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On Hopf and Fold Bifurcations of Jerk Systems

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Abstract: In this paper we consider a jerk system $\dot{x} = y, \dot{y} = z, \dot{z} = j(x, y, z, \alpha)$, where j is an arbitrary smooth function and α is a real parameter. Using the derivatives of j at an equilibrium point, we discuss the stability of that point, and we point out some local codim-1 bifurcations. Moreover, we deduce jerk approximate normal forms for the most common fold bifurcations.

Keywords: jerk systems; local stability; codim-1 bifurcations

MSC: 70K20; 70K45; 70K50

1. Introduction

A jerk equation is a third-order ordinary differential equation of the form

$$\dddot{x} = j(x, \dot{x}, \ddot{x}),$$

where the rate of change of acceleration of the motion, $\dddot{x}(t)$, is called jerk [1]. Its equivalent form is the jerk system

$$\dot{x} = y, \dot{y} = z, \dot{z} = j(x, y, z).$$

Jerk systems have triggered wide interest. On one hand, they have a simple form but display complex behavior, and on the other hand, jerk systems model, among other things, some oscillators, in particular chaotic oscillators (see, e.g., [2–5]). Jerk systems display different dynamic behaviors, which depend on the total number of terms and parameters of j, particularly on the number of nonlinearities (see, e.g., [6] and references therein). The variation in the parameters leads to some bifurcations and an asymptotically stable or periodic orbit can becomes chaotic. Bifurcations of particular jerk systems were analyzed (see, e.g., [7–11]). Moreover, new chaotic jerk systems were recently reported (see, e.g., [12–14]).

In this paper, we analyze the jerk system

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = j(x, y, z, \alpha) \end{cases}, \tag{1}$$

where j is smooth, and α is a real parameter under the assumption that there is $\alpha = \alpha_0$ such that $E(x_0, 0, 0)$ is an equilibrium point of (1), that is,

$$j(x_0, 0, 0, \alpha_0) = 0. \tag{2}$$

Of course, E can be an equilibrium point for every $\alpha \in \mathbb{R}$. For instance, if j is odd, i.e., $j(-x, -y, -z, \alpha) = -j(x, y, z, \alpha)$, for every x, y, z, α, then $j(0, 0, 0, \alpha) = 0$, for every α.

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We denote

$$\frac{\partial j}{\partial x}(x_0,0,0,\alpha) = a(\alpha), \frac{\partial j}{\partial y}(x_0,0,0,\alpha) = b(\alpha), \frac{\partial j}{\partial z}(x_0,0,0,\alpha) = c(\alpha), \tag{3}$$

and

$$a(\alpha_0) = a, b(\alpha_0) = b, c(\alpha_0) = c. \tag{4}$$

The Jacobian matrix of system (1) at the equilibrium E is

$$J(x_0,0,0,\alpha) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a(\alpha) & b(\alpha) & c(\alpha) \end{pmatrix}$$

and the characteristic equation is given by

$$\lambda^3 - c(\alpha)\lambda^2 - b(\alpha)\lambda - a(\alpha) = 0. \tag{5}$$

For $\alpha = \alpha_0$, it becomes

$$\lambda^3 - c\lambda^2 - b\lambda - a = 0, \tag{6}$$

which allows us to discuss the stability of E and some codim-1 local bifurcations, namely Hopf and fold bifurcations.

A fold bifurcation is a local bifurcation associated with the appearance of a zero eigenvalue. In the one-dimensional case, the generic fold bifurcation, also called the saddle-node bifurcation, has the normal form given by the equation

$$\dot{\eta} = \beta \pm \eta^2,$$

where β is a real parameter (see, e.g., [15]). The other common fold bifurcations are the transcritical bifurcation and the pitchfork bifurcation, and their normal forms are also given by one-dimensional equations. In higher dimensions, the same normal form for the saddle-node bifurcation is obtained by reduction on the center manifold. One of the goals of this work is to obtain (approximate) normal forms for these fold bifurcations that continue being jerk systems, that is, the simplest jerk systems that display such bifurcations. To obtain such normal forms, we first study the stability of an arbitrary jerk system, and we point out the framework that ensures the existence of the above-mentioned bifurcations.

The paper is organized as follows: In Section 2, we discuss the local stability of an equilibrium point E of system (1), taking into account the sign of a, b, c and $a + bc$. In Section 3, using again the sign of a, b, c and the behavior of the function $h(\alpha) = a(\alpha) + b(\alpha)c(\alpha)$, we prove that system (1) experiences a Hopf bifurcation. In Section 4, we establish conditions of existence for each above-mentioned fold bifurcation. In Section 5, we deduce approximate normal forms for the saddle-node bifurcation, the transcritical bifurcation, and the pitchfork bifurcation. These normal forms are also jerk systems. We study the stability of the proposed approximate normal form using the reduction on the center manifold. Also, we give bifurcation diagrams and some orbits of these normal forms. In Section 6, we present some conclusions.

2. Stability of a Jerk System

The equilibria of system (1) are of the form $E(x^*,0,0)$, where x^* fulfills the condition $j(x^*,0,0,\alpha_0) = 0$.

Theorem 1. *Consider jerk system* (1) *with property* (2) *and notations* (3) *and* (4).
(a) For $a \neq 0$, the equilibrium point $E(x_0,0,0)$

(i) is asymptotically stable if $a < 0, b < 0, c < 0$, and $bc + a > 0$;
(ii) is unstable if $a > 0$, or $b \geq 0$, or $c \geq 0$, or $bc + a < 0$.

(b) For $a = 0$, the equilibrium point $E(x_0,0,0)$ is unstable if $b > 0$ or $c > 0$.

Proof. Taking into account (6), the conclusions follow from the Routh–Hurwitz theorem (see, e.g., [16]). □

Remark 1. *For $a \neq 0$, the stability of the equilibrium E is discussed in all cases except $a < 0, b < 0, c < 0, bc + a = 0$. In this case, there is a pair of pure imaginary eigenvalues. Therefore, reduction on the center manifold will be necessary. In particular, if the first Lyapunov coefficient $l_1(0)$ is computed, then we can conclude that E is weakly asymptotically stable if $l_1(0) < 0$ and unstable if $l_1(0) > 0$.*

If $a = 0$, the follow cases remain: $b < 0, c < 0$ (a zero eigenvalue), $b = 0, c < 0$ (a pair of zero eigenvalues), $b < 0, c = 0$ (a zero eigenvalue and a pair of pure imaginary eigenvalues), and $b = c = 0$ (all eigenvalues are zero). In the first two cases, the stability of E results from the reduced form of the center manifold.

Some of the above-mentioned cases are related to some local codim-1 bifurcations that can occur in the dynamics of the considered system. Such bifurcations will be discussed in the next sections.

3. Hopf Bifurcation

In this section, we give conditions for which system (1) experiences a Hopf bifurcation.

Theorem 2. *Consider jerk system (1) with property (2) and notations (3) and (4), and let $h(\alpha) = a(\alpha) + b(\alpha)c(\alpha)$. Assume that $E(x_0, 0, 0)$ is an equilibrium point of system (1) for every α in the neighborhood of α_0.*

If $a < 0$, $b < 0$, $c < 0$, $h(\alpha_0) = 0$, and $h'(\alpha_0) \neq 0$, then system (1) displays a Hopf bifurcation at the point $E(x_0, 0, 0)$ when α passes through the critical value α_0.

Proof. For $h(\alpha_0) = 0$, that is, $a = -bc$, the roots of (6) are $\lambda_1 = c < 0$ and $\lambda_{2,3} = \pm i\sqrt{-b}$.

It remains to check the transversality condition from Hopf's bifurcation theorem (see, e.g., [17]). Using (5) and the implicit function theorem, we have

$$\frac{d\lambda}{d\alpha} = \frac{c'(\alpha)\lambda^2 + b'(\alpha)\lambda + a'(\alpha)}{3\lambda^2 - 2c(\alpha)\lambda - b(\alpha)}$$

and

$$Re\left(\frac{d\lambda}{d\alpha}(\alpha_0)\right) = \frac{a'(\alpha_0) + b'(\alpha_0)c + bc'(\alpha_0)}{2(b - c^2)} \neq 0,$$

which finishes the proof. □

Remark 2. *In some particular cases, we can calculate the first Lyapunov coefficient $l_1(0)$, which allows to see if the Hopf bifurcation is non-degenerate ($l_1(0) \neq 0$) or degenerate ($l_1(0) = 0$).*

Remark 3. *Since the Hopf bifurcation involves a change in the stability, the conditions $a < 0$, $b < 0, c < 0$ are necessary conditions. In this case, the change in the stability is given by $h(\alpha_0) = 0$ and $h'(\alpha_0) \neq 0$ (the transversality condition), where $h(\alpha) = a(\alpha) + b(\alpha)c(\alpha)$ (3).*

Taking into account the above results, we notice that one of the simplest nonlinear jerk system which experiences the Hopf bifurcation at $O(0, 0, 0)$ when the parameter α passes through the critical value $\alpha = 0$ is given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = (\alpha - bc)x + by + cz + yz \end{cases},$$

with $b < 0, c < 0$. In fact, it is easy to see that any jerk system (1) with

$$j(x, y, z, \alpha) = (\alpha - bc)x + by + cz + \psi(x, y, z)$$

has the same property, where ψ is smooth, $\psi(0, 0, 0) = 0, d\psi(0, 0, 0) = 0, b < 0, c < 0$.

4. Fold Bifurcations

In this section, we study the bifurcation of the considered system associated with the appearance of a zero eigenvalue, namely the fold bifurcation or zero bifurcation. The non-degenerate fold bifurcation is called the saddle-node bifurcation. The most known degenerate fold bifurcations are the transcritical bifurcation and the pitchfork bifurcation.

Let $\alpha \in \mathbb{R}$ be the bifurcation parameter such that at $\alpha = \alpha_0$, system (1) has the non-hyperbolic equilibrium $E(x_0, 0, 0)$ with only one zero eigenvalue. Using (6), it follows that $a = 0$ and $b \neq 0$.

In the following, we point out fold bifurcations of system (1) using Sotomayor's theorem [18] (also see [17,19]). First, we compute the expressions that are used in Sotomayor's theorem.

Proposition 1. *Let $F(x, y, z, \alpha) = (y, z, j(x, y, z, \alpha))$ given by jerk system (1) with property (2) and notations (3) and (4). If $a = 0$, then the following assertions hold:*

1. $DF(x_0, 0, 0, \alpha_0)$ *has a simple eigenvalue* $\lambda_1 = 0$ *with right eigenvector* $v = (1, 0, 0)^t$ *and left eigenvector* $w = (-b, -c, 1)^t$;
2. $F'_\alpha(x_0, 0, 0, \alpha_0) = \left(0, 0, \dfrac{\partial j}{\partial \alpha}(x_0, 0, 0, \alpha_0)\right)^t$ *and* $w^t F'_\alpha(x_0, 0, 0, \alpha_0) = \dfrac{\partial j}{\partial \alpha}(x_0, 0, 0, \alpha_0)$;
3. $DF'_\alpha(x_0, 0, 0, \alpha_0)(v) = \left(0, 0, \dfrac{\partial^2 j}{\partial x \partial \alpha}(x_0, 0, 0, \alpha_0)\right)^t$ *and*
   $w^t DF'_\alpha(x_0, 0, 0, \alpha_0)(v) = \dfrac{\partial^2 j}{\partial x \partial \alpha}(x_0, 0, 0, \alpha_0)$;
4. $D^2F(x_0, 0, 0, \alpha_0)(v, v) = \left(0, 0, \dfrac{\partial^2 j}{\partial x^2}(x_0, 0, 0, \alpha_0)\right)^t$ *and*
   $w^t D^2F(x_0, 0, 0, \alpha_0)(v, v) = \dfrac{\partial^2 j}{\partial x^2}(x_0, 0, 0, \alpha_0)$;
5. $D^3F(x_0, 0, 0, \alpha_0)(v, v, v) = \left(0, 0, \dfrac{\partial^3 j}{\partial x^3}(x_0, 0, 0, \alpha_0)\right)^t$ *and*
   $w^t D^3F(x_0, 0, 0, \alpha_0)(v, v, v) = \dfrac{\partial^3 j}{\partial x^3}(x_0, 0, 0, \alpha_0)$.

Now, using Proposition 1 and Sotomayor's theorem, we obtain the next results.

Theorem 3. *Let $\alpha \in \mathbb{R}$ be the bifurcation parameter of system (1) with the critical value $\alpha_0 \in \mathbb{R}$. If*

SN1. $j(x_0, 0, 0, \alpha_0) = 0$, $\dfrac{\partial j}{\partial x}(x_0, 0, 0, \alpha_0) = a = 0$, $\dfrac{\partial j}{\partial y}(x_0, 0, 0, \alpha_0) = b \neq 0$,
$\dfrac{\partial j}{\partial z}(x_0, 0, 0, \alpha_0) = c \neq 0$,

SN2. $\dfrac{\partial j}{\partial \alpha}(x_0, 0, 0, \alpha_0) \neq 0$,

SN3. $\dfrac{\partial^2 j}{\partial x^2}(x_0, 0, 0, \alpha_0) \neq 0$,

then system (1) displays a saddle-node bifurcation at the point $E(x_0, 0, 0)$.

More precisely, there are two hyperbolic equilibrium points of the system near E for $\alpha < \alpha_0$ (or $\alpha > \alpha_0$), which collide for $\alpha = \alpha_0$ into the non-hyperbolic equilibrium E and then disappear. In addition, the two equilibrium points have stable manifolds of dimensions k and $k + 1$ respectively, where k is the number of eigenvalues of $DF(x_0, 0, 0, \alpha_0)$ with a negative real part.

Remark 4. *If we consider $b < 0, c < 0$ in the above theorem, then the Jacobian matrix $DF(x_0, 0, 0, \alpha_0)$ has $k = 2$ eigenvalues with negative real part. Therefore, the two equilibrium points that collide are a saddle and a stable node respectively (as in the one-dimensional case of the saddle-node bifurcation). If $b < 0, c > 0$, then $k = 0$ and the points are an unstable node and a saddle. Finally, if $b > 0$, then $k = 1$ and both equilibrium points are saddles.*

Theorem 4. *Let $\alpha \in \mathbb{R}$ be the bifurcation parameter of system (1) with the critical value $\alpha_0 \in \mathbb{R}$. If*

T1. $j(x_0, 0, 0, \alpha_0) = 0$, $\dfrac{\partial j}{\partial x}(x_0, 0, 0, \alpha_0) = a = 0$, $\dfrac{\partial j}{\partial y}(x_0, 0, 0, \alpha_0) = b \neq 0$, $\dfrac{\partial j}{\partial z}(x_0, 0, 0, \alpha_0) = c \neq 0$,

T2. $\dfrac{\partial j}{\partial \alpha}(x_0, 0, 0, \alpha_0) = 0$, $\dfrac{\partial^2 j}{\partial x \partial \alpha}(x_0, 0, 0, \alpha_0) \neq 0$,

T3. $\dfrac{\partial^2 j}{\partial x^2}(x_0, 0, 0, \alpha_0) \neq 0$,

then system (1) displays a transcritical bifurcation at the point $E(x_0, 0, 0)$.

Theorem 5. *Let $\alpha \in \mathbb{R}$ be the bifurcation parameter of system (1) with the critical value $\alpha_0 \in \mathbb{R}$. If*

P1. $j(x_0, 0, 0, \alpha_0) = 0$, $\dfrac{\partial j}{\partial x}(x_0, 0, 0, \alpha_0) = a = 0$, $\dfrac{\partial j}{\partial y}(x_0, 0, 0, \alpha_0) = b \neq 0$, $\dfrac{\partial j}{\partial z}(x_0, 0, 0, \alpha_0) = c \neq 0$,

P2. $\dfrac{\partial j}{\partial \alpha}(x_0, 0, 0, \alpha_0) = 0$, $\dfrac{\partial^2 j}{\partial x \partial \alpha}(x_0, 0, 0, \alpha_0) \neq 0$,

P3. $\dfrac{\partial^2 j}{\partial x^2}(x_0, 0, 0, \alpha_0) = 0$, $\dfrac{\partial^3 j}{\partial x^3}(x_0, 0, 0, \alpha_0) \neq 0$,

then system (1) displays a pitchfork bifurcation at the point $E(x_0, 0, 0)$.

Remark 5. *The condition $SN1$ (T1, P1) implies that $E(x_0, 0, 0)$ is an equilibrium point of the considered jerk system with a simple zero eigenvalue and no other eigenvalues on the imaginary axis. The genericity conditions of the saddle-node bifurcation are the transversality condition $SN2$ and the nondegeneracy condition $SN3$. Transcritical and pitchfork bifurcations are obtained violating some of these genericity conditions.*

5. Approximate Normal Forms for Fold Bifurcations

In this section, we derive approximate normal forms for each of the above fold bifurcations. We mention that these normal forms are also jerk systems.

In the following, we consider $x_0 = 0$ and $\alpha_0 = 0$. Then, for the fold bifurcation we have

$$j(0,0,0,0) = 0, a = \frac{\partial j}{\partial x}(0,0,0,0) = 0, b = \frac{\partial j}{\partial y}(0,0,0,0) \neq 0, c = \frac{\partial j}{\partial z}(0,0,0,0) \neq 0.$$

5.1. Approximate Normal Forms for the Saddle-Node Bifurcation

We follow the steps used for the one-dimensional case (see, e.g., [15]). Assume that conditions $SN2$ and $SN3$ from Theorem 3 are fulfilled.

Consider a Taylor expansion of the function j with respect to (x, y, z) at $(0, 0, 0)$,

$$j(x,y,z) = j_0(\alpha) + j_1(\alpha)x + \frac{1}{2} j_2(\alpha)x^2 + b(\alpha)y + c(\alpha)z + B(\alpha)xy + C(\alpha)xz + \mathcal{O},$$

where $\mathcal{O}$ is the remainder of the expansion, and

$$
\begin{aligned}
&j_0(\alpha) = j(0,0,0,\alpha), j_1(\alpha) = \frac{\partial j}{\partial x}(0,0,0,\alpha), j_2(\alpha) = \frac{\partial^2 j}{\partial x^2}(0,0,0,\alpha),\\
&b(\alpha) = \frac{\partial j}{\partial y}(0,0,0,\alpha), c(\alpha) = \frac{\partial j}{\partial z}(0,0,0,\alpha),\\
&B(\alpha) = \frac{\partial^2 j}{\partial x \partial y}(0,0,0,\alpha), C(\alpha) = \frac{\partial^2 j}{\partial x \partial z}(0,0,0,\alpha).
\end{aligned}
$$

We do not explicitly use the term yz in the above Taylor expansion because such a term and its derivatives vanish at $(0,0,0,\alpha)$.

Step 1. Using the translation $x = \varepsilon - \delta$, $y = y$, $z = z$, where $\delta = \delta(\alpha)$, system (1) is written as

$$
\begin{array}{rcl}
\dot{\varepsilon} & = & y \\
\dot{y} & = & z \\
\dot{z} & = & \left[j_0(\alpha) - j_1(\alpha)\delta + \frac{1}{2}j_2(\alpha)\delta^2 + \mathcal{O}(\delta^3)\right] + \left[j_1(\alpha) - j_2(\alpha)\delta + \mathcal{O}(\delta^2)\right]\varepsilon \\
 & & +\frac{1}{2}[j_2(\alpha) + \mathcal{O}(\delta)]\varepsilon^2 + (b(\alpha) - B(\alpha)\delta)y + (c(\alpha) - C(\alpha)\delta)z + B(\alpha)\varepsilon y + C(\alpha)\varepsilon z + \mathcal{O}
\end{array}. \tag{7}
$$

We find δ such that the coefficient of ε vanishes. Let

$$F(\alpha,\delta) = j_1(\alpha) - j_2(\alpha)\delta + \psi(\alpha,\delta)\delta^2.$$

Then,

$$F(0,0) = j_1(0) = \frac{\partial j}{\partial x}(0,0,0,0) = 0 \; (SN1),$$

and

$$\frac{\partial F}{\partial \delta}(0,0) = -j_2(0) = -\frac{\partial^2 j}{\partial x^2}(0,0,0,0) \neq 0 \; (SN3).$$

Therefore, by using the implicit function theorem, there is a function $\delta = \delta(\alpha)$ with $\delta(0) = 0$ and $F(\alpha,\delta(\alpha)) = 0$ in the neighborhood of $\alpha = 0$. Moreover, since

$$\frac{\partial F}{\partial \alpha}(0,0) = \frac{\partial j_1}{\partial \alpha}(0) = \frac{\partial^2 j}{\partial x \partial \alpha}(0,0,0,0) := a'(0),$$

we obtain

$$\delta = \frac{a'(0)}{j_2(0)}\alpha + \mathcal{O}(\alpha^2).$$

Using δ and the following Taylor expansions

$$
\begin{aligned}
&j_0(\alpha) = \frac{\partial j}{\partial \alpha}(0,0,0,0)\alpha + \mathcal{O}(\alpha^2) = j_0'(0)\alpha + \mathcal{O}(\alpha^2),\\
&j_1(\alpha) = \frac{\partial^2 j}{\partial x \partial \alpha}(0,0,0,0)\alpha + \mathcal{O}(\alpha^2),\; j_2(\alpha) = j_2(0) + \mathcal{O}(\alpha^2),\\
&b(\alpha) = b + \mathcal{O}(\alpha),\; c(\alpha) = c + \mathcal{O}(\alpha), B(\alpha) = B(0) + \mathcal{O}(\alpha), C(\alpha) = C(0) + \mathcal{O}(\alpha),
\end{aligned}
$$

system (7) becomes

$$
\begin{array}{rcl}
\dot{\varepsilon} & = & y \\
\dot{y} & = & z \\
\dot{z} & = & \left[j_0'(0)\alpha + \mathcal{O}(\alpha^2)\right] + \frac{1}{2}[j_2(0) + \mathcal{O}(\alpha)]\varepsilon^2 + (b + \mathcal{O}(\alpha))y + (c + \mathcal{O}(\alpha))z \\
 & & +(B(0) + \mathcal{O}(\alpha))\varepsilon y + (C(0) + \mathcal{O}(\alpha))\varepsilon z + \mathcal{O}
\end{array}. \tag{8}
$$

Step 2. We denote

$$j_0'(0)\alpha + \psi(\alpha,\delta)\alpha^2 = \mu.$$

Let

$$G(\mu,\alpha) = j_0'(0)\alpha + \psi(\alpha,\delta)\alpha^2 - \mu$$

We have $G(0,0) = 0$ and

$$\frac{\partial G}{\partial \alpha}(0,0) = j_0'(0) = \frac{\partial j}{\partial \alpha}(0,0,0,0) \neq 0 (SN2).$$

Then, there is a function $\alpha = \alpha(\mu)$ such that $\alpha(0) = 0$ and $F(\mu,\alpha(\mu)) = 0$ in the neighborhood of $\mu = 0$. Consequently, system (8) becomes

$$\begin{array}{rcl} \dot{\varepsilon} & = & y \\ \dot{y} & = & z \\ \dot{z} & = & \mu + \frac{1}{2}[j_2(0) + \mathcal{O}(\mu)]\varepsilon^2 + (b + \mathcal{O}(\mu))y + (c + \mathcal{O}(\mu))z \\ & & +(B(0) + \mathcal{O}(\mu))\varepsilon y + (C(0) + \mathcal{O}(\mu))\varepsilon z + \mathcal{O} \end{array}. \qquad (9)$$

Step 3. We denote

$$A(\mu) = \frac{1}{2}[j_2(0) + \mathcal{O}(\mu)].$$

It follows that

$$A(0) = \frac{1}{2}j_2(0) = \frac{1}{2}\frac{\partial^2 j}{\partial x^2}(0,0,0,0) \neq 0 (SN3),$$

thus, $A(\mu) \neq 0$ in the neighborhood $\mathcal{V}$ of $\mu = 0$. Moreover,

$$sign(A(\mu)) := s = \mathrm{sign}(\frac{\partial^2 j}{\partial x^2}(0,0,0,0))$$

on $\mathcal{V}$.

Using the change in the parameter and variables given by

$$\beta = sA(\mu)\mu, X = sA(\mu)\varepsilon,\ Y = sA(\mu)y,\ Z = sA(\mu)z,$$

with $\mu = \frac{2s}{j_2(0)}\beta + \mathcal{O}(\beta^2)$, system (9) becomes

$$\begin{array}{rcl} \dot{X} & = & Y \\ \dot{Y} & = & Z \\ \dot{Z} & = & \beta + sX^2 + (b + \mathcal{O}(\beta))Y + (c + \mathcal{O}(\beta))Z \\ & & +(\frac{2sB(0)}{j_2(0)} + \mathcal{O}(\beta))XY + (\frac{2sC(0)}{j_2(0)} + \mathcal{O}(\beta))XZ + \mathcal{O} \end{array}, \qquad (10)$$

where the remainder $\mathcal{O}$ was properly changed, but it does not contain terms in the form of those already written in the third equation of (10).

In conclusion, we have proven the following result.

Theorem 6. *Consider jerk system* (1) *with smooth j such that $O(0,0,0)$ is an equilibrium at $\alpha = 0$ with a zero eigenvalue, that is, $j(0,0,0,0) = 0$, $\frac{\partial j}{\partial x}(0,0,0,0) = 0$. Suppose that*

SN1. $b = \frac{\partial j}{\partial y}(0,0,0,0) \neq 0$, $c = \frac{\partial j}{\partial z}(0,0,0,0) \neq 0$,

SN2. $\frac{\partial j}{\partial \alpha}(0,0,0,0) \neq 0$,

SN3. $\frac{\partial^2 j}{\partial x^2}(0,0,0,0) \neq 0$.

Then, there are invertible coordinate and parameter changes transforming system (1) *into system* (10).

Let $s = 1$. An equilibrium point of system (10) has the form $E(x_0, 0, 0)$, where x_0 is a solution of the equation $\beta + X^2 + \mathcal{O}(X^3) = 0$. Since the fold bifurcation takes place at $\beta = 0$ in O, we are concerned with equilibria close to O, thus x_0 is near 0. Recall that for any sufficiently small $\beta < 0$, the above equation has two solutions near the origin, which are close to the solutions of the equation $\beta + x^2 = 0$ for the same parameter value (see, e.g., the proof of Lemma 3.1 [15]). Therefore, in order to obtain a normal form for the fold bifurcation in $O(0, 0, 0)$ at $\beta = 0$, we can truncate system (10). Thus, we obtain the following truncated normal form for the generic fold bifurcation of a jerk system:

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = Z \\ \dot{Z} = \beta + sX^2 + bY + cZ + \frac{2sB(0)}{j_2(0)}XY + \frac{2sC(0)}{j_2(0)}XZ \end{cases}. \tag{11}$$

Moreover, we can consider the following approximate normal form for the generic fold bifurcation of a jerk system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = \alpha + sx^2 + by + cz \end{cases}, \tag{12}$$

where $\alpha \in \mathbb{R}$ is the parameter, $s = \pm 1$, and $b, c \in \mathbb{R}$ are fixed such that $b, c \neq 0$.

We note that the last two systems have the same equilibria and the corresponding characteristic polynomials are similar.

We will study system (12) near the origin and α in the neighborhood of 0.

Let $s = 1$. Then, system (12) has two equilibria $E^{\pm}(\pm\sqrt{-\alpha}, 0, 0)$ for $\alpha < 0$; one equilibrium point $O(0, 0, 0)$ for $\alpha = 0$; and no equilibria for $\alpha > 0$.

Proposition 2. *Let $O(0, 0, 0)$ and $E^{\pm}(\pm\sqrt{-\alpha}, 0, 0)$ be the equilibria of system (12), where $\alpha \leq 0$ such that $|\alpha|$ is sufficiently small.*

1. *Assume $b < 0$ and $c < 0$.*
   (a) *If $\alpha < 0$, then E^- is asymptotically stable and E^+ is unstable.*
   (b) *If $\alpha = 0$, then $O = E$ is unstable.*
2. *If $b > 0$ or $c > 0$, then O and $E^{\pm}$ are unstable for any α, provided they exist.*

Proof. According to Theorem 1, it only remains to study the case $\alpha = 0$ with $b < 0, c < 0$. Because the equilibrium point has a zero eigenvalue, we will reduce system (12) on the center manifold (see, e.g., [17,20]). Using the eigenvectors of the Jacobian matrix at O, we deduce the following transformation

$$\begin{cases} u = \dfrac{c + \sqrt{c^2 + 4b}}{2b\sqrt{c^2 + 4b}}\left(-\dfrac{c + \sqrt{c^2 + 4b}}{2}y + z\right) \\ v = \dfrac{c - \sqrt{c^2 + 4b}}{2b\sqrt{c^2 + 4b}}\left(\dfrac{c - \sqrt{c^2 + 4b}}{2}y + z\right) \\ w = x + \dfrac{c}{b}y - \dfrac{1}{b}z \end{cases}.$$

Then, system (12) becomes

$$\begin{cases} \dot{u} = \dfrac{c - \sqrt{c^2 + 4b}}{2}u + \dfrac{c + \sqrt{c^2 + 4b}}{2b\sqrt{c^2 + 4b}}(u + v + w)^2 \\ \dot{v} = \dfrac{c + \sqrt{c^2 + 4b}}{2}v - \dfrac{c - \sqrt{c^2 + 4b}}{2b\sqrt{c^2 + 4b}}(u + v + w)^2 \\ \dot{w} = -\dfrac{1}{b}(u + v + w)^2 \end{cases}. \tag{13}$$

Consider the center manifold

$$W_{\text{loc}}^{O} = \{(u, v, w)|u = F(w), v = G(w), |w| \ll 1, F(0) = F'(0) = G(0) = G'(0) = 0\},$$

where $F(w) = Aw^2 + \mathcal{O}(w^3), G(w) = Bw^2 + \mathcal{O}(w^3)$. Using (13), we obtain

$$A = \frac{(c + \sqrt{c^2 + 4b})^2}{4b^2\sqrt{c^2 + 4b}}\ ,\ B = -\frac{(c - \sqrt{c^2 + 4b})^2}{4b^2\sqrt{c^2 + 4b}},$$

and the reduced equation on W_{loc}^{O}, given by

$$\dot{w} = -\frac{1}{b}w^2 + \mathcal{O}(w^2).$$

Thus, O is unstable, and the proof is finished. □

For $s = -1$, we obtain a similar result.

Remark 6. *We notice that the results presented in Remark 4 are valid for the approximate normal form* (12)*.*

In Proposition 2, we discuss the stability of system (12) for α close to zero. The equilibrium points can be stable only if $b, c < 0$. In this case, when $\alpha < 0$ moves away from zero we deduce that E^+ remains unstable and E^- changes its stability for $\alpha = -\frac{b^2c^2}{4}$ (see Theorem 1). Moreover, a Hopf bifurcation occurs in the dynamics of system (12) (Theorem 2), and a stable limit cycle is born (see Figure 1c–e). The saddle-node and Hopf bifurcations are also pointed out in the bifurcation diagram obtained by numerical simulations using MatCont/MatLab (Figure 1a).

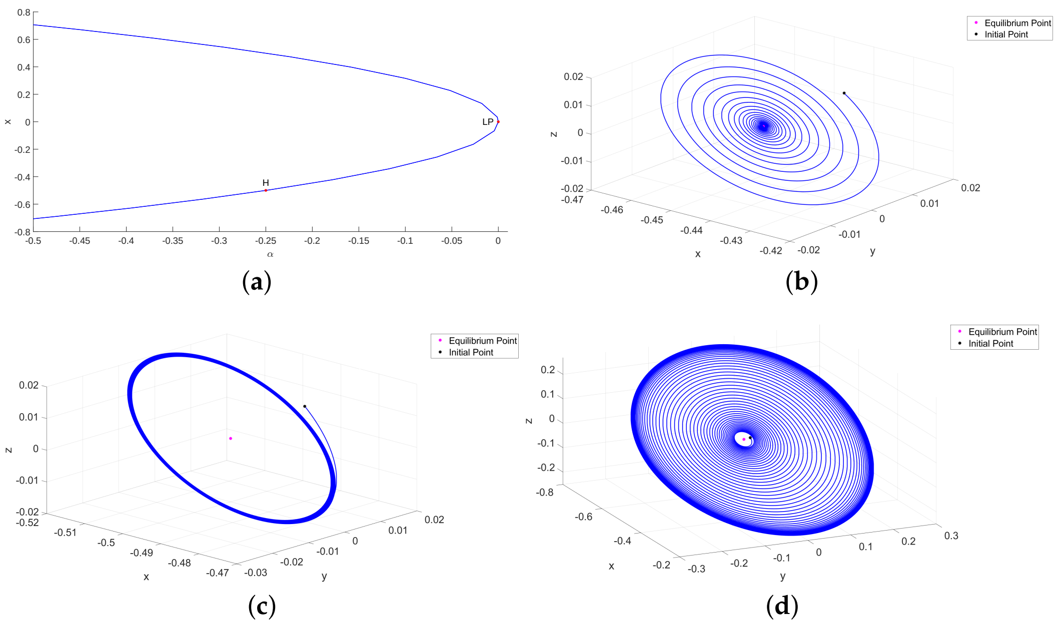
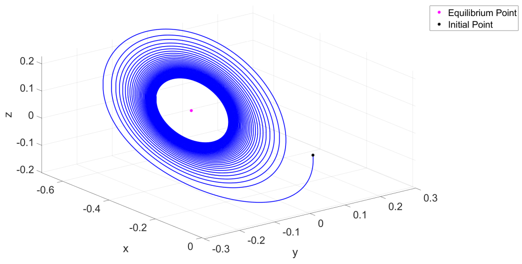
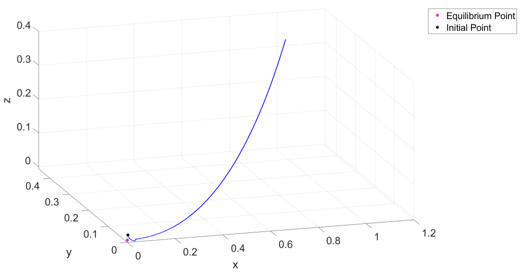


Figure 1. *Cont.*



(e)



(f)

Figure 1. The normalform (12), $s = 1, b = -1, c = -1$: (**a**) the bifurcation diagram (H stands for Hopf bifurcation and LP for the limit point, that is, the fold bifurcation); (**b**) an asymptotically stable orbit ($a = -0.2$); (**c**) a stable limit cycle ($a = -0.25$); (**d**,**e**) unstable orbits attracted by the limit cycle ($a = -0.28$ and $a = -0.25$, respectively); (**f**) an unbounded unstable orbit ($a = 0$). The initial conditions are close to E^- in (**d**) and close to O in the other cases.

5.2. Approximate Normal Forms for the Transcritical Bifurcation

The transcritical bifurcation is a non-generic fold bifurcation, and the method used in the above paragraph is useless.

Assume that hypotheses $T1$–$T3$ of Theorem 4 are satisfied. Therefore, we have the following Taylor expansion of the function j with respect to (x, y, z, α) at $(0, 0, 0, 0)$

$$j(x, y, z, \alpha) = by + cz + a_{11}x^2 + a_{12}xy + a_{13}xz + a_{14}\alpha x + \mathcal{O},$$

where $b, c, a_{11}, a_{14} \neq 0$. Let $s = sign(a_{11})$. Using the transformation

$$X = s \cdot a_{11}x, Y = s \cdot a_{11}y, Z = s \cdot a_{11}z, \beta = a_{14}\alpha,$$

system (1) becomes

$$\left\{ \begin{array}{l} \dot{X} = Y \\ \dot{Y} = Z \\ \dot{Z} = \beta X + sX^2 + bY + cZ + s\frac{a_{12}}{a_{11}}XY + s\frac{a_{13}}{a_{11}}XZ + \mathcal{O} \end{array} \right. . \tag{14}$$

An equilibrium point of system (14) has the form $E(x_0, 0, 0)$, where x_0 is a solution of the equation $\beta X + sX^2 + \mathcal{O}(X^3) = 0$. A solution is $x_1 = 0$, and the others satisfy $F(\beta, X) = 0$ near $(0, 0)$, where $F(\beta, X) = \beta + sX + X^2\varphi(X, \beta)$. From the implicit function theorem, there is a unique function $x = x(\beta)$ with $x(0) = 0$ and $F(\beta, x(\beta)) = 0$ in the neighborhood of $\beta = 0$. Moreover, $x'(0) = -s$, thus, $x(\beta) = -s\beta + \mathcal{O}(\beta^2)$ near 0, that is, for any sufficiently small β, the equation $F(\beta, X) = 0$ has a unique solution $X = X(\beta)$ near the origin, which is close to the solution of the equation $\beta + sx = 0$ for the same parameter value. In conclusion, near the origin, the equilibria of system (14) are close to the equilibria of its truncated normal forms

$$\left\{ \begin{array}{l} \dot{X} = Y \\ \dot{Y} = Z \\ \dot{Z} = \beta X + sX^2 + bY + cZ + s\frac{a_{12}}{a_{11}}XY + s\frac{a_{13}}{a_{11}}XZ \end{array} \right. \tag{15}$$

and

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = \alpha x + sx^2 + by + cz \end{array} \right. , \tag{16}$$

where $\alpha \in \mathbb{R}$ is the parameter, $s = \pm 1$, and $b, c \in \mathbb{R}$ are fixed such that $b, c \neq 0$.

The characteristic polynomials at the equilibria of the last two systems are similar. Therefore, we consider that system (16) is an approximate normal form for the transcritical bifurcation of a jerk system.

Let $s = -1$. Then, system (16) has two equilibria for every $\alpha \neq 0$, namely, $O(0,0,0)$ and $E(\alpha, 0, 0)$, which collide for $\alpha = 0$.

Proposition 3. *Let O and E be the equilibria of system (16) and $\alpha \in \mathbb{R}$ with sufficiently small $|\alpha|$.*

1. *Assume $b < 0$ and $c < 0$.*
   (a) *If $\alpha < 0$, then O is asymptotically stable and E is unstable.*
   (b) *If $\alpha = 0$, then $O = E$ is unstable.*
   (c) *If $\alpha > 0$, then E is asymptotically stable and O is unstable.*
2. *If $b > 0$ or $c > 0$, then O and E are unstable for any α.*

Proof. We immediately obtain the results $1(a)$, $1(c)$, and 2 via Theorem 1. For $\alpha = 0$ $(b < 0, c < 0)$, we proceed as in the proof of Proposition 2. In this case, the reduced equation on the center manifold is given by

$$\dot{w} = \frac{1}{b}w^2 + \mathcal{O}(w^3).$$

Thus, O is unstable, and the proof is finished. □

For $s = 1$, we obtain a similar result.

Remark 7. *We notice that the case $b < 0, c < 0$ explains the transcritical bifurcation, also named the exchange of stability: two equilibria collide when the parameter passes through the critical value and then split, exchanging their stability. In the other cases, even if both equilibria are unstable, there is an exchange of the dimensions of the stable manifolds at the bifurcation value.*

The bifurcation diagram of system (16) is presented in Figure 2. Again, we notice a Hopf bifurcation, which follows from Theorem 2. In this case, the orbits look similar to those depicted in Figure 1.

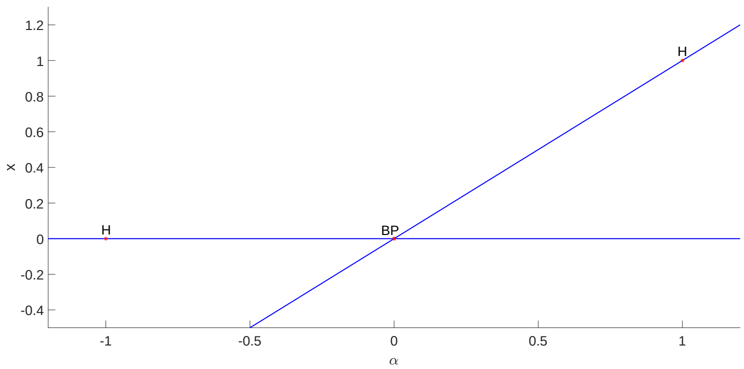


Figure 2. The bifurcation diagram of the normal form (16), $s = -1, b = -1, c = -1$ (BP stands for the branch point, that is, it highlights the transcritical bifurcation in this case).

5.3. Approximate Normal Forms for the Pitchfork Bifurcation

In this case, we work under the hypothesis $P1$–$P3$ of Theorem 5. It results in the following Taylor expansion of the function j with respect to (x, y, z, α) at $(0, 0, 0, 0)$

$$j(x, y, z, \alpha) = by + cz + a_{11}x^3 + a_{12}xy + a_{13}xz + a_{14}\alpha x + \mathcal{O},$$

where $b, c, a_{11}, a_{14} \neq 0$. Using the transformation

$$X = px, Y = py, Z = pz, \beta = a_{14}\alpha,$$

where $p = \sqrt{sa_{11}}, s = sign(a_{11})$, system (1) writes

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = Z \\ \dot{Z} = \beta X + sX^3 + bY + cZ + \frac{a_{12}}{p} XY + \frac{a_{13}}{p} XZ + \mathcal{O} \end{cases}. \tag{17}$$

An equilibrium point of system (17) has the form $E(x_0, 0, 0)$, where x_0 is a solution of the equation $\beta X + sX^3 + \mathcal{O}(X^4) = 0$. A solution is $x_1 = 0$, and the others satisfy $\beta + sX^2 + \mathcal{O}(X^3) = 0$ near $(0,0)$, which is the same equation as in the case of the saddle-node bifurcation. Thus, we can truncate the above expansion. Consequently, we obtain the following approximate normal forms for the pitchfork bifurcation of a jerk system:

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = Z \\ \dot{Z} = \beta X + sX^3 + bY + cZ + \frac{a_{12}}{p} XY + \frac{a_{13}}{p} XZ \end{cases} \tag{18}$$

and

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = \alpha x + sx^3 + by + cz \end{cases}, \tag{19}$$

where $\alpha \in \mathbb{R}$ is the parameter, $s = \pm 1$, and $b, c \in \mathbb{R}$ are fixed such that $b, c \neq 0$.

Proposition 4. *Let $s = -1$. Then, $O(0,0,0)$ is an equilibrium point of system (19) for every $\alpha \in \mathbb{R}$, and other equilibria are $E^{\pm}(\pm\sqrt{\alpha}, 0, 0)$ if $\alpha > 0$.*

Let $\alpha \in \mathbb{R}$ with sufficiently small $|\alpha|$.

1. *Assume $b < 0$ and $c < 0$.*
   (a) *If $\alpha < 0$, then O is asymptotically stable.*
   (b) *If $\alpha = 0$, then $O = E^+ = E^-$ is stable.*
   (c) *If $\alpha > 0$, then $E^{\pm}$ are asymptotically stable and O is unstable.*
2. *If $b > 0$ or $c > 0$, then O and $E^{\pm}$ are unstable for any α, provided they exist.*

Proof. For $\alpha = 0$ $(b < 0, c < 0)$, we proceed as in the proof of Proposition 2. In this case, the reduced equation of the center manifold is given by

$$\dot{w} = \frac{1}{b} w^3 + \mathcal{O}(w^4),$$

thus O is stable. The other conclusions result from Theorem 1. □

Similarly, we obtain:

Proposition 5. *Let $s = 1$. Then, $O(0,0,0)$ is an equilibrium point of system (19) for every $\alpha \in \mathbb{R}$, and other equilibria are $E^{\pm}(\pm\sqrt{-\alpha}, 0, 0)$ if $\alpha < 0$.*

Let $\alpha \in \mathbb{R}$ with sufficiently small $|\alpha|$.

1. *Assume $b < 0$ and $c < 0$.*
   (a) *If $\alpha < 0$, then O is asymptotically stable and $E^{\pm}$ are unstable.*
   (b) *If $\alpha = 0$, then $O = E^+ = E^-$ is unstable.*
   (c) *If $\alpha > 0$, then O is unstable.*
2. *If $b > 0$ or $c > 0$, then O and $E^{\pm}$ are unstable for any α, provided they exist.*

Proof. For $\alpha = 0, b < 0, c < 0$ the reduced equation on the center manifold is given by

$$\dot{w} = -\frac{1}{b} w^3 + \mathcal{O}(w^4),$$

thus O is unstable. $\square$

Remark 8. *The pitchfork bifurcation shows the following local phenomenon in the dynamics of a system: at the point of bifurcation, the stability of an equilibrium changes, and a pair of equilibria appears (see Figure 3) or disappears (see Figure 4). This situation is pointed out in the case $b < 0, c < 0$. More precisely, for $s = -1$ we have obtained the supercritical pitchfork bifurcation, that is, a stable equilibrium point becomes unstable and a pair of stable equilibria is born from it at the branch point BP (Figure 3). For $s = 1$, a stable equilibrium point collides with a pair of unstable equilibria and becomes unstable, which indicates a subcritical pitchfork bifurcation (Figure 4).*

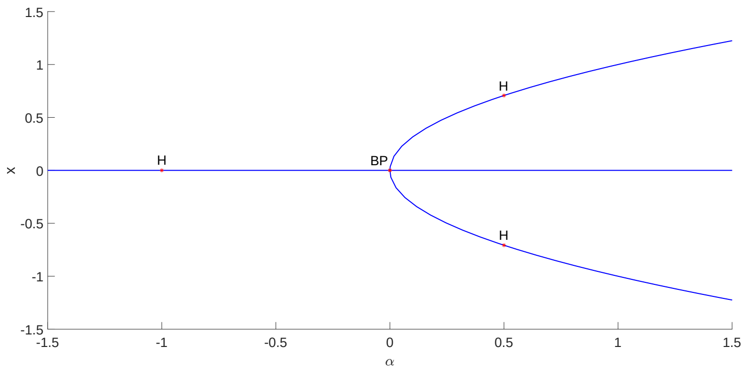


Figure 3. The bifurcation diagram of the normal form (19), $s = -1, b = -1, c = -1$.

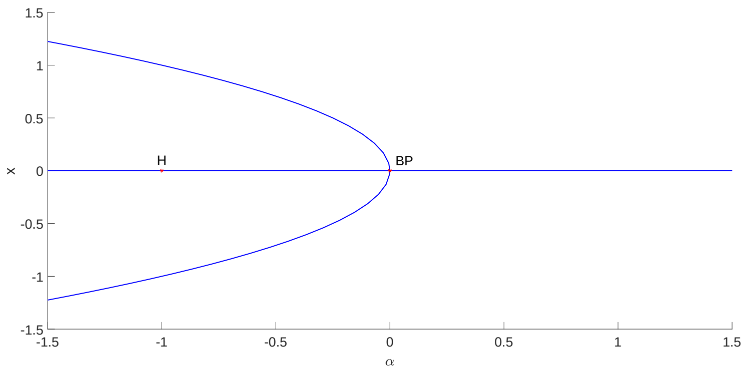


Figure 4. The bifurcation diagram of the normal form (19), $s = 1, b = -1, c = -1$.

The pitchfork bifurcation is specific to systems with symmetries, more precisely to those systems that are invariant under the transformation $(x, y, z) \to (-x, -y, -z)$. We notice that system (19) is such a system. Moreover, such systems can have symmetrical orbits and symmetrical chaotic attractors (see, e.g., [2,10]), as shown in Figure 5.

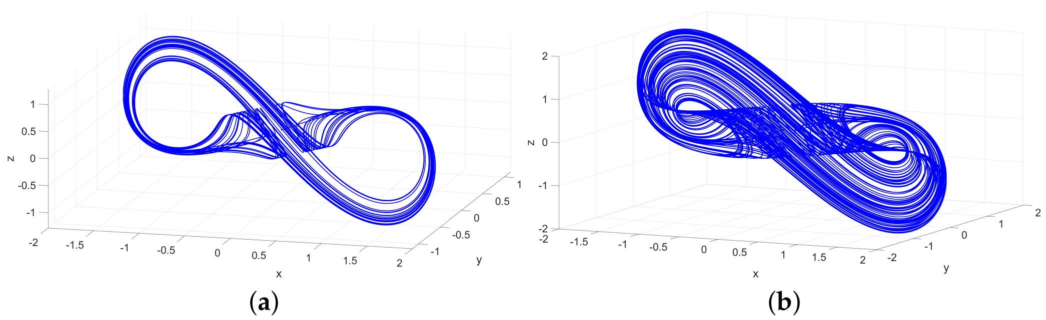


Figure 5. Chaotic attractors in the dynamics of system (19) for $s = -1, b = -1, c = -1$: (**a**) $\alpha = 1.37$; (**b**) $\alpha = 1.47$.

6. Conclusions

In this paper, we have studied the local stability and codim-1 bifurcations of an arbitrary one-parameter jerk system. We have established conditions for which Hopf or fold bifurcations occur in the dynamics of a jerk system. It is known that for an $n-$dimensional dynamical system, a normal form of a local bifurcation is obtained by reduction on the center manifold. The normal forms of the most well-known fold bifurcations, namely the saddle-node bifurcation, the transcritical bifurcation, and the pitchfork bifurcation, are given by one-dimensional equations. Our goal was to derive three-dimensional (approximate) normal forms, which are themselves jerk systems. We have obtained such a normal form for each above-mentioned fold bifurcation. Particularly, because the saddle-node bifurcation is a generic fold bifurcation, we have derived such an approximate normal form using invertible coordinate and parameter changes.

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