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Malliavin Calculus and Its Application to Robust Optimal Investment for an Insider

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Abstract: In the theory of portfolio selection, there are few methods that effectively address the combined challenge of insider information and model uncertainty, despite numerous methods proposed for each individually. This paper studies the problem of the robust optimal investment for an insider under model uncertainty. To address this, we extend the Itô formula for forward integrals by Malliavin calculus, and use it to establish an implicit anticipating stochastic differential game model for the robust optimal investment. Since traditional stochastic control theory proves inadequate for solving anticipating control problems, we introduce a new approach. First, we employ the variational method to convert the original problem into a nonanticipative stochastic differential game problem. Then we use the stochastic maximum principle to derive the Hamiltonian system governing the robust optimal investment. In cases where the insider information filtration is of the initial enlargement type, we derive the closed-form expression for the investment by using the white noise theory when the insider is 'small'. When the insider is 'large', we articulate a quadratic backward stochastic differential equation characterization of the investment. We present the numerical result and conduct an economic analysis of the optimal strategy across various scenarios.

Keywords: Malliavin calculus; forward integral; robust optimal investment; insider information; model uncertainty; stochastic maximum principle; white noise theory

MSC: 60H07; 60H40; 60E05; 93E20; 91G80

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1. Introduction

The optimal investment problem is a fundamental topic in financial mathematics, originally introduced by Merton [1,2]. Its primary objective is to choose an investment strategy π that maximizes the expected terminal utility as follows

$$\max_{\pi} \mathbb{E}[U(X_T^{\pi})], \tag{1}$$

where $U(X_T^{\pi})$ is the terminal utility, and $T > 0$ is some fixed terminal time. In continuous-time financial models, there are three conventional methods, rooted in classical Itô theory (see [3]), for addressing this problem: the martingale method, the dynamic programming method and the stochastic maximum principle (see [1,2,4]). Extended models and related problems have been explored over the past decades (see [5–7]).

Recently, there is a growing emphasis on the optimization problem of insider trading. That is, the investor who owns additional future information. In this setting, we naturally suppose the investment process π is adapted to the insider information filtration $\{\mathcal{H}_t\}$, which might contain the natural filtration $\{\mathcal{F}_t\}$ of the noise W. In other words, the relevant Itô stochastic differential equations (SDEs) should be replaced by anticipating SDEs, which implies that the above methods may not be directly applicable.

Pikovski and Karatzas [8] was the first to study the optimization problem of insider trading. They assumed the insider information is hidden in a random variable Y from the beginning of filtrations. Thus, the insider information filtration is of the initial enlargement type, i.e.,

$$\mathcal{H}_t = \bigcap_{s>t} (\mathcal{F}_s \vee Y). \tag{2}$$

They employed the technique of enlargement of filtration to address the issue (see [9]). The critical point is that W is indeed a semi-martingale with respect to the new filtration $\{\mathcal{H}_t\}$. Biagini and Øksendal [10] developed a method based on the theory of forward integrals to deal with the more general filtration $\{\mathcal{H}_t\}$. Kohatsu-Higa and Sulem [11] dug into the large insider-trading problem and derived a characterization theorem for the solution. Many other extensions could be found in, for example, [9,12–15], which are all based on the forward integral.

Although numerous insider-trading problems have been explored, the foundational theory of forward integrals, especially the Itô formula, remains incomplete. As a result in insider-trading models, certain conditions, such as the forward integrability of parameter processes, are both abstract and demanding to validate. In this context, Malliavin calculus offers a natural avenue for investigating the properties of forward integrals. This is facilitated by the comprehensive nature of its theory and its connection to the Skorohod integral (see [11,16–19]). To the best of our knowledge, only Nualart [19] used the Malliavin calculus to derive the Itô formula for forward integrals. However, the forward integral in his research is defined using Riemann sums, which necessitate an additional continuity condition. This condition may not be directly applicable to insider trading, as it contradicts the càglàd nature of the investment process π.

Most of the existing works in the literature on finance, including the articles mentioned above, presume that the parameters in models are accurate and the investors are ambiguity-neutral. However, as pointed out by Chen and Epstein [20], the risk-based models that constitute the paradigm have well documented empirical failures. Thus, a model uncertainty setup should be considered. In this situation, the investor is ambiguity-averse. She might not believe the model is accurate by empirical statistics, which forces her to choose the robust optimal investment under the worst-case probability. As a consequence, the optimization problem (1) becomes the following stochastic differential game (SDG) problem (see [20])

$$\max_{\pi} \min_{\theta} \mathbb{E}_{\mathcal{Q}^\theta}\left[U(X_T^\pi) + \int_0^T g(\theta_s)\mathrm{d}s\right], \tag{3}$$

where $\mathcal{Q}^\theta$ is the prior probability measure to describe the model uncertainty parametrized by θ, and g is viewed as a step adopted to penalize the difference between $\mathcal{Q}^\theta$ and original reference probability $\mathbb{P}$. We refer to [7,20–23] for further studies.

When we combine insider trading with model uncertainty, the nonanticipative SDG problem (3) becomes an anticipating SDG problem. Directly applying the forward integral method is infeasible, as it lacks relevant results for the variation associated with the other controlling process θ. An et al. [24] introduced a generalized stochastic maximum principle for the anticipating SDG problem using Malliavin calculus. However, the result is limited to the controlled Itô-Lévy processes due to the intricacies of Malliavin derivative. Peng et al. [14] used the Donsker δ functional technique in withe noise theory to transform the anticipating SDG problem into a nonanticipative SDG problem. Subsequently, they applied the stochastic maximum principle to resolve the problem. However, no closed form of solution was obtained since the problem could only be reduced to a nested linear backward stochastic differential equation (BSDE). Moreover, the filtration $\{\mathcal{H}_t\}$ in their research adheres to the special type (2), and the prior probability $\mathcal{Q}^\theta$ is not exact a probability measure when $\theta(y) = \theta(Y)$.

Inspired by these prior studies, this paper focuses on resolving the optimization challenge related to insider trading amid model uncertainty. The main contributions are as follows.

- On the aspect of basic mathematical theory, we enhance some properties of the forward integral using the Malliavin calculus, and extend the Itô formula for forward integrals by Malliavin calculus.
- We establish an implicit anticipating SDG model for the robust optimal investment strategy of an insider, and introduce a new approach combing the stochastic maximum principle with the variational method. In fact, we deduce the semi-martingale property of W with respect to $\{\mathcal{H}_t\}$ by taking the variation with respect to π. This allows us to transform the anticipating SDG problem into a nonanticipative one, which could be solved by the stochastic maximum principle.
- When the insider information is of the initial enlargement type, we are the first to derive the closed-form expression for the robust optimal investment strategy in the small insider case. In cases of large insider influence, we developed a quadratic BSDE characterization for the strategy. The core technique here involves the Donsker δ functional in the white noise theory, which is essentially different from that in [14]. In fact, the white noise technique in [14] was employed initially to convert the anticipating problem into a nonanticipative one with respect to the natural filtration $\{\mathcal{F}_t\}$. In contrast, our approach consistently centers on problems with respect to the larger filtration $\{\mathcal{H}_t\}$. We use the white noise methods ultimately to tackle the realm of generalized nonanticipative BSDE problems.
- We introduce the conception of the 'critical information time', which is the minimum amount of insider information needed by an ambiguity-averse investor to offset the loss in optimal expected utility arising from model uncertainty. Numerical experiments demonstrates that the impact of model uncertainty becomes more pronounced as the mean rate of return increases or the volatility decreases.

This paper is organized as follows. In Section 2, we introduce the basic theory of the Malliavin calculus and derive the Itô formula for forward integrals by Skorohod integrals. In Section 3, we formulate the robust optimization problem of insider trading. We give the initial characterization of the robust optimal investment strategy in Section 4 and the final characterization in Section 5. In Sections 6 and 7, we examine two cases: one for the small insider and the other for the large insider. The situation in which the insider information filtration is of the initial enlargement type is also considered in the two sections. Simulation and economic analysis are performed in Section 9. We summarize our conlusions in Section 10.

2. The Forward Integral by Malliavin Calculus

Malliavin theory is a new frontier field in stochastic analysis, which essentially involves an infinite-dimensional differential analysis on the Wiener space. In addition to addressing a vast array of problems at the intersection of probability theory and analysis, Malliavin theory has also proven to be highly successful in the field of finance (see [25–28]).

In this section, we briefly introduce the basic theory of the Skorohod integral in Malliavin calculus (see [19,29,30]). Subsequently, we leverage this understanding to enhance the theory of the forward integral. The main result of this section is that we extend certain propositions and derive the Itô formula for the forward integral via Malliavin calculus to suit our specific context.

2.1. The Basic Theory of Malliavin Calculus

Consider a filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, on which a standard Brownian motion $W = \{W_t\}_{0\le t\le T}$ is defined. Here, $\{\mathcal{F}_t\}_{0\le t\le T}$ is the $\mathbb{P}$-augmentation of the filtration generated by W, which satisfies the usual condition (see [3]). We also denote by H the real Hilbert space $L^2([0,T])$. Then $(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ is an irreducible Gaussian space (see [29]).

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions φ such that φ and all of its partial derivatives have polynomial growth. For a given separable Hilbert space E, denote by $\mathcal{S}(\Omega; E)$ the class of E-valued smooth random variables such that $X \in \mathcal{S}$ has the form

$$X = \sum_{j=1}^{m} \varphi_j\Big(\int_0^T h_{j,1}(s)\mathrm{d}W_s, \cdots, \int_0^T h_{j,n_j}(s)\mathrm{d}W_s\Big)e_j,$$

where $m, n_j \in \mathbb{N}_+$, $\varphi_j \in C_p^\infty(\mathbb{R}^{n_j})$, $h_{j,l} \in H$, and $e_j \in E$ for $l = 1, \cdots, n_j$ and $j = 1, \cdots, m$. Note that $\mathcal{S}(\Omega; E)$ is dense in $L^p(\Omega; E)$ for $p \geq 1$. The Malliavin gradient D_t of the E-valued smooth random variable X is defined as the $H \otimes E$-valued random variable $D_t X$ give by

$$D_t X := \sum_{j=1}^{m} \sum_{l=1}^{n_j} \frac{\partial \varphi_j}{\partial x_l}\Big(\int_0^T h_{j,1}(s)\mathrm{d}W_s, \cdots, \int_0^T h_{j,n_j}(s)\mathrm{d}W_s\Big)h_{j,l}(t) \otimes e_j.$$

For $k = 2, 3, \cdots$, the k-iteration of the operator D_t can be defined in such a way that for $X \in \mathcal{S}(\Omega; E)$, $D_t^k X$ is a random variable with values in $H^{\otimes k} \otimes E$.

We can check that D_t^k is a closable operator from $\mathcal{S}(\Omega; E) \subset L^p(\Omega; E)$ to $L^p(\Omega; H^{\otimes k} \otimes E)$ for $k \in \mathbb{N}_+$ and $p \geq 1$. Denote by $D^{k,p}(\Omega; E)$ the closure of the class of smooth random variables $\mathcal{S}(\Omega, E)$ with respect to the graph norm (see [31])

$$\|X\|_{D^{k,p}(\Omega;E)} := \Big[\|X\|^p_{L^p(\Omega;E)} + \sum_{j=1}^{k} \|D_t^j X\|^p_{L^p(\Omega; H^{\otimes j} \otimes E)}\Big]^{\frac{1}{p}}.$$

Then $(D_t^1, \cdots, D_t^k)'$ is a closed dense operator with dense domain $D^{k,p}(\Omega; E)$, which is a Banach space under the norm $\|\cdot\|_{D^{k,p}(\Omega;E)}$ and even a Hilbert space when $p = 2$. In addition, we define $D^{k,\infty}(\Omega; E) := \bigcap_{p\geq 1} D^{k,p}(\Omega; E)$ and $D^{\infty,\infty}(\Omega; E) := \bigcap_{k\in\mathbb{N}_+} \bigcap_{p\geq 1} D^{k,p}(\Omega; E)$, which are both locally convex space (see [29,31]).

When $E = \mathbb{R}$, $k = 1$ and $p = 2$, we define

$$\delta : L^2(\Omega \times [0, T]) \to L^2(\Omega)$$

with domain Dom δ as the adjoint of the closed dense operator

$$D_t : L^2(\Omega) \to L^2(\Omega \times [0, T]).$$

We call δ the Malliavin divergence operator. Denote by $L_a^2(\Omega \times [0, T])$ the set of all $\mathcal{F}_t$-adapted processes $u \in L^2(\Omega \times [0, T])$. Then we have $L_a^2(\Omega \times [0, T]) \subset$ Dom δ, and when $u \in L_a^2(\Omega \times [0, T])$, δu corresponds to the Itô integral $\int_0^T u_t \mathrm{d}W_t$. In this perspective, we call δu the Skorohod integral of u, and use $\int_0^T u_t \mathrm{d}W_t$ to represent it without causing ambiguity.

There are rich properties of D_t and δ (see [19]). Some of them can be found in Appendix A.

2.2. The Skorohod Integral

When u is Skorohod integrable (i.e., $u \in$ Dom δ), a natural question is that whether $\int_0^t u_s \mathrm{d}W_s := \delta(u_s 1_{[0,t]}(s))$ makes sense for a fixed $t \in [0, T]$. Unfortunately, $u_s 1_{[0,t]}(s)$ is not Skorohod integrable in general. However, since $D^{1,2}(\Omega; H) \subset$ Dom δ (see Lemma A3), $\int_0^t u_s \mathrm{d}W_s$ is well-defined for $u \in D^{1,2}(\Omega; H)$ by the chain rule (Lemma A1), and we can obtain more useful results in the subspaces of $D^{1,2}(\Omega; H)$.

Definition 1. *Define by $\mathcal{L}^{1,2}$ the space $D^{1,2}(\Omega; H)$, which is isomorphic to $L^2([0, T]; D^{1,2}(\Omega))$ (see [19]). For every $k \in \mathbb{N}_+$ and any $p \geq 2$, define by $\mathcal{L}^{k,p}$ the space $L^p([0, T]; D^{k,p}(\Omega))$, which is a subspace of $D^{k,p}(\Omega; H)$.*

Definition 2. *Let $u \in \mathcal{L}^{1,2}$, and let $q \in [1,2]$. We say that $u \in \mathcal{L}^{1,2,q-}$ (resp. $u \in \mathcal{L}^{1,2,q+}$) if there exists a (unique) process in $L^q(\Omega \times [0,T])$, denoted by D^-u (resp. D^+u), such that*

$$\lim_{\varepsilon \to 0^+} \int_0^T \sup_{(s-\varepsilon)\vee 0 \le t < s} \mathbb{E}(|D_s u_t - (D^- u)_s|^q) = 0.$$

$$(resp.\ \lim_{\varepsilon \to 0^+} \int_0^T \sup_{s < t \le (s+\varepsilon)\wedge T} \mathbb{E}(|D_s u_t - (D^+ u)_s|^q) = 0)$$

In particular, if $u \in \mathcal{L}^{1,2,q-} \cap \mathcal{L}^{1,2,q+}$, we say that $u \in \mathcal{L}^{1,2,q}$, and define $\nabla u := D^-u + D^+u$, which is also in $L^q(\Omega \times [0,T])$.

Remark 1. *In the earlier theory of the Skorohod integral, the space $\mathcal{L}^{1,2,C}$ was utilized (see [29,32]), allowing the existence of $(D^-u)_s := \lim_{\varepsilon\to 0^+} D_s u_{s-\varepsilon}$ and $(D^+u)_s := \lim_{\varepsilon\to 0^+} D_s u_{s+\varepsilon}$ in $L^2(\Omega)$ uniformly in s. However, this approach was considered overly restrictive. It couldn't adequately characterize convergence in $L^q(\Omega)$ and made certain proofs for sufficiency challenging. Furthermore, in our discussion of the forward integral in Section 2.3, we do not assume the existence of D^+u, which might not be feasible for a càglàd process in financial problems. As a result, we analyze within the more general spaces of $\mathcal{L}^{1,2,q-}$ and $\mathcal{L}^{1,2,q+}$ rather than relying on $\mathcal{L}^{1,2,C}$ introduced in [19], the second edition of [32].*

Similar to the Itô formula in classical Itô theory (as described in [3]), there exists a version of the Itô formula for the Skorohod integral. However, before presenting this formula, a localization technique is required, akin to the approach of the local martingale in Itô theory.

Definition 3. *If L is a class of random variables (or random fields), we denote by L_{loc} the set of random variables (or random fields) X such that there exists a sequence $\{(\Omega_n, X_n)\} \subset \mathcal{F} \times L$ with the following properties:*

(i) $\Omega_n \uparrow \Omega$, *a.s.*

(ii) $X_n = X$ *a.s. on* Ω_n.

Moreover, we can easily check that L_{loc} is a linear space if L is a linear space.

Due to the local properties of D_t and δ (Lemmas A4 and A5), the extensions of $D_t^k : D^{k,p}_{\mathrm{loc}}(\Omega; E) \to L^p_{\mathrm{loc}}(\Omega; H^{\otimes k} \otimes E)$ $(p \ge 1)$ and $\delta : \mathcal{L}^{1,2}_{\mathrm{loc}} \to L^2_{\mathrm{loc}}(\Omega)$ are well-defined, provided that E is a separable Hilbert space. The localizations for D^-, D^+ and ∇ follow a similar approach. The next proposition demonstrates that the Skorohod integral is also an extension of the generalized Itô integral in the sense of localization.

Proposition 1 (Proposition 1.3.18, [19])**.** *Let u be a measurable $\mathcal{F}_t$-adapted process such that $\int_0^T u_t^2 \mathrm{d}t < \infty$, a.s. Then u belongs to $(Dom\ \delta)_{loc}$ and δu is well-defined. Moreover, δu coincides with the Itô integral of u (with respect to the local martingale W). Thus, we can keep use of the notation $\int_0^T u_t \mathrm{d}W_t := \delta u$ without ambiguity when $u \in (Dom\ \delta)_{loc}$ and δu is well-defined.*

Now we can give the Itô formula for the Skorohod integral.

Theorem 1 (Theorem 3.2.2, [19])**.** *Consider a process of the form $X_t = X_0 + \int_0^t u_s \mathrm{d}W_s + \int_0^t v_s \mathrm{d}s$, where $X_0 \in D^{1,2}_{loc}(\Omega)$, $u \in (\mathcal{L}^{2,2} \cap \mathcal{L}^{1,4})_{loc}$, and $v \in \mathcal{L}^{1,2}_{loc}$. Then X_t is continuous and $X \in \mathcal{L}^{1,2,2}_{loc}$ by Lemma A6 and A7, respectively. Moreover, if $f \in C^2(\mathbb{R})$, then $f'(X_t)u_t \in \mathcal{L}^{1,2}_{loc}$ and we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)u_s^2 \mathrm{d}s + \int_0^t f''(X_s)(D^-X)_s u_s \mathrm{d}s. \quad (4)$$

2.3. The Forward Integral

The Skorohod integral process $\int_0^t u_s \mathrm{d}W_s$ is anticipating, meaning it is not adapted to the filtration $\mathcal{F}_t$. There is another anticipating integral called the forward integral, which was introduced by [33] and defined by [16]. This type of integral has been studied before and applied to insider trading in financial mathematics (see [10,34]). However, the sufficiency of the forward integrability and some related properties may be hard to obtain without the help of Malliavin calculus (see [11,17,18]). Given that some results in the above literature can be overly limiting in scope, we will study the forward integral by Malliavin calculus here completely. All proofs in this subsection can be found in Appendix B.

Definition 4. *Let $u \in L^2(\Omega \times [0,T])$. The forward integral of u is defined by*

$$\int_0^T u_t \mathrm{d}^- W_t := \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_0^T u_t \Big(W_{(t+\varepsilon)\wedge T} - W_t \Big) \mathrm{d}t, \tag{5}$$

if the limit exists in probability, in which case u is called forward integrable and we write $u \in$ Dom δ^-. If the limit exists also in $L^p(\Omega)$, we write $u \in \mathrm{Dom}_p\, \delta^-$.

Remark 2. *The forward integral is also an extension of the Itô integral. In other words, if there is a filtration $\{\mathcal{E}_t\}_{0 \le t \le T}$ satisfying the usual condition such that $\mathcal{E}_t \supset \mathcal{F}_t$ and W is a semi-martingale with respet to $\{\mathcal{E}_t\}$, $t \in [0,T]$, then $\int_0^T u_t \mathrm{d}^- W_t = \int_0^T u_t \mathrm{d}W_t$ for every $\mathcal{E}_t$-adapted process u such that u is Itô integrable with respect to W. We refer to [9] for the proof.*

It is worth noting that, akin to the Skorohod integral, the forward integrability of $u_s 1_{[0,t]}(s)$ for $t \in [0,T]$ cannot be inferred from that of u, which might be ignored in some literature. However, by Malliavin calculus, we can provide the sufficient condition for the aforementioned issue and elucidate the relationship between the Skorohod integral and the forward integral as the following two propositions, which has not been proved in our specific context.

Proposition 2. *Let $u \in \mathcal{L}^{1,2,1-}$. Then for all $t \in [0,T]$, we have $u_s 1_{[0,t]}(s) \in \mathrm{Dom}_1\, \delta^-$ and*

$$\int_0^t u_s \mathrm{d}^- W_s = \int_0^t u_s \mathrm{d}W_s + \int_0^t (D^- u)_s \mathrm{d}s. \tag{6}$$

Remark 3. *In [11], the condition $\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t u_s \mathrm{d}s = u_t$ in $\mathcal{L}^{1,2}$ which makes (6) hold is surplus (see Lemma A9), and the use of D_{t^+} is limiting by Remark 1. In [17], the condition $u \in \mathcal{L}^F$ requires the existence of the second derivative of u. In [19], the forward integral is defined by Riemann sums, and (6) needs an extra continuous condition which is contradict to the nature of the càglàd process in insider trading theory .*

Remark 4. *Proposition 2 illustrates that the forward integral can be extended to a linear operator δ^- from $\mathcal{L}^{1,2,1-}_{loc}$ into $L^1_{loc}(\Omega)$ as well.*

Proposition 3. *Let u be a process in $\mathcal{L}^{1,2,2-}$ and be L^2-bounded. Consider an $\mathcal{F}_t$-adapted process $\sigma \in \mathcal{L}^{1,2}$, which is L^2-bounded and left-continuous in the norm $L^2(\Omega)$. Assume further that σ and $D_s\sigma_t$ are bounded. Then $u\sigma \in \mathcal{L}^{1,2,1-}$, and for all $t \in [0,T]$, we have*

$$\int_0^t u_s \sigma_s \mathrm{d}^- W_s = \int_0^t u_s \sigma_s \mathrm{d}W_s + \int_0^t (D^- u)_s \sigma_s \mathrm{d}s. \tag{7}$$

Remark 5. *In [18], the condition that ensures the validity of Equation (7) necessitates the introduction of additional spaces and norms, leading to a rather complex proof. Furthermore, the constraints of the $\mathcal{L}^{1,2,C}$ space are overly limiting in this context.*

The Itô formula for the forward integral was first proved in [34] without Malliavin calculus. Here we use the Itô formula for the Skorohod integral (Theorem 1) to derive it, which can be viewed as an extension of [34]. Let $\mathcal{L}^f$ represent the linear space of processes $u \in \mathcal{L}^{2,2} \cap \mathcal{L}^{1,4} \cap \mathcal{L}^{1,2,1-}$ that are left-continuous in $L^2(\Omega)$, L^2-bounded, and for which $(D^- u) \in \mathcal{L}^{1,2}$. Then we have the following theorem.

Theorem 2. *Consider a process of the form $X_t = X_0 + \int_0^t u_s \mathrm{d}^- W_s + \int_0^t v_s \mathrm{d}s$, where $X_0 \in D^{1,2}_{loc}(\Omega)$, $u \in \mathcal{L}^f_{loc}$, and $v \in \mathcal{L}^{1,2}_{loc}$. Then X_t is continuous and $X \in \mathcal{L}^{1,2,2}_{loc}$. Moreover, if $f \in C^2(\mathbb{R})$, then $f'(X_t)u_t \in \mathcal{L}^{1,2,1^-}_{loc}$ and*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)u_s \mathrm{d}^- W_s + \int_0^t f'(X_s)v_s \mathrm{d}s + \frac{1}{2}\int_0^t f''(X_s)u_s^2 \mathrm{d}s. \tag{8}$$

3. Model Formulation

In this section, we will set up the model for insider trading under model uncertainty, and transform it into an implicit anticipating SDG problem.

We assume that all uncertainties arise from the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, on which a standard Brownian motion W is defined. Here, $\{\mathcal{F}_t\}_{t \ge 0}$ is the $\mathbb{P}$-augmentation of the filtration generated by W, and $\mathcal{F} = \mathcal{F}_\infty$. We fix a terminal time $T > 0$. Suppose all filtrations introduced in this section satisfy the usual condition.

3.1. Insider-Trading Model

Consider an investor who can invest in the financial market containing a risk-free asset (bond) B and a risky asset (stock) S. The price processes of the two assets are governed by the following anticipating SDEs

$$\begin{cases} \mathrm{d}B_t = r_t B_t \mathrm{d}t, \quad 0 \le t \le T, \\ \mathrm{d}S_t = \mu(t, \pi_t) S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}^- W_t, \quad 0 \le t \le T, \end{cases} \tag{9}$$

with constant initial values 1 and $S_0 > 0$, respectively. Here, the coefficients $r_t, \mu(t,x)$, and σ_t are all $\mathcal{F}_t$-adapted measurable stochastic processes for fixed $x \in \mathbb{R}$, and $\mu(t, \cdot)$ is C^1 for every $t \in [0, T]$.

Assume the investor is a large investor and has access to insider information characterized by another filtration $\{\mathcal{H}_t\}_{0 \le t \le T}$ with

$$\mathcal{F}_t \subset \mathcal{H}_t, \quad 0 \le t \le T. \tag{10}$$

Her investment strategy π could influence the mean rate of return μ of the risky asset. As a result, μ partly depends on π (see [9,11]).

The investment strategy π_t is defined as an $\mathcal{H}_t$-adapted càglàd process, which is L^2-bounded and belons to $\mathcal{L}^{1,2,2-}$. It represents the proportion of the investor's total wealth X_t invested in the risky asset S_t at time t. Since $\mu(t, \pi_t)$ is not adapted to $\mathcal{F}_t$, the stochastic integral in (9) should be interpreted as the forward integral (see [9,19]).

We make some assumptions on the coefficients:

- $r \in \mathcal{L}^{1,2}$. For each investment strategy π, $\mu(\cdot, \pi) - \frac{1}{2}\sigma^2 \in \mathcal{L}^{1,2}$. $\sigma \ge \epsilon > 0$ for some positive constant ϵ, and $\sigma \in \mathcal{L}^f$;
- σ and $D_s \sigma_t$ are bounded.

Given the conditions stated above, we can resolve the anticipating SDEs (9) by employing the Itô formula for forward integrals, as illustrated below (see Theorem 2)

$$S_t = S_0 \exp\left\{ \int_0^t \left(\mu(s, \pi_s) - \frac{1}{2}\sigma_s^2 \right) \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s \right\}, \tag{11}$$

which is no more an $\mathcal{F}_t$-semi-martingale, but an $\mathcal{H}_t$-adapted process.

Note that the investment strategy π of the investor can take negative values, which should be interpreted as engaging in short-selling of the risky asset. The wealth process X^π, associated with π, is governed by the following anticipating SDE (see [19]):

$$\mathrm{d}X_t^\pi = [r_t + (\mu(t,\pi_t) - r_t)\pi_t]X_t^\pi \mathrm{d}t + \sigma_t \pi_t X_t^\pi \mathrm{d}^- W_t, \quad 0 \le t \le T, \tag{12}$$

with constant initial value $X_0 > 0$. We can solve the anticipating SDE (12) by applying the Itô formula for forward integrals. Before that, we impose the following admissible conditions on π.

Definition 5. *We define $\mathcal{A}_1$ as the set of all investment strategies π satisfying the following conditions:*

(i) $\sigma\pi \in \mathcal{L}^f$;

(ii) $(\mu(\cdot,\pi) - r)\pi - \frac{1}{2}\sigma^2\pi^2 \in \mathcal{L}^{1,2}$;

(iii) $\int_0^T |r_t + (\mu(t,\pi_t) - r_t)\pi_t| \mathrm{d}t + \int_0^T |\sigma_t\pi_t|^2 \mathrm{d}t < \infty$.

Let $\pi \in \mathcal{A}_1$. By Theorem 2, the solution of (12) is given by

$$X_t^\pi = X_0 \exp\left\{ \int_0^t \left[r_s + (\mu(s,\pi_s) - r_s)\pi_s - \frac{1}{2}\sigma_s^2\pi_s^2 \right] \mathrm{d}s + \int_0^t \sigma_s \pi_s \mathrm{d}^- W_s \right\}. \tag{13}$$

3.2. Model Uncertainty Setup

Consider a model uncertainty setup. Assume that the investor is ambiguity-averse, implying that she is concerned about the accuracy of statistical estimation, and possible misspecification errors. Thus, a family of parametrized prior probability measures $\{\mathcal{Q}^\theta\}$ equivalent to the original probability measure $\mathbb{P}$ is assumed to exist in the real world. However, since the investor has insider information filtration $\{\mathcal{H}_t\}$ under which W might not be a semi-martingale, a generalization for the construction of $\{\mathcal{Q}^\theta\}$ needs to be considered by means of the forward integral.

Definition 6. *We define $\mathcal{A}_2$ as the set of all $\mathcal{H}_t$-adapted càglàd processes θ_t satisfying the following conditions:*

(i) $\theta \in \mathcal{L}^{1,2,1-}$;

(ii) *$\int_0^t \theta_s \mathrm{d}^- W_s$ is a continuous $\mathcal{H}_t$-semi-martingale, and the local martingale part (in the canonical decomposition) $\left(\int_0^t \theta_s \mathrm{d}^- W_s\right)^M$ satisfies the Novikov condition, i.e.,*

$$\mathbb{E}\left[\exp\left\{ \frac{1}{2} \left\langle \left(\int_0^\cdot \theta_s \mathrm{d}^- W_s \right)^M \right\rangle_t \right\} \right] < \infty, \quad \forall t \in [0,T].$$

For $\theta \in \mathcal{A}_2$, the Doléans-Dade exponential ε_t^θ is the unique $\mathcal{H}_t$-martingale with initial value 1 governed by

$$\mathrm{d}\varepsilon_t^\theta = \varepsilon_t^\theta \mathrm{d}\left(\int_0^t \theta_s \mathrm{d}^- W_s \right)^M, \quad 0 \le t \le T. \tag{14}$$

Thus, we have $\varepsilon_T^\theta > 0$ and $\int_\Omega \varepsilon_T^\theta \mathrm{d}\mathbb{P} = 1$, which induces a probability $\mathcal{Q}^\theta$ equivalent to $\mathbb{P}$ such that $\frac{\mathrm{d}\mathcal{Q}^\theta}{\mathrm{d}\mathbb{P}} = \varepsilon_T^\theta$. All such $\mathcal{Q}^\theta$ form a set of prior probability measures.

3.3. Robust Optimal Investment Problem

Taking into account the extra insider information and model uncertainty, the optimization problem for the investor can be formulated as an implicit anticipating (zero-sum) SDG. In other words, we need to solve the following problem.

Remark 6. *The local martingale part* $\left(\int_0^t \theta_s \mathrm{d}^- W_s\right)^M$ *in controlled system (14) could not be expressed analytically in general. Thus, the problem is implicit.*

Definition 7. *Define* $\mathcal{A}_1'$ *as the subset of* $\mathcal{A}_1$ *such that* $\mathbb{E}|\ln X_T^\pi|^2 < \infty$ *for all* $\pi \in \mathcal{A}_1'$*. Define* $\mathcal{A}_2'$ *as the subset of* $\mathcal{A}_2$ *such that* $\mathbb{E}\left[|\varepsilon_T^\theta|^2 + \int_0^T |g(\theta_s)|^2 \mathrm{d}s\right] < \infty$ *for all* $\theta \in \mathcal{A}_2'$*.*

Problem 1. *Select a pair* $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ *such that*

$$V := J(\pi^*, \theta^*) = \sup_{\pi \in \mathcal{A}_1'} \inf_{\theta \in \mathcal{A}_2'} J(\pi, \theta) = \inf_{\theta \in \mathcal{A}_2'} \sup_{\pi \in \mathcal{A}_1'} J(\pi, \theta), \tag{15}$$

where the performance function J *is given by*

$$J(\pi, \theta) := \mathbb{E}_{\mathcal{Q}^\theta}\left[\ln X_T^\pi + \int_0^T g(\theta_s)\mathrm{d}s\right] = \mathbb{E}\left[\varepsilon_T^\theta \ln X_T^\pi + \int_0^T \varepsilon_s^\theta g(\theta_s)\mathrm{d}s\right],$$

and the penalty function $g : \mathbb{R} \to \mathbb{R}$ *is a Fréchet differentiable convex function. We call* V *the value (or the robust optimal expected utility) of Problem 1.*

4. Initial Characterization of Investment: Variational Method

We use the variational method to give a first characterization of the optimal solution of Problem 1. Before that, we introduce the following notations.

Let $\pi \in \mathcal{A}_1'$ denote

$$m_t^\pi := \int_0^t \left(\mu(s, \pi_s) - r_s + \frac{\partial}{\partial x}\mu(s, \pi_s)\pi_s - \sigma_s^2 \pi_s\right)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s \tag{16}$$

with $t \in [0, T]$.

Assumption 1. *If* $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ *is optimal for Problem 1, then for all bounded* $\alpha \in \mathcal{A}_1'$*, there exists some* $\delta > 0$ *such that* $\pi^* + y\alpha \in \mathcal{A}_1'$ *for all* $|y| < \delta$*. Moreover, the following family of random variables*

$$\left\{\varepsilon_T^{\theta^*} (X_T^{\pi^* + y\alpha})^{-1} \frac{\mathrm{d}}{\mathrm{d}y} X_T^{\pi^* + y\alpha}\right\}_{y \in (-\delta, \delta)}$$

is $\mathbb{P}$*-uniformly integrable, where* $\frac{\mathrm{d}}{\mathrm{d}y} X_T^{\pi^* + y\alpha}$ *exists and the interchange of differentiation and integral with respect to* $\ln X_T^{\pi^* + y\alpha}$ *in (13) is justified.*

Assumption 2. *Let* $\alpha_s = \vartheta 1_{(t, t+h]}(s)$ *for fixed* $0 \le t < t + h \le T$*, where* ϑ *is an* $\mathcal{H}_t$*-measurable bounded random variable in* $D^{\infty,\infty}(\Omega)$*. Then we have* $\alpha \in \mathcal{A}_1'$*.*

Theorem 3. *Suppose* $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ *is optimal for Problem 1 under Assumptions 1 and 2. Then* $m_t^{\pi^*}$ *is an* $(\mathcal{H}_t, \mathcal{Q}^{\theta^*})$*-martingale.*

Proof. Suppose that the pair $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal. Then for any bounded $\alpha \in \mathcal{A}_1'$ and $|y| < \delta$, we have $J(\pi^* + y\alpha, \theta^*) \le J(\pi^*, \theta^*)$, which implies that $y = 0$ is a maximum point of the function $y \mapsto J(\pi^* + y\alpha, \theta^*)$. Thus, we have $\frac{\mathrm{d}}{\mathrm{d}y} J(\pi^* + y\alpha, \theta^*)|_{y=0} = 0$

once the differentiability is established. Thanks to Assumption 1, we can deduce by Proposition 3 that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}y} J(\pi^* + y\alpha, \theta^*)|_{y=0} \\
=&\mathbb{E}\left\{ \varepsilon_T^{\theta^*} \left[\int_0^T \alpha_s \left(\mu(s,\pi_s^*) - r_s + \frac{\partial}{\partial x}\mu(s,\pi_s^*)\pi_s^* - \sigma_s^2 \pi_s^* \right) \mathrm{d}s + \int_0^T \alpha_s \sigma_s \mathrm{d}^- W_s \right] \right\} \\
=&\mathbb{E}\left\{ \varepsilon_T^{\theta^*} \left[\int_0^T \alpha_s \left(\mu(s,\pi_s^*) - r_s + \frac{\partial}{\partial x}\mu(s,\pi_s^*)\pi_s^* - \sigma_s^2 \pi_s^* \right) \mathrm{d}s + \int_0^T \alpha_s \sigma_s \mathrm{d} W_s \right.\right. \\
&\left.\left. + \int_0^T (D^-\alpha)_s \sigma_s \mathrm{d}s \right] \right\} \\
=&0.
\end{aligned}
$$

Now fix $0 \le t < t+h \le T$. By Assumption 2, we can choose $\alpha \in \mathcal{A}_1'$ of the form

$$\alpha_s = \vartheta 1_{(t,t+h]}(s), \quad 0 \le s \le T,$$

where $\vartheta \in D^{\infty,\infty}(\Omega)$ is an $\mathcal{H}_t$-measurable bounded random variable. Then we have

$$(D^- \vartheta 1_{(t,t+h]})_s = D_s \vartheta 1_{(t,t+h]}(s).$$

By Lemma A2 we have

$$\mathbb{E}_{\mathcal{Q}^{\theta^*}} \left\{ \vartheta \left[\int_t^{t+h} \left(\mu(s,\pi_s^*) - r_s + \frac{\partial}{\partial x}\mu(s,\pi_s^*)\pi_s^* - \sigma_s^2 \pi_s^* \right) \mathrm{d}s + \int_t^{t+h} \sigma_s \mathrm{d}W_s \right] \right\} = 0.$$

Since this holds for all such ϑ, we can conclude that

$$\mathbb{E}_{\mathcal{Q}^{\theta^*}} \left[m_{t+h}^{\pi^*} - m_t^{\pi^*} \middle| \mathcal{H}_t \right] = 0.$$

Hence, $m_t^{\pi^*}$ is an $\mathcal{H}_t$-martingale under the probability measure $\mathcal{Q}^{\theta^*}$. □

Moreover, we have the following result under the original probability measure $\mathbb{P}$. Unless otherwise stated, all statements are back to $\mathbb{P}$ from now on.

Theorem 4. *Suppose $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal for Problem 1 under Assumptions 1 and 2. Then the following stochastic process*

$$\hat{m}_t^{\pi^*,\theta^*} := m_t^{\pi^*} - \int_0^t \varepsilon_s^{\theta^*} \mathrm{d}\left\langle (\varepsilon^{\theta^*})^{-1}, m^{\pi^*} \right\rangle_s, \quad 0 \le t \le T, \tag{17}$$

is an $\mathcal{H}_t$-local martingales. Here, $\langle \cdot, \cdot \rangle$ represents the covariance process (see [3]). We assume that $\left\langle (\varepsilon^{\theta^*})^{-1}, m^{\pi^*} \right\rangle_t$ is absolutely continuous.

Proof. If $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal, then by Theorem 3 we know that $m_t^{\pi^*}$ is an $(\mathcal{H}_t, \mathcal{Q}^{\theta^*})$-martingale. The conclusion is an immediate result from the Girsanov theorem (see [35]). □

Further, since $\hat{m}_t^{\pi^*,\theta^*}$ is an $\mathcal{H}_t$-local martingale, we can deduce from (16) that $\int_0^t \sigma_s \mathrm{d}W_s$ is a continuous $\mathcal{H}_t$-semi-martingale. Multiplying both sides of (17) by σ_t^{-1} and integrating, we have

$$\begin{aligned}\int_0^t \sigma_s^{-1}\mathrm{d}\hat{m}_s^{\pi^*,\theta^*} =& W_t + \int_0^t \sigma_s^{-1}\Big(\mu(s,\pi_s^*) - r_s + \frac{\partial}{\partial x}\mu(s,\pi_s^*)\pi_s^* - \sigma_s^2\pi_s^*\Big)\mathrm{d}s \\ &- \int_0^t \sigma_s^{-1}\varepsilon_s^{\theta^*}\mathrm{d}\big\langle(\varepsilon^{\theta^*})^{-1}, m^{\pi^*}\big\rangle_s.\end{aligned}$$

Since $\langle\hat{m}^{\pi^*,\theta^*}\rangle_t = \big\langle\int_0^\cdot \sigma_s \mathrm{d}W_s\big\rangle_t = \int_0^t \sigma_s^2\mathrm{d}s$, we have $\langle W\rangle_t = t$. Thus, by the Lévy theorem (see [3]), the canonical decomposition of the continuous $\mathcal{H}_t$-semi-martingale W_t can be given as $W_t = W_{\mathcal{H}}(t) + \int_0^t \phi_s \mathrm{d}s$, where $W_{\mathcal{H}}(t)$ is an $\mathcal{H}_t$-Brownian motion and ϕ_t is a measurable $\mathcal{H}_t$-adapted process. Moreover, by Remark 2, we have $\int_0^t \sigma_s \mathrm{d}W_s = \int_0^t \sigma_s \mathrm{d}W_{\mathcal{H}}(s) + \int_0^t \sigma_s\phi_s \mathrm{d}s$. In summary, we give the following theorem.

Theorem 5 (semi-martingale Decomposition)**.** *Suppose $(\pi^*,\theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal for Problem 1 under Assumptions 1 and 2. Then we have the following decomposition*

$$W_t = W_{\mathcal{H}}(t) + \int_0^t \phi_s \mathrm{d}s, \quad 0 \le t \le T, \tag{18}$$

where $W_{\mathcal{H}}(t)$ is an $\mathcal{H}_t$-Brownian motion, ϕ_t is a measurable $\mathcal{H}_t$-adapted process satisfying

$$\int_0^T |\phi_t|\mathrm{d}t < \infty.$$

Moreover, by the uniqueness of the canonical decomposition of a continuous semi-martingale, π^* solves the following equation

$$\begin{aligned}0 = &\int_0^t \Big(\mu(s,\pi_s^*) - r_s + \frac{\partial}{\partial x}\mu(s,\pi_s^*)\pi_s^* - \sigma_s^2\pi_s^*\Big)\mathrm{d}s + \int_0^t \sigma_s\phi_s \mathrm{d}s \\ &- \int_0^t \varepsilon_s^{\theta^*}\mathrm{d}\big\langle(\varepsilon^{\theta^*})^{-1}, m^{\pi^*}\big\rangle_s, \quad 0 \le t \le T.\end{aligned} \tag{19}$$

Further, by Theorem 5 and Remark 2, the dynamic of the $\mathcal{H}_t$-martingale ε_t^θ (see (14)) can be rewritten as

$$\mathrm{d}\varepsilon_t^\theta = \varepsilon_t^\theta \mathrm{d}\bigg(\int_0^t \theta_s \mathrm{d}W_{\mathcal{H}}(s)\bigg), \quad 0 \le t \le T, \tag{20}$$

for $\theta \in \mathcal{A}_2'$. By the Itô formula for Itô integrals (see [3]), we have

$$(\varepsilon_t^\theta)^{-1} = 1 + \int_0^t (\varepsilon_s^\theta)^{-1}\theta_s^2 \mathrm{d}s - \int_0^t (\varepsilon_s^\theta)^{-1}\theta_s \mathrm{d}W_{\mathcal{H}}(s). \tag{21}$$

For the optimal pair (π^*,θ^*), by Theorem 5, we can easily calculate the covariation process of $(\varepsilon_t^{\theta^*})^{-1}$ and $m_t^{\pi^*}$ as follows

$$\big\langle(\varepsilon^{\theta^*})^{-1}, m^{\pi^*}\big\rangle_t = -\int_0^t (\varepsilon_s^{\theta^*})^{-1}\theta_s^*\sigma_s \mathrm{d}s. \tag{22}$$

By substituting (22) into (19) in Theorem 5 we have the following theorem.

Theorem 6. *Suppose $(\pi^*, \theta^*) \in \mathcal{A}'_1 \times \mathcal{A}'_2$ is optimal for Problem 1 under Assumptions 1 and 2. Then π^* solves the following equation*

$$0 = \mu(t, \pi^*_t) - r_t + \frac{\partial}{\partial x}\mu(t, \pi^*_t)\pi^*_t - \sigma^2_t \pi^*_t + \sigma_t \phi_t + \sigma_t \theta^*_t, \quad 0 \le t \le T. \tag{23}$$

5. Final Characterization of Investment: Stochastic Maximum Principle

In the previous section, we give the characterization of π^* for the optimal pair (π^*, θ^*) by using the maximality of $J(\pi^*, \theta^*)$ with respect to π. Thus, we obtain the relationship between π^* and θ^* (see (23)). However, we have not used the minimality of $J(\pi^*, \theta^*)$ with respect to θ. Thus, we need the other half characterization of θ^*.

It is very difficult to give a direct characterization of θ^* due to the implicit nature of the controlled process ε^θ_t (see (14)). Fortunately, under Assumptions 1 and 2, we can decompose W_t into $W_{\mathcal{H}}(t)$ and $\int_0^t \phi_s \mathrm{d}s$ with respect to the filtration $\{\mathcal{H}_t\}$ using Theorem 5. Consequently, we transform the implicit anticipating SDE (14) with ε^θ_t into an explicit nonanticipative SDE (20). Furthermore, we can transform the anticipating SDE (12) with X^π_t into a nonanticipative SDE as follows

$$\mathrm{d}X^\pi_t = [r_t + (\mu(t, \pi_t) - r_t)\pi_t + \sigma_t \pi_t \phi_t] X^\pi_t \mathrm{d}t + \sigma_t \pi_t X^\pi_t \mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T. \tag{24}$$

Since (20) and (24) can be viewed as classical SDEs with respect to the $\mathcal{H}_t$-Brownian motion $W_{\mathcal{H}}(t)$, Problem 1 becomes a nonanticipative SDG problem with respect to the filtration $\{\mathcal{H}_t\}_{0 \le t \le T}$. Consequently, we can apply the stochastic maximum principle to resolve our problem.

Before delving into our methodology, we establish the following assumptions.

Assumption 3. *If $(\pi^*, \theta^*) \in \mathcal{A}'_1 \times \mathcal{A}'_2$ is optimal for Problem 1, then for all bounded $\beta \in \mathcal{A}'_2$, there exists some $\delta > 0$ such that $\theta^* + y\beta \in \mathcal{A}'_2$ for all $|y| < \delta$. Moreover, the following family of random variables*

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}y} \varepsilon^{\theta^* + y\beta}_T \ln X^{\pi^*}_T \right\}_{y \in (-\delta, \delta)}$$

is $\mathbb{P}$-uniformly integrable, and the following family of random fields

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}y} \varepsilon^{\theta^* + y\beta}_t g(\theta^* + y\beta) + \varepsilon^{\theta^* + y\beta}_t g'(\theta^* + y\beta)\beta \right\}_{y \in (-\delta, \delta)}$$

is $\mathrm{m} \times \mathbb{P}$-uniformly integrable, where m is the Borel-Lebesgue measure on $[0, T]$, and $\frac{\mathrm{d}}{\mathrm{d}y}\varepsilon^{\theta^* + y\beta}_t$ exists.

Assumption 4. *If $(\pi^*, \theta^*) \in \mathcal{A}'_1 \times \mathcal{A}'_2$ is optimal for Problem 1 under Assumptions 1–3, then for all bounded $(\alpha, \beta) \in \mathcal{A}'_1 \times \mathcal{A}'_2$, we can define $\tilde{\psi}^{\pi^*}_t := \frac{\mathrm{d}}{\mathrm{d}y} X^{\pi^* + y\alpha}_t |_{y=0}$ and $\psi^{\theta^*}_t := \frac{\mathrm{d}}{\mathrm{d}y} \varepsilon^{\theta^* + y\beta}_t |_{y=0}$ by Assumptions 1 and 3. Assume the following SDEs hold:*

$$\begin{cases} \mathrm{d}\tilde{\psi}^{\pi^*}_t = \left[\frac{\partial}{\partial x}\mu(t, \pi^*_t)\pi^*_t \alpha_t + (\mu(t, \pi^*_t) - r_t)\alpha_t + \sigma_t \phi_t \alpha_t \right] X^{\pi^*}_t \mathrm{d}t + \sigma_t \alpha_t X^{\pi^*}_t \mathrm{d}W_{\mathcal{H}}(t) \\ \qquad\qquad + \tilde{\psi}^{\pi^*}_t \left[r_t + (\mu(t, \pi^*_t) - r_t)\pi^*_t + \sigma_t \pi^*_t \phi_t \right] \mathrm{d}t + \tilde{\psi}^{\pi^*}_t \sigma_t \pi^*_t \mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ \tilde{\psi}^{\pi^*}_0 = 0; \end{cases}$$

$$\begin{cases} \mathrm{d}\psi^{\theta^*}_t = \varepsilon^{\theta^*}_t \beta_t \mathrm{d}W_{\mathcal{H}}(t) + \psi^{\theta^*}_t \theta^*_t \mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ \psi^{\theta^*}_0 = 0. \end{cases}$$

Assumption 5. *Let $\beta_s = \xi 1_{(t,t+h]}(s)$, $0 \le s \le T$, for fixed $0 \le t < t+h \le T$, where the random variable ξ is of the form 1_{A_t} for any $\mathcal{H}_t$-measurable set A_t. Then $\beta \in \mathcal{A}_2'$.*

Now we define the Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$ by

$$H(t,x,\varepsilon,\pi,\theta,p,q,\omega) := g(\theta)\varepsilon + [r_t + (\mu(t,\pi) - r_t)\pi + \sigma_t \pi \phi_t] x p_1 + \sigma_t \pi x q_1 + \varepsilon \theta q_2,$$

where $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, and $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. It is obvious that H is differentiable with respect to x, ε, π and θ. The corresponding BSDE system for the adjoint pair (p_t, q_t) is given by

$$\begin{cases} \mathrm{d}p_1(t) = -\dfrac{\partial H}{\partial x}(t)\mathrm{d}t + q_1(t)\mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ p_1(T) = \varepsilon_T^{\theta}(X_T^{\pi})^{-1}, \end{cases} \tag{25}$$

and

$$\begin{cases} \mathrm{d}p_2(t) = -\dfrac{\partial H}{\partial \varepsilon}(t)\mathrm{d}t + q_2(t)\mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ p_2(T) = \ln X_T^{\pi}, \end{cases} \tag{26}$$

where $p_i(t)$ is a continuous $\mathcal{H}_t$-semi-martingale, and $q_i(t)$ is an $\mathcal{H}_t$-adapted process with the following integrability

$$\int_0^T \left[\left|\frac{\partial H}{\partial x}(t)\right| + \left|\frac{\partial H}{\partial \varepsilon}(t)\right| + |q_i(t)|^2 \right] \mathrm{d}t < \infty,$$

$i = 1,2$. Here, $H(t) := H(t, X_t^{\pi}, \varepsilon_t^{\theta}, \pi_t, \theta_t, p_t, q_t, \omega)$, etc.

We give a necessary maximum principle and a sufficient maximum principle to characterize the optimal pair $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$.

Theorem 7. *Suppose $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal for Problem 1 under Assumptions 1–5, and (p^*, q^*) is the corresponding adjoint pair satisfying BSDEs (25) and (26). Then (π^*, θ^*) solves the following equations (the Hamiltonian system)*

$$\frac{\partial H^*}{\partial \pi}(t) = 0, \quad 0 \le t \le T, \tag{27}$$

and

$$\frac{\partial H^*}{\partial \theta}(t) = 0, \quad 0 \le t \le T, \tag{28}$$

given the following integrability conditions

$$\mathbb{E}\left\{ \int_0^T (\tilde{\psi}_t^{\pi^*})^2 \mathrm{d}\langle p_1^* \rangle_t + \int_0^T p_1^*(t)^2 \mathrm{d}\langle \tilde{\psi}^{\pi^*} \rangle_t \right\} < \infty,$$

and

$$\mathbb{E}\left\{ \int_0^T (\psi_t^{\theta^*})^2 \mathrm{d}\langle p_2^* \rangle_t + \int_0^T p_2^*(t)^2 \mathrm{d}\langle \psi^{\theta^*} \rangle_t \right\} < \infty,$$

for all bounded $(\alpha, \beta) \in \mathcal{A}_1' \times \mathcal{A}_2'$. Here, $H^*(t) := H(t, X_t^{\pi^*}, \varepsilon_t^{\theta^*}, \pi_t^*, \theta_t^*, p_t^*, q_t^*, \omega)$, etc.

Proof. Suppose that the pair $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal. Then for any bounded $\beta \in \mathcal{A}_2'$ and $|y| < \delta$, we have $J(\pi^*, \theta^* + y\beta) \ge J(\pi^*, \theta^*)$, which implies that $y = 0$ is a minimum point of the function $y \mapsto J(\pi^*, \theta^* + y\beta)$. By Assumptions 3 and 4 and Itô formula for Itô integrals, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}y} J(\pi^*, \theta^* + y\beta)|_{y=0} \\
=&\mathbb{E}\left[\psi_T^{\theta^*} \ln X_T^{\pi^*} + \int_0^T \psi_s^{\theta^*} g(\theta_s^*)\mathrm{d}s + \int_0^T \varepsilon_s^{\theta^*} g'(\theta_s^*)\beta_s \mathrm{d}s\right] \\
=&\mathbb{E}\left[\psi_T^{\theta^*} p_2^*(T) + \int_0^T \psi_s^{\theta^*} g(\theta_s^*)\mathrm{d}s + \int_0^T \varepsilon_s^{\theta^*} g'(\theta_s^*)\beta_s \mathrm{d}s\right] \\
=&\mathbb{E}\left[\int_0^T \psi_s^{\theta^*} \mathrm{d}p_2^*(s) + \int_0^T p_2^*(s)\mathrm{d}\psi_s^{\theta^*} + \langle p_2^*, \psi^{\theta^*}\rangle_T + \int_0^T \psi_s^{\theta^*} g(\theta_s^*)\mathrm{d}s + \int_0^T \varepsilon_s^{\theta^*} g'(\theta_s^*)\beta_s \mathrm{d}s\right] \\
=&\mathbb{E}\left[\int_0^T q_2^*(s)\varepsilon_s^{\theta^*}\beta_s \mathrm{d}s + \int_0^T \varepsilon_s^{\theta^*} g'(\theta_s^*)\beta_s \mathrm{d}s\right] = \mathbb{E}\left[\int_0^T \frac{\partial H^*}{\partial \theta}(s)\beta_s \mathrm{d}s\right] = 0.
\end{aligned}$$

By Assumption 5 and the same procedure in Theorem 3, we can deduce that $\frac{\partial H^*}{\partial \theta}(t) = 0$, $t \in [0, T]$. By similar arguments, we can conclude that $\frac{\partial H^*}{\partial \pi}(t) = 0$, $t \in [0, T]$. □

Theorem 8. *Assume that the semi-martingale decomposition (18) holds. Let $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ with the corresponding pair (p^*, q^*) satisfying BSDE (25) and (26). Suppose (u^*, v^*) satisfies the Hamiltonian system (27) and (28). Then (u^*, v^*) is optimal for Problem 1 given the following integrability conditions*

$$\mathbb{E}\left\{\int_0^T \left(|p_1^*(t)\sigma_t(\pi_t X_t^\pi - \pi_t^* X_t^{\pi^*})|^2 + |q_1^*(t)\tilde{X}_t|^2\right)\mathrm{d}t\right\} < \infty, \tag{29}$$

and

$$\mathbb{E}\left\{\int_0^T \left(|\tilde{\varepsilon}_t q_2(t)|^2\right)\mathrm{d}t + \int_0^T |p_2^*(t)|^2 \mathrm{d}\langle\tilde{\varepsilon}\rangle_t\right\} < \infty, \tag{30}$$

for all $(\pi, \theta) \in \mathcal{A}_1' \times \mathcal{A}_2'$. Here, we introduce the following denotations

$$\begin{cases}
\tilde{\pi}_t = \pi_t - \pi_t^*, \quad \tilde{X}_t = X_t^\pi - X_t^{\pi^*}, \quad H_1(t) = H(t, X_t^\pi, \varepsilon_t^{\theta^*}, \pi_t, \theta_t^*, p_t^*, q_t^*, \omega), \\
\tilde{\theta}_t = \theta_t - \theta_t^*, \quad \tilde{\varepsilon}_t = \varepsilon_t^\theta - \varepsilon_t^{\theta^*}, \quad H_2(t) = H(t, X_t^{\pi^*}, \varepsilon_t, \pi_t^*, \theta_t, p_t^*, q_t^*, \omega).
\end{cases}$$

Proof. For $\pi \in \mathcal{A}_1'$, by the Itô formula and Taylor formula, we have that

$$\begin{aligned}
J(\pi, \theta^*) - J(\pi^*, \theta^*) =& \mathbb{E}\left[\varepsilon^{\theta^*}\left(\ln X_T^\pi - \ln X_T^{\pi^*}\right)\right] \le \mathbb{E}\left[[\varepsilon^{\theta^*}\tilde{X}_T / X_T^{\pi^*}\right] = \mathbb{E}[p_1^*(T)\tilde{X}_T] \\
=& \mathbb{E}\left[\int_0^T p_1^*(t)\mathrm{d}\tilde{X}_t + \int_0^T \tilde{X}_t \mathrm{d}p_1^*(t) + \langle p_1^*, \tilde{X}\rangle_T\right] \\
=& \mathbb{E}\left[\int_0^T \left(H_1(t) - H^*(t) - \tilde{X}_t \frac{\partial H^*}{\partial x}(t)\right)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \frac{\partial H^*}{\partial \pi}(t)\tilde{\pi}_t \mathrm{d}t\right] \\
=& 0,
\end{aligned}$$

which induces that $\sup_{u \in \mathcal{A}_1'} J(u, v^*) = J(u^*, v^*)$. Similarly, we have $\inf_{v \in \mathcal{A}_2'} J(u^*, v) = J(u^*, v^*)$. Thus, we have

$$\begin{aligned}
\inf_{v \in \mathcal{A}_2'} \sup_{u \in \mathcal{A}_1'} J(u, v) \le& \sup_{u \in \mathcal{A}_1'} J(u, v^*) \le J(u^*, v^*) \\
\le& \inf_{v \in \mathcal{A}_2'} J(u^*, v) \le \sup_{u \in \mathcal{A}_1'} \inf_{v \in \mathcal{A}_2'} J(u, v).
\end{aligned}$$

Since $\inf_{v\in\mathcal{A}_2'}\sup_{u\in\mathcal{A}_1'} J(u,v) \geq \sup_{u\in\mathcal{A}_1'}\inf_{v\in\mathcal{A}_2'} J(u,v)$, we have

$$J(u^*,v^*) = \inf_{v\in\mathcal{A}_2'}\sup_{u\in\mathcal{A}_1'} J(u,v) = \sup_{u\in\mathcal{A}_1'}\inf_{v\in\mathcal{A}_2'} J(u,v).$$

Then (u^*,v^*) is optimal for Problem 1. □

Combining Theorems 7 and 8 with conclusions in Section 4, we can derive the total characterization of the optimal pair (π^*,θ^*) as the following theorem.

Theorem 9. *Suppose $(\pi^*,\theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal for Problem 1 with the corresponding pair (p^*,q^*) satisfying BSDEs (25) and (26) under the conditions in Theorem 7. Then (π^*,θ^*) solves Equations (23), (27) and (28). Conversely, if the semi-martingale decomposition (18) holds, $(\pi^*,\theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ with the corresponding pair (p^*,q^*) satisfying (25) and (26), and (u^*,v^*) satisfies Hamiltonian system (27) and (28). Then (u^*,v^*) is optimal for Problem 1 given the integrability conditions (29) and (30).*

Remark 7. *By combining equation (28) with equation (23), we could derive the optimal pair (π^*,θ^*). This combined method consistently provides a more comprehensive characterization of (π^*,θ^*) compared to relying solely on the Hamiltonian system (27) and (28). This is because the relationship between π^* and θ^* can be explicitly defined by equation (23). For instance, when the mean rate of return μ is dependent on π^*, as in the case of a large investor, obtaining the solution (π^*,θ^*) using only (27) and (28) is particularly challenging due to the non-homogeneous nature of X^π in such situations (see Section 7).*

The robust optimal investment π_t^* can be obtained from the Hamiltonian system (27) and (28). In the following two sections, we will explore two typical scenarios involving a small insider and a large insider to derive the expression for π_t^* in more detail.

6. The Small Insider Case

In this section, we will deduce a generalized linear BSDE that the robust optimal investment π_t^* entails in the case of a small insider. Subsequently, we will calculate the closed form of π_t^* based on this equation by white noise theory.

We assume that the mean rate of return function $\mu(t,x) = \mu_0(t)$ for some $\mathcal{F}_t$-adapted measurable processes $\mu_0(t)$, which means the insider's strategy could not influence the market. Thus, she is a small insider.

Put $\iota_t = \frac{\mu_0(t)-r_t}{\sigma_t}$ and $\tilde{\phi}_t = \iota_t + \phi_t$. Assume further the penalty function g is of the quadratic form, i.e., $g(\theta) = \frac{1}{2}\theta^2$. Then we have by the Girsanov theorem that

$$\begin{aligned}\mathbb{E}\left[\int_0^T \varepsilon_s^{\theta^*} g(\theta_s^*)\mathrm{d}s\right] = \mathbb{E}_{Q^{\theta^*}}\left[\int_0^T g(\theta_s^*)\mathrm{d}s\right] &= \mathbb{E}_{Q^{\theta^*}}\left[\int_0^T \theta_s^*\mathrm{d}W_{\mathcal{H}}(s) - \ln\varepsilon_T^{\theta^*}\right] \\ &= \mathbb{E}_{Q^{\theta^*}}\left[\int_0^T (\theta_s^*)^2\mathrm{d}s - \ln\varepsilon_T^{\theta^*}\right] \\ &= 2\mathbb{E}_{Q^{\theta^*}}\left[\int_0^T g(\theta_s^*)\mathrm{d}s\right] - \mathbb{E}_{Q^{\theta^*}}\left[\ln\varepsilon_T^{\theta^*}\right],\end{aligned}$$

which implies that

$$\mathbb{E}\left[\int_0^T \varepsilon_s^{\theta^*} g(\theta_s^*)\mathrm{d}s\right] = \mathbb{E}_{Q^{\theta^*}}\left[\ln\varepsilon_T^{\theta^*}\right] = \mathbb{E}\left[\varepsilon_T^{\theta^*}\ln\varepsilon_T^{\theta^*}\right]. \quad (31)$$

We make the following assumption.

Assumption 6. *Suppose the following integrability condition holds*

$$\int_0^T |\tilde{\phi}_t|^2 \mathrm{d}t < \infty.$$

In order to calculate the robust optimal investment, we give the following lemmas.

Lemma 1. *Assume that $\mu(t,x) = \mu_0(t)$ for some $\mathcal{F}_t$-adapted measurable process $\mu_0(t)$, and $g(\theta) = \frac{1}{2}\theta^2$. Suppose $(\pi^*, \theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal for Problem 1 under the conditions in Theorem 7. Suppose Assumption 6 holds. Then we have*

$$\varepsilon_T^{\theta^*} = \frac{1}{\mathbb{E}\left[(X_T^{\pi^*})^{-1} | \mathcal{H}_0\right] X_T^{\pi^*}}. \tag{32}$$

Proof. Utilizing the Hamiltonian system (28) in Theorem 7, we have

$$\frac{\partial H^*}{\partial \theta}(t) = \theta_t^* + q_2^*(t) = 0, \tag{33}$$

Substituting (33) into the adjoint BSDE (26) with respect to $p_2^*(t)$, we have

$$\begin{cases} \mathrm{d}p_2^*(t) = \dfrac{(\theta_t^*)^2}{2}\mathrm{d}t - \theta_t^* \mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ p_2^*(T) = \ln X_T^{\pi^*}. \end{cases} \tag{34}$$

The SDE (20) of $\varepsilon_t^{\theta^*}$ implies that

$$\mathrm{d}\ln \varepsilon_t^{\theta^*} = -\frac{(\theta_t^*)^2}{2}\mathrm{d}t + \theta_t^* \mathrm{d}W_{\mathcal{H}}(t). \tag{35}$$

By comparing (34) with (35), the solution of the BSDE (34) can be given as

$$p_2^*(t) = p_2^*(0) - \ln \varepsilon_t^{\theta^*}. \tag{36}$$

Substituting the terminal condition in (34), i.e., $p_2^*(T) = \ln X_T^{\pi^*}$, into (36) with $t = T$, we have

$$\ln\left(\varepsilon_T^{\theta^*} X_T^{\pi^*}\right) = c_2^*, \tag{37}$$

where $c_2^* := p_2^*(0)$. Since $\varepsilon_t^{\theta^*}$ is an $\mathcal{H}_t$-martingale, we have

$$\varepsilon_t^{\theta^*} = \mathbb{E}\left[\varepsilon_T^{\theta^*} | \mathcal{H}_t\right] = \mathbb{E}\left[e^{c_2^*}(X_T^{\pi^*})^{-1} | \mathcal{H}_t\right] \tag{38}$$

by (37). Considering that $\varepsilon_0^{\theta^*} = 1$, we obtain

$$e^{c_2^*} = \frac{1}{\mathbb{E}\left[(X_T^{\pi^*})^{-1} | \mathcal{H}_0\right]}. \tag{39}$$

Substituting (39) into (37), we obtain

$$\varepsilon_T^{\theta^*} = \frac{1}{\mathbb{E}\left[(X_T^{\pi^*})^{-1} | \mathcal{H}_0\right] X_T^{\pi^*}}. \tag{40}$$

□

Lemma 2. *Assume the conditions in Lemma 1 hold. Then we have*

$$X_T^{\pi^*} = \frac{1}{\sqrt{c_3^* \Pi^*(0,T)}}, \tag{41}$$

where $c_3^* := c_1^* \mathbb{E}\Big[(X_T^{\pi^*})^{-1} | \mathcal{H}_0\Big]$, $c_1^* := p_1^*(0)$, *and*

$$\Pi^*(t_1,t_2) := \exp\left\{ -\int_{t_1}^{t_2} r_s \mathrm{d}s - \int_{t_1}^{t_2} \tilde{\phi}_s \mathrm{d}W_{\mathcal{H}}(s) - \frac{1}{2}\int_{t_1}^{t_2} \tilde{\phi}_s^2 \mathrm{d}s \right\}. \tag{42}$$

Proof. Utilizing the Hamiltonian system (27) in Theorem 7, we have

$$\frac{\partial H^*}{\partial \pi}(t) = (\mu_0(t) - r_t + \sigma_t \phi_t) X_t^{\pi^*} p_1^*(t) + \sigma_t X_t^{\pi^*} q_1^*(t) = 0, \tag{43}$$

which implies that

$$(\mu_0(t) - r_t + \sigma_t \phi_t) p_1^*(t) + \sigma_t q_1^*(t) = 0. \tag{44}$$

Substituting (44) into the adjoint BSDE (25) with respect to $p_1^*(t)$ yields

$$\begin{cases} \mathrm{d}p_1^*(t) = -r_t p_1^*(t)\mathrm{d}t - \tilde{\phi}_t p_1^*(t) \mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ p_1^*(T) = \varepsilon_T^{\theta^*} (X_T^{\pi^*})^{-1}. \end{cases} \tag{45}$$

Then the unique solution of (45) is given by

$$p_1^*(t) = c_1^* \Pi^*(0,t). \tag{46}$$

Substituting (46) into (45) with $t = T$, we have

$$X_T^{\pi^*} = \frac{\varepsilon_T^{\theta^*}}{c_1^* \Pi^*(0,T)}. \tag{47}$$

Combining (47) with (32), we have

$$X_T^{\pi^*} = \frac{1}{\sqrt{c_3^* \Pi^*(0,T)}}. \tag{48}$$

□

Lemmas 1 and 2 give the terminal values of the controlled process $\varepsilon_t^{\theta^*}$ and $X_t^{\pi^*}$. Thus, we can apply the generalized BSDE method to our solution.

Put $z_t^* = \sigma_t \pi_t^* X_t^{\pi^*}$. Then we have

$$\pi_t^* = \frac{z_t^*}{\sigma_t X_t^{\pi^*}}. \tag{49}$$

Combining SDE (24) with (49) leads to the following generalized linear BSDE

$$\begin{cases} \mathrm{d}X_t^{\pi^*} = -f_{\mathrm{L}}(t, X_t^{\pi^*}, z_t^*, \omega)\mathrm{d}t + z_t^* \mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ X_T^{\pi^*} = \dfrac{1}{\sqrt{c_3^* \Pi^*(0,T)}}, \end{cases} \tag{50}$$

where the generator (or the driver) $f_{\mathrm{L}} : [0,T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is given by

$$f_{\mathrm{L}}(t,x,z,\omega) = -r_t x - \tilde{\phi}_t z. \tag{51}$$

Remark 8. *Note that the filtration $\{\mathcal{H}_t\}$ in (50) is not necessarily the filtration generated by the noise, which is different from the assumption in the classical theory of BSDEs.*

It's worth noting that the terminal value condition in (50) is implicit, as the $\mathcal{H}_0$-measurable random variable c_3^* is dependent on $X_T^{\pi^*}$.

Inspired by the classical theory of linear BSDE, we can deduce the expression for c_3^* from the following lemma.

Lemma 3. *Assume the conditions in Lemma 1 hold. Suppose the following integrability condition holds*

$$\mathbb{E}\left[\left(\int_0^T \Pi^*(0,t)^2\left(z_t^* - \tilde{\phi}_t X_t^{\pi^*}\right)^2 \mathrm{d}t\right)^{\frac{1}{2}} + (X_T^{\pi^*})^2\right] < \infty. \tag{52}$$

Then we have

$$X_t^{\pi^*} = \frac{X_0 \mathbb{E}\left[\sqrt{\Pi^*(t,T)}|\mathcal{H}_t\right]}{\mathbb{E}\left[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0\right]\sqrt{\Pi^*(0,t)}}, \tag{53}$$

and

$$X_T^{\pi^*} = \frac{X_0}{\mathbb{E}\left[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0\right]\sqrt{\Pi^*(0,T)}}. \tag{54}$$

Proof. By the Itô formula for Itô integrals, we have

$$\begin{aligned} \mathrm{d}\left(\Pi^*(0,t)X_t^{\pi^*}\right) &= \Pi^*(0,t)\mathrm{d}X_t^{\pi^*} + X_t^{\pi^*}\mathrm{d}\Pi^*(0,t) + \mathrm{d}\langle X^{\pi^*}, \Pi^*(0,\cdot)\rangle_t \\ &= \Pi^*(0,t)\left(z_t^* - \tilde{\phi}_t X_t^{\pi^*}\right)\mathrm{d}W_{\mathcal{H}}(t). \end{aligned} \tag{55}$$

By the Burkholder–Davis–Gundy inequality (see [3]) and the integrability condition (52), we deduce that $\Pi^*(0,t)X_t^{\pi^*}$ is an $\mathcal{H}_t$-martingale. Taking the expectation in (55), we have

$$X_t^{\pi^*} = \frac{1}{\sqrt{c_3^*}}\mathbb{E}\left[\frac{\Pi^*(t,T)}{\sqrt{\Pi^*(0,T)}}|\mathcal{H}_t\right]. \tag{56}$$

Substituting (56) into the initial value condition $X_0^{\pi^*} = X_0$ with $t = 0$ yields

$$\frac{1}{\sqrt{c_3^*}} = \frac{X_0}{\mathbb{E}\left[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0\right]}. \tag{57}$$

Combining (56) with (57), we obtain (53) and (54). □

From Lemma 3, we can characterize the robust optimal investment by a generalized linear BSDE.

Theorem 10. *Assume the conditions in Lemma 3 hold. Then π^* and $X_t^{\pi^*}$ are given by (49)–(53), respectively. θ^* is given by*

$$\theta_t^* = \frac{z_t^*}{X_t^{\pi^*}} - \tilde{\phi}_t. \tag{58}$$

Here, Π^* is given by (42), and (X^{π^*}, z^*) solves the following generalized linear BSDE with respect to $\{\mathcal{H}_t\}$

$$\begin{cases} dX_t^{\pi^*} = -f_L(t, X_t^{\pi^*}, z_t^*, \omega)dt + z_t^* dW_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ X_T^{\pi^*} = \dfrac{X_0}{\mathbb{E}\Big[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0\Big]\sqrt{\Pi^*(0,T)}}. \end{cases} \tag{59}$$

Generator $f_L : [0, T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is given by

$$f_L(t, x, z, \omega) = -r_t x - \tilde{\phi}_t z. \tag{60}$$

The value V is given by

$$V = \ln X_0 - 2\mathbb{E}\Big[\ln \mathbb{E}(\sqrt{\Pi^*(0,T)}|\mathcal{H}_0)\Big]. \tag{61}$$

Furthermore, suppose that $\{\mathcal{H}_t\}_{0\le t\le T}$ is the augmentation of the natural filtration of $W_{\mathcal{H}}(t)$, the right hand of the terminal value condition in (59) is L^2-integrable, and r and $\tilde{\phi}$ are bounded. Then (59) is a classical linear BSDE with a unique strong solution, and z^* is given by

$$z_t^* = D_t X_t^{\pi^*} \tag{62}$$

under mild conditions.

Proof. θ_t^* can be calculated by Equation (23) in Theorem 6. Substituting $X_T^{\pi^*}$ in Lemma 3 into (50) yields the BSDE (59). By Lemma 1 and the terminal value condition in (59), we can calculate the value of Problem 1 as follows

$$\begin{aligned} V &= \mathbb{E}\Big[\varepsilon_T^{\theta^*} \ln\Big(\varepsilon_T^{\theta^*} X_T^{\pi^*}\Big)\Big] = \mathbb{E}\Big[-\varepsilon_T^{\theta^*} \ln \mathbb{E}[(X_T^{\pi^*})^{-1}|\mathcal{H}_0]\Big] \\ &= -\mathbb{E}\Big[\mathbb{E}\Big[\varepsilon_T^{\theta^*} \ln \mathbb{E}[(X_T^{\pi^*})^{-1}|\mathcal{H}_0]|\mathcal{H}_0\Big]\Big] = -\mathbb{E}\Big[\ln \mathbb{E}[(X_T^{\pi^*})^{-1}|\mathcal{H}_0]\mathbb{E}\Big[\varepsilon_T^{\theta^*}|\mathcal{H}_0\Big]\Big] \\ &= -\mathbb{E}\Big[\ln \mathbb{E}[(X_T^{\pi^*})^{-1}|\mathcal{H}_0]\Big] = \ln X_0 - 2\mathbb{E}\Big[\ln \mathbb{E}(\sqrt{\Pi^*(0,T)}|\mathcal{H}_0)\Big]. \end{aligned}$$

Further, if the filtration $\{\mathcal{H}_t\}_{0\le t\le T}$ is the augmentation of the natural filtration of $W_{\mathcal{H}}(t)$, then $\mathcal{H}_0$ is generated by the trivial σ-algebra and all $\mathbb{P}$-negligible sets. By [6] (Theorem 4.8), the linear BSDE (59) has a unique strong solution (X^{π^*}, z^*). In other words, $X_t^{\pi^*}$ is a continuous $\mathcal{H}_t$-adapted process with $\mathbb{E}\Big[\sup_{0\le t\le T} |X_t^{\pi^*}|^2\Big] < \infty$, z_t^* is a measurable $\mathcal{H}_t$-adapted process with $\mathbb{E}\Big[\int_0^T |z_t^*|^2 dt\Big] < \infty$, and (X^{π^*}, z^*) satisfies the BSDE (59). Under mild conditions, we can obtain the formulae for z_t^* as follows (see [36] (Proposition 3.5.1))

$$z_t^* = D_t X_t^{\pi^*},$$

where D_t is the Malliavin gradient operator from the Sobolev space $D^{1,2}(\Omega)$ to $L^2(\Omega \times [0, T])$. □

Remark 9. *If the filtration $\{\mathcal{H}_t\}_{0\le t\le T}$ in Theorem 10 is not the augmentation of the natural filtration of $W_{\mathcal{H}}(t)$, or the coefficients of the generator f_L are not necessarily bounded, we refer to [37–41] for further results. In those cases, the existence and uniqueness of the solution to the BSDE (59) still hold under mild conditions when a general martingale representation property was assumed, or a transposition solution was considered, or a stochastic Lipschitzs condition was considered.*

6.1. Without Insider Information

If the investor has no insider information, i.e., $\mathcal{H}_t = \mathcal{F}_t$, we have $\phi = 0$.

Assume further that all the parameter processes are assumed to be deterministic bounded functions. Then we can derive the closed-form expression for the investment as the following corollary.

Corollary 1. *Assume the conditions in Lemma 3 hold. Assume further that $\mathcal{H}_t = \mathcal{F}_t$ and all parameter processes are deterministic bounded functions. Then (π^*, θ^*) is given by*

$$\begin{cases} \pi_t^* = \dfrac{\mu_0(t) - r_t}{2\sigma_t^2}, \\ \theta_t^* = -\dfrac{\mu_0(t) - r_t}{2\sigma_t}. \end{cases} \tag{63}$$

The value V is given by

$$V = \ln X_0 + \int_0^T r_t \mathrm{d}t + \frac{1}{4}\int_0^T \left(\frac{\mu_0(t) - r_t}{\sigma_t}\right)^2 \mathrm{d}t. \tag{64}$$

Proof. By Theorem 10, we have

$$\begin{aligned} z_t^* &= D_t X_t^{\pi^*} \\ &= \frac{X_0 \mathbb{E}\left[D_t \sqrt{\Pi^*(t,T)} | \mathcal{F}_t\right]}{\mathbb{E}\sqrt{\Pi^*(0,T)}\sqrt{\Pi^*(0,t)}} - \frac{1}{2}\frac{X_0 \mathbb{E}\left[\sqrt{\Pi^*(t,T)} | \mathcal{F}_t\right]}{\mathbb{E}\sqrt{\Pi^*(0,T)}\Pi^*(0,t)^{\frac{3}{2}}} D_t^1 \Pi^*(0,t) \\ &= \frac{1}{2}\frac{X_0 \mathbb{E}\left[\sqrt{\Pi^*(t,T)} | \mathcal{F}_t\right]}{\mathbb{E}\sqrt{\Pi^*(0,T)}\sqrt{\Pi^*(0,t)}} \iota_t. \end{aligned}$$

Then (π^*, θ^*) can be calculated by (49) and (58). Moreover, the value of Problem 1 can be calculated by (61) as follows

$$\begin{aligned} V &= \ln X_0 - \ln\left(\mathbb{E}\sqrt{\Pi^*(0,T)}\right)^2 \\ &= \ln X_0 + \int_0^T r_t \mathrm{d}t + \frac{1}{4}\int_0^T \left(\frac{\mu_0(t) - r_t}{\sigma_t}\right)^2 \mathrm{d}t. \end{aligned}$$

□

6.2. Insider Information of Initial Enlargement Type

Next, we give a particular case to derive the closed-form expression for the robust optimal investment. Assume that the filtration is of initial enlargement type, i.e.,

$$\mathcal{H}_t = \bigcap_{s>t} (\mathcal{F}_s \vee Y_0) := \bigcap_{s>t} \left(\mathcal{F}_s \vee \int_0^{T_0} \varphi_u \mathrm{d}W_u\right), \quad 0 \le t \le T, \tag{65}$$

for some $T_0 > T$, and all the parameter processes are assumed to be deterministic bounded functions. Here, φ_t is some deterministic function satisfying $\|\varphi\|_{[s,t]}^2 := \int_s^t \varphi_u^2 \mathrm{d}u < \infty$ for all $0 \le s \le t \le T_0$, and $\|\varphi\|_{[T,T_0]}^2 > 0$.

In this situation, each $\mathcal{H}_t$-adapted process x_t has the form $x_t = x_1(t, Y_0, \omega)$ for some function $x_1 : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $x_1(t,y)$ is $\mathcal{F}_t$-adapted for every $y \in \mathbb{R}$. For simplicity, we write x instead of x_1 in the sequel. To get the explicit expression for ϕ_t and solve the generalized linear BSDE (59) in Theorem 10, we need to introduce some white noise techniques (see [9,13,29]).

Definition 8 (Donsker δ functional)**.** *Let $Y:\Omega\to\mathbb{R}$ be a random variable, which belongs to the distribution space $(\mathcal{S})^{-1}$ (see [29] for the definition). Then a continuous linear operator $\delta_{\cdot}(Y):\mathbb{R}\to(\mathcal{S})^{-1}$ is called a Donsker δ functional of Y if it has the property that*

$$\int_{\mathbb{R}} f(y)\delta_y(Y)\mathrm{d}y = f(Y)$$

for all Borel measurable functions $f:\mathbb{R}\to\mathbb{R}$ such that the integral converges in $(\mathcal{S})^{-1}$.

The following lemma gives a sufficient condition for the existence of the Donsker δ functional. The proof can be found in [9].

Lemma 4. *Let $Y:\Omega\to\mathbb{R}$ be a Gaussian random variable with mean $\bar{\mu}$ and variance $\bar{\sigma}^2>0$. Then its Donsker δ functional $\delta_y(Y)$ exists and is uniquely given by*

$$\delta_y(Y) = \frac{1}{\sqrt{2\pi\bar{\sigma}^2}}\exp^{\diamond}\left\{-\frac{(y-Y)^{\diamond 2}}{2\bar{\sigma}^2}\right\} \in (\mathcal{S})' \subset (\mathcal{S})^{-1},$$

where $(\mathcal{S})'$ is the Hida distribution space, and $\diamond$ denotes the Wick product. We refer to [29] for relevant definitions.

By Lemma 4 and the Lévy theorem, the Donsker δ functional of Y_0 in (65) is given by

$$\delta_y(Y_0) = \frac{1}{\sqrt{2\pi\|\varphi\|^2_{[0,T_0]}}}\exp^{\diamond}\left\{-\frac{(y-Y_0)^{\diamond 2}}{2\|\varphi\|^2_{[0,T_0]}}\right\},$$

and we have

$$G_t := \mathbb{E}[\delta_y(Y_0)|\mathcal{F}_t] = \frac{1}{\sqrt{2\pi\|\varphi\|^2_{[t,T_0]}}}\exp\left\{-\frac{(y-\int_0^t\varphi_s\mathrm{d}W_s)^2}{2\|\varphi\|^2_{[t,T_0]}}\right\}.$$

Using the Donsker δ functional technique, we can obtain the explicit expression for ϕ by the following lemma, which was first proposed by Draouil and Øksendal [42].

Lemma 5 (Enlargement of filtration)**.** *Suppose Y is an $\mathcal{F}_{T_0}$-measurable random variable for some $T_0>T$ and belongs to $(\mathcal{S})'$. The Donsker δ functional of Y exists and satisfies $\mathbb{E}[\delta_{\cdot}(Y)|\mathcal{F}_t]\in L^2(\mathrm{m}\times\mathbb{P})$ and $\mathbb{E}[D_t\delta_{\cdot}(Y)|\mathcal{F}_t]\in L^2(\mathrm{m}\times\mathbb{P})$, where D_t is the (extended) Hida–Malliavin derivative (see [9]). Assume further that $\mathcal{H}_t=\bigcap_{s>t}(\mathcal{F}_s\vee Y)$, which satisfies the usual condition, and W is an $\mathcal{H}_t$-semi-martingale with the decomposition (18). Then we have*

$$\phi_t = \frac{\mathbb{E}[D_t\delta_y(Y)|\mathcal{F}_t]|_{y=Y}}{\mathbb{E}[\delta_y(Y)|\mathcal{F}_t]|_{y=Y}}.$$

If $\{\mathcal{H}_t\}$ is of the form (65), we have by Lemma 5 that

$$\phi_t = \phi_t(Y_0) = \frac{Y_0-\int_0^t\varphi_s\mathrm{d}W_s}{\|\varphi\|^2_{[t,T_0]}}\varphi_t. \tag{66}$$

In order to transform the generalized BSDE (59) into a classical BSDE, we need to rewrite $\Pi^*(0,t)$ and $X_T^{\pi^*}$ as functions of Y_0.

Lemma 6. *Assume the conditions in Lemma 3 hold. Assume further that* $\{\mathcal{H}_t\}$ *is given by (65) and all parameter processes are deterministic bounded functions. Then we have*

$$\Pi^*(t_1, t_2) = \frac{G_{t_1}}{G_{t_2}} \Pi_a^*(t_1, t_2)\big|_{y=Y_0}, \tag{67}$$

where

$$\Pi_a^*(t_1, t_2) := \exp\left\{ -\int_{t_1}^{t_2} r_s \mathrm{d}s - \int_{t_1}^{t_2} \iota_s \mathrm{d}W_s - \frac{1}{2}\int_{t_1}^{t_2} \iota_s^2 \mathrm{d}s \right\} \tag{68}$$

is an $\mathcal{F}_t$*-adapted semi-martingale. Moreover, the terminal value* $X_T^{\pi^*}$ *is given by*

$$X_T^{\pi^*} = \tilde{c}_3^*(y)\sqrt{\frac{G_T}{\Pi_a^*(0,T)}}\Bigg|_{y=Y_0}, \tag{69}$$

where $\tilde{c}_3^*(y) := \frac{X_0}{\mathbb{E}\left[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0\right](y)\sqrt{G_0}}$ *is a Borel measurable function with respect to* y.

Proof. Substituting (66) into (42) and using the Itô formula, we can rewrite the expression for $\Pi^*(t_1, t_2)$, $0 \le t_1 \le t_2 \le T$, as follows

$$\begin{aligned}
\Pi^*(t_1, t_2) &= \Pi^*(t_1, t_2, y)|_{y=Y_0} \\
&= \exp\left\{ -\int_{t_1}^{t_2} \phi_s(Y_0)\mathrm{d}W_s + \frac{1}{2}\int_{t_1}^{t_2} \phi_s(Y_0)\mathrm{d}y \right\} \Pi_a^*(t_1, t_2) \\
&= \frac{G_{t_1}}{G_{t_2}} \Pi_a^*(t_1, t_2)\big|_{y=Y_0}.
\end{aligned}$$

From the terminal value condition of $X_T^{\pi^*}$ in (59), we have

$$\begin{aligned}
X_T^{\pi^*} = X_T^{\pi^*}(y)|_{y=Y_0} &= \frac{X_0}{\mathbb{E}\left[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0\right](y)} \sqrt{\frac{G_T}{G_0 \Pi_a^*(0,T)}}\Bigg|_{y=Y_0} \\
&= \tilde{c}_3^*(y)\sqrt{\frac{G_T}{\Pi_a^*(0,T)}}\Bigg|_{y=Y_0}.
\end{aligned}$$

□

By Definition 8, the generalized linear BSDE (59) with respect to $\{\mathcal{H}_t\}$ can be rewritten as

$$\begin{aligned}
\int_{\mathbb{R}} X_t^{\pi^*}(y)\delta_y(Y_0)\mathrm{d}y = &\int_{\mathbb{R}} X_T^{\pi^*}(y)\delta_y(Y_0)\mathrm{d}y \\
&+ \int_{\mathbb{R}}\int_t^T \left[-r_s X_s^{\pi^*}(y) - \iota_s z_s^*(y)\right]\mathrm{d}s\,\delta_y(Y_0)\mathrm{d}y \\
&- \int_{\mathbb{R}}\int_t^T z_s^*(y)\mathrm{d}W_s\delta_y(Y_0)\mathrm{d}y.
\end{aligned} \tag{70}$$

It is obvious that (59) holds if and only if $(X_t^{\pi^*}(y), z_t^*(y))$ is the solution of the following classical linear BSDE with respect to the natural filtration $\{\mathcal{F}_t\}$ for each y

$$\begin{cases}
\mathrm{d}X_t^{\pi^*}(y) = -\bar{f}_{\mathrm{L}}(t, X_t^{\pi^*}(y), z_t^*(y))\mathrm{d}t + z_t^*(y)\mathrm{d}W_t, \quad 0 \le t \le T, \\
X_T^{\pi^*}(y) = \tilde{c}_3^*(y)\sqrt{\dfrac{G_T}{\Pi_a^*(0,T)}},
\end{cases} \tag{71}$$

where the generator $\bar{f}_{\mathrm{L}} : [0, T] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\bar{f}_{\mathrm{L}}(t, x, z) = -r_t x - \iota_t z. \tag{72}$$

Utilizing the classical theory of linear linear BSDEs, we can calculate the robust optimal investment, as stated in the following theorem.

Theorem 11. *Assume the conditions in Lemma 6 hold. Then* (π^*, θ^*) *is given by*

$$\begin{cases} \pi_t^* = \dfrac{\iota_t}{2\sigma_t} + \dfrac{Y_0 - \int_0^t \varphi_s \mathrm{d}W_s + \frac{1}{2}\int_t^T \varphi_s \iota_s \mathrm{d}s}{\sigma_t\left(\|\varphi\|^2_{[t,T_0]} + \|\varphi\|^2_{[T,T_0]}\right)}\varphi_t, \\ \theta_t^* = -\dfrac{1}{2}\iota_t + \dfrac{Y_0 - \int_0^t \varphi_s \mathrm{d}W_s + \frac{1}{2}\int_t^T \varphi_s \iota_s \mathrm{d}s}{\left(\|\varphi\|^2_{[t,T_0]} + \|\varphi\|^2_{[T,T_0]}\right)}\varphi_t - \dfrac{Y_0 - \int_0^t \varphi_s \mathrm{d}W_s}{\|\varphi\|^2_{[t,T_0]}}\varphi_t. \end{cases} \tag{73}$$

Proof. By [6] (Theorem 4.8), the unique strong solution of (71) is given by

$$X_t^{\pi^*}(y) = \mathbb{E}\left[\Pi_a^*(t, T)\tilde{c}_3^*(y)\sqrt{\frac{G_T}{\Pi_a^*(0, T)}}\middle|\mathcal{F}_t\right]. \tag{74}$$

As per the initial value condition $X_0^{\pi^*}(y) = X_0$, the Borel measurable function $\tilde{c}_3^*(y)$ in (74) is given by

$$\tilde{c}_3^*(y) = \frac{X_0}{\mathbb{E}\sqrt{\Pi_a^*(0, T)G_T}} = \frac{X_0}{\mathbb{E}\left[\sqrt{\Pi^*(0, T)}|\mathcal{H}_0\right](y)\sqrt{G_0}}. \tag{75}$$

The last equation in (75) is the definition of $\tilde{c}_3^*(y)$. Substituting (75) into (74), we obtain

$$X_t^{\pi^*}(y) = \frac{X_0\mathbb{E}\left[\sqrt{\Pi_a^*(t, T)G_T}|\mathcal{F}_t\right]}{\mathbb{E}\sqrt{\Pi_a^*(0, T)G_T}\sqrt{\Pi_a^*(0, t)}}. \tag{76}$$

By [36] (Proposition 3.5.1), we have

$$\begin{aligned} z_t^*(y) &= D_t X_t^{\pi^*}(y) \\ &= \frac{X_0\mathbb{E}\left[D_t\sqrt{\Pi_a^*(t, T)G_T}|\mathcal{F}_t\right]}{\mathbb{E}\sqrt{\Pi_a^*(0, T)G_T}\sqrt{\Pi_a^*(0, t)}} - \frac{1}{2}\frac{X_0\mathbb{E}\left[\sqrt{\Pi_a^*(t, T)G_T}|\mathcal{F}_t\right]}{\mathbb{E}\sqrt{\Pi_a^*(0, T)G_T}\Pi_a^*(0, t)^{\frac{3}{2}}}D_t\Pi_a^*(0, t) \\ &= \frac{1}{2}\frac{X_0\mathbb{E}\left[\sqrt{\Pi_a^*(t, T)G_T}\left(y - \int_0^T \varphi_s \mathrm{d}W_s\right)|\mathcal{F}_t\right]}{\|\varphi\|^2_{[T,T_0]}\mathbb{E}\sqrt{\Pi_a^*(0, T)G_T}\sqrt{\Pi_a^*(0, t)}}\varphi_t + \frac{1}{2}\frac{X_0\mathbb{E}\left[\sqrt{\Pi_a^*(t, T)G_T}|\mathcal{F}_t\right]}{\mathbb{E}\sqrt{\Pi_a^*(0, T)G_T}\sqrt{\Pi_a^*(0, t)}}\iota_t. \end{aligned}$$

Substituting the above equation into (49), we obtain the robust optimal investment strategy

$$\begin{aligned} \pi_t^* &= \pi_t^*(y)|_{y=Y_0} \\ &= \frac{\iota_t}{2\sigma_t} + \frac{1}{2}\frac{\mathbb{E}\left[\tilde{\Pi}_a^*(t, T)\sqrt{G_T}\left(y - \int_0^T \varphi_s \mathrm{d}W_s\right)|\mathcal{F}_t\right]}{\sigma_t\|\varphi\|^2_{[T,T_0]}\mathbb{E}\left[\tilde{\Pi}_a^*(t, T)\sqrt{G_T}|\mathcal{F}_t\right]}\varphi_t\Bigg|_{y=Y_0}, \end{aligned}$$

where

$$\tilde{\Pi}_a^*(0, t) := \exp\left\{-\int_0^t \frac{\iota_s}{2}\mathrm{d}W_s - \frac{1}{8}\int_0^t \iota_s^2 \mathrm{d}s\right\}.$$

Then, by the Girsanov theorem, $W_{\mathbb{Q}}(t) := W_t + \int_0^t \frac{\iota_s}{2}\mathrm{d}s$ is an $\mathcal{F}_t$-Brownian motion under the new equivalent probability measure $\mathbb{Q}$ defined by $\mathrm{d}\mathbb{Q} = \tilde{\Pi}_a^*(0, T)\mathrm{d}\mathbb{P}$. Thus, by the Bayes rule (see [3]), we can rewrite the robust optimal investment strategy as follows

$$\pi_t^* = \frac{1}{2}\frac{\mathbb{E}_{\mathbb{Q}}\left[\sqrt{G_T}\left(y - \int_0^T \varphi_s \mathrm{d}W_s\right)|\mathcal{F}_t\right]}{\sigma_t \|\varphi\|^2_{[T,T_0]}\mathbb{E}_{\mathbb{Q}}\left[\sqrt{G_T}|\mathcal{F}_t\right]}\varphi_t\Bigg|_{y=Y_0} + \frac{\iota_t}{2\sigma_t}$$
$$= \frac{1}{2}\frac{\varphi_t}{\sigma_t\|\varphi\|^2_{[T,T_0]}}\left\{y - \frac{\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{-\frac{(y-\tilde{I}_T+\tilde{\iota}_T)^2}{4\|\varphi\|^2_{[T,T_0]}}\right\}\left(\tilde{I}_T - \tilde{\iota}_T\right)|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{-\frac{(y-\tilde{I}_T+\tilde{\iota}_T)^2}{4\|\varphi\|^2_{[T,T_0]}}\right\}|\mathcal{F}_t\right]}\right\}\Bigg|_{y=Y_0}$$
$$+ \frac{\iota_t}{2\sigma_t},$$

where $\tilde{\iota}_t := \frac{1}{2}\int_0^t \varphi_s \iota_s \mathrm{d}s$, $t \in [0,T]$, and $\tilde{I}_t := \int_0^t \varphi_s \mathrm{d}W_{\mathbb{Q}}(s)$. On the other hand, the conditional $\mathbb{Q}$ law of $\tilde{I}_T$, given $\mathcal{F}_t$, is normal with mean $\tilde{I}_t$ and variance $\|\varphi\|^2_{[t,T]}$ due to the Markov property of Itô diffusion processes (see [3]). Thus, the above formula leads to

$$\pi_t^* = \frac{1}{2}\frac{\varphi_t}{\sigma_t\|\varphi\|^2_{[T,T_0]}}\left\{y + \tilde{\iota}_T - \frac{\int_{\mathbb{R}} \frac{x}{\sqrt{2\pi\|\varphi\|^2_{[t,T]}}}\exp\left\{-\frac{(y-x+\tilde{\iota}_T)^2}{4\|\varphi\|^2_{[T,T_0]}} - \frac{(x-z)^2}{2\|\varphi\|^2_{[t,T]}}\right\}\mathrm{d}x}{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\|\varphi\|^2_{[t,T]}}}\exp\left\{-\frac{(y-x+\tilde{\iota}_T)^2}{4\|\varphi\|^2_{[T,T_0]}} - \frac{(x-z)^2}{2\|\varphi\|^2_{[t,T]}}\right\}\mathrm{d}x}\right\}\Bigg|_{\substack{z=\tilde{I}_t\\ y=Y_0}}$$
$$+ \frac{\iota_t}{2\sigma_t}$$
$$= \frac{Y_0 - \int_0^t \varphi_s \mathrm{d}W_s + \frac{1}{2}\int_t^T \varphi_s \iota_s \mathrm{d}s}{\sigma_t\left(\|\varphi\|^2_{[t,T_0]} + \|\varphi\|^2_{[T,T_0]}\right)}\varphi_t + \frac{\iota_t}{2\sigma_t}.$$

By Theorem 10, we have

$$\theta_t^* = -\frac{1}{2}\iota_t + \frac{Y_0 - \int_0^t \varphi_s \mathrm{d}W_s + \frac{1}{2}\int_t^T \varphi_s \iota_s \mathrm{d}s}{\left(\|\varphi\|^2_{[t,T_0]} + \|\varphi\|^2_{[T,T_0]}\right)}\varphi_t - \frac{Y_0 - \int_0^t \varphi_s \mathrm{d}W_s}{\|\varphi\|^2_{[t,T_0]}}\varphi_t.$$

□

When $\varphi = 1$, the investor possesses the insider information W_{T_0} regarding the future price of the risky asset. This leads us to the following corollary.

Corollary 2. *Suppose the conditions in Lemma 6 hold. Assume further that $\varphi = 1$ in (65) . Then (π^*, θ^*) is given by*

$$\begin{cases} \pi_t^* = \dfrac{\mu_0(t) - r_t}{2\sigma_t^2} + \dfrac{W_{T_0} - W_t + \frac{1}{2}\int_t^T \frac{\mu_0(s)-r_s}{\sigma_s}\mathrm{d}s}{\sigma_t(T_0 - t + T_0 - T)}, \\ \theta_t^* = -\dfrac{\mu_0(t) - r_t}{2\sigma_t} + \dfrac{W_{T_0} - W_t + \frac{1}{2}\int_t^T \frac{\mu_0(s)-r_s}{\sigma_s}\mathrm{d}s}{(T_0 - t + T_0 - T)} - \dfrac{W_{T_0} - W_t}{T_0 - t}. \end{cases}$$

The value V is given by

$$V = \ln X_0 + \int_0^T r_t \mathrm{d}t + \frac{1}{4}\int_0^T \left(\frac{\mu_0(t) - r_t}{\sigma_t}\right)^2 \mathrm{d}t + \frac{1}{2}\ln\left(1 - \frac{T^2}{(2T_0 - T)^2}\right)^{-1}$$
$$+ \frac{T}{2(2T_0 - T)} + \frac{1}{4(2T_0 - T)}\left(\int_0^T \frac{\mu_0(t) - r_t}{\sigma_t}\mathrm{d}t\right)^2.$$

Proof. We have

$$\mathbb{E}(\sqrt{\Pi^*(0,T)}|\mathcal{H}_0) = \left(\mathbb{E}\sqrt{\Pi_a^*(0,T)\frac{G_T}{G_0}}\right)\Bigg|_{y=Y_0}$$

by (75). Substituting the above equation into (61), we can calculate by Girsanov theorem that

$$\begin{aligned} V &= \ln X_0 - 2\mathbb{E}\left[\left(\ln \mathbb{E}\sqrt{\Pi_a^*(0,T)\frac{G_T}{G_0}}\right)\Bigg|_{y=Y_0}\right] \\ &= \ln X_0 + \int_0^T r_t \mathrm{d}t + \frac{1}{4}\int_0^T \left(\frac{\mu_0(t)-r_t}{\sigma_t}\right)^2 \mathrm{d}t - 2\mathbb{E}\left[\left(\ln \mathbb{E}_{\mathbb{Q}}\sqrt{\frac{G_T}{G_0}}\right)\Bigg|_{y=Y_0}\right] \\ &= \ln X_0 + \int_0^T r_t \mathrm{d}t + \frac{1}{4}\int_0^T \left(\frac{\mu_0(t)-r_t}{\sigma_t}\right)^2 \mathrm{d}t + \frac{1}{2}\ln\left(1 - \frac{T^2}{(2T_0-T)^2}\right)^{-1} \\ &\quad + \frac{T}{2(2T_0-T)} + \frac{1}{4(2T_0-T)}\left(\int_0^T \frac{\mu_0(t)-r_t}{\sigma_t}\mathrm{d}t\right)^2 \end{aligned}$$

□

7. The Large Insider Case

In this section, we will deduce a generalized quadratic BSDE that the robust optimal investment π_t^* entails in the case of a large insider, and transform it into a classical quadratic BSDE using the white noise theory.

Assume that the mean rate of return $\mu(t,x) = \mu_0(t) + \rho_t x$ for some $\mathcal{F}_t$-adapted measurable processes $\mu_0(t)$ and ρ_t with $0 \le \rho_t < \frac{1}{2}\sigma_t^2$. Note that the insider is 'small' when $\rho_t = 0$.

Put $\iota_t = \frac{\mu_0(t)-r_t}{\sigma_t}$, $\tilde{\sigma}_t = \sigma_t - \frac{2\rho_t}{\sigma_t}$, and $\tilde{\phi}_t = \iota_t + \phi_t$. Assume further the penalty function g is given by $g(\theta) = \frac{1}{2}\theta^2$. Then we have

$$\mathbb{E}\left[\int_0^T \varepsilon_s^{\theta^*} g(\theta_s^*)\mathrm{d}s\right] = \mathbb{E}\left[\varepsilon_T^{\theta^*} \ln \varepsilon_T^{\theta^*}\right]. \tag{77}$$

If we follow the method in Section 6, the terminal condition in BSDE (50) will depend on z_t^*, which makes the BSDE (50) irregular and very hard to solve. The reason is that SDE (24) for $X_t^{\pi^*}$ is not homogeneous if $\rho_t \neq 0$.

However, we could use a generalized quadratic BSDE to characterize the robust optimal investment.

Theorem 12. *Assume that $\mu(t,x) = \mu_0(t) + \rho_t x$ for some $\mathcal{F}_t$-adapted measurable processes $\mu_0(t)$ and ρ_t with $0 \le \rho_t \le \frac{1}{2}\sigma_t^2$, and $g(\theta) = \frac{1}{2}\theta^2$. Suppose $(\pi^*,\theta^*) \in \mathcal{A}_1' \times \mathcal{A}_2'$ is optimal for Problem 1 under the conditions in Theorem 7. Then (π^*,θ^*) is given by*

$$\begin{cases} \pi_t^* = \dfrac{z_t^* + \tilde{\phi}_t}{\sigma_t + \tilde{\sigma}_t}, \\ \theta_t^* = \dfrac{\tilde{\sigma}_t z_t^* - \sigma_t \tilde{\phi}_t}{\sigma_t + \tilde{\sigma}_t}, \end{cases} \tag{78}$$

where $L_t^* := \ln\left(\varepsilon_t^{\theta^*} X_t^{\pi^*}\right)$ and z_t^* solve the following generalized quadratic BSDE with respect to $\{\mathcal{H}_t\}$

$$\begin{cases} \mathrm{d}L_t^* = -f_Q(t,z_t^*,\omega)\mathrm{d}t + z_t^* \mathrm{d}W_{\mathcal{H}}(t), \quad 0 \le t \le T, \\ L_T^* = c_2^*. \end{cases} \tag{79}$$

Here, the generator $f_Q : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is given by

$$f_Q(t,z,\omega) = \frac{z^2}{4} - \frac{\tilde{\phi}_t}{2}z - r_t - \frac{\tilde{\phi}_t^2}{4} - \frac{(\sigma_t - \tilde{\sigma}_t)}{4(\sigma_t + \tilde{\sigma}_t)}(z + \tilde{\phi}_t)^2, \tag{80}$$

and the $\mathcal{H}_0$-measurable random variable c_2^* can be determined by $L_0^* = \ln X_0$ under some integrability conditions (see Remark 10). The value V is given by

$$V = \mathbb{E}L_T^*. \tag{81}$$

Furthermore, if $\{\mathcal{H}_t\}_{0\le t\le T}$ is the augmentation of the natural filtration of $W_{\mathcal{H}}(t)$, and c_2^*, $\tilde{\phi}$, r, σ and $\tilde{\sigma}$ are bounded, then the quadratic BSDE (79) has a unique strong solution and z^* is given by

$$z_t^* = D_t L_t^* \tag{82}$$

under mild conditions, where c_2^* can be determined by traversing all constants such that the condition $L_0^* = \ln X_0$ holds.

Remark 10. *In fact, integrating (79) from t to T yields $L_T^* - L_t^* = -\int_t^T f_Q(s, z_s^*, \omega)\mathrm{d}s + \int_t^T z_s^* \mathrm{d}W_{\mathcal{H}}(s)$. Taking conditional expectation and assuming the Itô integrals are L^2-martingales, we get $L_t^* = \mathbb{E}\left[\int_t^T f_Q(s, z_s^*, \omega)\mathrm{d}s + L_T^* \middle| \mathcal{H}_t\right]$. Taking $t = 0$ and using the initial value condition we have $c_2^* = \ln X_0 - \mathbb{E}\left[\int_0^T f_Q(s, z_s^*, \omega)\mathrm{d}s \middle| \mathcal{H}_0\right]$.*

Proof. By a similar procedure in Section 6 with respect to the Hamiltonian system (28), we have (see (37))

$$\ln\left(\varepsilon_T^{\theta^*} X_T^{\pi^*}\right) = c_2^*, \tag{83}$$

where $c_2^* = p_2^*(0)$ is an $\mathcal{H}_0$-measurable random variable. Combining the Itô formula for Itô integrals with the expressions for $\varepsilon_t^{\theta^*}$ and $X_t^{\pi^*}$ yields the following SDE

$$\begin{aligned} \mathrm{d}\ln\left(\varepsilon_t^{\theta^*} X_t^{\pi^*}\right) =& [r_t + (\mu_0(t) - r_t)\pi_t^* + \sigma_t \pi_t^* \phi_t]\mathrm{d}t \\ & - \frac{1}{2}\left[(\sigma_t \pi_t^*)^2 + (\theta_t^*)^2 - \sigma_t(\sigma_t - \tilde{\sigma}_t)(\pi_t^*)^2\right]\mathrm{d}t \\ & + (\sigma_t \pi_t^* + \theta_t^*)\mathrm{d}W_{\mathcal{H}}(t). \end{aligned} \tag{84}$$

Put $L_t^* = \ln\left(\varepsilon_t^{\theta^*} X_t^{\pi^*}\right)$ and $z_t^* = \sigma_t \pi_t^* + \theta_t^*$. From (23) in Theorem 6, we obtain

$$\theta_t^* = \tilde{\sigma}_t \pi_t^* - \tilde{\phi}_t. \tag{85}$$

Then we have

$$\begin{cases} \pi_t^* = \dfrac{z_t^* + \tilde{\phi}_t}{\sigma_t + \tilde{\sigma}_t}, \\ \theta_t^* = \dfrac{\tilde{\sigma}_t z_t^* - \sigma_t \tilde{\phi}_t}{\sigma_t + \tilde{\sigma}_t}. \end{cases} \tag{86}$$

Combining SDE (84) with (83) and (86) yields the BSDE (79). By (77) and (83), the value V can be calculated by

$$V = \mathbb{E}\left[\varepsilon_T^{\theta^*} \ln\left(\varepsilon_T^{\theta^*} X_T^{\pi^*}\right)\right] = \mathbb{E}c_2^* = \mathbb{E}L_T^*. \tag{87}$$

If the filtration $\{\mathcal{H}_t\}$ is the augmentation of the natural filtration of $W_{\mathcal{H}}(t)$, then c_2^* is a constant. Suppose that c_2^*, $\tilde{\phi}$, r, σ and $\tilde{\sigma}$ are bounded. Then, by [43] (Theorem 4.1), the quadratic BSDE (79) has a unique strong solution (L^*, z^*). In other words, L_t^* is a bounded continuous $\mathcal{H}_t$-adapted process, z_t^* is a measurable $\mathcal{H}_t$-adapted process with $\mathbb{E}\int_0^T |z_t^*|^2 \mathrm{d}t < \infty$ and $\int_0^t z_s^* \mathrm{d}W_{\mathcal{H}}(s)$ is an $\mathcal{H}_t$-$\mathcal{BMO}$-martingale (see [44]), and (L^*, z^*) satisfies the BSDE (79). Under mild conditions on the Malliavin derivative, we can calculate z_t^* by (82) (see Corollary 5.1 in [43]). □

Remark 11. *If the filtration $\{\mathcal{H}_t\}_{0\le t\le T}$ in Theorem 12 is not the augmentation of the natural filtration of $W_{\mathcal{H}}(t)$, or the coefficients of the generator f_Q is not necessarily bounded, we refer to [13,37–40] for further results. Meanwhile, the $\mathcal{H}_0$-measurable random variable c_2^* can be* *determined by traversing all $\mathcal{H}_0$-measurable random variable such that the condition $L_0^* = \ln X_0$ holds. Moreover, if $\mathcal{H}_0$ is generated by a random variable F and all $\mathbb{P}$-negligible sets, then by the monotone class theorem of functional forms (see [44]), there exists a Borel measurable function f such that $c_2^* = f(F)$, a.s. Thus, c_2^* can be determined by traversing all Borel measurable functions f such that the initial value condition $L_0^* = \ln X_0$ holds.*

7.1. Without Insider Information

If the investor has no insider information, i.e., $\mathcal{H}_t = \mathcal{F}_t$, we have $\phi = 0$.

Assume further that all the parameter processes are assumed to be deterministic bounded functions. Then we can deduce the following corollary.

Corollary 3. *Assume the conditions in Theorem 12 hold. Assume further that $\mathcal{H}_t = \mathcal{F}_t$ and all parameter processes are deterministic bounded functions. Then (π^*, θ^*) is given by*

$$\begin{cases} \pi_t^* = \dfrac{z_t^* + \iota_t}{\sigma_t + \tilde{\sigma}_t}, \\ \theta_t^* = \dfrac{\tilde{\sigma}_t z_t^* - \sigma_t \iota_t}{\sigma_t + \tilde{\sigma}_t}, \end{cases} \tag{88}$$

where $L_t^* := \ln\left(\varepsilon_t^{\theta^*} X_t^{\pi^*}\right)$ and z_t^* solve the following classical quadratic BSDE with respect to $\{\mathcal{F}_t\}$

$$\begin{cases} \mathrm{d}L_t^* = -f_Q(t, z_t^*, \omega)\mathrm{d}t + z_t^* \mathrm{d}W_t, \quad 0 \le t \le T, \\ L_T^* = \ln X_0 - \mathbb{E}\left[\displaystyle\int_0^T f_Q(s, z_s^*, \omega)\mathrm{d}s\right]. \end{cases} \tag{89}$$

Here, the generator $f_Q : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is given by

$$f_Q(t, z, \omega) = \frac{z^2}{4} - \frac{\iota_t}{2} z - r_t - \frac{\iota_t^2}{4} - \frac{(\sigma_t - \tilde{\sigma}_t)}{4(\sigma_t + \tilde{\sigma}_t)}(z + \iota_t)^2. \tag{90}$$

The value V is given by

$$V = \ln X_0 - \mathbb{E}\left[\int_0^T f_Q(s, z_s^*, \omega)\mathrm{d}s\right]. \tag{91}$$

Proof. The result is an immediate consequence of Theorem 12. □

7.2. Insider Information of Initial Enlargement Type

Next, we consider a particular case when the filtration is of initial enlargement type, i.e.,

$$\mathcal{H}_t = \bigcap_{s>t} (\mathcal{F}_s \vee Y_0) := \bigcap_{s>t} \left(\mathcal{F}_s \vee \int_0^{T_0} \varphi_u \mathrm{d}W_u\right), \quad 0 \le t \le T, \tag{92}$$

for some $T_0 > T$, and all the parameter processes are assumed to be deterministic bounded functions. Here, φ_t is some deterministic function satisfying $\|\varphi\|^2_{[s,t]} := \int_s^t \varphi_u^2 \mathrm{d}u < \infty$ for all $0 \le s \le t \le T_0$, and $\|\varphi\|^2_{[T,T_0]} > 0$.

By the Donsker δ functional $\delta_y(Y_0)$ and a similar procedure in Section 6.2, we have

$$\phi_t = \phi_t(y)|_{y=Y_0} = \frac{y - \int_0^t \varphi_s \mathrm{d}W_s}{\|\varphi\|^2_{[t,T_0]}} \varphi_t \Bigg|_{y=Y_0}. \tag{93}$$

Then we have the following theorem.

Theorem 13. *Assume the conditions in Theorem 12 hold. Assume further that* $\{\mathcal{H}_t\}$ *is given by (92) and all parameter processes are deterministic bounded functions. Then* (π^*, θ^*) *is given by (78), where* $(L^*(y), z^*(y))$ *solves the classical quadratic BSDE*

$$\begin{cases} \mathrm{d}L_t^*(y) = -\bar{f}_Q(t, z_t^*(y), y, \omega)\mathrm{d}t + z_t^*(y)\mathrm{d}W_t, \quad 0 \le t \le T, \\ L_T^*(y) = c_2^*(y). \end{cases} \tag{94}$$

Here, the generator $\bar{f}_Q : [0, T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ *is given by*

$$\bar{f}_Q(t, z, y, \omega) = \frac{z^2}{4} - \frac{\iota_t - \phi_t(y)}{2} z - r_t - \frac{(\iota_t + \phi_t(y))^2}{4} - \frac{(\sigma_t - \tilde{\sigma}_t)}{4(\sigma_t + \tilde{\sigma}_t)}(z + \iota_t + \phi_t(y))^2, \tag{95}$$

$\phi_t(y)$ *is given by (93), and the* $\mathcal{H}_0$*-measurable random variable* c_2^* *can be determined by traversing all Borel measurable functions* $c_2^*(y)$ *such that* $L_0^* = \ln X_0$*. Moreover, the value V is given by*

$$V = \mathbb{E}(L_T^*(y)|_{y=Y_0}). \tag{96}$$

Proof. The result is an immediate consequence of Theorem 12. □

8. Optimal Investment Without Model Uncertainty

We focus on the specific scenario that excludes model uncertainty. The findings in this section are also documented in [9]. Nonetheless, we retain this section to maintain the cohesiveness of this paper and facilitate the numerical experiments in the subsequent section.

When there is no model uncertainty, that is, $\mathcal{A}_2' = \{(0)\}$, Problem 1 degenerates to the following anticipating stochastic control problem.

Problem 2. *Select* $\pi^* \in \mathcal{A}_1'$ *such that*

$$\tilde{V} := \tilde{J}(\pi^*) = \sup_{\pi \in \mathcal{A}_1'} \tilde{J}(\pi), \tag{97}$$

where $\tilde{J}(\pi) := \mathbb{E}[\ln X_T^\pi]$*. We call* $\tilde{V}$ *the value (or the optimal expected utility) of Problem 2.*

Suppose that $\mu(t, x) = \mu_0(t) + \rho_t x$ for some $\mathcal{F}_t$-adapted measurable processes $\mu_0(t)$ and ρ_t with $0 \le \rho_t < \frac{1}{2}\sigma_t^2$. Put $\iota_t = \frac{\mu_0(t) - r_t}{\sigma_t}$, $\tilde{\sigma}_t = \sigma_t - \frac{2\rho_t}{\sigma_t}$, and $\tilde{\phi}_t = \iota_t + \phi_t$.

By Theorem 6 in the initial characterization of the solution, we can easily obtain the following result.

Theorem 14. *Assume that* $\mu(t, x) = \mu_0(t) + \rho_t x$ *for some* $\mathcal{F}_t$*-adapted measurable processes* $\mu_0(t)$ *and* ρ_t *with* $0 \le \rho_t < \frac{1}{2}\sigma_t^2$*, and no model uncertainty is considered. Suppose* $\pi^* \in \mathcal{A}_1'$ *is optimal for Problem 2 under Assumptions 1 and 2. Then* π^* *is given by*

$$\pi_t^* = \frac{\mu_0(t) - r_t}{\sigma_t \tilde{\sigma}_t} + \frac{\phi_t}{\tilde{\sigma}_t}. \tag{98}$$

The value $\tilde{V}$ *is given by*

$$\tilde{V} = \ln X_0 + \mathbb{E}\int_0^T r_t \mathrm{d}t + \frac{1}{2}\mathbb{E}\int_0^T \frac{\sigma_t}{\tilde{\sigma}_t}\left(\frac{\mu_0(t) - r_t}{\sigma_t} + \phi_t\right)^2 \mathrm{d}t. \tag{99}$$

Proof. Since $\mathcal{A}_2' = \{0\}$, which implies that $\theta^* = 0$ in Theorem 6, we have that

$$\pi_t^* = \frac{\tilde{\phi}_t}{\tilde{\sigma}_t} = \frac{\mu_0(t) - r_t}{\sigma_t \tilde{\sigma}_t} + \frac{\phi_t}{\tilde{\sigma}_t}. \tag{100}$$

Substitute (100) into (97). Then (99) is a result from (23) and tedious calculation. □

8.1. Without Insider Information

If the investor has no insider information, i.e., $\mathcal{H}_t = \mathcal{F}_t$, we have $\phi = 0$.

Assume further that all the parameter processes are assumed to be deterministic bounded functions. Then we can deduce the following corollary.

Corollary 4. *Suppose that the conditions in Theorem 14 hold. Assume further that $\mathcal{H}_t = \mathcal{F}_t$ and all parameter processes are deterministic bounded functions. Then the optimal investment π^* is given by*

$$\pi_t^* = \frac{\mu_0(t) - r_t}{\sigma_t \tilde{\sigma}_t}. \tag{101}$$

The value $\tilde{V}$ is given by

$$\tilde{V} = \ln X_0 + \int_0^T r_t \mathrm{d}t + \frac{1}{2}\int_0^T \frac{\sigma_t}{\tilde{\sigma}_t}\left(\frac{\mu_0(t) - r_t}{\sigma_t}\right)^2 \mathrm{d}t. \tag{102}$$

Proof. The corollary is an immediate consequence of Theorem 14. □

8.2. Insider Information of Initial Enlargement Type

We consider the particular case when the filtration is of initial enlargement type, i.e.,

$$\mathcal{H}_t = \bigcap_{s>t} (\mathcal{F}_s \vee W_{T_0}), \quad 0 \le t \le T, \tag{103}$$

for some $T_0 > T$, and all the parameter processes are assumed to be deterministic bounded functions.

The enlargement of filtration technique can be applied to give the explicit expression for ϕ in (98). We give the following lemma, the proof of which can be found in [9] (p. 327).

Lemma 7 (Enlargement of filtration)**.** *The process W_t, $t \in [0, T]$, is a semi-martingale with respect to the filtration $\{\mathcal{H}_t\}$ given by (103). Its semi-martingale decomposition is*

$$W_t = W_{\mathcal{H}}(t) + \int_0^t \frac{W_{T_0} - W_s}{T_0 - s} \mathrm{d}s, \quad 0 \le t \le T,$$

where $W_{\mathcal{H}}(t)$, $t \in [0, T]$, is an $\mathcal{H}_t$-Brownian motion.

From Lemma 7, we can easily deduce the following corollary.

Corollary 5. *Suppose that the conditions in Theorem 14 hold. Assume further that $\mathcal{H}_t$ is given by (103) and all parameter processes are deterministic bounded functions. Then the optimal investment π^* is given by*

$$\pi_t^* = \frac{\mu_0(t) - r_t}{\sigma_t \tilde{\sigma}_t} + \frac{W_{T_0} - W_t}{\tilde{\sigma}_t (T_0 - t)}. \tag{104}$$

The value $\tilde{V}$ is given by

$$\tilde{V} = \ln X_0 + \int_0^T r_t \mathrm{d}t + \frac{1}{2}\int_0^T \frac{\sigma_t}{\tilde{\sigma}_t}\left(\frac{\mu_0(t) - r_t}{\sigma_t}\right)^2 \mathrm{d}t + \frac{1}{2}\int_0^T \frac{\sigma_t}{\tilde{\sigma}_t}\frac{1}{T_0 - t} \mathrm{d}t. \tag{105}$$

Proof. By Lemma 7, the $\mathcal{H}_t$-adapted process ϕ_t in the semi-martingale decomposition is of the form

$$\phi_t = \frac{W_{T_0} - W_t}{T_0 - t}, \quad 0 \le t \le T.$$

The result is an immediate consequence of Theorem 14. □

9. Numerical Analysis

In this section, we present numerical experiments for Sections 6 and 8 by comparing the optimal terminal expected utilities (i.e., the values) of six types of investors under varying parameters.

The six types of investors are as follows. R_I_S and R_NI_S are small robust (i.e., ambiguity-averse) investors under model uncertainty, with and without access to insider information, respectively. NR_I_S and NR_NI_S are small investors using an accurate model, with and without access to insider information, respectively. NR_I_L and NR_NI_L are large investors using an accurate model, with and without access to insider information, respectively.

In our experiments, we follow the usual parameters selection in simulation without loss of generality (see [45–47]). We set $r = 0.02$, $X_0 = 1$, $T = 1$, $\rho = \frac{1}{4}\sigma^2$ for all examples. As pointed out by [48], a risky asset typically has a volatility between 15% and 60%. Thus, we set $\sigma = 0.2, 0.35, 0.5$ and $\mu_0 \in [0.04, 0.2]$. As the insider information time T_0 exceeds T, we set $T_0 \in (1, 100]$.

Figure 1 shows the values of six types of investors under varying T_0.

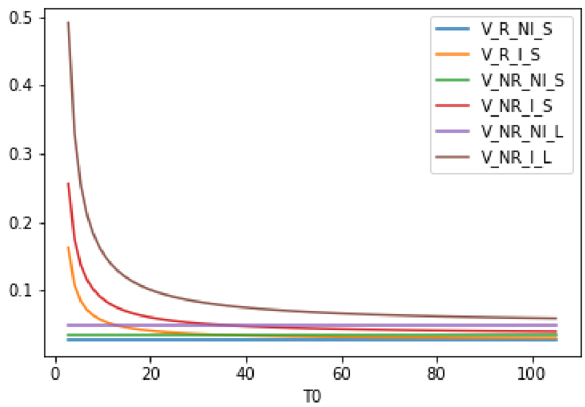

Figure 1. Values of six types of investors under varying T_0 ($\mu_0 = 0.15$, $\sigma = 0.35$).

When the insider information time T_0 is close to the terminal time T, NR_I_L has a significantly highest value, followed by NR_I_S and R_I_S. This implies that insider information has a substantial positive impact on the value. Without insider information, high influence on the financial market has a positive impact on the value, while model uncertainty has a negative impact. However, neither impact is significant.

As the insider information time T_0 increase , the additional information of the insider decrease and the profit from insider trading decays. By Corollaries 4 and 5 in Section 8, the insider information rent of NR_I_L or NR_I_S is given by

$$\Delta\tilde{V} = \frac{1}{2}\int_0^T \frac{\sigma_t}{\tilde{\sigma}_t}\frac{1}{T_0 - t}\mathrm{d}t.$$

By Corollaries 1 and 2 in Section 6, the insider information rent of R_I_S is given by

$$\Delta V^i = \frac{1}{2}\ln\left(1 - \frac{T^2}{(2T_0 - T)^2}\right)^{-1} + \frac{T}{2(2T_0 - T)} + \frac{1}{4(2T_0 - T)}\left(\int_0^T \frac{\mu_0(t) - r_t}{\sigma_t}\mathrm{d}t\right)^2.$$

Thus, the insider information rent is inversely proportional to T_0. If T_0 is 10 times larger than T, the value of insider trading is economically insignificant.

Figure 2 illustrates the requisite insider information for R_I_S to offset the value loss arising from model uncertainty under varying μ_0 and σ.

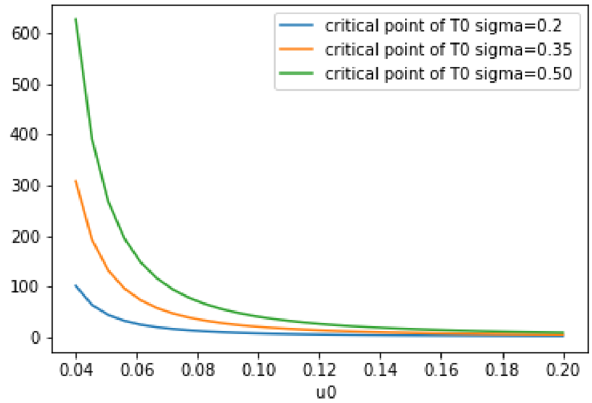

Figure 2. Insider information T_0^* needed for R_I_S to compensate for the loss of value due to model uncertainty with different μ_0 and σ.

On one hand, the insider information rent is represented as ΔV^i. On the other hand, robust investment mitigates the risk associated with model uncertainty at the cost of expected utility, which can be calculated by Corollaries 1 and 4 as follows

$$\Delta V^r = \frac{1}{4}\int_0^T \left(\frac{\mu_0(t) - r_t}{\sigma_t}\right)^2 \mathrm{d}t.$$

The minimum amount of insider information required for R_I_S could be quantified by the value T_0 that satisfies the following equation

$$\Delta V^i = \Delta V^r.$$

Denote the above T_0 by T_0^*. We refer to T_0^* as the critical information time.

Furthermore, the impact of model uncertainty becomes more pronounced as the mean rate of return μ_0 increases or the volatility σ decreases.

Figure 3 displays the (robust) optimal investment strategies π_t^* for three types of insiders, each associated with varying current risky asset prices S_t at time $t = 0.5$. We only show the curves of insiders since the strategy of investors without insider information will be trivial on the assumption of constant model parameters.

As risk W_t increases, all types of insiders will inevitably reduce their position in the risky asset. When we consider $W_{T_0} = W_t$, if becomes apparent that all types of insiders will maintain their positions in the risky asset, primarily due to the positive drift term $\mu_0(t)$.

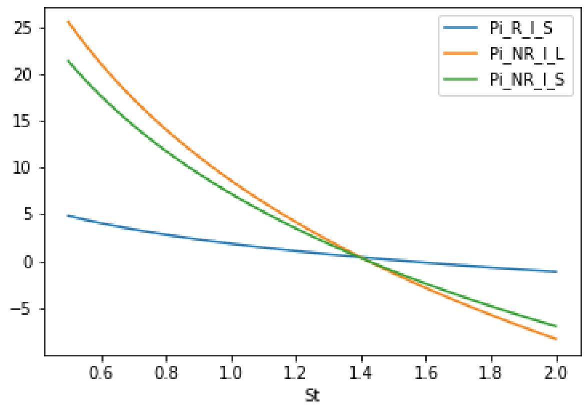

Figure 3. Investment strategies π_t^* in the risky asset for three types of insiders with different S_t ($\mu_0 = 0.08$, $\sigma = 0.35$, $t = 0.5$).

Among all insiders, R_I_S is dramatically less aggressive, and the derivative of π_t^* with respect to W_t is $\frac{-1}{\sigma_t(T_0-t+t_0-T)}$. Concerned about the risks associated with model uncertainty, R_I_S responds less vigorously to changes in the disparity between market conditions and insider information. In contrast, NR_I_L is the most aggressive, with the derivative of π_t^* with respect to W_t being $\frac{-1}{\tilde{\sigma}_t(T_0-t)}$. For NR_I_S, the derivative of π_t^* is $\frac{-1}{\sigma_t(T_0-t)}$.

10. Conclusions

In this paper, we enhance certain properties of forward integrals and extend the Itô formula for forward integrals, which was originally proposed by [34], through the application of Malliavin calculus.

We use the anticipating Itô formula to transform robust optimal investment problem for an insider under model uncertainty into an implicit anticipating SDG model. This represents a significant expansion of the model originally introduced by [14].

Given that traditional stochastic control theory cannot not be directly applied to solve the anticipating SDG problem, we introduce a new method. First, we utilize the variational method to establish the semi-martingale property of the noise in relation to the insider information filtration. Subsequently, we convert the anticipating SDG problem into a nonanticipative SDG problem, enabling us to make use of the stochastic maximum principle.

We consider two scenarios where the insider is categorized as 'small' and 'large', and provide the corresponding BSDEs to characterize the robust optimal investment strategies. In the small insider case, we derive the closed-form expression for the strategy when insider information filtration is of the initial enlargement type. The core technique here involves the Donsker $delta$ functional in the white noise theory. It's worth noting that a similar issue in [14] remains unsolved, as the approach presented in [14] only leads to a nested linear BSDE, which is hard to solve. In the large insider case, the strategy is involved in a quadratic BSDE.

We use numerical experiments to compare the optimal expected utilities among various investor types. The results highlight the substantial positive impact of insider information on utility. Additionally, a strong influence in the market also contributes positively to utility, while model uncertainty exerts a negative influence.

For further work, since the quadratic BSDE corresponding to the robust optimal investment strategy for a large insider has no analytical solution at present, we could only resort to the numerical methods. Moreover, extending the optimization problem to other models, like the jump-diffusion model and the fractional Brownian motion model (see [49]), is also a subject of ongoing research.

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Appendix A. Some Results in Malliavin Calculus

The following lemma is the chain rule of D_t, which is an extension of Lemma A.1 in [50].

Lemma A1. *Let* $\mathbf{X} = (X_1, \cdots, X_n) \in D^{1,p}(\Omega; \mathbb{R}^n)$ *and* $f \in C^1(\mathbb{R}^n)$ *for some* $n \in \mathbb{N}_+$ *and* $p \geq 1$. *Assume that*

$$\|f(\mathbf{X})\|_{L^p(\Omega)} + \left\| \sum_{l=1}^{n} \frac{\partial f}{\partial x_l}(\mathbf{X}) \cdot D_t X_l \right\|_{L^p(\Omega;H)} < \infty. \tag{A1}$$

Then $f(\mathbf{X}) \in D^{1,p}(\Omega)$ *and* $D_t f(\mathbf{X}) = \sum_{l=1}^{n} \frac{\partial f}{\partial x_l}(\mathbf{X}) \cdot D_t X_l$.

Proof. The proof is similar with that of Lemma A.1 in [50]. □

The following two lemmas with respect to δ are the multiplication formula, and the boundedness property, respectively.

Lemma A2. (Proposition 1.3.3, [19]) *Let* $X \in D^{1,2}(\Omega)$ *and* $u \in Dom\ \delta$. *Suppose that* $Xu \in L^2(\Omega; H)$. *Then* $Xu \in Dom\ \delta$ *and we have*

$$\delta(Xu) = X\delta u - \int_0^T D_t X \cdot u_t \mathrm{d}t, \tag{A2}$$

provided the right-hand side of (A2) is square integrable.

Lemma A3. (Proposition 1.3.1, [19]) $D^{1,2}(\Omega; H) \subset Dom\ \delta$. *Moreover,* δ *is bounded from* $D^{1,2}(\Omega; H)$ *into* $L^2(\Omega)$.

The following two lemmas characterize the local properties of D_t and δ.

Lemma A4. (Proposition 3.8, [29]) *Let* X *be a random variable in the space* $D^{1,1}$ *such that* $X = 0$ *a.s. on some set* $A \in \mathcal{F}$. *Then* $DX = 0$ *a.s. on* A.

Lemma A5. (Proposition 3.9, [29]) *Let* $u \in \mathcal{L}^{1,2}$ *and* $A \in \mathcal{F}$, *such that* $u = 0$ *a.a. on* $[0, T] \times A$. *Then* $\delta u = 0$ *a.s. on* A.

When $u \in L_a^2(\Omega \times [0, T])$, its Itô integral $\int_0^t u_s \mathrm{d}W_s$ is a continuous process (see [3]). A similar result in the context of the Skorohod integral can be provided by the following lemma.

Lemma A6. (Proposition 3.2.2, [19]) *Let* $u \in \mathcal{L}^{1,p}$, $p > 2$. *Then the Skorohod integral* $\int_0^t u_s \mathrm{d}W_s$ *is continuous (in the sense of modification).*

Similar to semi-martingales (or Itô processes) in Itô theory, here we provide a parallel characterization within the context of the Skorohod integral.

Lemma A7. (Proposition 3.1.1, [19]) *Consider a process of the form* $X_t = X_0 + \int_0^t u_s \mathrm{d}W_s + \int_0^t v_s \mathrm{d}s$, *where* $X_0 \in D^{1,2}(\Omega)$, $u \in \mathcal{L}^{2,2}$, *and* $v \in \mathcal{L}^{1,2}$. *Then we have* $X \in \mathcal{L}^{1,2,2}$, *and* $(D^- X)_s = D_s X_0 + \int_0^s D_s u_r \mathrm{d}W_r + \int_0^s D_s v_r \mathrm{d}r$ *(resp.* $(D^+ X)_s = u_s + D_s X_0 + \int_0^s D_s u_r \mathrm{d}W_r + \int_0^s D_s v_r \mathrm{d}r$*).*

Before presenting the approximation property $u \in \mathcal{L}^{1,2}$, which will be employed in the theory of the forward integral (Proposition 2), it is essential to introduce an important inequality in Harmonic Analysis.

Lemma A8. (Hardy-Littlewood, Theorem 2.5, [51]) *If* f *is locally integrable on* $\mathbb{R}^n$, *we define its Hardy–Littlewood maximal function* Mf *by* $Mf(x) := \sup_{r>0} \frac{1}{\mathrm{m}(B_x(r))} \int_{B_x(r)} |f(y)| \mathrm{d}y, \quad x \in \mathbb{R}^n$, *where* $B_x(r)$ *is a Euclidean ball of radius* r *centered at* x, *and* m *is the Lebesgue measure on* $\mathbb{R}^n$. *Then for all* $p \in (1, \infty]$, *we have*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \tag{A3}$$

where constant C_p depends only on p. Moreover, for a locally integrable function f, we have

$$\lim_{r\to 0^+} \frac{1}{\mathrm{m}(B_x(r))} \int_{B_r(x)} f(y)\mathrm{d}y = f(x) \tag{A4}$$

for a.e. x. The above conclusions still hold when we take some types of cubes containing x instead of $B_x(r)$.

Lemma A9. *If $u \in L^2(\Omega \times [0,T])$, then $\lim_{\varepsilon\to 0^+} \int_{(r-\varepsilon)\vee 0}^{r} \frac{u_s}{\varepsilon}\mathrm{d}s = u_r$ in $L^2(\Omega \times [0,T])$. Furthermore, the convergence also holds in $\mathcal{L}^{1,2}$ whenever $u \in \mathcal{L}^{1,2}$.*

Proof. For convenience, we only prove the convergence in $L^2(\Omega \times [0,T])$. By Lemma A8, we have $\int_{(r-\varepsilon)\vee 0}^{r} \frac{u_s}{\varepsilon}\mathrm{d}s \to u_r$ for a.a. $(\omega, r) \in \Omega \times [0,T]$. Since

$$\left| \int_{(r-\varepsilon)\vee 0}^{r} \frac{u_s}{\varepsilon}\mathrm{d}s \right| \le \sup_{0\le\varepsilon\le T} \int_{(r-\varepsilon)\vee 0}^{r} \frac{|u_s|}{\varepsilon}\mathrm{d}s, \tag{A5}$$

and

$$\mathbb{E}\left\| \sup_{0\le\varepsilon\le T} \int_{(r-\varepsilon)\vee 0}^{r} \frac{|u_s|}{\varepsilon}\mathrm{d}s \right\|^2_{L^2([0,T])} \le C\mathbb{E}\|u_r\|^2_{L^2([0,T])} < \infty \tag{A6}$$

by Lemma A8, we have $\lim_{\varepsilon\to 0^+} \int_{(r-\varepsilon)\vee 0}^{r} \frac{u_s}{\varepsilon}\mathrm{d}s = u_r$ in $L^2(\Omega \times [0,T])$ by the dominated convergence theorem. □

The following two lemmas give the Multiplication formulae of D_t and D^-, which will be utilized in the theory of the forward integral (Proposition 3).

Lemma A10. *Let $u, \sigma \in \mathcal{L}^{1,2}$. Also, assume that σ and $D_s\sigma_t$ are bounded. Then $u\sigma \in \mathcal{L}^{1,2}$ and $D_s(u_t\sigma_t) = \sigma_t D_s u_t + u_t D_s \sigma_t$.*

Proof. For each $t \in [0,T]$, $\mathbb{E}|u_t\sigma_t|^2 \le C\mathbb{E}|u_t|^2$, $\mathbb{E}\int_0^T |\sigma_t D_s u_t|^2\mathrm{d}s \le C\mathbb{E}\int_0^T |D_s u_t|^2\mathrm{d}s$, and $\mathbb{E}\int_0^T |u_t D_s\sigma_t|^2\mathrm{d}s \le C\mathbb{E}\int_0^T |u_t|^2\mathrm{d}s$. Thus $u_t\sigma_t \in D^{1,2}(\Omega)$ and $D_s(u_t\sigma_t) = \sigma_t D_s u_t + u_t D_s\sigma_t$ by Lemma A1. Since all the above norms are controlled, we deduce that $u\sigma \in \mathcal{L}^{1,2}$. □

Lemma A11. *Let u, σ be processes in $\mathcal{L}^{1,2,2-}$ which are L^2-bounded and left-continuous in the norm $L^2(\Omega)$. Also, assume that σ and $D_s\sigma_t$ are bounded. Then $u\sigma \in \mathcal{L}^{1,2,1-}$ and $(D^-(u\sigma))_s = \sigma_s(D^-u)_s + u_s(D^-\sigma)_s$.*

Proof. First, we have $u\sigma \in \mathcal{L}^{1,2}$ by Lemma A10. By the definition of the operator D^-, it suffices to estimate $\mathbb{E}|\sigma_t D_s u_t - \sigma_s(D^-u)_s|$ and $\mathbb{E}|u_t D_s\sigma_t - u_s(D^-\sigma)_s|$. For the first term, we have

$$\begin{aligned}
&\int_0^T \sup_{(s-\varepsilon)\vee 0\le t<s} \mathbb{E}|\sigma_t D_s u_t - \sigma_s (D^- u)_s|\mathrm{d}s \\
\le &\int_0^T \sup_{(s-\varepsilon)\vee 0\le t<s} \mathbb{E}|\sigma_t D_s u_t - \sigma_t (D^- u)_s|\mathrm{d}s \\
&+ \int_0^T \sup_{(s-\varepsilon)\vee 0\le t<s} \mathbb{E}|\sigma_t (D^- u)_s - \sigma_s (D^- u)_s|\mathrm{d}s \\
\le &\left(\int_0^T \sup_{(s-\varepsilon)\vee 0\le t<s} \mathbb{E}|\sigma_t|^2\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^T \sup_{(s-\varepsilon)\vee 0\le t<s} \mathbb{E}|D_s u_t - (D^- u)_s|^2\mathrm{d}s\right)^{\frac{1}{2}} \\
&+ \left(\int_0^T \sup_{(s-\varepsilon)\vee 0\le t<s} \mathbb{E}|\sigma_t - \sigma_s|^2\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^T \mathbb{E}|(D^- u)_s|^2\mathrm{d}s\right)^{\frac{1}{2}},
\end{aligned}$$

which converges to 0 when $\varepsilon \to 0^+$ due to the fact that $u \in \mathcal{L}^{1,2,2-}$ and σ is L^2-bounded and left-continuous. The convergence of the second term is in a similar way. □

Remark A1. *Notice that if an $\mathcal{F}_t$-adapted process $v \in \mathcal{L}^{1,2}$, then $v \in \mathcal{L}^{1,2,2-}$ with $(D^- v)_s = 0$. Thus, the condition '$\sigma \in \mathcal{L}^{1,2,2-}$ and u is left-continuous in the norm $L^2(\Omega)$' in Lemma A11 can be replaced by 'σ is $\mathcal{F}_t$-adapted and belongs to $\mathcal{L}^{1,2}$'.*

Appendix B. Proofs of Main Results

Proof of Proposition 2. By the multiplication formula (Lemma A2) and the Fubini theorem (Exercise 3.2.7 in [19]), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \varepsilon^{-1} \int_0^t u_s (W_{(s+\varepsilon)\wedge T} - W_s)\mathrm{d}s \\
=& \lim_{\varepsilon\to 0^+} \int_0^t u_s \int_s^{(s+\varepsilon)\wedge T} \frac{1}{\varepsilon}\mathrm{d}W_r\mathrm{d}s \\
=& \lim_{\varepsilon\to 0^+} \left\{ \int_0^t \int_s^{(s+\varepsilon)\wedge T} \frac{u_s}{\varepsilon}\mathrm{d}W_r\mathrm{d}s + \int_0^t \int_s^{(s+\varepsilon)\wedge T} \frac{D_r u_s}{\varepsilon}\mathrm{d}r\mathrm{d}s \right\} \\
=& \lim_{\varepsilon\to 0^+} \left\{ \int_0^T \int_{(r-\varepsilon)\vee 0}^r \frac{u_s 1_{[0,t]}(s)}{\varepsilon}\mathrm{d}s\mathrm{d}W_r + \int_0^T \int_{(r-\varepsilon)\vee 0}^r \frac{D_r u_s 1_{[0,t]}(s)}{\varepsilon}\mathrm{d}s\mathrm{d}r \right\}.
\end{aligned} \tag{A7}$$

For the first term on the right side of (A7), we have

$$\lim_{\varepsilon\to 0^+} \int_{(r-\varepsilon)\vee 0}^r \frac{u_s 1_{[0,t]}(s)}{\varepsilon}\mathrm{d}s = u_r 1_{[0,t]}(r) \tag{A8}$$

in $\mathcal{L}^{1,2}$ by Lemma A9. Then the convergence of the first term in $L^2(\Omega)$ follows from the boundedness of δ (Lemma A3). For the second term, we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^T\int_{(r-\varepsilon)\vee 0}^r \frac{D_r u_s 1_{[0,t]}(s)}{\varepsilon}\mathrm{d}s\mathrm{d}r - \int_0^t (D^- u)_r \mathrm{d}r\right| \\
\leq &\mathbb{E}\int_0^T\int_{r-\varepsilon}^r \frac{|D_r u_s 1_{[(r-\varepsilon)\vee 0,r]}(s) - (D^- u)_r|}{\varepsilon} 1_{[0,t]}(r)\mathrm{d}s\mathrm{d}r \\
&+ \mathbb{E}\int_0^T\int_{(r-\varepsilon)\vee 0}^r \frac{|D_r u_s||1_{[0,t]}(r) - 1_{[0,t]}(s)|}{\varepsilon}\mathrm{d}s\mathrm{d}r \\
\leq &\int_0^t \sup_{r-\varepsilon\leq s<r} \mathbb{E}|D_r u_s 1_{[(r-\varepsilon)\vee 0,r]}(s) - (D^- u)_r|\mathrm{d}r \\
&+ \int_0^T \sup_{(r-\varepsilon)\vee 0\leq s<r} \mathbb{E}|D_r u_s| \int_{(r-\varepsilon)\vee 0}^r \frac{|1_{[0,t]}(r) - 1_{[0,t]}(s)|}{\varepsilon}\mathrm{d}s\mathrm{d}r \\
\leq &\int_0^t \sup_{(r-\varepsilon)\vee 0\leq s<r} \mathbb{E}|D_r u_s - (D^- u)_r|\mathrm{d}r + \int_0^\varepsilon \mathbb{E}|(D^- u)_r|\mathrm{d}r \\
&+ \int_t^{(t+\varepsilon)\wedge T} \sup_{(r-\varepsilon)\vee 0\leq s<r} \mathbb{E}|D_r u_s|\mathrm{d}r.
\end{aligned}$$

The above convergence in $L^1(\Omega)$ follows from the definition of $\mathcal{L}^{1,2,1-}$. □

Proof of Proposition 3. It is an immediate consequence of Proposition 2, Lemma A11 and Remark A1. □

Proof of Theorem 2. By means of localization we can assume that the processes $f(X_t)$, $f'(X_t)$, $f''(X_t)$ and $\int_0^T u_t^2 \mathrm{d}t$ are uniformly bounded, $X_0 \in D^{1,2}(\Omega)$, $u \in \mathcal{L}^f$ and $v \in \mathcal{L}^{1,2}$ (see [19,29]). By Proposition 2, the process X_t has the following decomposition

$$X_t = X_0 + \int_0^t u_s \mathrm{d}W_s + \int_0^t v_s \mathrm{d}s + \int_0^t (D^- u)_s \mathrm{d}s. \tag{A9}$$

This process verifies the conditions in Theorem 1. Consequently, we can apply Itô formula to X and deduce that

$$\begin{aligned}
f(X_t) =& f(X_0) + \int_0^t f'(X_s)u_s \mathrm{d}W_s + \int_0^t f'(X_s)v_s \mathrm{d}s + \int_0^t f'(X_s)(D^- u)_s \mathrm{d}s \\
&+ \frac{1}{2}\int_0^t f''(X_s)u_s^2 \mathrm{d}s + \int_0^t f''(X_s)(D^- X)_s u_s \mathrm{d}s.
\end{aligned} \tag{A10}$$

The process $f'(X_t)$ belongs to $\mathcal{L}^{1,2,1-}$. In fact, notice first that as in the proof of Lemma A10 the boundedness of $f'(X_t)$, $f''(X_t)$ and $\int_0^T u_t^2 \mathrm{d}t$ and the fact that $u_t \in \mathcal{L}^{2,2} \cap \mathcal{L}^{1,4}$, $v, D^- u \in \mathcal{L}^{1,2}$ and $X_0 \in D^{1,2}(\Omega)$ imply that this process belongs to $\mathcal{L}^{1,2}$ and

$$D_s(f'(X_t)u_t) = f'(X_t)D_s u_t + f''(X_t)D_s X_t u_t. \tag{A11}$$

On the other hand, as in the proof of Lemma A11, using that $u \in \mathcal{L}^{1,2,1-}$, u_t is left-continuous in $L^2(\Omega)$, u is L^2-bounded, X_t is continuous and $X \in \mathcal{L}^{1,2,2}$ (by Theorem 1), we deduce that $f'(X_t)u_t \in \mathcal{L}^{1,2,1-}$ and that

$$(D^-(f'(X)u))_s = f'(X_s)(D^- u)_s + f''(X_s)(D^- X)_s u_s. \tag{A12}$$

Substituting it into (A10) and using Proposition 2, we obtain

$$
\begin{aligned}
f(X_t) =& f(X_0) + \int_0^t f'(X_s)u_s \mathrm{d}W_s + \int_0^t (D^-(f'(X)u))_s \mathrm{d}s + \int_0^t f'(X_s)v_s \mathrm{d}s \\
& + \frac{1}{2}\int_0^t f''(X_s)u_s^2 \mathrm{d}s \\
=& f(X_0) + \int_0^t f'(X_s)u_s \mathrm{d}^- W_s + \int_0^t f'(X_s)v_s \mathrm{d}s + \frac{1}{2}\int_0^t f''(X_s)u_s^2 \mathrm{d}s.
\end{aligned}
$$

□

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