independent, then there exists $u_{x_1,y}, u_{x_2,y}$, $u_{x_1+x_2,y}$ such that

$$\Psi(x_1 \otimes y) = x_1 \otimes u_{x_1,y} + \lambda_{x_1,y} I,$$
$$\Psi(x_2 \otimes y) = x_2 \otimes u_{x_2,y} + \lambda_{x_2,y} I$$

and

$$\Psi((x_1 + x_2) \otimes y) = (x_1 + x_2) \otimes u_{x_1+x_2,y} + \lambda_{x_1+x_2,y} I.$$

From the additivity of Ψ, we know that

$$x_1 \otimes (u_{x_1+x_2,y} - u_{x_1,y}) + x_2 \otimes (u_{x_1+x_2,y} - u_{x_2,y}) = (\lambda_{x_1+x_2,y} - \lambda_{x_1,y} - \lambda_{x_2,y}) I.$$

As $\dim \mathcal{H} > 2$, $u_{x_1,y} = u_{x_2,y}$. Therefore, $u_{x,y}$ does not depend on x, that is,

$$\Psi(x \otimes y) = x \otimes \lambda_y y + \lambda_{x,y} I.$$

If y_1 and y_2 are linearly dependent, then there exists $y_3 \in \mathcal{H}$, such that y_3 is linearly independent on y_1 and y_2, respectively. Similarly, for any $x_1, x_2 \in \mathcal{H}$ there exists λ_y, which does not depend on y, such that

$$\Psi(x \otimes y) = \lambda x \otimes y + \lambda_{x,y} I.$$

Note that $\Psi(x \otimes x) = x \otimes x$ holds for all unit vectors x. Therefore, there exists $\lambda_{x,y} \in \mathbb{R}$ for any $x, y \in \mathcal{H}$ such that

$$\Psi(x \otimes y) = x \otimes y + \lambda_{x,y} I.$$

Claim 7 For any $A \in \mathcal{B}(\mathcal{H})$, we have $\Psi(A) = A$.

If $\dim \mathcal{H} < \infty$, then $\Psi(A) = A + \lambda_A I$ holds for all $A \in \mathcal{B}(\mathcal{H})$, where $\lambda_A \in \mathbb{R}$ depends on A. If $\ker A^* \neq \{0\}$, then for any non-zero vector $x \in \ker A^*$ there exists a nonzero vector $y \in \mathcal{H}$ that satisfies $x \perp y$. We have

$$A^*(x \otimes y) + (x \otimes y)^* A = 0.$$

Evidently from Claim 4,

$$(A + \lambda_A I)^*(x \otimes y + \lambda_{x,y} I) + (x \otimes y + \lambda_{x,y} I)^*(A + \lambda_A I) = 0.$$

The above equality can be reduced to

$$\lambda_{x,y}(A - A^*) = \lambda_A x \otimes y - \lambda_A y \otimes x.$$

If $\lambda_{x,y} = 0$, then $\lambda_A = 0$; thus, $\Psi(A) = A$.

If $\lambda_{x,y} \neq 0$, then

$$A - A^* = \frac{\lambda_A}{\lambda_{x,y}} x \otimes y - \frac{\lambda_A}{\lambda_{x,y}} y \otimes x.$$

As $\dim \mathcal{H} \geq 3$, then there exists a non-zero vector $z \in \mathcal{H}$, such that $z \perp x$ and $z \perp y$. Similarly, if $\lambda_{x,z} = 0$, then $\lambda_A = 0$. If $\lambda_{x,z} \neq 0$, then

$$A - A^* = \frac{\lambda_A}{\lambda_{x,z}} x \otimes z - \frac{\lambda_A}{\lambda_{x,z}} z \otimes x.$$

We know that

$$x \otimes (\frac{\lambda_A}{\lambda_{x,y}} y - \frac{\lambda_A}{\lambda_{x,z}} z) = (\frac{\lambda_A}{\lambda_{x,y}} y - \frac{\lambda_A}{\lambda_{x,z}} z) \otimes x.$$

Thus, $\lambda_A = 0$.

If $\ker A^* = \{0\}$, there exists a unit vector $x \in \mathcal{H}$ such that $\ker B \neq \{0\}$, where $B = A - (x \otimes x)A$. Thus,

$$\Psi(B) = B, \Psi((x \otimes x)A) = (x \otimes x)A,$$

that is,

$$\Psi(A) = \Psi(B) - \Psi((x \otimes x)A) = A.$$

If $\dim \mathcal{H} = \infty$, then each $A \in \mathcal{B}(\mathcal{H})$ can be written as the sum of five idempotents (Theorem 4). Therefore, we only need to prove that $\Psi(Q) = Q$ for all idempotents $Q \in \mathcal{B}(\mathcal{H})$.

First, it is known that $\Psi(P) = P$ holds for all $P \in \mathcal{P}(\mathcal{H})$. For any non-zero projection P, there exists a nonzero vector x such that $Px = 0$, and considering the non-zero vector y, such that $x \perp y$. It is easy to verify that $P^*(x \otimes y) + (x \otimes y)^* P = 0$; therefore, from Claim 3, it is known that

$$(P + h(P)I)(x \otimes y) + (x \otimes y)^*(P + h(P)I) = 0.$$

The above equality can be simplified to $h(P)x \otimes y = h(P)y \otimes x$, indicating that $h(P) = 0$. Note that for any $P \in \mathcal{P}(\mathcal{H})$ we have

$$\Psi(P) = dU^*\Phi(P)U = P.$$

Therefore,

$$I = \Phi(I) = \Phi(P) + \Phi(I - P) = d^{-1}I,$$

that is, $d = 1$. Hence, $\Psi(I) = I$ can be obtained using Equation (11).

Second, we prove that $\Psi(Q) = Q$ holds for all idempotents $Q \in \mathcal{B}(\mathcal{H})$. For any idempotent Q, if Q is a finite rank operator then there exits $\Psi(Q) = Q$, else there should be $\ker Q^* \neq \{0\}$. Let P be a projection on $\ker Q^*$. Notably, $Q^*P + P^*Q = 0$ and $\Psi(P) = P$. Therefore, $\Psi(Q)^*P + P\Psi(Q) = 0$, that is,

$$\Psi(Q)^* = Q^*\Psi(Q)^*Q^* + Q^*\Psi(Q)^*(I - Q)^* + (I - Q)^*\Psi(Q)^*(I - Q)^*.$$

Similarly, for idempotent I-Q we can obtain

$$\Psi(I - Q)^* = (I - Q)^*\Psi(I - Q)^*(I - Q)^* + (I - Q)^*\Psi(I - Q)^*Q^* + Q^*\Psi(I - Q)^*Q^*.$$

Therefore,

$$\begin{aligned} I &= \Psi(I) = \Psi(Q)^* + \Psi(I - Q)^* \\ &= Q^*\Psi(Q)^*Q^* + Q^*\Psi(Q)^*(I - Q)^* + (I - Q)^*\Psi(Q)^*(I - Q)^* \\ &+ (I - Q)^*\Psi(I - Q)^*(I - Q)^* + (I - Q)^*\Psi(I - Q)^*Q^* \\ &+ Q^*\Psi(I - Q)^*Q^* \end{aligned}. \quad (13)$$

By multiplying the equality in Equation (13) by Q^* and $(I - Q)^*$ from the left and right, respectively, we obtain $Q^*\Psi(Q)^*(I - Q)^* = 0$. Thus, there exits

$$\Psi(Q)^* = Q^*\Psi(Q)^*Q^* + (I - Q)^*\Psi(Q)^*(I - Q)^*.$$

For any unit vector $x \in \ker Q^*$, we have $Q^*(x \otimes x) + (x \otimes x)^*Q = 0$. Evidently, from Claim 2,

$$\Psi(Q)^*(x \otimes x) + (x \otimes x)\Psi(Q) = 0.$$

This indicates that there exists $\mu_x \in \mathbb{C}$ such that

$$\Psi(Q)^*x = \mu_x x, \forall x \in \ker Q^*.$$

Then,

$$\Psi(Q)^*|_{\ker Q^*} = \mu_Q I|_{\ker Q^*}.$$

Thus,

$$\Psi(Q)^* = Q^*\Psi(Q)^*Q^* + \mu_Q(I-Q)^*.$$

Similarly, for idempotent $I-Q$ there exists $\gamma_Q \in \mathbb{C}$ such that

$$\Psi(I-Q)^* = (I-Q)^*\Psi(I-Q)^*(I-Q)^* + \mu_{I-Q}Q^*.$$

It can again be inferred from Equation (13) that

$$I = Q^*\Psi(Q)^*Q^* + \mu_Q(I-Q)^* + (I-Q)^*\Psi(I-Q)^*(I-Q)^* + \mu_{I-Q}Q^*. \tag{14}$$

By multiplying the equality in Equation (14) by Q^* from the left and right, respectively, we obtain

$$\Psi(Q)^* = (1-\mu_{I-Q}-\mu_Q)Q^* + \mu_Q I.$$

Thus, there is

$$\Psi(Q) = \alpha_Q Q + \beta_Q I, \alpha_Q = 1-\overline{\mu_{I-Q}}-\overline{\mu_Q}, \beta_Q = \overline{\mu_Q}.$$

If Q is a finite rank operator, then $\Psi(Q) = Q$. If Q is not a finite rank operator, then let Q_1 be a non-zero finite rank idempotent such that $QQ_1 = Q_1Q = Q_1$. Then, we have $Q = Q_1 + Q_2$ and $Q_1Q_2 = Q_2Q_1 = 0$, where $Q_2 = Q - Q_1$ is an idempotent. Therefore,

$$\Psi(Q) = \Psi(Q_1+Q_2) = \alpha_Q Q + \beta_Q I = Q_1 + \beta_{Q_1} I + \alpha_{Q_2} Q_2 + \beta_{Q_2} I,$$

that is,

$$(\alpha_Q - 1)Q_1 + (\alpha_Q - \alpha_{Q_1})Q_2 = (\beta_{Q_1} + \beta_{Q_2} - \beta_Q)I.$$

This shows that $\alpha_Q = 1 = \alpha_{Q_2}$, that is, $\Psi(Q) = Q + \beta_Q I$. Considering any two non-zero vectors $x \in \ker Q^*$ and $y \in \mathcal{H}$ such that $x \perp y$, we obtain

$$0 = Q^*(x \otimes y) = (x \otimes y)^* Q.$$

Thus,

$$(Q + \beta_Q I)^*(x \otimes y) = (x \otimes y)^*(Q + \beta_Q I),$$

essentially meaning that $\overline{\beta_Q} x \otimes y = \beta_Q y \otimes x$; thus, $\beta_Q = 0$. □

6. Conclusions

In this study, the isomorphism invariant zero-$*$ products on $\mathcal{B}(\mathcal{H})$ were found using the linear preservation method. We have demonstrated that: (1) if Φ satisfies $A^*B = 0 \Leftrightarrow \Phi(A)^*\Phi(B) = 0$ for any $A, B \in \mathcal{B}(\mathcal{H})$, then Φ is a unitary or conjugate unitary isomorphism; (2) if Φ satisfies $A^*B - B^*A = 0 \Leftrightarrow \Phi(A)^*\Phi(B) - \Phi(B)^*\Phi(A) = 0$ for any $A, B \in \mathcal{B}(\mathcal{H})$, then Φ is a unitary or conjugate unitary isomorphism; and (3) if Φ satisfies $A^*B + B^*A = 0 \Leftrightarrow \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A) = 0$ for any $A, B \in \mathcal{B}(\mathcal{H})$, then Φ is a unitary or conjugate unitary isomorphism. This indicates that the zero-$*$, $*$-Lie, and $*$-Jordan products can be used as isomorphism invariants on $\mathcal{B}(\mathcal{H})$, retaining the basic structure and properties of the algebra, which has crucial implications for the study of algebraic classification. Furthermore, the elements on $\mathcal{B}(\mathcal{H})$ with zero-$*$ products completely determine the basic structure and properties of this algebra. This finding has important implications for the study of algebraic categorization. Lastly, to retain the structure and properties of two algebras, one need only consider a small fraction of elements, namely, those elements with zero-$*$ products, which greatly reduces the workload. Additionally, this approach can be applied to the study of other subjects, such as quantum information. Therefore, the results of this paper have considerable research value and significance.

In terms of research methods, the traditional research idea of the preserving problem was originally adopted. The detailed technique of the proof has its own originality; its core is to use $A^*B = 0$, $A^*B - B^*A = 0$ and $A^*B + B^*A = 0$ to construct special operators. A key step in the proof is using the two conclusions from reference [34].

Another most important point is that in the process of proving Theorem 3 we have used the relationship $A^*B = B^*A = 0$ to construct operators in several places. This relation is a partial special operator satisfying $A^*B + B^*A = 0$. Therefore, we guess that if we can find a new proof method that takes full advantage of the properties of the zero-$*$ Jordan products, then the conditions of the mapping in Theorem 3 may be weakened, thereby narrowing the range of invariants and yielding better results. Although we predict that this will be a very difficult task, we intend to continue exploring it.

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