mathematics

Article

On a New Class of Bi-Close-to-Convex Functions with Bounded Boundary Rotation

Daniel Breaz [1,*], Prathviraj Sharma [2], Srikandan Sivasubramanian [2] and Sheza M. El-Deeb [3,4]

[1] Department of Mathematics, "1 Decembrie 1918" University of Alba-Iulia, 510009 Alba Iulia, Romania
[2] Department of Mathematics, University College of Engineering Tindivanam, Anna University, Tindivanam 604001, India; sirprathvi99@gmail.com (P.S.); sivasubramanian@aucet.in (S.S.)
[3] Department of Mathematics, College of Science and Arts, Al-Badaya, Qassim University, Buraidah 52571, Saudi Arabia; s.eldeeb@qu.edu.sa
[4] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
* Correspondence: dbreaz@uab.ro

Abstract: In the current article, we introduce a new class of bi-close-to-convex functions with bounded boundary rotation. For this new class, the authors obtain the first three initial coefficient bounds of the newly defined bi-close-to-convex functions with bounded boundary rotation. By choosing special bi-convex functions, the authors obtain the first three initial coefficient bounds in the last section. The authors also verify the special cases where the familiar Brannan and Clunie's conjecture is satisfied. Furthermore, the famous Fekete–Szegö inequality is also obtained for this new class of functions. Apart from the new interesting results, some of the results presented here improves the earlier results existing in the literature.

Keywords: analytic; univalent; close-to-convex function of order γ; close-to-convex function; bounded boundary rotations; coefficient estimates; convolution

MSC: 30C45; 33C50; 30C80

check for updates

Citation: Breaz, D.; Sharma, P.; Sivasubramanian, S.; El-Deeb, S.M. On a New Class of Bi-Close-to-Convex Functions with Bounded Boundary Rotation. *Mathematics* **2023**, *11*, 4376. https://doi.org/10.3390/math11204376

Academic Editor: Sergei Sitnik

Received: 14 September 2023
Revised: 16 October 2023
Accepted: 19 October 2023
Published: 21 October 2023

1. Introduction

Let $\mathscr{A}$ be the class of all functions defined by

$$h(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \tag{1}$$

normalized by the conditions $h(0) = 0$ and $h'(0) - 1 = 0$ which are analytic in $\mathbb{D} = \{\zeta : |\zeta| < 1\}$. Furthermore, let us denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ where the functions in $\mathscr{S}$ are also univalent in $\mathbb{D}$. Let $\mathscr{S}^*(\gamma)$ and $\mathscr{C}(\gamma)$ be the subclasses of $\mathscr{S}$ consisting of functions that are starlike of order γ and convex of order γ, $0 \leq \gamma < 1$. The analytic descriptions of the above two classes are, respectively, given by

$$\mathscr{S}^*(\gamma) = \left\{ h \in \mathscr{S} : \Re\left(\frac{\zeta h'(\zeta)}{h(\zeta)}\right) > \gamma,\ 0 \leq \gamma < 1 \right\} \tag{2}$$

$$\mathscr{C}(\gamma) = \left\{ h \in \mathscr{S} : \Re\left(1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right) > \gamma,\ 0 \leq \gamma < 1 \right\}. \tag{3}$$

A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under the complete circuit. Let $h(\zeta)$ map $\mathbb{D}$ onto a domain $\mathscr{G}$. If $\mathscr{G}$ is a schlicht domain with a continuously differentiable boundary curve, let $\pi\mu(t)$ denote the angle of the tangent vector at the point $f(e^{it})$ to the boundary curve with respect to the positive real axis. The boundary rotation of $\mathscr{G}$ is equal to $\pi \int_0^{2\pi} |d\mu(t)|$. If $\mathscr{G}$ does not have a sufficiently smooth boundary curve, the boundary rotation is defined by a limiting process. Let us start with following definitions.

Definition 1 ([1])**.** *Let $k \geq 2$ and $0 \leq \gamma < 1$. Let $\mathscr{P}_k(\gamma)$ denote the class of functions p, that are analytic and normalized with $p(0) = 1$, satisfying the condition*

$$\int_0^{2\pi} \left| \frac{\Re(p(\zeta)) - \gamma}{1 - \gamma} \right| d\theta \leq k\pi. \tag{4}$$

where $\zeta = re^{i\theta} \in \mathbb{D}$.

If $\gamma = 0$, we denote $\mathscr{P}_k(0)$ as $\mathscr{P}_k$. Hence, the class $\mathscr{P}_k$ (defined by Pinchuk [2]) represents the class of analytic functions $p(\zeta)$, with $p(0) = 1$. Therefore, the functions $p \in \mathscr{P}_k$ will be having a representation

$$p(\zeta) = \int_0^{2\pi} \left| \frac{1 - \zeta e^{it}}{1 + \zeta e^{it}} \right| d\mu(t), \tag{5}$$

where μ is a real-valued function with bounded variation satisfying

$$\int_0^{2\pi} d\mu(t) = 2 \text{ and } \int_0^{2\pi} |d\mu(t)| \leq k, \; k \geq 2. \tag{6}$$

Let $\mathscr{U}_k(\gamma)$ represent the class of analytic functions $h(\zeta)$ in $\mathbb{D}$ with $h(0) = 0$, $h'(0) = 1$, and satisfying

$$\frac{\zeta h'(\zeta)}{h(\zeta)} \in \mathscr{P}_k(\gamma). \tag{7}$$

This class generalizes the class $\mathscr{S}^*(\gamma)$ of starlike functions of the order γ, which are also investigated by Robertson [3]. Let $\mathscr{V}_k(\gamma)$ denote the class of all functions $h(\zeta)$ in $\mathbb{D}$ normalized by $h(0) = 0$ and $h'(0) = 1$, satisfying

$$1 + \frac{\zeta h''(\zeta)}{h'(\zeta)} \in \mathscr{P}_k(\gamma), \; 0 \leq \gamma < 1. \tag{8}$$

For $\gamma = 0$, we obtain the class $\mathscr{V}_k(0) \equiv \mathscr{V}_k$, which is the class of all functions of bounded boundary rotation studied by Paatero [4]. This class $\mathscr{V}_k(\gamma)$ generalizes the class of all convex functions $\mathscr{C}(\gamma)$ of order γ introduced by Robertson [3]. An interesting connection for the classes $\mathscr{V}_k(\gamma)$ and $\mathscr{U}_k(\gamma)$ with $\mathscr{P}_k(\gamma)$ is stated below and being given by

$$h(\zeta) \in \mathscr{V}_k(\gamma) \Longleftrightarrow 1 + \frac{\zeta h''(\zeta)}{h'(\zeta)} \in \mathscr{P}_k(\gamma) \tag{9}$$

and

$$h(\zeta) \in \mathscr{U}_k(\gamma) \Longleftrightarrow \frac{\zeta h'(\zeta)}{h(\zeta)} \in \mathscr{P}_k(\gamma) \tag{10}$$

was established by Pinchuk [2]. Pinchuk [2] also proved that functions in $\mathscr{V}_k$ are close-to-convex in $\mathbb{D}$ if $2 \leq k \leq 4$ and hence univalent. Brannan [5] proved that the function $h(\zeta)$ of the form

(1), belongs to $\mathscr{V}_k$ if and only if there are two function $s_1(\zeta)$ and $s_2(\zeta)$ normalized and starlike $\mathbb{D}$, such that

$$h'(\zeta) = \frac{\left(\frac{s_1(\zeta)}{\zeta}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{s_2(\zeta)}{\zeta}\right)^{\frac{k}{4}-\frac{1}{2}}}.$$

Paatero [4] gave the distortion bounds for the functions $h \in \mathscr{V}_k$, such that for $|\zeta| = r < 1$,

$$\frac{(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}} \leq |h'(\zeta)| \leq \frac{(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}. \tag{11}$$

Both bounds in (11) are sharp for each $r \in (0,1)$ for the function

$$H_k(\zeta) = \frac{1}{k}\left(\frac{1+\zeta}{1-\zeta}\right)^{\frac{k}{2}} - \frac{1}{k}.$$

Remark 1 ([6]). *$\mathscr{P} \equiv \mathscr{P}_2$ is the normalized analytic function with the positive real part in $\mathbb{D}$, which is familiarly called the class of Carathéodory functions.*

It is already known that every function $h \in \mathscr{S}$ and of the form (1) has an inverse h^{-1} that is defined by

$$(h^{-1} \odot h)(\zeta) = \zeta(\zeta \in \mathbb{D}) \tag{12}$$

and

$$(h \odot h^{-1})(w) = w(|w| < r_0(h);\ r_0(h) \geq 1/4). \tag{13}$$

For details, see [7]. It is to be remarked here that for $h \in \mathscr{S}$ and of the form (1), the inverse h^{-1} may have an analytic continuation to $\mathbb{D}$, where

$$h^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_3^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{14}$$

A function $h \in \mathscr{S}$ is said to be bi-univalent in $\mathbb{D}$ if there exists a function $g \in \mathscr{S}$ such that $g(\zeta)$ is an univalent extension of h^{-1} on $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$. The functions $\frac{\zeta}{1-\zeta}$, $\frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right)$ and $-\log(1-\zeta)$ are in the class Σ. It is interesting to note that the famous Koebe function $\frac{\zeta}{(1-\zeta)^2}$ is not bi-univalent. Lewin [8] investigated the class of bi-univalent function Σ and obtained a bound $|a_2| < 1.51$. Furthermore, Brannan and Clunie [9], Brannan and Taha [10] also worked on certain subclasses of the bi-univalent function class Σ and obtained bounds for their initial coefficients. The study of bi-univalent functions gained concentration as well as thrust mainly due to the investigation of Srivastava et al. [11]. Different classes of bi-univalent functions were introduced and considered in the current period. In this direction, a conjecture that $|a_2| \leq \sqrt{2}$ was wished for by Brannan and Clunie [9]. An exact upper bound $|a_2| = \frac{4}{3}$ for a subclass Σ_1 of Σ where Σ_1 consists of the functions that are bi-univalent and the range of each function in Σ_1 contains the unit disk $\mathbb{D}$ was obtained by Netanyahu [12]. If $h \in \Sigma$, the exact lower bound of $|a_2|$ and also subsequent higher coefficient bounds $|a_n|$ $(n > 2)$ are still unknown and eluding in the researcher point of view. Analogous to $\mathscr{S}^*(\gamma)$ and $\mathscr{C}(\gamma)$ defined by (2) and (3), Brannan and Taha [10] defined the classes $\mathscr{S}^*_\Sigma(\gamma)$ and $\mathscr{C}_\Sigma(\gamma)$ of bi-starlike functions of order γ and bi-convex functions of order γ. The bounds on $|a_n|(n = 2,3)$ for the class $\mathscr{S}^*_\Sigma(\gamma)$ and $\mathscr{C}_\Sigma(\gamma)$ (for details, see [10]) were established and non-sharp. Subsequent to Brannan and Taha [10], lots of researchers (see [13–22]) in recent times have introduced and investigated several interesting subclasses of the class Σ. They have obtained bounds on the initial two Taylor–Maclaurin coefficients for the new bi-univalent classes they introduced and were identified as non-sharp.

To obtain the main results in this article, we seek few definitions and lemmas which are stated below which already exist in the literature. Let $0 < \gamma \leq 1$. Let $\mathscr{K}_\gamma$ denote the family consisting of analytic function h as represented in (1) with $h'(\zeta) \neq 0$, on $\mathbb{D}$. A function $h \in \mathscr{A}$ is said to be close to convex if there exists a convex function ψ such that

$$\left|\arg\left(\frac{h'(\zeta)}{\psi'(\zeta)}\right)\right| < \frac{\gamma\pi}{2}. \tag{15}$$

The class $\mathscr{K}_\gamma$ was introduced by Kaplan [23] and investigated further by Reade [24]. In particular, $\mathscr{K}_0$ and $\mathscr{K}_1$ are, respectively, the family of convex univalent functions and the family of close-to-convex functions. It is to be observed that $\mathscr{K}_{\gamma_1} \subset \mathscr{K}_{\gamma_2}$ on any $\gamma_1 < \gamma_2$. We denote by $\mathcal{R}$ the subfamily of $\mathcal{A}$ consisting of all functions f such that

$$\Re(f'(z)) > 0 \quad z \in \mathbb{D}. \tag{16}$$

Functions in $\mathcal{R}$ are called functions of bounded turning.

Definition 2 ([20]). *Let $0 \leq \gamma < 1$. A function $h \in \mathscr{A}$ of the form given in (1) with $h'(\zeta) \neq 0$ on $\mathbb{D}$ is in the class of close-to-convex function of order γ if there exists a convex function ψ satisfying*

$$\Re\left(\frac{h'(\zeta)}{\psi'(\zeta)}\right) > \gamma. \tag{17}$$

Definition 3 ([21]). *Let $\mathscr{A}_\Sigma$ denote the class of functions of the form (1) defined on $\mathbb{D}$, for which the function $h \in \mathscr{A}$ and its inverse h^{-1} with Taylor series expansion as in (14) and both are univalent in $\mathbb{D}$, i.e., $h \in \mathscr{A}_\Sigma$, then both h and h^{-1} are univalent in $\mathbb{D}$.*

It is to be pointed out at this juncture that $\mathscr{A}_\Sigma \subset \mathscr{A}$.

Definition 4 ([21]). *Let $0 \leq \gamma < 1$. A function $h \in \mathscr{A}_\Sigma$ of the form given in (1) is said to be bi-close-to-convex function of order γ if there exists bi-convex functions ψ and ϕ satisfying*

$$\Re\left(\frac{h'(\zeta)}{\psi'(\zeta)}\right) > \gamma \tag{18}$$

and

$$\Re\left(\frac{g'(w)}{\phi'(w)}\right) > \gamma. \tag{19}$$

Here, g is the analytic continuation of h^{-1} on $\mathbb{D}$. The class of bi-close-to-convex functions is denoted by $\mathscr{K}_\Sigma(\gamma)$.

Presume that if h is given by (1), then

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_3^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{20}$$

For

$$\psi(\zeta) = \zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 + \cdots, \tag{21}$$

one may obtain,

$$\phi(w) = w - c_2w^2 + (2c_2^2 - c_3)w^3 - (5c_3^3 - 5c_2c_3 + c_4)w^4 + \cdots. \tag{22}$$

Here, $\psi^{-1}(w) = \phi(w)$.

Lemma 1 ([6,25]). *Let $\Psi(\zeta) = 1 + \sum_{n=1}^{\infty} B_n\zeta^n$, $\zeta \in \mathbb{D}$ be such that $\Psi \in \mathscr{P}_k(\gamma)$. Then*

$$|B_n| \leq k(1-\gamma), \ n \geq 1. \tag{23}$$

Let us consider that the functions $p, q \in \mathscr{P}_k(\gamma)$, with

$$p(\zeta) = 1 + \sum_{n=1}^{\infty} p_n \zeta^n \tag{24}$$

and

$$q(\zeta) = 1 + \sum_{n=1}^{\infty} q_n \zeta^n. \tag{25}$$

Then, from Lemma 1, we have

$$|p_n| \leq k(1-\gamma), \ \forall n \geq 1 \tag{26}$$

and

$$|q_n| \leq k(1-\gamma), \ \forall n \geq 1. \tag{27}$$

Let us now state two lemmas which gives the Fekete–Szegö inequality for convex functions and a bi-convex function.

Lemma 2 ([26])**.** *If $\Psi(\zeta) = \zeta + \sum_{n=2}^{\infty} D_n \zeta^n, \ \zeta \in \mathbb{D}$ is a bi-convex function, then for $\Lambda \in \mathbb{R}$,*

$$\left| D_3 - \Lambda D_2^2 \right| \leq \begin{cases} 1 - \Lambda & \text{for } \Lambda < \frac{2}{3}, \\ \frac{1}{3} & \text{for } \frac{2}{3} \leq \Lambda \leq \frac{4}{3}, \\ \Lambda - 1 & \text{for } \Lambda > \frac{4}{3}. \end{cases} \tag{28}$$

Lemma 3 ([27])**.** *If $\Psi(\zeta) = \zeta + \sum_{n=2}^{\infty} D_n \zeta^n, \ \zeta \in \mathbb{D}$ is a convex function, then for $\Lambda \in \mathbb{R}$,*

$$\left| D_3 - \Lambda D_2^2 \right| \leq \begin{cases} 1 - \Lambda & \text{for } \Lambda < \frac{2}{3}, \\ 1 & \text{for } \frac{2}{3} \leq \Lambda \leq \frac{4}{3}, \\ \Lambda - 1 & \text{for } \Lambda > \frac{4}{3}. \end{cases} \tag{29}$$

In the current article, we introduce a new class of bi-close- to-convex functions with bounded boundary rotation. For this new class, we obtain first three initial coefficient bounds. We also verify the special cases where the familiar Brannan and Clunie's conjecture are satisfied. Furthermore, the famous Fekete–Szegö inequalities are also obtained for this new class of functions. The results of this article give few interesting corollaries. Apart from this, some of the results presented here generalize the result of Sivasubramanian et al. [21] and improves the results of Srivastava et al. [11].

2. Coefficient Bounds for $\mathscr{K}_\Sigma(k,\gamma)$

Definition 5. *Suppose $0 \leq \gamma < 1$ and $2 \leq k \leq 4$. A function $h \in \mathscr{A}_\Sigma$ given by (1) is such that $h'(\zeta) \neq 0$ on $\mathbb{D}$. Then, we call h a bi-close-to-convex function with a bounded boundary rotation of order γ denoted by $\mathscr{K}_\Sigma(k,\gamma)$ if there exists bi-convex functions $\psi, \phi \in \mathscr{C}_\Sigma$ satisfying*

$$\frac{h'(\zeta)}{\psi'(\zeta)} \in \mathscr{P}_k(\gamma) \tag{30}$$

and

$$\frac{g'(w)}{\phi'(w)} \in \mathscr{P}_k(\gamma) \tag{31}$$

where g is the analytic continuation of h^{-1} to $\mathbb{D}$.

Let g, ψ and ϕ be Taylor expansions as in (20), (21) and (22).

Theorem 1. *Suppose $0 \leq \gamma < 1$ and $2 \leq k \leq 4$. Let h given by (1) be in the class $\mathscr{K}_\Sigma(k,\gamma)$. Then*

$$|a_2| \leq \sqrt{1+k(1-\gamma)}, \tag{32}$$

$$|a_3| \leq 1+k(1-\gamma) \tag{33}$$

and

$$|a_4| \leq 1+\frac{3}{2}k(1-\gamma). \tag{34}$$

Furthermore, if Λ is real, then

$$\left|a_3-\Lambda a_2^2\right| \leq \begin{cases} (1-\Lambda)(1+\delta) & \text{for } \Lambda < 0, \\ \frac{1}{3}[(1-\Lambda)(2\delta+3)+\delta] & \text{for } 0 \leq \Lambda < \frac{2}{3}, \\ \frac{1}{3}[1+3\delta-2\delta\Lambda] & \text{for } \frac{2}{3} \leq \Lambda < 1, \\ \frac{1}{3}[1+2\delta\Lambda-\delta] & \text{for } 1 \leq \Lambda \leq \frac{4}{3}, \\ \frac{1}{3}[(\Lambda-1)(2\delta+3)+\delta] & \text{for } \frac{4}{3} < \Lambda < 2, \\ (\Lambda-1)(\delta+1) & \text{for } \Lambda \geq 2. \end{cases} \tag{35}$$

where

$$\delta \leq k(1-\gamma). \tag{36}$$

Proof. Let g, ψ and ϕ be given in the form (20), (21) and (22). Since $h \in \mathscr{K}_\Sigma(k,\gamma)$, there exists analytic functions p, q with

$$p(\zeta) = 1+p_1\zeta+p_2\zeta^2+\cdots \tag{37}$$

and

$$q(\zeta) = 1+q_1\zeta+q_2\zeta^2+\cdots \tag{38}$$

and satisfying

$$\frac{h'(\zeta)}{\psi'(\zeta)} = p(\zeta), \quad \frac{g'(w)}{\phi'(w)} = q(w). \tag{39}$$

Therefore,

$$h'(\zeta) = p(\zeta)\psi'(\zeta) \tag{40}$$

and

$$g'(w) = q(w)\phi'(w). \tag{41}$$

From Equations (40) and (41), we obtain

$$2a_2 = p_1+2c_2, \tag{42}$$

$$3a_3 = 2p_1c_2+p_2+3c_3, \tag{43}$$

$$-2a_2 = -2c_2+q_1, \tag{44}$$

$$6a_2^2-3a_3 = -2q_1c_2+q_2+6c_2^2-3c_3 \tag{45}$$

and

$$4a_4 = p_3 + 2p_2c_2 + 3p_1c_3 + 4c_4. \tag{46}$$

Then, from (42) and (44), we obtain $p_1 = -q_1$. The addition of (43) and (45) implies

$$6a_2^2 = 6c_2^2 + 2c_2(p_1 - q_1) + (p_2 + q_2). \tag{47}$$

By virtue of bound for convex functions, we have each $|c_n| \leq 1$. By the relation $q_1 = -p_1$, $|c_n| \leq 1$ and using (26), (27) and applying in (47), we obtain

$$|a_2|^2 \leq 1 + k(1-\gamma). \tag{48}$$

This essentially gives (32). Using $|c_n| \leq 1$, an application of (26) and (27) in (43) at once gives (33). To obtain (34), we apply the same technique in relation (46).

Now, by (43) and (47), for all $\Lambda \in \mathbb{R}$,

$$a_3 - \Lambda a_2^2 = c_3 - \Lambda c_2^2 + \frac{2}{3}c_2p_2(1-\Lambda) + \frac{1}{6}(2-\Lambda) - \frac{1}{2}q_2\Lambda. \tag{49}$$

Hence,

$$\left|a_3 - \Lambda a_2^2\right| \leq \left|c_3 - \Lambda c_2^2\right| + \frac{2}{3}k(1-\gamma)|1-\Lambda| + \frac{1}{6}k(1-\gamma)[|2-\Lambda| + |\Lambda|]. \tag{50}$$

By using Lemma 2, we obtain (35). This completes the proof of Theorem 1. □

For the particular choice of $\gamma = 0$, we have $\mathscr{K}_\Sigma(k,0) \equiv \mathscr{K}_\Sigma(k)$, which represents the class of all bi-close-to-convex functions with bounded boundary rotation.

Corollary 1. *Let h given by (1) belong to the class $\mathscr{K}_\Sigma(k)$ and $2 \leq k \leq 4$. Then*

$$|a_2| \leq \sqrt{1+k}, \tag{51}$$

$$|a_3| \leq 1 + k \tag{52}$$

and

$$|a_4| \leq 1 + \frac{2}{3}k. \tag{53}$$

For the particular choice of $k = 2$, we have $\mathscr{K}_\Sigma(2,\gamma) \equiv \mathscr{K}_\Sigma(\gamma)$, which represents the class of all bi-close-to-convex function of order γ, and Theorem 1 reduces to the following corollary.

Corollary 2. *Let h given by (1) belong to the class $\mathscr{K}_\Sigma(\gamma)$. Then,*

$$|a_2| \leq \sqrt{3-2\gamma}, \tag{54}$$

$$|a_3| \leq 3 - 2\gamma \tag{55}$$

and

$$|a_4| \leq \frac{7-4\gamma}{3}. \tag{56}$$

The bounds of $|a_2|$ and $|a_3|$ in Corollary 2 verify the result given in [21] for the subclass $\mathscr{K}_\Sigma(2,\gamma)$, $0 \leq \gamma < 1$. Finally, we will verify whether Brannan and Clunie's conjecture is satisfied for the class $\mathscr{K}_\Sigma(k,\gamma)$, and it is stated in the following Corollary 3.

Corollary 3. *If $h \in \mathscr{K}_\Sigma(k,\gamma)$, then for $1-\frac{1}{k} \leq \gamma < 1, 2 \leq k \leq 4$,*

$$|a_2| \leq \sqrt{2}.$$

Therefore, it is evident to note that the familiar Brannan and Clunie's conjecture is true for all $2 \leq k \leq 4$ satisfying the condition $1-\frac{1}{k} \leq \gamma < 1$. The following Corollary 4 easily follows from Theorem 1 if $\frac{1}{2} \leq \gamma < 1$.

Corollary 4. *Let h given by (1) belong to the class $\mathscr{K}_\Sigma(k,\gamma)$ and $\frac{1}{2} \leq \gamma < 1$. Then*

$$|a_2| \leq \sqrt{1+\frac{k}{2}}. \tag{57}$$

Since each bi-convex function is convex, then the following Remark 2 can be stated.

Remark 2. *Instead of applying Lemma 2, if we use Lemma 3, inequality (35) becomes*

$$\left|a_3 - \Lambda a_2^2\right| \leq \begin{cases} (1-\Lambda)(1+\delta) & \text{for } \Lambda < 0, \\ [(1-\Lambda)(2\delta+3)+\delta] & \text{for } 0 \leq \Lambda < \dfrac{2}{3}, \\ [1+3\delta-2\delta\Lambda] & \text{for } \dfrac{2}{3} \leq \Lambda < 1, \\ [1+2\delta\Lambda-\delta] & \text{for } 1 \leq \Lambda \leq \dfrac{4}{3}, \\ [(\Lambda-1)(2\delta+3)+\delta] & \text{for } \dfrac{4}{3} < \Lambda < 2, \\ (\Lambda-1)(\delta+1) & \text{for } \Lambda \geq 2. \end{cases} \tag{58}$$

where

$$\delta \leq k(1-\gamma).$$

Remark 3. *For the choice of $k = 2$, (58) reduces to the result of Sivasubramanian et al.* [21].

For the particular choice of the function $\psi(\zeta) = \zeta$, we can obtain the following Theorem 2. However, the calculation needs to be reworked, and we omit the details involved.

Theorem 2. *Suppose $0 \leq \gamma < 1$ and $2 \leq k \leq 4$. Let h given by (1) belong to the class $\mathscr{K}_\Sigma[k,\gamma]$. Then*

$$|a_2| \leq \min\left\{\frac{k(1-\gamma)}{2}, \sqrt{\frac{k(1-\gamma)}{3}}\right\}, \tag{59}$$

$$|a_3| \leq \frac{k(1-\gamma)}{3}, \tag{60}$$

$$|a_4| \leq \frac{k(1-\gamma)}{4}, \tag{61}$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{k(1-\gamma)}{3} \tag{62}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{k(1-\gamma)}{3}. \tag{63}$$

If $k = 2$ in Theorem 2, we can obtain the following corollary.

Corollary 5. *Let h given by (1) belong to the class $\mathscr{K}_\Sigma[2,\gamma]$, $0 \le \gamma < 1$. Then*

$$|a_2| \le \min\left\{(1-\gamma), \sqrt{\frac{2(1-\gamma)}{3}}\right\}, \tag{64}$$

$$|a_3| \le \frac{2(1-\gamma)}{3} < \frac{(1-\gamma)(5-3\gamma)}{3}, \tag{65}$$

$$|a_4| \le \frac{(1-\gamma)}{2}, \tag{66}$$

$$\left|a_3 - 2a_2^2\right| \le \frac{2(1-\gamma)}{3} \tag{67}$$

and

$$\left|a_3 - a_2^2\right| \le \frac{2(1-\gamma)}{3}. \tag{68}$$

Remark 4.

(i) *Since*

$$|a_3| \le \frac{2(1-\gamma)}{3} < \frac{(1-\gamma)(5-3\gamma)}{3},$$

Inequality (65) verifies that the bounds of $|a_3|$ is less than that of the bound given by Srivastava et al. [11].

(ii) *The inequality (68) coincides with the result in* [26].

3. Coefficient Bounds for $\mathscr{K}_\Sigma(k,\gamma)$ for Some Particular Choices of Functions

In this section, we give special choices for the convex function $\psi(\zeta)$ and obtain the coefficient bounds. For the particular choice of the function $\psi(\zeta) = \zeta - \frac{1}{2}\zeta^2$, we denote the class $\mathscr{K}_\Sigma(k,\gamma)$ by $\mathscr{K}_{1,\Sigma}(k,\gamma)$. For the above class of functions, we can obtain the following Theorem 3, and it is stated as below.

Theorem 3. *Suppose $0 \le \gamma < 1$ and $2 \le k \le 4$. Let h given by (1) be in the class $\mathscr{K}_{1,\Sigma}(k,\gamma)$. Then, $\exists\ \psi(\zeta) = \zeta - \frac{1}{2}\zeta^2$ such that*

$$|a_2| \le \sqrt{\frac{4k(1-\gamma)+1}{12}}, \tag{69}$$

$$|a_3| \le \frac{2k(1-\gamma)}{3} \tag{70}$$

and

$$\left|a_3 - \Lambda a_2^2\right| \le \begin{cases} \dfrac{2\rho}{3} + \dfrac{\Lambda(2\rho+3)}{6} & \text{for } \Lambda < 0, \\ \dfrac{2\rho}{3} + \dfrac{\Lambda}{12} & \text{for } 0 \le \Lambda < 2, \\ \dfrac{\Lambda\rho}{3} + \dfrac{1}{12} & \text{for } \Lambda \ge 2 \end{cases} \tag{71}$$

where

$$\rho \le k(1-\gamma). \tag{72}$$

Proof. Suppose $0 \le \gamma < 1$ and $2 \le k \le 4$. Let h given by (1) be in the class $\mathscr{K}_{1,\Sigma}(k,\gamma)$. Then

$$\frac{h'(\zeta)}{\psi'(\zeta)} = p(\zeta), \ \frac{g'(w)}{\phi'(w)} = q(w). \tag{73}$$

Hence

$$h'(\zeta) = p(\zeta)\psi'(\zeta) \tag{74}$$

and

$$g'(w) = q(w)\phi'(w). \tag{75}$$

Since $\psi(\zeta) = \zeta - \dfrac{1}{2}\zeta^2$, $\phi(w) = w - \frac{1}{2}w^2 + \frac{1}{2}w^3 - \cdots$.
From Equations (74) and (75), we obtain

$$2a_2 = p_1 - 1, \tag{76}$$

$$3a_3 = p_2 - p_1, \tag{77}$$

$$-2a_2 = q_1 - 1 \tag{78}$$

and

$$6a_2^2 - 3a_3 = q_2 - q_1 + \frac{3}{2}. \tag{79}$$

From (76) and (78) we obtain

$$p_1 + q_1 = 2. \tag{80}$$

From (77) and (79), we obtain

$$6a_2^2 = q_2 - q_1 + p_2 - p_1 + \frac{3}{2}. \tag{81}$$

Now, by using (80) in (81), we have

$$6a_2^2 = q_2 + p_2 - \frac{1}{2}. \tag{82}$$

Using (26) and (27) in (82), we obtain

$$|a_2|^2 \leq \frac{4k(1-\gamma)+1}{12}. \tag{83}$$

This gives (69). Applying (26) and (27) into (77) gives (70).
Now, by (77) and (82), for all real Λ,

$$a_3 - \Lambda a_2^2 = \frac{p_2}{3}(2-\Lambda) - \frac{p_1}{3} + \frac{\Lambda q_2}{6} - \frac{1}{12}. \tag{84}$$

Hence

$$\left|a_3 - \Lambda a_2^2\right| \leq \frac{k(1-\gamma)}{3} + |\Lambda|\frac{k(1-\gamma)}{6} + \frac{|\Lambda|}{12} + \frac{k(1-\gamma)}{6}|2-\Lambda|. \tag{85}$$

Now, by using different ranges of Λ, we obtain (71). This completes the proof of Theorem 3. □

For the particular choice of $\gamma = 0$, we have $\mathscr{K}_{1,\Sigma}(k,0) \equiv \mathscr{K}_{1,\Sigma}(k)$, representing the class of all bi-close-to-convex functions with bounded boundary rotation with respect to the function $\psi(\zeta) = \zeta - \dfrac{1}{2}\zeta^2$. For the class $\mathscr{K}_{1,\Sigma}(k)$, we have the following corollary.

Corollary 6. *Suppose* $2 \le k \le 4$. *Let* h *given by* (1) *belong to the class* $\mathscr{K}_{1,\Sigma}(k)$ *with respect to function* $\psi(\zeta) = \zeta - \frac{1}{2}\zeta^2$. *Then*

$$|a_2| \le \sqrt{\frac{4k+1}{12}} \tag{86}$$

and

$$|a_3| \le \frac{2k}{3}. \tag{87}$$

For the particular choice of $k = 2$, we have $\mathscr{K}_{1,\Sigma}(2,\gamma)$, which represents the class of all bi-close-to-convex functions of order γ with respect to the function $\psi(\zeta) = \zeta - \frac{1}{2}\zeta^2$. For the class $\mathscr{K}_{1,\Sigma}(2,\gamma)$, Theorem 3 reduces to the following corollary.

Corollary 7. *Suppose* $0 \le \gamma < 1$. *Let* h *given by* (1) *belong to the class* $\mathscr{K}_{1,\Sigma}(2,\gamma)$. *Then,*

$$|a_2| \le \sqrt{\frac{9-8\gamma}{12}} \tag{88}$$

and

$$|a_3| \le \frac{4(1-\gamma)}{3}. \tag{89}$$

Next, let us fix the convex function as $\psi(\zeta) = \dfrac{\zeta}{1-\zeta}$ and for this choice, we denote the class $\mathscr{K}_\Sigma(k,\gamma)$ by $\mathscr{K}_{2,\Sigma}(k,\gamma)$. For the above class of functions $\mathscr{K}_{2,\Sigma}(k,\gamma)$, we can obtain the following Theorem 4, and it is stated as below.

Theorem 4. *Suppose* $0 \le \gamma < 1$ *and* $2 \le k \le 4$. *Let* h *given by* (1) *be in the class* $\mathscr{K}_{2,\Sigma}(k,\gamma)$. *Then,* $\exists\ \psi(\zeta) = \dfrac{\zeta}{1-\zeta}$ *such that*

$$|a_2| \le \sqrt{k(1-\gamma)+1}, \tag{90}$$

$$|a_3| \le k(1-\gamma)+1 \tag{91}$$

and

$$\left|a_3 - \Lambda a_2^2\right| \le \begin{cases} (1+\rho)(1-\Lambda) & \text{for } \Lambda < 0, \\ \rho - \Lambda + 1 - \dfrac{2\rho\Lambda}{3} & \text{for } 0 \le \Lambda < 1, \\ \Lambda - 1 - \dfrac{\rho}{3} + \dfrac{2\rho\Lambda}{3} & \text{for } 1 \le \Lambda < 2, \\ (1+\rho)(\Lambda - 1) & \text{for } \Lambda \ge 2 \end{cases} \tag{92}$$

where

$$\rho \le k(1-\gamma). \tag{93}$$

Proof. Suppose $0 \le \gamma < 1$ and $2 \le k \le 4$. Let h given by (1) be in the class $\mathscr{K}_{2,\Sigma}(k,\gamma)$. Then

$$\frac{h'(\zeta)}{\psi'(\zeta)} = p(\zeta), \ \frac{g'(w)}{\phi'(w)} = q(w). \tag{94}$$

Hence

$$h'(\zeta) = p(\zeta)\psi'(\zeta) \tag{95}$$

and

$$g'(w) = q(w)\phi'(w). \tag{96}$$

Since $\psi(\zeta) = \dfrac{\zeta}{1-\zeta}$, $\phi(w) = w - w^2 + w^3 - \cdots$.

From the equations (95) and (96), we obtain

$$2a_2 = p_1 + 2, \tag{97}$$

$$3a_3 = p_2 + 2p_1 + 3, \tag{98}$$

$$-2a_2 = q_1 - 2 \tag{99}$$

and

$$6a_2^2 - 3a_3 = q_2 - 2q_1 + 3. \tag{100}$$

From (97) and (99) we obtain

$$p_1 + q_1 = 0. \tag{101}$$

From (98), (100) and (101) we obtain

$$6a_2^2 = q_2 + p_2 + 4p_1 + 6. \tag{102}$$

Using (26) and (27) in (102), we obtain

$$|a_2|^2 \leq m(1-\gamma) + 1. \tag{103}$$

This gives (90). An application of (26) and (27) in (98) gives (91).

Now, by (98) and (102), for all real Λ,

$$a_3 - \Lambda a_2^2 = \frac{p_2}{6}(2-\Lambda) + \left(\frac{2p_1}{3} + 1\right)(1-\Lambda) - \frac{q_2}{6}\Lambda. \tag{104}$$

Hence

$$\left|a_3 - \Lambda a_2^2\right| \leq \frac{k(1-\gamma)}{6}|2-\Lambda| + \left(\frac{2k(1-\gamma)}{3} + 1\right)|1-\Lambda| + \frac{k(1-\gamma)}{6}|\Lambda|. \tag{105}$$

Now, using different ranges of Λ essentially gives (92). This completes the proof of Theorem 4. □

For the particular choice of $\gamma = 0$, we have $\mathscr{K}_{2,\Sigma}(k,0) \equiv \mathscr{K}_{2,\Sigma}(k)$, representing the class of all bi-close-to-convex functions with bounded boundary rotation with respect to the function $\psi(\zeta) = \dfrac{\zeta}{1-\zeta}$.

Corollary 8. *Suppose* $2 \leq k \leq 4$. *Let* h *given by* (1) *belong to the class* $\mathscr{K}_{2,\Sigma}(k)$ *with respect to function* $\psi(\zeta) = \dfrac{\zeta}{1-\zeta}$ *and* $k \geq 2$. *Then,*

$$|a_2| \leq \sqrt{k+1} \tag{106}$$

and

$$|a_3| \leq k+1. \tag{107}$$

For the particular choice of $k = 2$, we have $\mathscr{K}_{2,\Sigma}(2,\gamma)$ representing the class of all bi-close-to-convex functions of order γ with respect to the function $\psi(\zeta) = \dfrac{\zeta}{1-\zeta}$, and Theorem 4 reduces to the following corollary.

Corollary 9. *Suppose $0 \le \gamma < 1$. Let h given by (1) belong to the class $\mathscr{K}_{2,\Sigma}(2,\gamma)$. Then,*

$$|a_2| \le \sqrt{3-2\gamma} \tag{108}$$

and

$$|a_3| \le 3-2\gamma. \tag{109}$$

The bounds of $|a_2|$ and $|a_3|$ in Corollary 9 verify the result given in [21] for the subclass $\mathscr{K}_{2,\Sigma}(2,\gamma)$ with respect to the function $\psi(\zeta) = \dfrac{\zeta}{1-\zeta}$ and $0 \le \gamma < 1$. Finally, we will verify whether the Brannan and Clunie's conjecture is satisfied for the class $\mathscr{K}_{2,\Sigma}(k,\gamma)$ and it is stated in the following Corollary 10.

Corollary 10. *If $h \in \mathscr{K}_{2,\Sigma}(k,\gamma)$, then for $1 - \frac{1}{k} \le \gamma < 1$, $k \ge 2$,*

$$|a_2| \le \sqrt{2}.$$

Therefore, it is evident to note that the familiar Brannan and Clunie's conjecture is true for all $k \ge 2$, satisfying the condition $1 - \frac{1}{k} \le \gamma < 1$.

Finally, we chose the choice of the function as $\psi(\zeta) = -\log(1-\zeta)$. We denote this class $\mathscr{K}_{\Sigma}(k,\gamma)$ by $\mathscr{K}_{3,\Sigma}(k,\gamma)$. For the above class of functions, we can obtain the following Theorem 5, and it is shown as below.

Theorem 5. *Let h given (1) be in the class $\mathscr{K}_{3,\Sigma}(k,\gamma)$, $0 \le \gamma < 1$. Then, $\exists\, \psi(\zeta) = -\log(1-\zeta)$ such that*

$$|a_2| \le \sqrt{\frac{10k(1-\gamma)+1}{12}}, \tag{110}$$

$$|a_3| \le \frac{2k(1-\gamma)+1}{3} \tag{111}$$

and

$$|a_3 - 4a_2^2| \le \frac{2+8k(1-\gamma)}{3}. \tag{112}$$

Proof. Suppose $0 \le \gamma < 1$ and $2 \le k \le 4$. Let h given by (1) be in the class $\mathscr{K}_{3,\Sigma}(k,\gamma)$. Then

$$\frac{h'(\zeta)}{\psi'(\zeta)} = p(\zeta), \ \frac{g'(w)}{\phi'(w)} = q(w). \tag{113}$$

Hence

$$h'(\zeta) = p(\zeta)\psi'(\zeta) \tag{114}$$

and

$$g'(w) = q(w)\phi'(w) \tag{115}$$

Since $\psi(\zeta) = -\log(1-\zeta) = \sum_{n=1}^{\infty} \dfrac{z^n}{n}$ then $\phi(w) = w - \frac{1}{2}w^2 + \frac{1}{6}w^3 - \cdots$.

From Equations (114) and (115), we obtain

$$2a_2 = p_1 + 1, \tag{116}$$

$$3a_3 = p_2 + p_1 + 1, \tag{117}$$

$$-2a_2 = q_1 - 2 \tag{118}$$

and

$$3(2a_2^2 - a_3) = q_2 - 2q_1 + \frac{1}{2}. \tag{119}$$

From (116) and (118), we obtain

$$p_1 + q_1 = 1. \tag{120}$$

From (117), (119) and (120), we obtain

$$6a_2^2 = q_2 + p_2 + 3p_1 - \frac{1}{2}. \tag{121}$$

Applying (26) and (27) in (121), we obtain

$$|a_2|^2 \leq \frac{10k(1-\gamma)+1}{12}. \tag{122}$$

This gives, (110). An application of (26) and (27) in (117) gives (111). Now, from (117) and (121), we obtain

$$3a_3 - 12a_2^2 = 2 - 2q_2 - p_2 - 5p_1. \tag{123}$$

An application of (26) and (27) in (123) now gives (112). This completes the proof of Theorem 5. □

For the particular choice of $\gamma = 0$, we have $\mathscr{K}_{3,\Sigma}(k,0) \equiv \mathscr{K}_{3,\Sigma}(k)$, representing the class of all bi-close-to-convex functions with bounded boundary rotation with respect to the function $\psi(\zeta) = -\log(1-\zeta)$.

Corollary 11. *Suppose $2 \leq k \leq 4$. Let h given by (1) belong to the class $\mathscr{K}_{3,\Sigma}(k)$ with respect to function $\psi(\zeta) = -\log(1-\zeta)$. Then*

$$|a_2| \leq \sqrt{\frac{10k+1}{12}}, \tag{124}$$

$$|a_3| \leq \frac{2k+1}{3} \tag{125}$$

and

$$|a_3 - 4a_2^2| \leq \frac{2+8k}{3}. \tag{126}$$

For the particular choice of $k = 2$, we have $\mathscr{K}_{3,\Sigma}(2,\gamma)$ representing the class of all bi-close-to-convex functions of order γ with respect to the function $\psi(\zeta) = -\log(1-\zeta)$, and Theorem 5 reduces to the following corollary.

Corollary 12. *Suppose $0 \leq \gamma < 1$. Let h given by (1) belong to the class $\mathscr{K}_{3,\Sigma}(2,\gamma)$. Then,*

$$|a_2| \leq \sqrt{\frac{20k(1-\gamma)+1}{12}}, \tag{127}$$

$$|a_3| \leq \frac{4(1-\gamma)+1}{3} \tag{128}$$

and

$$|a_3 - 4a_2^2| \leq \frac{2+16(1-\gamma)}{3}. \tag{129}$$

4. Concluding Remarks and Observations

In this article, we investigated the estimates of second and third Taylor–Maclaurin coefficients for bi-close-to-convex functions of order γ with bounded boundary rotation. Also, interesting Fekete–Szegö coefficient estimates for functions in this class are obtained. The authors have verified the special cases where the familiar Brannan and Clunie's conjecture are satisfied. Apart from these remarks which are given in the present article, more corollaries and remarks can be stated for the choice of $\gamma = 0$, and those details are omitted. The authors also investigated the estimates of second and third Taylor–Maclaurin coefficients for some special functions and obtain a few interesting results.

Author Contributions: Conceptualization, D.B., S.S. and S.M.E.-D.; Methodology, P.S. and S.M.E.-D.; Formal analysis, P.S. and S.S.; Investigation, S.S. and S.M.E.-D.; Data curation, D.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No Datas are used in this research.

Acknowledgments: The authors would like to thank the referees for their comments and suggestions on the original manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Padmanabhan, K.S.; Parvatham, R. Properties of a class of functions with bounded boundary rotation. *Ann. Pol. Math.* **1976**, *31*, 311–323. [CrossRef]
2. Pinchuk, B. Functions of bounded boundary rotation. *Isr. J. Math.* **1971**, *10*, 6–16. [CrossRef]
3. Robertson, M.S. On the theory of univalent functions. *Ann. Math.* **1936**, *337*, 74–408. [CrossRef]
4. Paatero, V. Über Gebiete von beschrankter Randdrehung. *Ann. Acad. Sci. Fenn. Ser. A* **1933**, *37*, 9.
5. Brannan, D.A. On functions of bounded boundary rotation-I. *Proc. Edinb. Math. Soc.* **1969**, *16*, 339–347. [CrossRef]
6. Li Y.; Vijaya, K.; Murugusundaramoorthy, G.; Tang, H. On new subclasses of bi-starlike functions with bounded boundary rotation. *AIMS Math.* **2020**, *5*, 3346–3356. [CrossRef]
7. Duren, P.L. *Univalent Functions (Grundlehren der Mathematischen Wissenschaften, 259)*; Springer: New York, NY, USA, 1983.
8. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]
9. Brannan, D.A.; Clunie, J.G. *Aspects of Contemporary Complex Analysis*; Academic Press, Inc.: London, UK, 1980.
10. Brannan D.A.; Taha, Ṫ.S. On some classes of bi-univalent functions. In *Mathematical Analysis and Its Applications (Kuwait, 1985)*; KFAS Proc. Ser. 3; Pergamon: Oxford, UK, 1988; pp. 53–60.
11. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
12. Netanyahu, E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|\zeta| < 1$. *Arch. Ration. Mech. Anal.* **1969**, *32*, 100–112.
13. Ali, R.M.; Lee, S.K.; Ravichandran, V.; Supramanium, S. Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. *Appl. Math. Lett.* **2012**, *25*, 344–351. [CrossRef]
14. Darus, M.; Shanmugam, T.N.; Sivasubramanian, S. Fekete–Szegö inequality for a certain class of analytic functions. *Mathematica* **2007**, *49*, 29–34.
15. Frasin B.A.; Aouf, Ṁ.K. New subclasses of bi-univalent functions. *Appl. Math. Lett.* **2011**, *24*, 1569–1573. [CrossRef]
16. Goyal S.P.; Goswami, Ṗ. Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives. *J. Egypt. Math. Soc.* **2012**, *20*, 179–182. [CrossRef]
17. Hayami, T.; Owa, Ṡ. Coefficient bounds for bi-univalent functions. *PanAm. Math. J.* **2012**, *22*, 15–26.
18. Xu, Q.H.; Gui, Y.-C.; Srivastava, H.M. Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Appl. Math. Lett.* **2012**, *25*, 990–994. [CrossRef]
19. Xu, Q.H.; Srivastava, H.M.; Xiao, H.-G. A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. *Appl. Math. Comput.* **2012**, *218*, 11461–11465. [CrossRef]
20. Xu, Q.H.; Srivastava, H.M.; Zhou, L. A certain subclass of analytic and close-to-convex functions. *Appl. Math. Lett.* **2011**, *24*, 396–401. [CrossRef]
21. Sivasubramanian, S.; Sivakumar, R.; Kanas, S.; Kim, S. Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions. *Ann. Pol. Math.* **2015**, *113*, 295–304. [CrossRef]
22. Kanas, S.; Sivasankari, V.; Karthiyayini, R.; Sivasubramanian, S. Second Hankel determinant for a certain subclass of bi-close to convex functions defined by Kaplan. *Symmetry* **2021**, *13*, 567. [CrossRef]
23. Kaplan, W. Close-to-convex schlicht functions. *Michigan Math. J.* **1953**, *1*, 169–185. [CrossRef]

24. Reade, M.O. The coefficients of close-to-convex functions. *Duke Math. J.* **1956**, *23*, 459–462. [CrossRef]
25. Alkahtani, B.S.T.; Goswami P.; Bulboacă, Ṫ. Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions. *Miskolc Math. Notes* **2016**, *17*, 739–748. [CrossRef]
26. Zaprawa, P. On the Fekete-Szegö problem for classes of bi-univalent functions. *Bull. Belg. Math. Soc. Simon Stevin* **2014**, *21*, 169–178. [CrossRef]
27. Kanas, S. An unified approach to the Fekete-Szegö problem. *Appl. Math. Comput.* **2012**, *218*, 8453–8461. [CrossRef]