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Article

On System of Root Vectors of Perturbed Regular Second-Order Differential Operator Not Possessing Basis Property

Makhmud Sadybekov [1,2] and Nurlan Imanbaev [1,3,*]

1 Institute of Mathematics and Mathematical Modeling, Almaty 050010, Kazakhstan; sadybekov@math.kz
2 Depatment of Mechanics and Mathematics, Al-Farabi Kazakh National University, Almaty 050040, Kazakhstan
3 Faculty of Physics and Mathematics, South-Kazakhstan State Pedagogical University, Shymkent 160012, Kazakhstan
* Correspondence: imanbaev@math.kz

Abstract: This article delves into the spectral problem associated with a multiple differentiation operator that features an integral perturbation of boundary conditions of one specific type, namely, regular but not strengthened regular. The integral perturbation is characterized by the function $p(x)$, which belongs to the space $L_2(0,1)$. The concept of problems involving integral perturbations of boundary conditions has been the subject of previous studies, and the spectral properties of such problems have been examined in various early papers. What sets the problem under consideration apart is that the system of eigenfunctions for the unperturbed problem (when $p(x) \equiv 0$) lacks the property of forming a basis. To address this, a characteristic determinant for the spectral problem has been constructed. It has been established that the set of functions $p(x)$, for which the system of eigenfunctions of the perturbed problem does not constitute an unconditional basis in $L_2(0,1)$, is dense within the space $L_2(0,1)$. Furthermore, it has been demonstrated that the adjoint operator shares a similar structure.

Keywords: second-order differential operator; eigenvalue; system of root vectors; basis property; integral perturbation of boundary conditions; characteristic determinant

MSC: 34B09; 34B10; 34B37; 34L10

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1. Introduction and Formulation of Problem

It is widely acknowledged that the operator defined by a formally self-adjoint differential expression with arbitrary self-adjoint boundary conditions exhibits a discrete spectrum, and its eigenfunctions collectively form an orthonormal basis in the space L_2. The question of whether this basis property is preserved under certain (weakened in a specific sense) perturbations of the original operator has been the subject of investigation in numerous research papers.

In the case of the self-adjoint original operator, a similar question has been addressed in prior works such as [1] through [2], while for non-self-adjoint operators, research efforts can be found in references such as [3] through [4].

In this paper, we delve into a spectral problem that closely relates to the investigations conducted in [5]:

$$l(u) \equiv -u''(x) = \lambda u(x),\ 0 < x < 1, \tag{1}$$

$$U_1(u) \equiv u'(0) + u'(1) - \alpha u(1) = 0,\ \alpha > 0, \tag{2}$$

$$U_2(u) \equiv u(0) = 0. \tag{3}$$

Here, the parameter $\alpha > 0$ is any positive number of your choice.

Given that Equation (1) is a straightforward equation with constant coefficients, allowing for an explicit solution, there is no doubt about the Fredholm property of problem (1)–(3). Therefore, the eigenvalue problem, and the eigenfunction problem of this problem, can be considered.

It is widely recognized that the system of root functions of an ordinary differential operator, subject to arbitrary strengthened regular boundary conditions, constitutes a Riesz basis in the space $L_2(0,1)$. You can refer to examples in literature like [6] through [7] for further details on this topic. Today, it is broadly accepted that the completeness property of a system of root functions (in cases of degenerate boundary conditions), and the basis property of root functions (when the boundary conditions are regular but not strongly regular), are influenced not only by the type of boundary conditions, but also by the values of the coefficients within the equation itself. This effect was first observed by V.A. Il'in in [8], who constructed a relevant example for a second-order differential operator of a general form. The research reveals that the presence of the basis property is contingent upon both the boundary conditions and the coefficient values of the differential operator. Furthermore, this property can change with even the slightest variations in the coefficients within the specified classes.

Suppose L_1 is an operator in $L_2(0,1)$, given by expression (1), boundary condition (2) and "perturbed" condition (1) in the form:

$$U_2(u) \equiv u(0) = \int_0^1 \overline{p(x)} u(x) dx, \ p(x) \in L_2(0,1). \tag{4}$$

The operator corresponding to problem (1)–(3) in the case where $p(x) \equiv 0$ is denoted as L_0 and is represented as the *unperturbed operator* .

In [9], the stability of the basis property of root vectors for the spectral problem is explored when $\alpha = 0$. In this particular case, where $\alpha = 0$, the system of eigen- and associated functions of the unperturbed problem constitutes an unconditional basis in the space $L_2(0,1)$.

In our previous paper [10], we studied various options for the integral perturbation of boundary conditions. In that paper, under the assumption that the unperturbed operator L_0 possesses a system of eigen- and associated functions that forms a Riesz basis in $L_2(0,1)$, a characteristic determinant for the spectral problem associated with the operator L_1 was developed. Using the derived formula, conclusions were drawn regarding the stability or instability of the Riesz basis properties of the eigen- and associated functions in the context of the problem with the integral perturbation of the boundary condition.

The pivotal distinction of this paper lies in the fact that the system of eigenfunctions of the unperturbed problem (1)–(3) is *complete*, yet *it does not constitute a basis* in $L_2(0,1)$, as highlighted in [11]. Therefore, the previously used method from our previous papers cannot be applied in this case.

In our paper [5], we studied the "perturbation" of the operator L_0, when the first boundary condition $U_1(u)$ is perturbed. In this paper, we study the case of "perturbation" of the second boundary condition (condition $U_2(u)$).

Studies on the stability of the spectral properties of an operator under (limited or subordinate) perturbation of the boundary condition are close to the perturbation of the action of operators by certain (subordinate in a certain sense) operators. Among such works, the work [12] is close.

2. Construction of Auxiliary Basis

In the unperturbed problem (1)–(3), the boundary conditions are regular but not strengthened regular [13]. The system of root functions of the operator L_0 is a complete system, but in $L_2(0,1)$ it does not form even an ordinary basis [11].

Nonetheless, as demonstrated in [14], it is possible to construct a basis using these eigenfunctions, which enables the application of the method of separation of variables for solving an initial-boundary value problem with the boundary condition (2). And we use the concept from [14] for addressing our specific problem.

By expressing a general solution of Equation (1) as

$$u(x,\lambda) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \sin\left(\sqrt{\lambda}x\right),$$

and substituting this expression into the boundary conditions (4), we obtain a linear system in terms of the coefficients C_k:

$$\begin{cases} -C_1\left(\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) + \alpha\cos\left(\sqrt{\lambda}\right)\right) + C_2\left(\sqrt{\lambda}\left(1+\cos\left(\sqrt{\lambda}\right)\right) - \alpha\sin\left(\sqrt{\lambda}\right)\right) = 0, \\ C_1\left(1 - \int_0^1 \overline{p(x)}(x)\cos\left(\sqrt{\lambda}x\right) dx\right) - C_2\int_0^1 \overline{p(x)}\sin\left(\sqrt{\lambda}x\right)dx = 0. \end{cases} \tag{5}$$

Therefore, the characteristic determinant of problem (1), (4) has the following form:

$$\begin{aligned} \Delta_1(\lambda) &= \left(\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) + \alpha\cos\left(\sqrt{\lambda}\right)\right)\int_0^1 \overline{p(x)}\sin\left(\sqrt{\lambda}x\right)dx \\ &- \left(\sqrt{\lambda}\left(1+\cos\left(\sqrt{\lambda}\right)\right) - \alpha\sin\left(\sqrt{\lambda}\right)\right)\left(1 - \int_0^1 \overline{p(x)}\cos\left(\sqrt{\lambda}x\right)dx\right). \end{aligned} \tag{6}$$

In the case where $p(x) \equiv 0$, we derive the characteristic determinant of the unperturbed problem (1)–(3). This characteristic determinant is denoted as

$$\Delta_0(\lambda) = \alpha\sin\left(\sqrt{\lambda}\right) - \sqrt{\lambda}\left(1+\cos\left(\sqrt{\lambda}\right)\right).$$

By solving the equation

$$\Delta_0(\lambda) \equiv 2\cos\left(\frac{\sqrt{\lambda}}{2}\right)\left[\alpha\sin\left(\frac{\sqrt{\lambda}}{2}\right) - \sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}}{2}\right)\right] = 0,$$

we identify two series of eigenvalues for the unperturbed problem (1)–(3):

$$\begin{aligned} \lambda_k^{(1)} &= ((2k+1)\pi)^2, \\ \lambda_k^{(2)} &= (2\beta_k)^2. \end{aligned}$$

In the provided context, where $k = 0, 1, 2, \ldots$, the values of β_k are the roots of the equation

$$\cot\beta = \frac{\alpha}{2\beta},\ \beta > 0, \tag{7}$$

and these roots are positive while satisfying the following inequalities:

$$\frac{\pi}{2} + \pi k < \beta_k < \pi + \pi k,\ k = 0, 1, 2, \ldots.$$

For sufficiently large values of k, the asymptotic behavior of δ_k, defined as

$$\delta_k = \beta_k - \left(\frac{\pi}{2} + \pi k\right),$$

follows the relation

$$\delta_k = O\left(\frac{1}{k}\right). \tag{8}$$

This indicates that δ_k diminishes as the integer k becomes larger.

The eigenfunctions of the unperturbed problem (1)–(3) take the following forms:

$$u_k^{(1)}(x) = \sin((2k+1)\pi)x,$$
$$u_k^{(2)} = \sin(2\beta_k x).$$

Here, $k = 0, 1, 2, \ldots$ in both cases.

Let us demonstrate that the system $\left\{u_k^{(1)}, u_k^{(2)}\right\}$ does not constitute an unconditional basis in $L_2(0,1)$.

The adjoint problem to (1)–(3) is the boundary value problem given by

$$l^*(u) \equiv -v''(x) = \overline{\lambda} v(x),\ v(0) + v(1) = 0,\ v'(1) + \alpha v(0) = 0. \quad (9)$$

Its eigenvalues can be calculated explicitly:

$$v_k^{(1)}(x) = C_k^{(1)}\left(\cos((2k+1)\pi)x - \frac{\alpha}{\pi + 2\pi k}\sin((2k+1)\pi)x\right),\ k = 0, 1, \ldots,$$
$$v_k^{(2)}(x) = C_k^{(2)}\left(\cos(2\beta_k x) - \frac{\alpha}{2\beta_k}\sin(2\beta_k x)\right),\ k = 0, 1, \ldots.$$

Let us choose $C_k^{(1)}, C_k^{(2)}$ from the biorthonormalization relations

$$\left(u_k^{(1)}, v_k^{(1)}\right) = 1,\ \left(u_k^{(2)}, v_k^{(2)}\right) = 1.$$

Hence, it is easy to see

$$C_k^{(1)} = -\frac{2(\pi + 2\pi k)}{\alpha},\ C_k^{(2)} = -\frac{4}{\alpha}\beta_k + O\left(\frac{1}{k}\right) = -\frac{4}{\alpha}\left(\frac{\pi}{2} + \pi k\right) + O\left(\frac{1}{k}\right). \quad (10)$$

Therefore,

$$\lim_{k\to\infty}\left\|u_k^{(1)}\right\|\left\|v_k^{(1)}\right\| = \lim_{k\to\infty}\left|C_k^{(1)}\right|\frac{1}{2}\sqrt{1 + \left(\frac{\alpha}{\pi + 2\pi k}\right)^2} = \lim_{k\to\infty}\frac{\pi + 2\pi k}{|\alpha|} = \infty. \quad (11)$$

Indeed, the uniform minimality condition, as given by

$$\lim_{k\to\infty}\|u_k\|\|v_k\| < \infty,$$

is not satisfied. This condition is a necessary requirement for the unconditional basis property of the system, as indicated in reference [5]. In this case, the system $\left\{u_k^{(1)}, u_k^{(2)}\right\}$ does not meet this condition, further supporting the conclusion that it does not form an unconditional basis in $L_2(0,1)$.

Thus, the following lemma is proven.

Lemma 1. *The system of eigenfunctions* $\left\{u_k^{(1)}, u_k^{(2)}\right\}$ *does not constitute an unconditional basis in* $L_2(0,1)$.

Next, we will use the idea of the paper [14], which proposes a method of the construction of a basis from the system of eigenfunctions of the problem, similar to the one under study.

Consider an auxiliary system

$$u_{2k}(x) = u_k^{(1)}(x),\ k = 0, 1, 2, \ldots,$$
$$u_{2k-1}(x) = \left(u_k^{(2)}(x) - u_k^{(1)}(x)\right)\cdot(2\delta_k)^{-1},\ k = 1, 2, \ldots.$$

It is easily seen that the system

$$v_{2k}(x) = v_k^{(2)}(x) + v_k^{(1)}(x),\ k = 0,1,2,\ldots,$$
$$v_{2k-1}(x) = 2\delta_k v_k^{(2)}(x),\ k = 1,2,\ldots.$$

which is constructed from the eigenfunctions of problem (9), is biorthogonal to it.

We show that the system $\{u_k\}$, and consequently, the system $\{v_k\}$ form the Riesz basis in $L_2(0,1)$. It is generally known that a system that is quadratically close to the Riesz basis is also the Riesz basis. As a known system, let us choose the following system:

$$\phi_{2k}(x) = \sin((2k+1)\pi x),\ \phi_{2k-1}(x) = x\cos((2k+1)\pi)x,\ k = 0,1,2,\ldots.$$

These functions are the eigenfunctions (ϕ_{2k}) and the associated functions (ϕ_{2k-1}) of the Samarskii–Ionkin-type problem

$$-\phi''(x) = \lambda\phi(x),\ 0 < x < 1,$$
$$\phi(0) = 0,\ \phi'(0) + \phi'(1) = 0.$$

As demonstrated in [11], the system of eigenfunctions and associated functions of this problem constitutes a Riesz basis in the space $L_2(0,1)$. Let us show that the systems of functions $\{u_k(x)\}$ and $\{\phi_k(x)\}$ are quadratically close. Indeed, we have

$$\sum_{k=1}^{\infty} \|\phi_k - u_k\|^2 = \sum_{k=1}^{\infty} \|\phi_{2k-1} - u_{2k-1}\|^2.$$

By direct calculation, we have

$$u_{2k-1}(x) = \frac{\sin(\delta_k x)}{\delta_k}\cos((2k+1)\pi + \delta_k)x.$$

Therefore,

$$\begin{aligned}
&\|\phi_{2k-1} - u_{2k-1}\| = \left\| x\cos((2k+1)\pi x) - \frac{\sin(\delta_k x)}{\delta_k}\cos((2k+1)\pi + \delta_k)x \right\| \\
&\le \left\| x\cos((2k+1)\pi x) - \frac{\sin(\delta_k x)}{\delta_k}\cos((2k+1)\pi x) \right\| \\
&+ \left\| \frac{\sin(\delta_k x)}{\delta_k}\cos((2k+1)\pi x) - \frac{\sin(\delta_k x)}{\delta_k}\cos((2k+1)\pi + \delta_k)x \right\| \\
&\le \left\| x - \frac{\sin(\delta_k x)}{\delta_k} \right\| \|\cos((2k+1)\pi x)\| \\
&+ \left\| \frac{\sin(\delta_k x)}{\delta_k} \right\| \|\cos((2k+1)\pi)x - \cos((2k+1)\pi + \delta_k)x\| \\
&\le \left\| x - \frac{\sin(\delta_k x)}{\delta_k} \right\| + 2\left\| \sin\left((2k+1)\pi + \frac{\delta_k}{2}\right)x \right\| \left\| \sin\left(\frac{\delta_k}{2}x\right) \right\|.
\end{aligned}$$

From here, taking into account that $\sin z = z + O(z^3)$, we obtain

$$\|\phi_{2k-1} - u_{2k-1}\| \le O(\delta_k).$$

Taking into account assymptotics (8), we find that

$$\sum_{k=1}^{\infty} \|\phi_k - u_k\|^2 = C\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

That is, the systems of functions $\{u_k(x)\}$ and $\{\phi_k(x)\}$ are quadratically close. Thus, we have proved:

Lemma 2. *The auxiliary system $\{u_k(x)\}$, and consequently, the system $\{v_k(x)\}$ form the Riesz basis in $L_2(0,1)$.*

3. Characteristic Determinant of the Spectral Problem (1), (2), (4)

The function $p(x)$, belonging to $L_2(0,1)$, can be expressed as a Fourier series with respect to the auxiliary system $\{v_k(x)\}$:

$$p(x) = \sum_{k=0}^{\infty} \alpha_k v_k(x), \tag{12}$$

where α_k are the Fourier coefficients.

We calculate the integrals included in (6):

$$\int_0^1 \overline{p(x)} \sin\left(\sqrt{\lambda}x\right)dx = \left[\sqrt{\lambda}\left(1+\cos\left(\sqrt{\lambda}\right)\right) - \alpha\sin\left(\sqrt{\lambda}\right)\right]$$
$$\cdot\left[\sum_{k=1}^{\infty}\alpha_k\cdot\left(\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}+\frac{C_k^{(1)}}{\lambda-((2k+1)\pi)^2}\right)+\sum_{k=1}^{\infty}\alpha_k\cdot 2\delta_k\cdot\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}\right],$$
$$\int_0^1 \overline{p(x)} \cos\left(\sqrt{\lambda}x\right)dx = \left[\alpha\cdot\left(1-\cos\left(\sqrt{\lambda}\right)\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)\right]$$
$$\cdot\left[\sum_{k=1}^{\infty}\alpha_k\cdot\left(\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}+\frac{C_k^{(1)}}{\lambda-((2k+1)\pi)^2}\right)+\sum_{k=1}^{\infty}\alpha_k\cdot 2\delta_k\cdot\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}\right].$$

We use the obtained result and reduce determinant (6) to the form:

$$\Delta_1(\lambda) = \left(\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)+\alpha\cdot\cos\left(\sqrt{\lambda}\right)\right)\cdot\left[\sqrt{\lambda}\left(1+\cos\left(\sqrt{\lambda}\right)\right)-\alpha\sin\left(\sqrt{\lambda}\right)\right]$$
$$\cdot\left[\sum_{k=1}^{\infty}\alpha_k\cdot\left(\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}+\frac{C_k^{(1)}}{\lambda-((2k+1)\pi)^2}\right)+\sum_{k=1}^{\infty}\alpha_k\cdot 2\delta_k\cdot\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}\right]$$
$$-\left[\sqrt{\lambda}\left(1+\cos\left(\sqrt{\lambda}\right)\right)-\alpha\sin\left(\sqrt{\lambda}\right)\right]\cdot\left(1-\left(\alpha\left(1-\cos\sqrt{\lambda}\right)\right)-\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)\right)$$
$$\cdot\left[\sum_{k=0}^{\infty}\alpha_k\cdot\left(\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}+\frac{C_k^{(1)}}{\lambda-((2k+1)\pi)^2}\right)+\sum_{k=1}^{\infty}\alpha_k\cdot 2\delta_k\cdot\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}\right].$$

After straightforward simplifications, the characteristic determinant $\Delta_1(\lambda)$ of the spectral problem (1), (2), (4) can be expressed as

$$\Delta_1(\lambda) = \Delta_0(\lambda)\cdot\left[1-\alpha\cdot\sum_{k=0}^{\infty}\alpha_k\cdot\left(\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}+\frac{C_k^{(1)}}{\lambda-((2k+1)\pi)^2}\right)\right]$$
$$+\Delta_0(\lambda)\sum_{k=1}^{\infty}\alpha_k\cdot 2\delta_k\cdot\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}, \tag{13}$$

where $\Delta_0(\lambda) = \alpha\sin\sqrt{\lambda} - \sqrt{\lambda}\left(1+\cos\sqrt{\lambda}\right)$.

We denote the following expression by $A(\lambda)$:

$$A(\lambda) = 1-\alpha\cdot\sum_{k=0}^{\infty}\alpha_k\cdot\left(\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}+\frac{C_k^{(1)}}{\lambda-((2k+1)\pi)^2}\right)+\sum_{k=1}^{\infty}\alpha_k\cdot 2\delta_k\cdot\frac{C_k^{(2)}}{\lambda-(2\beta_k)^2}.$$

The obtained result is formally presented as a theorem:

Theorem 1. *The characteristic determinant of the spectral problems* (1), (2), (4) *with perturbed boundary conditions can be expressed in the form of* (13), *where* $\Delta_0(\lambda)$ *represents the characteristic determinant of the unperturbed problem* (1)–(3), *and* α_k *denote the Fourier coefficients from the expansion* (12) *of the function* $p(x)$ *with respect to the biorthogonal system* $\{v_k(x)\}$, *which is specially constructed from the eigenfunctions of the adjoint problem* (9).

The function $A(\lambda)$ exhibits first-order poles when $\lambda = ((2k+1)\pi)^2$ and $\lambda = (2\beta_k)^2$, and the function $\Delta_0(\lambda)$ possesses first-order zeros at these same points. Consequently, $\Delta_1(\lambda) = \Delta_0(\lambda) \cdot A(\lambda)$ forms an entire analytic function of the variable λ.

In the case where $p(x)$ is represented as a finite sum in (12), resulting in a situation where there exists a number N such that $\alpha_k = 0$ for all $k > N$, the function $A(\lambda)$ takes the following form:

$$A(\lambda) = 1 - \alpha \cdot \sum_{k=0}^{N} \alpha_k \cdot \left(\frac{C_k^{(2)}}{\lambda - (2\beta_k)^2} + \frac{C_k^{(1)}}{\lambda - ((2k+1)\pi)^2} \right) + \sum_{k=1}^{N} \alpha_k \cdot 2\delta_k \cdot \frac{C_k^{(2)}}{\lambda - (2\beta_k)^2}. \tag{14}$$

This expression represents the simplified form of the characteristic determinant (13) when the Fourier coefficients α_k are zero for $k > N$.

From Formula (14), it is evident that $\Delta_1\left(\lambda_k^{(1)}\right) = 0$ and $\Delta_1\left(\lambda_k^{(2)}\right) = 0$ for all $k > N$. Therefore, all the eigenvalues $\lambda_k^{(1)}$, $\lambda_k^{(2)}$, $k > N$ of the unperturbed problem (1)–(3) (that is, problems for $p(x) = 0$) are indeed eigenvalues of the perturbed problem (4). In this scenario, the multiplicity of the eigenvalues $\lambda_k^{(1)}$, $\lambda_k^{(2)}$, $k > N$ is preserved.

Furthermore, the biorthogonality condition for the system of eigenfunctions $\{u_k(x)\}$ and $\{v_j(x)\}$ implies

$$\int_0^1 \overline{p(x)} u_k(x) dx = 0, \ k > N.$$

This indicates that, for eigenvalues beyond a certain index N, the eigenfunctions are orthogonal to the perturbation term, which is an interesting property in the context of spectral problems.

Hence, the eigenfunctions $\left\{u_k^{(1)}, u_k^{(2)}\right\}$, $k > N$ of the unperturbed problem (1)–(3) (i.e., the problem for $p(x) = 0$) satisfy the boundary conditions of the perturbed problem (1), (2), (4). As a result, the system of eigenfunctions of the perturbed problem (1), (2), (4), and the system of eigenfunctions of the unperturbed problem (1)–(3) (for $p(x) = 0$) coincide, except for a finite number of the first terms. This demonstrates the interplay between the eigenfunctions of the perturbed and unperturbed problems in this specific context.

As demonstrated earlier, the system of eigenfunctions of the unperturbed problem (1)–(3) does not constitute an unconditional basis in $L_2(0,1)$. Consequently, in this specific case, the system of eigenfunctions of the perturbed problem (1), (2), (4) is also not a basis in L_2(0,1).

However, since the set of functions $p(x)$ represented as a finite sum in (12) is dense in $L_2(0,1)$, this leads to the following conclusion.

Theorem 2. *The set of functions* $p(x)$, *for which the system of eigenfunctions of problem* (1), (2), (4) *does not constitute an unconditional basis in* $L_2(0,1)$, *is dense in* $L_2(0,1)$.

This theorem highlights the density of functions leading to systems of eigenfunctions that do not form an unconditional basis in the given space.

4. Related Results for the Loaded Equation

One distinctive characteristic of the perturbed spectral problem being discussed is that the spectral problem for the loaded differential equation:

$$L_1^* v \equiv -v''' - p(x)v'(0) = \overline{\lambda} v(x),\ 0 < x < 1 \tag{15}$$

with the boundary conditions, (9) for $\alpha > 0$ serves as the adjoint problem to (1), (2), (4).

For further research, first of all we will find a general solution to Equation (15). Assuming that $v'(0)$ is a fixed constant, we can establish that the general solution to Equation (15) can be expressed in the following form:

$$\begin{aligned} v(x) &= C_1 \cos\left(\sqrt{\overline{\lambda}} x\right) + C_2 \sin\left(\sqrt{\overline{\lambda}} x\right) \\ &+ v'(0) \int_0^x p(\xi) \sin\left(\sqrt{\overline{\lambda}}(x-\xi)\right) d\xi. \end{aligned} \tag{16}$$

Starting with the assumption that $x = 0$ and subsequently substituting (16) into the boundary conditions (9), we arrive at a system of three equations that can be presented in vector-matrix form as follows:

$$\begin{aligned} &\begin{bmatrix} 0 & 1 & -1 \\ \alpha - \sqrt{\overline{\lambda}} \sin\sqrt{\overline{\lambda}} & \sqrt{\overline{\lambda}} \cos\sqrt{\overline{\lambda}} & \sqrt{\overline{\lambda}} \int_0^1 P(\xi) \cos\left(\sqrt{\overline{\lambda}}(1-\xi)\right) d\xi \\ 1 + \cos\sqrt{\overline{\lambda}} & \sin\sqrt{\overline{\lambda}} & \int_0^1 P(\xi) \sin\left(\sqrt{\overline{\lambda}}(1-\xi)\right) d\xi \end{bmatrix} \\ &\cdot \begin{bmatrix} C_1 \\ C_2 \\ v'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{17}$$

After performing standard calculations, we determine that the characteristic determinant $\Delta_1(\overline{\lambda})$ of the spectral problem for the loaded differential Equation (15) with the boundary conditions (9) is represented in the form (6), which is ultimately reduced to the form (13). Consequently, it can be deduced that:

Corollary 1. *The characteristic determinant of the loaded spectral problem* (15) *with the boundary conditions* (9) *can be expressed as* (13)*, where* $\Delta_0(\overline{\lambda})$ *represents the characteristic determinant of the unperturbed problem, and* a_k *are the Fourier coefficients obtained from the expansion* (12) *of the function* $p(x)$ *with respect to the biorthogonal system* $\{v_k(x)\}$*, which is constructed from the eigenfunctions of the adjoint perturbed problem* (9).

It is important to emphasize that the adjoint operator L_1^* exhibits a similar structure, particularly in cases where $p(x)$ is represented as a finite sum in (12). In these scenarios, we can draw conclusions regarding the basis property of the system of eigenfunctions of the adjoint problem (15) with boundary conditions (9). The following lemma is proven in a manner analogous to the previous explanations.

Lemma 3. *If the function* $p(x)$ *is represented as a finite sum in* (12)*, then the system of eigenfunctions of the adjoint problem* (15) *with the boundary conditions* (9) *do not constitute a basis in* $L_2(0,1)$.

From this specific case, we can readily establish the following:

Corollary 2. *The set of functions* $p(x)$*, for which the system of eigenfunctions of problem* (15), (9) *for a loaded differential equation in* $L_2(0,1)$ *does not constitute an unconditional basis, is dense in* $L_2(0,1)$.

The basis properties of root vector-functions of loaded differential operators have also been explored in the works of I.S. Lomov, particularly in the papers [15,16]. In these works, the method of spectral expansions introduced by V.A. Il'in was extended to the case of loaded differential operators. Furthermore, the ideas proposed by V.A. Il'in for the scenario of a non-self-adjoint perturbation of a self-adjoint periodic problem were developed in the research of A.S. Makin [1] and in our studies regarding the anti-periodic problem, as mentioned in [17]. In our specific investigations, the operator was modified through an integral perturbation of one of the boundary conditions.

5. Concluding Remarks

In conclusion, it is well-established that the system of root functions of a spectral problem with strongly regular boundary conditions forms a Riesz basis, and this basis property remains unchanged when the boundary condition is perturbed. However, when dealing with regular but not strongly regular boundary conditions, the situation becomes more complex. Even if the system of root functions for such a problem initially forms a basis, the preservation of this property upon perturbation of the boundary condition is not guaranteed. In our previous research, we have explored various scenarios and options when the system of root functions forms a basis and how this property may or may not be preserved when subjected to perturbations in the boundary conditions.

The problem examined in this work differs significantly from our previous research. Here, we addressed a problem with unstrengthened regular boundary conditions, and its system of root functions does not form a basis. Our focus was on the perturbation of the boundary condition in this specific scenario. To address this, we had to update the methodology we previously employed in our earlier works. In this study, we have successfully constructed the characteristic determinant of the perturbed problem. Based on the formula derived, we have deduced that the set of functions $p(x)$ (the kernel of the integral perturbation), for which the system of root functions of the perturbed problem also does not form a basis, constitutes a dense set in L_2. This stands as the primary outcome of this research.

In our future research, we intend to investigate the question of the density in L_2 of the set of functions $p(x)$ for which the system of root functions of the perturbed problem forms a basis, while simultaneously the system of the unperturbed problem does not form a basis. This represents a promising avenue for further exploration in this field.

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