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Probability Spaces Identifying Ordinal and Cardinal Utilities in Problems of an Economic Nature: New Issues and Perspectives

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Abstract: Prevision bundles identifying expected returns on risky assets are established. A probability space associated with risky assets is defined. In this research work, the optimization principle is based on the notion of distance. This is because problems of an economic nature are not handled in an axiomatic or intrinsic way, but they are investigated with regard to a given coordinate system. The latter is shown to be invariant. The notion of mathematical expectation applied to summarizing both monetary values and utilities is treated. Such a notion is extended to study portfolios of financial assets. Objective conditions of coherence connected with the notion of mathematical expectation are extended. Rational behaviors towards risk are based on them. A model representing diagrams considered inside the same coordinate system is shown. Such a model identifies as many optimal choices as pair comparisons it is possible to take into account in order to obtain a multilinear measure. The latter is the expected return on a specific portfolio of financial assets.

Keywords: moral expectation; optimization principle; behavior towards risk; Fréchet class; prevision bundle; multilinear index

MSC: 60A05; 60B05; 91B24; 91B16; 91B06; 91B08

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1. Introduction

This paper focuses on a precise point of view that goes back to Daniel Bernoulli. It leads to considering a specific behavior towards risk as rational when there exists a function that behaves additively. Whenever the right to having uncertain situations denoted by P_1, $P_2, \ldots, P_n$, where their probabilities are expressed by $p_1, p_2, \ldots, p_n$, is considered to be equivalent to the right to having the certain situation denoted by P, the following weighted average takes place:

$$u(P) = \sum_{h=1}^{n} u(P_h)\, p_h. \tag{1}$$

One of the utility indices for a given decision maker is denoted by u. Their number is infinite. One of the utility indices for him or her is uniquely determined by (1), where changes of origin and unit of measurement are inessential. Any increasing function of one of them can subjectively be taken into account. A linear or concave or convex function can then be taken into account. Uncertain situations denoted by $P_1, P_2, \ldots, P_n$ are the possible values for a single random quantity. Such values, together with their probabilities, give rise to a nonparametric distribution of probability. In the first stage, probabilities change in such a way that an infinite number of probability distributions appears. One of them is chosen in the second stage based on a further hypothesis of an empirical nature. It is the mathematical expectation of the single random quantity taken into account such that P can be less than or greater than it or equal to it. P is intrinsically connected to a specific utility function. In this paper, two random quantities at a time are studied inside the budget set of the decision maker. They give rise to nonparametric joint distributions of probability. Their number is infinite in the first stage. One of them is chosen in the second stage based on a further hypothesis of an empirical nature. An aggregate measure is obtained by considering four ordered pairs of random quantities. Four nonparametric joint distributions of probability arise. Their events are exchangeable. Their evaluations of probability are symmetrical or invariant with respect to permutations. The notion of α-product deals with them. The mathematical expectation of a multiple random quantity of order 2 is an aggregate measure obtained by considering four ordered pairs of random quantities. If the mathematical expectation of a multiple random quantity of order 2 is coherent, then it is a point belonging to the union of convex sets (see [1]). The properties of the barycenter of masses are extended in this way, as well as the ones of the notion of expected utility given by (1). With regard to a multiple random quantity of order 2, the certain situation can be less than or greater than its mathematical expectation or equal to it. These are objective conditions of coherence. Analogous conditions are established with regard to a multiple random quantity of order m, with $m > 2$ being an integer (see [2]). The criteria of coherent decision making under uncertainty and riskiness also consist of fixing as their goal the maximization of the expected utility. A unique value cannot be provided a priori. If a unique value is provided by means of the introduction of arbitrary mathematical conventions without reference to the objective conditions of coherence, then this is unjustified and inadmissible. What will be said in this paper is in agreement with existing research works in the fields of economics, probability, and statistics (see [3–7]). The subjective opinion is not directly observable. Nevertheless, it is something objective. This is because it is something known by a given decision maker. For this reason, the subjective opinion can be a reasonable object of an accurate study (see [8,9]). I do not agree with the proliferation of research works characterized by partial views. I would prefer an increase in the number of research works like this one characterized by unitary and enlarged views. Problems of an economic nature contemplating infinite outcomes in number are illusory. Only a part of such outcomes is observable to establish if each explicit outcome is true or false, so their main goal is to base mathematical methods permitting the derivation of a uniquely determined answer to an issue even when it is indeterminate between given limits.

The Objectives of This Paper

In this paper, m risky assets are m random goods studied inside the budget set of the investor. The components of an m-risky asset portfolio, where the latter is a multiple random good of order m, are m risky assets. I use a quadratic metric, so I do not study more than two goods at a time. Any two risky assets identify two nonparametric marginal distributions of mass. Also, they always identify a nonparametric joint distribution of mass. The budget set of the investor is a subset of $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R} \times \mathbb{R}$ is a linear space over $\mathbb{R}$. The space where a given investor rationally chooses has an infinite number of points. Each point of it is a prevision bundle. A given investor chooses one of the infinite prevision bundles existing inside his or her budget set. He or she chooses that prevision bundle which maximizes his or her subjective utility. This research work focuses on how to maximize the utility of the expected return on an m-risky asset portfolio. The notion of ordinal utility is shown to be a distance. The notion of risk is validated to be of a subjective nature.

In Section 2, a probability space associated with a risky asset is defined. Problems of an economic nature are studied using the notion of utility. In Section 3, after focusing on the properties of the notion of mathematical expectation, revealed preference theory is applied to studying prevision bundles. In Section 4, the optimization principle is treated using the notion of distance. Section 5 focuses on the study of two scales. They are the monetary scale and the one of utility. In Section 6, a specific utility function is considered inside the budget set of the investor in the case of his or her rigidity in the face of risk. An enlargement of the notion of moral expectation is shown. Section 7 studies some indices of a multilinear nature. Section 8 outlines future perspectives.

2. Problems of an Economic Nature Treated by Means of Subjective Tools

2.1. Outcomes Underlying a Random Process and Their Probabilities

With regard to a random process, the basic thing that is being chosen by a given decision maker is a probability distribution. The latter consists of a list of different objects of decision maker choice and the probability associated with each object. Different objects of decision maker choice are different outcomes underlying a random process. Each outcome is an event. Its meaning is not collective, but it is atomistic. Since it is possible to allude to a specific result in a single and well-defined case, an event is a proposition susceptible of being true or false at the right time. The right time is when uncertainty characterizing any random process whatsoever ceases (see [10]). Uncertainty about an event has to be meant as ignorance by a given individual (see [11]). In this paper, events are not studied without reference to the space in which their set is naturally embedded. Accordingly, a finite number of possible events identifies the possible values for a random good. A random good is a random quantity. It is studied inside a Euclidean space, so all the outcomes underlying a random process are real numbers in the space of random quantities. All the outcomes are the components of a vector belonging to a linear space over $\mathbb{R}$ provided with a specific dimension. The probability of an outcome depends on what a given individual feels (see [12]). The notion of probability is not undefined within this context. Its nature is not objective. It does not exist on its own, independently of the evaluations a given individual makes of it, but it is subjective (see [13]). Probability evaluations can be based on a priori probabilities characterizing a random process of a symmetric nature. They can also be based on statistical probabilities. Nevertheless, they always exist according to a subjective judgment being made by a given individual. Probability evaluations being made by a given individual have only to be reasonable (see [14]). All reasonable evaluations are admissible. This characterizes nonparametric distributions of probability (see [15]). Reasonable evaluations express conditions of coherence. They are conditions of rationality underlying the decision maker's choice behavior. Conditions of coherence also concern subjective preferences described by the notion of utility. Such conditions never depend on subjective opinions. They never depend on subjective judgments. They are of a normative nature. Conditions of coherence are the weakest because it is appropriate to want them to be the strongest in terms of absolute validity. For this reason, only absolutely inadmissible choices have to be excluded in the first stage. A choice is absolutely inadmissible if it does not belong to a convex set containing all admissible choices, whose number is infinite.

2.2. A Risky Asset and Its Probability Space

A risky asset is a random good. A random good is a random entity. If uncertainty ceases, then the true value of such an entity is unique. Nevertheless, since a given investor does not know it at the present time, he or she is in doubt between at least two possible values. A random good is a random variable X. The latter is a function. I write

$$X\colon \Omega \to \mathbb{R}, \tag{2}$$

so its domain is a sample space denoted by Ω. Its codomain is $\mathbb{R}$. An assignment of numerical values meant as real numbers to the points in $\Omega = I(X) = \{x_1, x_2, \ldots, x_n\}$ identifies X. For this reason, each point in Ω is a one-point set. Given an orthonormal basis of E^n, where the latter is an n-dimensional linear space over $\mathbb{R}$ with a Euclidean structure, a random good is an ordered pair of two n-dimensional vectors. The components of the first one are the points of $I(X) = \Omega$ identifying a finite partition into elementary events. Such points are later transferred on a one-dimensional straight line, so a reduction of dimension happens. Since $I(X)$ denotes what is not sure, $I(X)$ with the assignment of probabilities is a probability triple expressed by

$$(\Omega, \mathcal{F}, \mathbf{P}). \tag{3}$$

The components of the second vector identifying a random good are probabilities. In my approach, Ω is embedded in a linear space over $\mathbb{R}$ provided with a specific dimension. I write $\mathcal{F} = \{\emptyset, \Omega\}$ to denote an algebra of events. $\mathbf{P}$ is a function of probability.

In this paper, I study m risky assets, where each of them is characterized by n possible values. They coincide with n possible outcomes underlying a random process of an economic nature. If I have $n > m$, where m risky assets are logically independent, then there are n^m possible outcomes for them. In particular, given any two risky assets, there are n^2 possible outcomes for them. They coincide with the Cartesian product of two sets. Let X_i, $i = 1, \ldots, m$, be a risky asset whose possible values are given by $I(X_i) = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$. I write $x_{i1} < x_{i2} < \ldots < x_{in}$. This is because I handle a finite partition of mutually exclusive events. Since $\inf I(X_i) = x_{i1}$ and $\sup I(X_i) = x_{in}$, X_i is bounded from above and below. It is possible to think of X_i, $i = 1, \ldots, m$, as being m investments in mutual funds that buy stocks. If the stock markets perform well, then these investments will perform well. Conversely, if the stock markets perform poorly, then these investments will perform poorly. Hence, with regard to each investment, there is uncertainty, as well as riskiness. This is because there is no fixed return. Each risky asset is identified with two n-dimensional vectors belonging to E^n. While the components of the first vector represent n objective outcomes underlying a random process of an economic nature, the components of the second one represent n nonnegative masses. Two different aspects of uncertainty are considered in this way. They are an objective aspect and a subjective one (see [16]). In this paper, I always distinguish two stages. In the first stage, all the weighted averages of n values given by $x_{i1}, x_{i2}, \ldots, x_{in}$ are handled. Their number is infinite. They give rise to a convex set. In the second stage, one of them is chosen based on a further hypothesis of an empirical nature. I refer to Bayes' theorem in this way. It is always characterized by two stages. In the first stage, all the subjective opinions are admissible. They have to be coherent. If they are not in contradiction among themselves, then they give rise to a convex set. In the second stage, they converge towards a specific point belonging to the same set. From the first stage to the second one, a further hypothesis of an empirical nature appears. Finally, in order to obtain a formulation of a choice problem that is economically satisfactory, it is necessary to consider the notion of utility.

2.3. The Expected Return on a Risky Asset Viewed as a Subjective Price: Its Connection with a Particular Investor's Scale of Preference

Given a risky asset denoted by X whose possible values are expressed by $I(X) = \{x_1, x_2, \ldots, x_n\}$, with $x_1 < x_2 < \ldots < x_n$, the price of X for a given investor is denoted by $\mathbf{P}(X)$, where $\mathbf{P}(X)$ is the expected return on X. It is the prevision or mathematical expectation of X. In the first stage, all the weighted averages of n values given by $x_1, x_2, \ldots, x_n$ are handled. Their number is infinite. They are fair evaluations or estimations of X. They give rise to a convex set. In the second stage, one of them is chosen based on a further hypothesis of an empirical nature. Picking up on what Bruno de Finetti said, $\mathbf{P}(X)$ is called the price of X for a given investor (see [17]). This is because $\mathbf{P}(X)$ represents the certainty equivalent to X according to a given individual whenever his or her subjective scale of preference is a linear utility function. This function is the 45-degree line, whose nature is cardinal. The size of the utility difference between two different wealth levels has then a specific meaning. This will be seen more in detail in other sections of this paper. I write

$$\mathbf{P}(X) = x_1\, p_1 + \ldots + x_n\, p_n, \tag{4}$$

where I have

$$0 \leq p_i \leq 1, \tag{5}$$

as well as

$$\sum_{i=1}^{n} p_i = 1. \tag{6}$$

In general, I denote by x the certainty equivalent to X with regard to a specific $u(x)$ associated with a given investor, where $u(x)$ is a cardinal utility function living in the Cartesian plane. Since I observe $x = \mathbf{P}(X)$ on the subjective investor's scale of preference, I deal with a risk-neutral investor. In this case, the monetary scale characterizing $I(X) = \{x_1, x_2, \ldots, x_n\}$ coincides with the one of utility. Among those decisions under uncertainty and riskiness leading to different risky assets, a given investor makes that decision leading to the risky asset with the highest subjective price denoted by $\mathbf{P}$. This is because he or she is only interested in the criterion of mathematical expectation. I note the following:

Remark 1. *Let X be a random good. Its possible values are given by $I(X) = \{x_1, \ldots, x_n\}$. The elements of $I(X)$ are of a monetary nature. The expected return on X coincides with the certainty equivalent to X according to a given individual. It is expressed by $\mathbf{P}(X)$. If this happens, then the scale of utility is given by the 45-degree line. Such a scale coincides with the monetary one. Similarly, the certainty equivalent to a unit gain depending on the occurrence of an event denoted by E_i, $i = 1, \ldots, n$, is expressed by $\mathbf{P}(E_i) = p_i$, with $0 \leq p_i \leq 1$. It is the probability of E_i for a given individual. The notion of prevision is unique. The symbol $\mathbf{P}$ serves for it. In the case of events, the prevision of one of them coincides with the notion of probability.*

Remark 2. *Let $|E|$ be the indicator of E. It is a random quantity. Whenever uncertainty ceases, it takes 1 when E is true, whereas it takes 0 when E is false. Given a finite partition of mutually exclusive events, I write*

$$X = x_1|E_1| + \ldots + x_n|E_n|, \tag{7}$$

where X is a random quantity being bounded from above and below.

By definition, any subjective price whatsoever denoted by $\mathbf{P}$ is always additive (see [18]). I write

$$\mathbf{P}(X_1' + X_1'') = \mathbf{P}(X_1') + \mathbf{P}(X_1''), \tag{8}$$

where X_1' and X_1'' are two random goods. Each mass associated with a possible value for X_1' belonging to $I(X_1') = \{x_1', x_2', \ldots, x_n'\}$ is the same as the one associated with a possible value for X_1'' belonging to $I(X_1'') = \{x_1'', x_2'', \ldots, x_n''\}$. Otherwise, (8) does not work. Whenever $\mathbf{P}$ is additive, if one buys a random good denoted by X_1' at the price $\mathbf{P}(X_1')$ and a random good denoted by X_2'' at the price $\mathbf{P}(X_2'')$, then one buys both of them at the price $\mathbf{P}(X_1') + \mathbf{P}(X_1'')$. The purchase of one of them does not affect the desirability of the other. Nevertheless, this is not always true. The additivity property of $\mathbf{P}$ is then a simplifying hypothesis with respect to decision-theoretic criteria of an economic nature. Such a hypothesis coincides with the rigidity in the face of risk by a given decision maker. Anyway, the additivity property of $\mathbf{P}$ is admissible in problems of an economic nature whenever the monetary sums taken into account are not too large in relation to the global wealth possessed by a given decision maker. If the possible values for a random good X are not of a monetary nature, then talking about price is no longer appropriate. $\mathbf{P}(X)$ is therefore the mean value of X.

2.4. Other Scales of Preference

Let $u(x)$ be a cardinal utility function living in the Cartesian plane. Thus, $u(x)$ can identify a risk-averse investor or a risk-loving one. Let X be a random good. Its possible values can be of a monetary nature, so X is a random gain. Given a subjective investor's scale of preference expressed by $u(x)$, X is preferred to the certain gain denoted by x if and only if x is greater than $\mathbf{P}(X)$. This means that there is a risk-loving investor, whose cardinal utility function $u(x)$ is a convex utility function. Its slope gets steeper as his or her wealth increases. Given another subjective investor's scale of preference expressed by $u(x)$, the certain gain denoted by x is preferred to X if and only if x is less than $\mathbf{P}(X)$ (see [19]). This means that there is a risk-averse investor, whose cardinal utility function $u(x)$ is a concave utility function. Its slope gets flatter as his or her wealth increases. For every individual, in any given choice problem under uncertainty and riskiness, the desirability of a random gain X is included into his or her subjective scale of the certain gains denoted by $u(x)$. This is a necessary condition of all economic decision making criteria under uncertainty and riskiness (see [20]).

3. Coherence Properties of the Notion of Expected Return on Risky Assets Studied Inside the Budget Set of the Investor

Additivity, convexity, and linearity of $\mathbf{P}$ are coherence properties of the notion of expected return on risky assets. An evaluation or estimation of a risky asset is fair if and only if $\mathbf{P}$ is coherent. Let X_1 and X_2 be two risky assets. The possible values for X_1 are found on the horizontal axis, whereas the possible values for X_2 are found on the vertical one. This is because a reduction of dimension later happens. I note the following:

Remark 3. *Let X_1 and X_2 be two risky assets. Their possible values are finite in number. Since I can write $X_1 = Y_1 - Z_1$, with $Y_1 = X_1\,(X_1 \geq 0)$ and $Z_1 = -X_1\,(X_1 \leq 0)$, as well as $X_2 = Y_2 - Z_2$, with $Y_2 = X_2\,(X_2 \geq 0)$ and $Z_2 = -X_2\,(X_2 \leq 0)$, the possible values for Y_1 and Z_1 are nonnegative. This is because $Y_1 = X_1$ if $X_1 > 0$ and zero otherwise, and $Z_1 = -X_1$ if $X_1 < 0$ and zero otherwise. The same is true with regard to Y_2 and Z_2, where $X_2 = Y_2 - Z_2$.*

Since it is always possible to handle nonnegative values for X_1 and X_2, I write

$$\mathbf{P}(aX_1) = a\mathbf{P}(X_1), \tag{9}$$

as well as

$$\mathbf{P}(aX_2) = a\mathbf{P}(X_2), \tag{10}$$

where a is any real number. It follows that $\mathbf{P}$ is linear. X_1 and X_2 are two random goods studied inside the budget set of the investor, so two half-lines are handled. If two perpendicular half-lines are handled, then the space where a given investor chooses his or her best expected return on risky assets is the same as the one where a given consumer chooses his or her best consumption bundle. Given $X_1 = Y_1 - Z_1$ and $X_2 = Y_2 - Z_2$, I write

$$\mathbf{P}(Y_1 - Z_1) = \mathbf{P}(Y_1) - \mathbf{P}(Z_1) \tag{11}$$

and

$$\mathbf{P}(Y_2 - Z_2) = \mathbf{P}(Y_2) - \mathbf{P}(Z_2) \tag{12}$$

because $\mathbf{P}$ is additive. The left-hand side of (11) has the same probabilities as the right-hand one, where the latter contains two summarized probability distributions, $\mathbf{P}(Y_1)$ and $\mathbf{P}(Z_1)$. The left-hand side of (12) has the same probabilities as the right-hand one, where the latter contains two summarized probability distributions, $\mathbf{P}(Y_2)$ and $\mathbf{P}(Z_2)$. Otherwise, (11) and (12) do not work. I write

$$\inf I(Y_1 - Z_1) \leq \mathbf{P}(Y_1 - Z_1) \leq \sup I(Y_1 - Z_1), \tag{13}$$

with $Y_1 - Z_1 = X_1$, and

$$\inf I(Y_2 - Z_2) \leq \mathbf{P}(Y_2 - Z_2) \leq \sup I(Y_2 - Z_2), \tag{14}$$

with $Y_2 - Z_2 = X_2$, because $\mathbf{P}$ is convex (see [21]). Since $\mathbf{P}$ is convex, closed line segments appear (see [22]). This means that admissible expected returns on a risky asset are infinite in number. The length of each closed line segment depends on the range of possibility only. The latter expresses the possible values for a risky asset. All the results of probability theory are consequential to coherence properties. An individual who contradicts a theorem of probability theory is not coherent. $\mathbf{P}$ can be both linear and bilinear. If $\mathbf{P}$ is bilinear, then a joint distribution of probability is summarized.

3.1. An Orthogonal Projection of Joint Expected Returns onto Two Mutually Perpendicular Axes

X_1 and X_2 are two random goods studied inside the budget set of the investor. In this paper, X_1 and X_2 always determine $X_1\,X_2$, where $X_1\,X_2$ is a joint random good. This happens because an event is not a set. With an event, the subdivision might be continued all the time. However, it should be stopped when the goal is reached. Hence, it should stop when $X_1\,X_2$ arises. Given n possible outcomes for each random good, a joint random good denoted by $X_1\,X_2$ has then $n \times n = n^2$ possible alternatives. A subdivision of the notion of event takes place in this way. Nevertheless, such a subdivision is not final. This is because the budget line needs to be drawn, so its horizontal and vertical intercepts have to be determined. This means that n possible outcomes for each random good are not considered any more, but $n+1$ possible alternatives are now treated. The focus then shifts to different linear spaces over $\mathbb{R}$, where each of them is provided with a specific dimension. From a linear inequality expressed by

$$c_1\,X_1 + c_2\,X_2 \leq c, \tag{15}$$

it follows that it is possible to write

$$c_1\,\mathbf{P}(X_1) + c_2\,\mathbf{P}(X_2) \leq c, \tag{16}$$

where (16) represents the budget constraint of the investor. The budget line is given by

$$c_1\,\mathbf{P}(X_1) + c_2\,\mathbf{P}(X_2) = c, \tag{17}$$

where c denotes the total amount of money the investor has to spend. The slope of (17) is expressed by $-\frac{c_1}{c_2}$. Its horizontal intercept is denoted by $\frac{c}{c_1}$. Its vertical intercept is denoted by $\frac{c}{c_2}$. $I(X_1) \cup \{\frac{c}{c_1}\}$ and $I(X_2) \cup \{\frac{c}{c_2}\}$ are studied. The values of $I(X_1) \cup \{\frac{c}{c_1}\}$ and $I(X_2) \cup \{\frac{c}{c_2}\}$ are the components of two $(n+1)$-dimensional vectors. They are uniquely expressed as linear combinations of $n+1$ basis vectors of E^{n+1}. I write $I(X_1) \cup \{\frac{c}{c_1}\} = I^*(X_1)$ and $I(X_2) \cup \{\frac{c}{c_2}\} = I^*(X_2)$. I denote by $\mathcal{P}$ the convex set of all coherent expected returns expressed by $\mathbf{P}$ connected with two risky assets that are studied inside the budget set of the investor. The number of all coherent expected returns expressed by $\mathbf{P}$ is infinite. $\mathcal{P}$ is the closed convex hull of $I^*(X_1) \times I^*(X_2)$ (see [23]). $I^*(X_1) \times I^*(X_2)$ with the attribution of masses is a probability space denoted by $(\Omega,\, \mathcal{F},\, \mathbf{P})$. The set of all possible alternatives is denoted by $\Omega = I^*(X_1) \times I^*(X_2)$. I write $\mathcal{F} = \{\emptyset,\, \Omega\}$ to denote an algebra of events. $\mathbf{P}$ stands for probability or prevision. The line expressed by (17) is a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$. I do not consider a linear space over $\mathbb{R}$ denoted by E^{n+1}, but I consider $\mathbb{R}$. It is itself a linear space. Its dimension is equal to 1. A reduction of dimension takes place in this way. In fact, there exists a one-to-one correspondence between a one-dimensional linear subspace of E^{n+1} and a one-dimensional straight line. Two one-dimensional linear subspaces of E^{n+1} are handled because two random goods are studied inside the budget set of the investor. Their possible values are found on two straight lines. More specifically, their possible values are found on two half-lines.) By definition, $\mathbf{P} \in \mathcal{P}$ is a coherent expected return if and only if (17) never separates any point $\mathbf{P} \in \mathcal{P}$ from $I^*(X_1) \times I^*(X_2)$. Also, (17) does not separate any point $\mathbf{P} \in \mathcal{P}$ from $I^*(X_1)$, nor from $I^*(X_2)$. If there exist $[(n+1)\cdot(n+1)]$ possible values for $X_1\,X_2$, then X_1 and X_2 are logically independent. Nevertheless, only n^2+2 out of $[(n+1)\cdot(n+1)]$ points are firstly uncertain. Secondly, only n^2 out of n^2+2 points remain possible for a given investor. This means that more recent information and knowledge determine a limitation of expectations.

Example 1. *X_1 and X_2 are two random goods studied inside the budget set of the investor. Their possible values are pure numbers. They belong to the sets $I(X_1) = \{0, 1, 4\}$ on the horizontal axis and $I(X_2) = \{0, 2, 3\}$ on the vertical one, so I write $n = 3$. The Cartesian product given by $I(X_1) \times I(X_2)$ determines $3^2 = 9$ ordered pairs of possible values for a joint random good denoted* *by $X_1\,X_2$. They are $(0,0)$, $(0,2)$, $(0,3)$, $(1,0)$, $(1,2)$, $(1,3)$, $(4,0)$, $(4,2)$, and $(4,3)$. The point given by*

$$(sup\, I(X_1),\, sup\, I(X_2)) = (4,3)$$

belongs to the budget line, so its horizontal intercept is expressed by $(7,0)$, whereas its vertical one is given by $(0,6)$. The budget set of the investor contains infinite points in number. They are found between two closed line segments and the budget line. Two closed intervals are treated. This is because coherent previsions of two random goods are established in the first stage. Two closed intervals are two subsets of $\mathbb{R}$. A closed interval from 0 to 7 denoted by $[0,7]$ is considered on the horizontal axis, whereas a closed interval from 0 to 6 denoted by $[0,6]$ is considered on the vertical one. $[(3+1)\cdot(3+1)] = 16$ ordered pairs of possible values for $X_1\,X_2$ are not handled. After drawing the budget line, only $3^2+2 = 11$ out of 16 points are treated. This means that the points given by $(7,2)$, $(7,3)$, $(7,6)$, $(1,6)$, and $(4,6)$ are now impossible. From the following Table 1, it follows that the possible values for $X_1\,X_2$ are $3^2 = 9$ only. Given three marginal masses for random good 1 and random good 2, the joint ones can arbitrarily vary. $n \geq 10$ is usually treated.

Table 1. The possible values for a joint random good and their probabilities.

Random Good 1 \ Random Good 2	0	2	3	Sum
0	p_{11}	p_{12}	p_{13}	$p_{1\cdot}$
1	p_{21}	p_{22}	p_{23}	$p_{2\cdot}$
4	p_{31}	p_{32}	p_{33}	$p_{3\cdot}$
Sum	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	1

3.2. A Full Analogy between Properties Connected with Expected Returns on Risky Assets and the Ones Associated with Well-Behaved Preferences

I pass from $I(X_1) = \{x_{11}, \ldots, x_{1n}\} \cup \{\frac{c}{c_1}\}$ to $\mathbf{P}(X_1)$ using $n+1$ marginal nonnegative masses. $\mathbf{P}(X_1)$ belongs to a closed line segment containing infinite previsions of X_1. This closed line segment is a convex set on the horizontal axis. Possible choices of marginal nonnegative masses are ∞^n. With regard to the vertical axis, I pass from $I(X_2) = \{x_{21}, \ldots, x_{2n}\} \cup \{\frac{c}{c_2}\}$ to $\mathbf{P}(X_2)$ in the same way. I pass from $I^*(X_1) \times I^*(X_2)$ to $\mathbf{P}(X_1\,X_2) \equiv (\mathbf{P}(X_1), \mathbf{P}(X_2))$ using joint masses. Only one point belonging to $\mathcal{P}$ is the one chosen by a given investor in the second stage characterized by a further hypothesis of an empirical nature. Prevision bundles denoted by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ are the objects that can be chosen by a given investor inside his or her budget set. If $(\mathbf{P}(X_1), \mathbf{P}(X_2)) \in \mathcal{P}$, then this choice is rational (see [24]).

I note the following:

Remark 4. *The prices denoted by c_1 and c_2 are formally the two real coefficients of a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$. Such coefficients identify its negative slope. It is appropriate to consider the possible values for X_1, X_2, as well as $X_1\,X_2$, because the budget line is a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$.*

A given investor can rank various prevision bundles (see [25]). How he or she subjectively ranks all prevision bundles under consideration describes his or her preferences (see [26]). This paper focuses on well-behaved preferences. More of good 1 and good 2 is better, so well-behaved preferences are monotonic. Averages are weakly preferred to extremes, so well-behaved preferences are convex. In this paper, revealed preference theory is used. This is because preferences are not directly observable, but it is possible to discover an investor's preferences from observing his or her choice behavior. I consider indifference curves. They are lines with the same slope as the budget line. Coherent expected returns on risky assets and well-behaved preferences have the same formal properties. This is established by the following:

Theorem 1. *Let X_1 and X_2 be two risky assets that are studied inside the budget set of the investor. Their possible values are expressed by $I(X_1) = \{x_{11} = 0, \ldots, x_{1n}\} \cup \{\frac{c}{c_1}\}$ and $I(X_2) = \{x_{21} = 0, \ldots, x_{2n}\} \cup \{\frac{c}{c_2}\}$. All coherent expected returns on $X_1\, X_2$ are denoted by $\mathbf{P}(X_1\, X_2)$. The formal properties of $\mathbf{P}(X_1\, X_2)$ are the same as the ones of well-behaved preferences.*

This theorem has been demonstrated in another paper of mine (see [27]). Indifference curves can never cross. Given any two prevision bundles belonging to two different indifference curves, a given investor can order them as to their Euclidean distance from the point $(0,0)$. This distance is measured along the 45-degree line. The distance of $\mathbf{P}$ from $O = (0,0)$ is given by

$$^{2}d(O,\mathbf{P}) = \sqrt{\sum_{i=1}^{2} \mathbf{P}(X_i)^2}, \tag{18}$$

where $\mathbf{P}$ is a two-dimensional vector expressed by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. I could also write

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}(X_1) \\ \mathbf{P}(X_2) \end{pmatrix}. \tag{19}$$

The bundles for which a given investor is indifferent to $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ form the indifference curve. Preferences for perfect substitutes can be imagined inside the budget set of the investor. The indifference curve intersects the 45-degree line in one point only. In my approach, the notion of utility has an independent meaning. I found one way given by (18) based on the Pythagorean theorem to assign utility numbers to objects of decision maker choice. I wonder whether it is possible to find an infinite number of ways. Since I refer to the notion of distance, I can find an infinite number of ways to do it. This is because the notion of distance is invariant. An objective way of describing all the admissible preferences is provided by my approach. Objective conditions of coherence are established. Preferred bundles have a higher distance from the point O than less preferred bundles. The idea of preference is based on the decision maker's behavior. For this reason, preference relations have to be meant as notions of an operational nature. I write

$$^{2}d(O,\mathbf{P}) = \|O - \mathbf{P}\| = \sqrt{\langle O - \mathbf{P},\, O - \mathbf{P}\rangle'} \tag{20}$$

with regard to the orthonormal basis denoted by $\mathcal{B}_2'$. I write

$$^{2}d(O,\mathbf{P}) = \|O - \mathbf{P}\| = \sqrt{\langle O - \mathbf{P},\, O - \mathbf{P}\rangle''} \tag{21}$$

with regard to the orthonormal basis denoted by $\mathcal{B}_2''$. It follows that I observe

$$\sqrt{\langle O - \mathbf{P},\, O - \mathbf{P}\rangle'} = \sqrt{\langle O - \mathbf{P},\, O - \mathbf{P}\rangle''} \tag{22}$$

when I pass from an orthonormal basis to another one. Given a specific basis of a linear space over $\mathbb{R}$, every vector of it depends on one and only one set of components.

I write

$$\mathbf{P}(X_1) = \{\mathbf{P}(X_1)[(c_1,\, c_2,\, c)]\} \tag{23}$$

and

$$\mathbf{P}(X_2) = \{\mathbf{P}(X_2)[(c_1,\, c_2,\, c)]\} \tag{24}$$

to denote that $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ depend on objective and subjective elements (see [28]). Given $(\mathbf{P}(X_1), \mathbf{P}(X_2))$, a given investor also chooses a summarized element of the Fréchet class such that $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ never vary. Each point of the budget set of the investor is an element of the Fréchet class (see [29]). With regard to it, there are three paradigmatic cases. Two of them are extreme cases. The other case is an intermediate one. They correspond to three specific values that can be taken by the correlation coefficient written in the form

$$\rho_{X_1, X_2} = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}}. \tag{25}$$

Such a coefficient is equal to $+1$ if a perfect direct linear relationship between X_1 and X_2 appears. It is equal to -1 if a perfect inverse linear relationship between X_1 and X_2 is observed. Finally, if X_1 and X_2 are independent, then the correlation coefficient is equal to zero. X_1 and X_2 are uncorrelated. The numerator of (25) contains joint masses of a joint distribution of probability (see [30]). Such masses identify an element of the Fréchet class if the marginal masses underlying a joint distribution are held fixed.

All coherent expected returns on X_1 and X_2 can also be established by

$$\frac{c_1}{c_1+c_2}\,\mathbf{P}(X_1) + \frac{c_2}{c_1+c_2}\,\mathbf{P}(X_2) \leq \frac{c}{c_1+c_2}. \tag{26}$$

A given investor divides his or her relative wealth given by

$$\frac{c_1}{c_1+c_2} \tag{27}$$

and

$$\frac{c_2}{c_1+c_2} \tag{28}$$

between the two risky assets taken into account. I observe

$$\frac{c_1}{c_1+c_2} + \frac{c_2}{c_1+c_2} = 1. \tag{29}$$

The two real coefficients expressed by $\frac{c_1}{c_1+c_2}$ and $\frac{c_2}{c_1+c_2}$ are the two prices of the two random goods taken into account. I then pass from (23) and (24) to

$$\mathbf{P}(X_1) = \left\{\mathbf{P}(X_1)\left[\left(\frac{c_1}{c_1+c_2}, \frac{c_2}{c_1+c_2}, \frac{c}{c_1+c_2}\right)\right]\right\} \tag{30}$$

and

$$\mathbf{P}(X_2) = \left\{\mathbf{P}(X_2)\left[\left(\frac{c_1}{c_1+c_2}, \frac{c_2}{c_1+c_2}, \frac{c}{c_1+c_2}\right)\right]\right\} \tag{31}$$

respectively.

3.3. Utility Functions Whose Arguments belong to the Budget Set of the Investor

The possible values for each marginal random good studied inside the budget set of the investor can be transferred on a one-dimensional straight line. A theorem proving this is contained in another paper of mine, which is now under review by an international journal. The budget set of the investor is a subset of a linear space over $\mathbb{R}$. Let E^2 be a linear space over $\mathbb{R}$. Its dimension is equal to 2. Its structure is Euclidean. The set of all $\mathbf{x} \in E^2$, with $x_1 = \mathbf{P}(X_1) \geq 0$ and $x_2 = \mathbf{P}(X_2) \geq 0$, is expressed by E^2_+. The set of all $\mathbf{x} \in E^2$, with $x_1 = \mathbf{P}(X_1) > 0$ and $x_2 = \mathbf{P}(X_2) > 0$, is expressed by E^2_{++}. All expected returns on joint risky assets that are chosen by a given investor in the second stage identify a sequence denoted by

$$\{\mathbf{x}^k \,|\, k = 1, \ldots, K\}. \tag{32}$$

For each k, it is possible to write (x_1^k, x_2^k). For instance, if $K = 2$, then

$$(x_1^1, x_2^1), (x_1^2, x_2^2). \tag{33}$$

The budget set of the investor is expressed by E^2_+. I handle a collection expressed by $\mathcal{U}$ of utility functions given by

$$U\colon E^2_+ \to \mathbb{R}, \tag{34}$$

where $U \in \mathcal{U}$. Each expected return on a joint risky asset is a two-dimensional vector $\mathbf{x} \in E^2_+$ obtained from the budget set of the investor expressed by

$$B(\mathbf{c}, c) = \{\mathbf{x} \in E^2_+ \,|\, \mathbf{c} \cdot \mathbf{x} \leq c\}, \tag{35}$$

so (35) is the same as (16). Also, $\mathbf{c} \cdot \mathbf{x}$ is a scalar or inner product. A generic pair denoted by

$$(\mathbf{x}, \mathbf{c}) \in E^2_+ \times E^2_{++} \tag{36}$$

shows both an expected return on a joint risky asset and a price vector. A collection of pairs given by

$$\{(\mathbf{x}^1, \mathbf{c}^1), \ldots, (\mathbf{x}^K, \mathbf{c}^K)\} \tag{37}$$

denotes a dataset. (In my approach, the role of possible outcomes for a random process is essential. With regard to economic problems under uncertainty and riskiness treated in this paper, such outcomes are directly observed. If they are not directly observed, then they can always be estimated. If $\mathbf{P}(X_1)$ is already an element contained in the dataset given by (37), then it is possible to identify $n+1$ possible outcomes belonging to a closed neighborhood of $\mathbf{P}(X_1)$ expressed by $[\mathbf{P}(X_1) - \epsilon\,;\, \mathbf{P}(X_1) + \epsilon']$ on the horizontal axis, where ϵ and ϵ' are two small positive quantities. The same is true with regard to a closed neighborhood of $\mathbf{P}(X_2)$ expressed by $[\mathbf{P}(X_2) - \epsilon\,;\, \mathbf{P}(X_2) + \epsilon']$ on the vertical axis. It follows that $n+1$ possible outcomes are summarized in such a way that $\mathbf{P}(X_1)$ is actually chosen. A nonparametric marginal distribution of mass is accordingly estimated. Solutions of a linear equation, whose unknowns are probabilities, can be established. The same is true with regard to $n+1$ possible outcomes associated with random good 2. It is not crucial that one of $n+1$ possible outcomes coincides with $\mathbf{P}(X_1)$. The same is true with regard to $\mathbf{P}(X_2)$. A nonparametric joint distribution of mass derives from two marginal distributions of mass. They are all estimated before being summarized.) Utility functions with an ordinal nature identifying a preference ordering can be concave or convex or linear. This is because each point of the budget set of the investor is an element of the Fréchet class. A given investor can be risk averse. Given a collection $\mathcal{U}$ of strictly increasing and concave utility functions, a dataset expressed by (37) is $\mathcal{U}$-rational if there exists $U \in \mathcal{U}$ such that it can be observed, for each k, that

$$\mathbf{x}^k \in \operatorname{argmax}\,\{U(\mathbf{x}) \,|\, \mathbf{x} \in B(\mathbf{c}^k, \mathbf{c}^k \cdot \mathbf{x}^k)\}. \tag{38}$$

A given investor can be risk lover. Given a collection $\mathcal{U}'$ of strictly increasing and convex utility functions, a dataset given by (37) is $\mathcal{U}'$-rational if there exists $U' \in \mathcal{U}'$ such that it can be observed, for each k, that

$$\mathbf{x}^k \in \operatorname{argmax}\,\{U'(\mathbf{x}) \,|\, \mathbf{x} \in B(\mathbf{c}^k, \mathbf{c}^k \cdot \mathbf{x}^k)\}. \tag{39}$$

If a given investor is risk neutral, then his or her linear utility function is the 45-degree line. A dataset expressed by (37) is $\mathcal{U}''$-rational if there exists one and one only $U'' \in \mathcal{U}''$ such that, for each k,

$$\mathbf{x}^k \in \operatorname{argmax}\,\{U''(\mathbf{x}) \,|\, \mathbf{x} \in B(\mathbf{c}^k, \mathbf{c}^k \cdot \mathbf{x}^k)\}. \tag{40}$$

Given $\mathbf{x}$, $U''(\mathbf{x})$ can always be depicted inside the budget set of the investor unlike $U(\mathbf{x})$ and $U'(\mathbf{x})$. This is because to depict $U(\mathbf{x})$ and $U'(\mathbf{x})$ three axes are necessary instead of two. The budget set of the investor is characterized by two axes only. A given investor maximizes his or her subjective utility associated with each prevision bundle belonging to (37) when and only when his or her choices lie on the budget lines. This is because the distance from the point O of the prevision bundles lying on the budget lines is the highest. In particular, a given investor could maximize his or her subjective utility associated with each prevision bundle when and only when his or her choices lie on the budget lines where the 45-degree line exactly crosses them. (Nevertheless, where the 45-degree line exactly crosses the budget lines it is not always sure that there are objective alternatives being chosen as prevision bundles in the second stage.)

4. Revealed Expected Returns on Risky Assets

With regard to the budget set of the investor, I do not study more than two risky assets at a time for a reason of a metric nature. Among all points formally admissible in terms of coherence, the point actually chosen by a given investor is given by $(r_1, r_2) \in \mathcal{P}$, with $r_1 = \mathbf{P}(X_1)$ and $r_2 = \mathbf{P}(X_2)$. This point intrinsically depends on a further hypothesis of an empirical nature. However, its distance from $O = (0,0)$ has to be the highest. The optimization principle is then satisfied. Given the budget (c_1, c_2, c), if (r_1, r_2) represents an optimal choice for a given investor, then I write

$$c_1 r_1 + c_2 r_2 = c, \tag{41}$$

where the budget line given by (41) is a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$. Given (c_1, c_2, c), a given investor can choose, if he or she wants, the prevision bundle given by (s_1, s_2), where $s_1 = \mathbf{P}(X_1)$ and $s_2 = \mathbf{P}(X_2)$, with $s_1 \neq r_1$ and $s_2 \neq r_2$, and he or she can even spend a smaller amount of money. When I say that the investor can choose the prevision bundle expressed by (s_1, s_2) at the prices (c_1, c_2) and income c, I mean that (s_1, s_2) satisfies the budget constraint. I then observe

$$c_1 s_1 + c_2 s_2 \leq c. \tag{42}$$

Putting together (41) and (42), it is possible to write

$$c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2. \tag{43}$$

In other terms, I state the following principle of revealed expected return:

Remark 5. *Let (r_1, r_2) be the chosen prevision bundle by a given investor at the prices (c_1, c_2), where $r_1 = \mathbf{P}(X_1)$ and $r_2 = \mathbf{P}(X_2)$, and let (s_1, s_2) be another prevision bundle such that it is possible to write $c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2$, where $s_1 = \mathbf{P}(X_1)$ and $s_2 = \mathbf{P}(X_2)$, with $s_1 \neq r_1$ and $s_2 \neq r_2$. If a given investor is maximizing his or her subjective utility, then the r-bundle is strictly preferred to the s-bundle. The distance from the point O of the r-bundle is greater than the one of the s-bundle.*

If the inequality $c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2$ is satisfied and (s_1, s_2) is actually a different prevision bundle with respect to (r_1, r_2), then (r_1, r_2) is directly revealed preferred to (s_1, s_2) in the sense that (r_1, r_2) is chosen under uncertainty and riskiness instead of (s_1, s_2) (see [31]). Its ordinal utility is therefore greater. All prevision bundles that have been rejected in favor of (r_1, r_2) are revealed worse than (r_1, r_2). I state the following weak axiom of revealed expected return:

Remark 6. *If the r-bundle is directly revealed preferred to the s-bundle and the two prevision bundles are not the same, then it is not possible that the s-bundle is directly revealed preferred to the r-bundle.*

Now, I suppose that the prevision bundle expressed by (s_1, s_2) is chosen at the prices (d_1, d_2) and that it is revealed preferred to another prevision bundle given by (t_1, t_2), where $t_1 = \mathbf{P}(X_1)$ and $t_2 = \mathbf{P}(X_2)$, with $t_1 \neq s_1$ and $t_2 \neq s_2$. I write

$$d_1 s_1 + d_2 s_2 \geq d_1 t_1 + d_2 t_2. \tag{44}$$

If the r-bundle is strictly preferred to the s-bundle and if the s-bundle is strictly preferred to the t-bundle, then the r-bundle is indirectly revealed preferred to the t-bundle (see [32]). An assumption of transitivity is clearly used. I therefore state the following strong axiom of revealed expected return:

Remark 7. *If the r-bundle is revealed preferred to the s-bundle (either directly or indirectly) and the s-bundle is different from the r-bundle, then the s-bundle cannot be directly or indirectly revealed preferred to the r-bundle.*

In general, the budget line changes its negative slope with respect to any pair comparison of risky assets (see [33]). The optimal choice under uncertainty and riskiness being made by a given investor is that prevision bundle inside his or her budget set which is found on the highest indifference curve (see [34]). In the first stage, the highest indifference curve for a given investor coincides with (17). His or her optimal choice under uncertainty and riskiness would be any point of the line given by (17). He or she can freely move along it according to the equality given by

$$\frac{\Delta\mathbf{P}(X_2)}{\Delta\mathbf{P}(X_1)} = -\frac{c_1}{c_2}. \tag{45}$$

If a given investor increases $\mathbf{P}(X_1)$, then he or she must decrease $\mathbf{P}(X_2)$ and vice versa in order to move along (17). In the same Cartesian plane, I consider diagrams identifying as many lines like the one given by (17) as pair comparisons it is possible to take into account (see [35]). These diagrams identify as many optimal choices under uncertainty and riskiness as pair comparisons it is possible to take into account. Given m risky assets, the number of distinct pair comparisons is given by

$$C^r_{m,2} = \frac{1}{2}m\,(m+1). \tag{46}$$

The number of all pair comparisons is overall equal to m^2. It is accordingly possible to study m^2 nonparametric joint distributions of mass identifying a multiple random good of order m. It is possible to determine all optimal choices connected with m^2 nonparametric joint distributions of mass. Each optimal choice of m^2 optimal choices is a two-dimensional point belonging to the budget set of the investor, where such a point corresponds to a bilinear measure obtained by summarizing a nonparametric joint distribution of mass. Given X_1 and X_2, the optimal choice associated with the ordered pair of risky assets given by $(X_1,\, X_2)$ coincides with the two-dimensional point expressed by $(\mathbf{P}(X_1),\, \mathbf{P}(X_2))$. The latter corresponds to the bilinear measure denoted by $\mathbf{P}(X_1\, X_2)$. The prevision bundle being chosen by a given investor inside his or her budget set maximizes his or her subjective utility if and only if it is found on the budget line. This is because its distance from O measured along the 45-degree line is the highest. The budget line contains boundary points, where each of them produces the highest ordinal utility.

5. The Monetary Scale and the One of Utility Connected with Two Risky Assets Studied Outside the Budget Set of the Investor

In this section, I do not stay inside the budget set of the investor. I do not study a joint risky asset, but I separately study two risky assets only. After focusing on the two axes of a two-dimensional Cartesian coordinate system, I take two different scales into account. They are the monetary scale and the one of utility. Given X_1 and X_2, where the possible values for X_1 and X_2 identifying $I(X_1) = \{x_{11}, x_{12}, \ldots, x_{1n}\}$ and $I(X_2) = \{x_{21}, x_{22}, \ldots, x_{2n}\}$ are, respectively, represented on the horizontal axis and on the vertical one, $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ are chosen in the second stage based on a further hypothesis of an empirical nature. $\mathbf{P}(X_1)$ is chosen with respect to one of the two axes taken into account. $\mathbf{P}(X_2)$ is chosen with respect to the other axis. If a given investor is risk neutral, then the monetary scale and the one of utility coincide. I observe $u(x) = x$. The degree of preferability of an uncertain wealth is then inserted into the scale of the sure one. I consider X_1' and X_1'' on the horizontal axis of the Cartesian plane. Given $\mathbf{P}(X_1')$ and $\mathbf{P}(X_1'')$, with $\mathbf{P}(X_1') \neq \mathbf{P}(X_1'')$, a given investor prefers X_1' to X_1'' if $\mathbf{P}(X_1')$ is higher than $\mathbf{P}(X_1'')$. He or she prefers X_1'' to X_1' if $\mathbf{P}(X_1'')$ is higher than $\mathbf{P}(X_1')$. The same is true with regard to $\mathbf{P}(X_2')$ and $\mathbf{P}(X_2'')$, where X_2' and X_2'' are studied on the vertical axis of the same Cartesian plane.

If a given investor is risk averse, then the monetary scale and the one of utility do not coincide. Because of risk aversion, successive increments of equal monetary value have smaller and smaller utility for him or her. In all cases, a risk-averse investor will prefer the certain alternative expressed by x to the uncertain one denoted by $M = \mathbf{P}(X_1)$ on the horizontal axis and by $M = \mathbf{P}(X_2)$ on the vertical one. $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ are chosen in the second stage based on a further hypothesis of an empirical nature. In the first stage, all fair evaluations or estimations of X_1 and X_2 are made by a given risk-averse investor. Their number is infinite. They give rise to two convex sets. Cardinal utility is needed to describe choice behavior. A cardinal utility framework is established in the following way:

Remark 8. *If a given investor is risk averse, then to arrive at the real indifference with regard to his or her scale of utility he or she would content himself or herself with receiving with certainty an amount of money given by x, which is less than $M = \mathbf{P}(X_1)$, in exchange for the hypothetical gain given by $2M$. Such a gain is an event whose probability is equal to $1/2$. Equal levels of utility on the vertical axis are observed in passing from 0 to x and from x to $2M$ on the horizontal one, where 0 expresses the other event of the finite partition of events characterizing every judgment of indifference expressed by a given risk-averse investor. Its probability is equal to $1/2$, so $1/2 + 1/2 = 1$. The same is true with regard to another risky asset. Its possible values are found on the other axis of the same Cartesian plane.*

Since equal levels of utility on the vertical axis are observed in passing from 0 to x and from x to $2M$ on the horizontal one, the notion of cardinal utility is itself a distance as well. In this section, I consider x on the horizontal axis and $u(x)$ on the vertical one with regard to X_1, where $u(x)$ is a cardinal utility function identifying a specific attitude towards risk characterizing a given investor. Conversely, I consider x on the vertical axis and $u(x)$ on the horizontal one with regard to X_2, where $u(x)$ is the same cardinal utility function identifying the same attitude towards risk associated with the same investor.

The two scales coincide whenever the passage to the limit is mathematically considered. Given a concave utility function denoted by $u(x)$ and identifying a risk-averse investor, if I write

$$\mathbf{P}(X_1) = \lim_{a \to 0} \left(\frac{1}{a}\right) \mathbf{P}^*(aX_1), \tag{47}$$

with respect to a risky asset denoted by X_1 whose possible values lie on the horizontal axis, and

$$\mathbf{P}(X_2) = \lim_{a \to 0} \left(\frac{1}{a}\right) \mathbf{P}^*(aX_2), \tag{48}$$

with respect to a risky asset denoted by X_2 whose possible values lie on the vertical one, then $\mathbf{P}$ is considered linear and, in particular, additive in the case that the monetary values taken into account are not too large with regard to his or her global wealth. It is not, however, appropriate to define $\mathbf{P}$ by taking (47) and (48) into account because if a is too small then an evaluation of all probabilities into account loses any reliability. I am interested in showing (47) and (48) because they show that the scale of utility associated with a given investor (to which $\mathbf{P}^*$ is referred) approximately coincides with the monetary one (to which $\mathbf{P}$ is referred) whenever his or her investment remains within appropriate limits. From (47) and (48), it follows that it is analytically possible to replace the portion of $u(x)$ that is of interest (the smaller it is, the better the approximation) with the tangent at the starting point.

Given a concave utility function denoted by $u(x)$ and identifying a risk-averse investor, it is possible to write

$$x = u^{-1}\left\{\sum_{k=1}^{n} u(x_{1k})p_{1k}\right\} \tag{49}$$

with regard to the horizontal axis, where $x < \mathbf{P}(X_1)$, and

$$x = u^{-1}\left\{\sum_{k=1}^{n} u(x_{2k})p_{2k}\right\} \tag{50}$$

with regard to the vertical one, where $x < \mathbf{P}(X_2)$. The expected utility (moral expectation) of X_1 is given by

$$u(x) = \sum_{k=1}^{n} u(x_{1k})p_{1k}, \tag{51}$$

whereas the expected utility (moral expectation) of X_2 is given by

$$u(x) = \sum_{k=1}^{n} u(x_{2k})p_{2k}. \tag{52}$$

However, with regard to the monetary scale, a fair transaction is such for everyone agreeing on the same evaluation of all probabilities taken into account, even for the other contracting party (such as a bettor with respect to a bookmaker). With regard to the scale of utility, an indifferent transaction is not such in the case that $u(x)$ varies. Anyway, it cannot be indifferent for a given individual studied together with his or her opponent. It follows that the monetary scale is invariant unlike the investor's scale of utility (see [36]). In fact, the latter always depends on the global wealth possessed by him or her, his or her temperament, his or her current mood, and some other circumstances. I note the following:

Remark 9. *The monetary scale is linear and, in particular, additive. The scale of utility is not necessarily linear. A concave utility function identifying a risk-averse investor is a scale of utility whose nature is not linear. A convex utility function identifying a risk-loving investor is a nonlinear scale of utility as well.*

6. A Utility Function Considered Inside the Budget Set of the Investor in the Case of His or Her Rigidity in the Face of Risk

Let X_1 and X_2 be two risky assets. They are studied inside the budget set of the investor. A joint risky asset then arises. Their possible values are respectively given by $I(X_1) = \{x_{11} = 0, \ldots, x_{1n}\} \cup \{\frac{c}{c_1}\}$ and $I(X_2) = \{x_{21} = 0, \ldots, x_{2n}\} \cup \{\frac{c}{c_2}\}$, with $x_{11} < \ldots < x_{1n}$ and $x_{21} < \ldots < x_{2n}$. A given investor can estimate $[(n+1)\cdot(n+1)]$ joint masses in such a way that there is no linear correlation between X_1 and X_2. It is possible to represent the 45-degree line identifying a risk-neutral investor inside his or her budget set. I observe $x = \mathbf{P}(X_1)$ on the horizontal axis, and $x = \mathbf{P}(X_2)$ on the vertical one, so the 45-degree line is a cardinal utility function. The following way can be used to assign cardinal utilities:

Remark 10. *A given risk-neutral investor is faced with two possible cases, which he or she judges equally probable. Two equiprobable alternatives are Heads (H) or Tails (T) when a coin is tossed in the air. This is written as $\mathbf{P}(H) = \mathbf{P}(T) = 1/2$. Since $\mathbf{P}(H + T) = \mathbf{P}(H) + \mathbf{P}(T) = 1$, H and T are two mutually exclusive events of a finite partition. A given risk-neutral investor is indifferent between receiving with certainty an amount of money denoted by M or twice this amount if one of the two eventualities denoted by H and T occurs. In other terms, if $\mathbf{P}(X_1) = M$, where M is an arbitrary amount of money that is different from 0, then a risk-neutral investor is indifferent between receiving with certainty M or $2M = M + M$ in the case that T occurs with a probability equal to $1/2$. It is possible to extend this. It is possible to divide an interval into four parts instead of two, so if $\mathbf{P}(X_1) = 2M$, then a risk-neutral investor is indifferent between receiving with certainty $2M$ or*

$4M = M + M + M + M$ in the case that T occurs with a probability equal to $1/2$. This means that $\mathbf{P}(X_1) = 2M$ coincides with $\mathbf{P}(4MT) = 4M\mathbf{P}(T) = 4M\frac{1}{2} = 2M$. It is also possible to divide an interval into 8 parts, 16 parts, and so on, in order to construct the scale of utility identifying a risk-neutral investor. The same is true with respect to X_2, whose possible values are on the vertical axis. It is possible to construct the scale of utility identifying a risk-averse investor by dividing an interval into 4, 8, 16, . . ., parts in a similar way. The same is true with regard to the construction of the scale of utility identifying a risk-loving investor.

I write

$$u(x) = x \tag{53}$$

to identify the 45-degree line. In particular, the point expressed by $(0, 0)$ is a point of $u(x)$. The scale of utility identifying a risk-neutral investor represents his or her rigidity in the face of risk. Such a rigidity is also admissible whenever it is possible to assume that his or her investment in any two risky assets has an outcome whose effect is not considerable with respect to his or her global wealth. Such an effect does not give origin to remarkable improvements in his or her situation, nor to heavy losses, so the hypothesis of rigidity in the face of risk is realistic enough within quite a wide range. $\mathbf{P}(X_1\, X_2)$ is always decomposed into $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$. Given the convex set containing all expected returns on $X_1\, X_2$, $y = x$ is written whenever all the certain gains equivalent to X_1 are considered, whereas all the certain gains equivalent to X_2 are recognized by $x = y$. Two functions expressed by $y = x$ and $x = y$ have the same diagram depicted in the Cartesian plane. I note the following:

Remark 11. *After observing that the utility function taken into account intersects the budget line, the optimal choice being made by a given risk-neutral investor is that point expressed by*

$$(sup\; I(X_1),\, sup\; I(X_2)) \tag{54}$$

corresponding to a specific prevision bundle. Such a point is orthogonally projected onto the two axes of the Cartesian plane.

The certainty equivalent to X_1 is a point on the horizontal axis. It coincides with the expected utility of X_1 given by

$$u(x) = \sum_{k=1}^{n+1} u(x_{1k})p_{1k} = \sum_{k=1}^{n+1} x_{1k}p_{1k}, \tag{55}$$

where (55) is an additive function identifying a rational behavior associated with a given risk-neutral investor (see [37,38]). The certainty equivalent to X_2, which is optimally chosen by a given risk-neutral investor, is a point on the vertical axis. It coincides with the corresponding expected utility of X_2 given by

$$u(x) = \sum_{k=1}^{n+1} u(x_{2k})p_{2k} = \sum_{k=1}^{n+1} x_{2k}p_{2k}, \tag{56}$$

where (56) is an additive function identifying a rational behavior associated with a given risk-neutral investor. In the same Cartesian plane, diagrams identifying all intersections of the utility function taken into account and every line connected with a comparison of any two risky assets are considered. Such lines depict the corresponding budget lines. The optimal choice maximizes the investor's utility. Such a choice is always a point lying on the budget line crossed by the 45-degree line. Its distance from O is the highest (see [39]).

An Enlargement of the Notion of Moral Expectation

Given X_1 and X_2, the mathematical expectation of X_1 is $\mathbf{P}(X_1)$, whereas the mathematical expectation of X_2 is $\mathbf{P}(X_2)$. Since X_1 and X_2 are jointly studied inside the budget set of the investor, the mathematical expectation of X_{12} denoted by $\mathbf{P}(X_{12})$ has to be determined. An extension of the notion of barycenter of masses then appears. I write

$$\mathbf{P}(X_{12}) = \begin{vmatrix} \mathbf{P}(X_1\, X_1) & \mathbf{P}(X_1\, X_2) \\ \mathbf{P}(X_2\, X_1) & \mathbf{P}(X_2\, X_2) \end{vmatrix}, \tag{57}$$

where (57) is the determinant of a 2×2 matrix. Given X_1 and X_2, it is possible to consider four ordered pairs of risky assets given by (X_1, X_1), (X_1, X_2), (X_2, X_1), and (X_2, X_2). Four nonparametric joint distributions of probability are considered. With regard to each nonparametric joint distribution of probability, subjective probabilities intrinsically connected with exchangeable or symmetric events are treated. I establish the following:

Definition 1. *X_{12} is a multiple good of order 2, whose constitutive elements are X_1 and X_2. $I(X_1)$ and $I(X_2)$ are the sets of possible values for X_1 and X_2, respectively. $I(X_1)$ and $I(X_2)$ contain pure numbers.*

The optimal choice associated with the ordered pair of risky assets given by (X_1, X_2) coincides with the point expressed by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ belonging to the budget set of the investor. This point lies on the budget line. It is a boundary point. Such a point corresponds to the bilinear measure denoted by $\mathbf{P}(X_1\, X_2)$. The same is true with respect to all other ordered pairs of risky assets. Since a prevision bundle is a list of two numbers, these two numbers can be equal. In particular, this implies that two nonparametric marginal distributions of mass are the same. It follows that $\mathbf{P}(X_1\, X_1)$ and $\mathbf{P}(X_2\, X_2)$ are always decomposed into two equal linear measures. With regard to $\mathbf{P}(X_1\, X_1)$, $\mathbf{P}(X_1)$ and $\mathbf{P}(X_1)$ are observed. Similarly, with regard to $\mathbf{P}(X_2\, X_2)$, $\mathbf{P}(X_2)$ and $\mathbf{P}(X_2)$ are observed. I note the following:

Remark 12. *How to establish $\mathbf{P}(X_1\, X_1) \equiv (\mathbf{P}(X_1)\, \mathbf{P}(X_1))$ and $\mathbf{P}(X_2\, X_2) \equiv (\mathbf{P}(X_2), \mathbf{P}(X_2))$ is intrinsically different from how to establish $\mathbf{P}(X_1\, X_2)$ and $\mathbf{P}(X_2\, X_1)$. How to establish $\mathbf{P}(X_1\, X_1)$ and $\mathbf{P}(X_2\, X_2)$ is constrained, unlike how to establish $\mathbf{P}(X_1\, X_2)$ and $\mathbf{P}(X_2\, X_1)$. How to establish $\mathbf{P}(X_1\, X_1)$ and $\mathbf{P}(X_2\, X_2)$ is constrained because their corresponding joint masses are fixed.*

Starting from $\mathbf{P}(X_1\, X_2)$, $\mathbf{P}(X_2\, X_1)$, $\mathbf{P}(X_1\, X_1)$, and $\mathbf{P}(X_2\, X_2)$ are obtained. It follows that joint and marginal masses never change.

Example 2. *Concerning Example 1, in addition to Table 1, I handle the following Table 2,*

Table 2. A joint random good obtained from a permutation of an ordered basic pair of goods.

Random Good 2 / Random Good 1	0	1	4	Sum
0	p_{11}	p_{12}	p_{13}	$p_{1\cdot}$
2	p_{21}	p_{22}	p_{23}	$p_{2\cdot}$
3	p_{31}	p_{32}	p_{33}	$p_{3\cdot}$
Sum	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	1

the following Table 3,

Table 3. Random good 1 studied together with itself.

Random Good 1 \ Random Good 2	**0**	**1**	**4**	**Sum**
0	p_{11}	0	0	$p_{1\cdot}$
1	0	p_{22}	0	$p_{2\cdot}$
4	0	0	p_{33}	$p_{3\cdot}$
Sum	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	1

and the following Table 4,

Table 4. Random good 2 studied together with itself.

Random Good 1 \ Random Good 2	**0**	**2**	**3**	**Sum**
0	p_{11}	0	0	$p_{1\cdot}$
2	0	p_{22}	0	$p_{2\cdot}$
3	0	0	p_{33}	$p_{3\cdot}$
Sum	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	1

in order to obtain $\mathbf{P}(X_{12})$*. Tables 1 and 2 use the same joint masses. Tables 3 and 4 use marginal masses only.*

In general, given m risky assets, after considering m^2 intersection points of two straight lines, m^2 optimal choices lie on m^2 budget lines. It is possible to study m^2 optimal choices lying on m^2 budget lines in order to obtain an aggregate measure of a multilinear nature. After considering m^2 points, I write

$$\mathbf{P}(X_{12\,\dots\,m}) = \begin{vmatrix} \mathbf{P}(X_1\,X_1) & \mathbf{P}(X_1\,X_2) & \dots & \mathbf{P}(X_1\,X_m) \\ \mathbf{P}(X_2\,X_1) & \mathbf{P}(X_2\,X_2) & \dots & \mathbf{P}(X_2\,X_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}(X_m\,X_1) & \mathbf{P}(X_m\,X_2) & \dots & \mathbf{P}(X_m\,X_m) \end{vmatrix}, \tag{58}$$

where $X_{12\,\dots\,m}$ is an m-risky asset portfolio viewed as a multiple random good of order m, whereas $\mathbf{P}(X_{12\,\dots\,m})$ is the expected return on it. Hence, (58) is an aggregate measure of a multilinear nature. It is obtained using a multilinear metric. It is a quadratic metric. Multilinear relationships between m risky assets are studied through (58), where the latter is the determinant of a square matrix of order m given by

$$\begin{pmatrix} \mathbf{P}(X_1\,X_1) & \mathbf{P}(X_1\,X_2) & \dots & \mathbf{P}(X_1\,X_m) \\ \mathbf{P}(X_2\,X_1) & \mathbf{P}(X_2\,X_2) & \dots & \mathbf{P}(X_2\,X_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}(X_m\,X_1) & \mathbf{P}(X_m\,X_2) & \dots & \mathbf{P}(X_m\,X_m) \end{pmatrix}. \tag{59}$$

In general, it is possible to observe

$$\mathbf{P}(X_{12\,\dots\,m}) \neq \mathbf{P}(X_1\,X_2\,\dots\,X_m). \tag{60}$$

In this subsection, the expected return on $X_{12\,\dots\,m}$ coincides with its expected utility (moral expectation). This is because I use the 45-degree line. The knowledge hypothesis being made by a given investor is such that any two different risky assets studied inside his or her budget set are stochastically independent. If another portfolio consisting of m risky assets is identified with $Y_{12\ldots m}$, then its expected return is expressed by

$$\mathbf{P}(Y_{12\ldots m}) = \begin{vmatrix} \mathbf{P}(Y_1\,Y_1) & \mathbf{P}(Y_1\,Y_2) & \ldots & \mathbf{P}(Y_1\,Y_m) \\ \mathbf{P}(Y_2\,Y_1) & \mathbf{P}(Y_2\,Y_2) & \ldots & \mathbf{P}(Y_2\,Y_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}(Y_m\,Y_1) & \mathbf{P}(Y_m\,Y_2) & \ldots & \mathbf{P}(Y_m\,Y_m) \end{vmatrix}, \tag{61}$$

where the corresponding matrix is given by

$$\begin{pmatrix} \mathbf{P}(Y_1\,Y_1) & \mathbf{P}(Y_1\,Y_2) & \ldots & \mathbf{P}(Y_1\,Y_m) \\ \mathbf{P}(Y_2\,Y_1) & \mathbf{P}(Y_2\,Y_2) & \ldots & \mathbf{P}(Y_2\,Y_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}(Y_m\,Y_1) & \mathbf{P}(Y_m\,Y_2) & \ldots & \mathbf{P}(Y_m\,Y_m) \end{pmatrix}. \tag{62}$$

It is possible to observe if

$$\mathbf{P}(X_{12\ldots m}) \geq \mathbf{P}(Y_{12\ldots m}) \tag{63}$$

or

$$\mathbf{P}(X_{12\ldots m}) < \mathbf{P}(Y_{12\ldots m}), \tag{64}$$

where the scale used to carry out the comparisons given by (63) and (64) is of a monetary nature. This is because it is possible to go away from the budget set of the investor whenever an aggregate measure is studied. It is possible to depict a cardinal utility function $u(x)$ living in the Cartesian plane. If it is depicted outside the budget set of the investor, then $\mathbf{P}(X_{12\ldots m})$ is a point belonging to the union of one-dimensional convex sets. Such sets are found on the horizontal axis. The arithmetical product of two possible values for two risky assets is considered together with their joint probability on the horizontal axis. With regard to aggregate measures studied outside the budget set of the investor, cardinal utility functions identifying a risk-averse investor or a risk-loving one can be handled as well. The two-dimensional barycenter is given by

$$[\mathbf{P}(X_{12\ldots m}),\, \mathbf{P}(u(X_{12\ldots m}))], \tag{65}$$

where the moral expectation of $X_{12\ldots m}$ is denoted by $\mathbf{P}(u(X_{12\ldots m}))$. The latter is a point belonging to the union of one-dimensional convex sets. Such sets are found on the vertical axis. The same utility function denoted by u is associated with the arithmetical product of two possible values for two risky assets on the vertical axis. This product of two values is always considered together with their joint probability. I write

$$\begin{vmatrix} \sum_{i=1}^{n+1} u(x_{1i}\,x_{1i})p_{ii} & \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} u(x_{1i}\,x_{2j})p_{ij} & \ldots & \sum_{i=1}^{n+1}\sum_{k=1}^{n+1} u(x_{1i}\,x_{mk})p_{ik} \\ \sum_{j=1}^{n+1}\sum_{i=1}^{n+1} u(x_{2j}\,x_{1i})p_{ji} & \sum_{j=1}^{n+1} u(x_{2j}\,x_{2j})p_{jj} & \ldots & \sum_{j=1}^{n+1}\sum_{k=1}^{n+1} u(x_{2j}\,x_{mk})p_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n+1}\sum_{i=1}^{n+1} u(x_{mk}\,x_{1i})p_{ki} & \sum_{k=1}^{n+1}\sum_{j=1}^{n+1} u(x_{mk}\,x_{2j})p_{kj} & \ldots & \sum_{k=1}^{n+1} u(x_{mk}\,x_{mk})p_{kk} \end{vmatrix} \tag{66}$$

in order to obtain $\mathbf{P}[(u(X_{12\ldots m}))]$, with $\mathbf{P}[(u(X_{12\ldots m}))] > 0$. This is because $\mathbf{P}[(u(X_{12\ldots m}))]$ is the α-norm of a tensor obtained by calculating the determinant of a square matrix. I note the following:

Remark 13. *Since I write*

$$x_{12\ldots m} = u^{-1}\{\mathbf{P}[(u(X_{12\ldots m}))]\}, \tag{67}$$

$x_{12\ldots m}$ is the certain gain, which is considered to be equivalent to $X_{12\ldots m}$ by a given investor based on his or her cardinal utility function. In particular, by virtue of risk aversion, it is possible to

observe equal levels in the scale where the investor's judgments of indifference appear. These equal levels are observed with respect to the vertical axis. Equal levels or distances are vertically observed in passing from 0 *to* $x_{12\dots m}$ *and from* $x_{12\dots m}$ *to* $2M$ *with respect to the horizontal axis, where* $x_{12\dots m} < M = \mathbf{P}(X_{12\dots m})$*. With respect to the horizontal axis, an interval is divided into two indifferent increments and in the same way it is possible to obtain subdivisions into* 4, 8, 16, …, *parts in order to construct the scale of utility identifying a risk-averse investor.*

7. Other Aggregate Measures of a Multilinear Nature

Given the two marginal distributions of mass identifying $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, a given investor can estimate $[(n+1)\cdot(n+1)]$ joint masses such that the Bravais–Pearson correlation coefficient is equal to 0. The correlation coefficient can be written in the following form expressed by

$$\cos\gamma = \frac{\langle \mathbf{d}_1, \mathbf{d}_2\rangle_\alpha}{\sqrt{\langle \mathbf{d}_1, \mathbf{d}_1\rangle_\alpha}\sqrt{\langle \mathbf{d}_2, \mathbf{d}_2\rangle_\alpha}}, \tag{68}$$

where $\mathbf{d}_1$ and $\mathbf{d}_2$ are two vectors belonging to E^{n+1} whose components identify all possible deviations of the possible values for X_1 and X_2 from $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, respectively. The coefficient given by (68) depends on the size of the angle between $\mathbf{d}_1$ and $\mathbf{d}_2$ denoted by γ. The symbol α contained as a subscript in (68) means that joint masses are considered with respect to the numerator of (68) in order to obtain the covariance of X_1 and X_2. (It is a measure of the joint variability of X_1 and X_2. In this paper, it always depends on what a given investor knows or does not know. This means that the joint variability of X_1 and X_2 is not standardized.) These masses allow for the consideration of the metric notion of α-product. It is a scalar or inner product. Marginal masses viewed as particular joint masses are treated with respect to the denominator of (68) in order to obtain the standard deviation of X_1 and X_2, respectively. The covariance of X_1 and X_2, as well as the standard deviation of X_1 and X_2 are obtained by means of α-products. The elements on the right-hand side of (57), (58), and (61) are all α-products. If (68) is equal to 0, then there exists no correlation between two different risky assets of m risky assets. I also write

$$\begin{aligned}\mathrm{Var}(X_{12\dots m}) &= \begin{vmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \dots & \mathrm{Cov}(X_1, X_m) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \dots & \mathrm{Cov}(X_2, X_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_m, X_1) & \mathrm{Cov}(X_m, X_2) & \dots & \mathrm{Var}(X_m) \end{vmatrix} \\ &= \begin{vmatrix} \langle \mathbf{d}_1, \mathbf{d}_1\rangle_\alpha & \langle \mathbf{d}_1, \mathbf{d}_2\rangle_\alpha & \dots & \langle \mathbf{d}_1, \mathbf{d}_m\rangle_\alpha \\ \langle \mathbf{d}_2, \mathbf{d}_1\rangle_\alpha & \langle \mathbf{d}_2, \mathbf{d}_2\rangle_\alpha & \dots & \langle \mathbf{d}_2, \mathbf{d}_m\rangle_\alpha \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{d}_m, \mathbf{d}_1\rangle_\alpha & \langle \mathbf{d}_m, \mathbf{d}_2\rangle_\alpha & \dots & \langle \mathbf{d}_m, \mathbf{d}_m\rangle_\alpha \end{vmatrix}\end{aligned} \tag{69}$$

in order to obtain the riskiness of an m-risky asset portfolio, where the corresponding matrix is given by

$$\begin{pmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \dots & \mathrm{Cov}(X_1, X_m) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \dots & \mathrm{Cov}(X_2, X_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_m, X_1) & \mathrm{Cov}(X_m, X_2) & \dots & \mathrm{Var}(X_m) \end{pmatrix}. \tag{70}$$

If a Treasury bill that pays a fixed rate of interest regardless of what happens is denoted by r_f, then it is also possible to consider the Sharpe ratio given by

$$\mathrm{SR} = \frac{\mathbf{P}(X_{12\dots m}) - r_f}{\sqrt{\mathrm{Var}(X_{12\dots m})}}, \tag{71}$$

where $\mathbf{P}(X_{12\ldots m})$ and $\sqrt{\mathrm{Var}(X_{12\ldots m})}$ are two multilinear measures. Within this context, the standard deviation of the return is a "bad". It is not a "good". It is obtained by means of an approach showing the subjective nature of the notion of risk (see [40]). $\mathbf{P}(X_{12\ldots m}) > r_f$, so the slope of the budget line is positive (see [41]). Surveys based on nonparametric distributions of mass can use data that are observed inside the budget set of the investor (see [42]). Data that are observed inside the budget set of the investor can vectorially be handled outside it by taking an appropriate number of dimensions into account (see [43]). Also, it is possible to consider the following index expressed by

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_m)}{\mathrm{Var}(r_m)}, \tag{72}$$

where the return on the stock i is denoted by r_i, whereas the expected market return is denoted by r_m. The latter can be obtained using a multilinear measure analogous to (58).

8. Conclusions, Discussion, and Future Perspectives

Fair evaluations of risky assets being made inside the budget set of the investor are studied using subjective tools. Since the notion of ordinal utility is a distance, this paper does not admit that coherence is investigated based on qualitative considerations. It does not admit this thing even if such considerations are made in such a way that they do not contradict basic axioms about investor preference. Given any two prevision bundles, a given investor is coherent if he or she evaluates their distance from the point O. A given investor is coherent if he or she chooses that prevision bundle such that its distance from the point O is greater. In this paper, unique utility indices are used based on quantitative considerations. A precise point of view that goes back to Daniel Bernoulli is taken into account with respect to the notion of cardinal utility. It is itself a distance as well. The expected return on an m-risky asset portfolio is obtained via an aggregate measure of a multilinear nature. An extension of the notion of barycenter of masses is then treated. The notion of expected utility (moral expectation) is extended. Rational choices under uncertainty and riskiness are based on objective conditions of coherence. The latter are extended. My multilinear approach can be extended by studying other types of issues. A stochastic approach to bound choices extending the least-squares model can be shown. Significant issues of statistical inference treated in econometrics can be based on these elements. To study them is useful in order to obtain new results.

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