mathematics

Article

Fourth-Order Neutral Differential Equation: A Modified Approach to Optimizing Monotonic Properties

Amany Nabih [1,2,*], Osama Moaaz [1,3,*], Sameh S. Askar [4], Ahmad M. Alshamrani [4] and Elmetwally M. Elabbasy [1]

1 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; emelabbasy@mans.edu.eg
2 Department of Mathematics and Basic Sciences, Higher Future Institute of Engineering and Technology, Mansoura 35516, Egypt
3 Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
4 Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; saskar@ksu.edu.sa (S.S.A.); ahmadm@ksu.edu.sa (A.M.A.)
* Correspondence: amanynabih89@std.mans.edu.eg (A.N.); o_moaaz@mans.edu.eg (O.M.)

Abstract: In this article, we investigate some qualitative properties of solutions to a class of functional differential equations with multi-delay. Using a modified approach, we first derive a number of optimized relations and inequalities that relate the solution $x(s)$ to its corresponding function $z(s)$ and its derivatives. After classifying the positive solutions, we follow the Riccati approach and principle of comparison, where fourth-order differential equations are compared with first-order differential equations to obtain conditions that exclude the positive solutions. Then, we introduce new oscillation conditions. With regard to previous relevant results, our results are an extension and complement to them. This work has theoretical significance in that it uncovers some new relationships that aid in developing the oscillation theory of higher-order equations in addition to the applied relevance of neutral differential equations.

Keywords: neutral differential equation; fourth order; monotonic properties; oscillation; canonical case

MSC: 34C10; 34K11

Citation: Nabih, A.; Moaaz, O.; Askar, S.S.; Alshamrani, A.M.; Elabbasy, E.M. Fourth-Order Neutral Differential Equation: A Modified Approach to Optimizing Monotonic Properties. *Mathematics* **2023**, *11*, 4380. https://doi.org/10.3390/math11204380

Academic Editor: Gennadii Demidenko

Received: 2 June 2023
Revised: 21 July 2023
Accepted: 16 October 2023
Published: 21 October 2023

1. Introduction

Diverse areas of pure and applied mathematics, physics, and engineering all involve the study of differential equations (DEs); these disciplines are all interested in the characteristics of various forms of DEs. Applied and pure mathematics emphasize the existence and uniqueness of solutions, as well as the precise justification of the methods for approximate solutions. Nearly every physical, technological, or biological process, including celestial motion, bridge design, and neuron interactions, is modeled in large part using DEs. DEs that are intended to address real-world issues may not always be directly solved; for instance, they may lack closed-form solutions. Alternative methods to this include approximating the results using numerical techniques [1].

Understanding these problems and events requires knowing how these equations are solved. However, DEs used to address real-world issues may not always be directly solvable, that is, they may not have closed-form solutions (see [2,3]). For this reason, the study of the qualitative theory, which is concerned with differential equation behavior through methods other than finding solutions, has been highly utilized. It evolved from Henri Poincaré's and Alexandre Lyapunov's works. Although there are relatively few DEs that can be solved directly, one can "solve" them in a qualitative sense by employing techniques from analysis and topology to learn more about their characteristics [4].

Fourth-order delay DEs are used to numerically depict biological, chemical, and physical phenomena. Problems of elasticity, the deformation of structures, or soil settlement are a few examples of these applications. A fourth-order oscillatory equation with delay can be used to simulate the oscillatory traction of a muscle, which occurs when the muscle is subjected to an inertial force [5].

In the theory of linear DEs, the oscillation theory has numerous significant applications. It can be used, for instance, to examine the stability of solutions to linear DEs. The theorem can be used, in particular, to demonstrate the lack of nontrivial solutions that converge to zero as time reaches infinity. In order to analyze eigenvalue problems, the oscillation theorem is very useful. The Schrodinger equation in quantum physics is one example of a DE whose eigenvalues and eigenfunctions can be studied using the theorem. The oscillation theorem can be used to estimate the number of eigenvalues present in a particular interval and to learn more about how eigenfunctions behave [6,7].

The study of oscillation for ordinary and fractional DE solutions with delay, neutral, mixed, or damping terms is currently included in the oscillation theory, which has recently seen significant growth and development. For instance, you can find delay equations in [8–11], mixed equations in [12–16], and neutral equations in [17–20].

The goal of this paper is to delve into the oscillation of the fourth-order neutral delay DE

$$(r(\mathrm{s})(z'''(\mathrm{s}))^{\gamma})' + \sum_{j=1}^{\ell} q_j(\mathrm{s})x^{\gamma}(\sigma_j(\mathrm{s})) = 0,\ \mathrm{s} \geq \mathrm{s}_0, \tag{1}$$

where ℓ is a positive integer and $z(\mathrm{s}) = x(\mathrm{s}) + p(\mathrm{s})x(\tau(\mathrm{s}))$, through which, under the following assumptions:

(A_1) $\gamma > 0$ is a quotient of odd positive integers;
(A_2) $r \in \mathbf{C}([\mathrm{s}_0, \infty), (0, \infty))$ and satisfies

$$\pi_0(\mathrm{s}) := \int_{\mathrm{s}_0}^{\mathrm{s}} r^{-1/\gamma}(\mathrm{u})\mathrm{du} \to \infty \text{ as } \mathrm{s} \to \infty; \tag{2}$$

(A_3) $\tau, \sigma_j \in \mathbf{C}([\mathrm{s}_0, \infty), (0, \infty))$ satisfy $\tau(\mathrm{s}) < \mathrm{s}, \sigma_j(\mathrm{s}) < \mathrm{s}, \lim_{\mathrm{s}\to\infty} \tau(\mathrm{s}) = \infty$ and $\lim_{\mathrm{s}\to\infty} \sigma_j(\mathrm{s}) = \infty$ for $j = 1, 2, \ldots, \ell$;
(A_4) $p, q_j \in \mathbf{C}([\mathrm{s}_0, \infty), (0, \infty))$, $p(\mathrm{s}) \leq p_0$, where p_0 is a constant and q_j does not gradually vanish .

As a solution to (1), we represent a real-valued function x that is four times differentiable and satisfies (1) for all sufficiently large s. Our attention is restricted to those solutions of (1) that satisfy the condition

$$\sup\{|x(\mathrm{s})| : \mathrm{s} \geq \mathrm{U}\} > 0,$$

for all $\mathrm{U} \geq \mathrm{s}_0$. A solution x to (1) is referred to as oscillatory or non-oscillatory depending on whether it is essentially positive or negative. If all of the solutions to an equation oscillate, the equation is said to be oscillatory.

In order to understand the asymptotic and oscillatory behavior of solutions to neutral DEs, it is crucial to understand the relationship between the solution x and its associated function z. The authors were able to determine a number of additional criteria that simplified and enhanced their previous research findings as a result of this relationship.

We now list some of the relationships that have been inferred in the literature.

For the second order, in the canonical case, the usual relationship

$$x(\mathrm{s}) > (1 - p(\mathrm{s}))z(\mathrm{s}) \tag{3}$$

is typically employed, and in the noncanonical case, the relationship

$$x(\mathrm{s}) > (1 - p(\mathrm{s})\pi(\tau(\mathrm{s}))/\pi(\mathrm{s}))z(\mathrm{s}) \tag{4}$$

is usually used (see [21,22]).

Moaaz et al. [23] took into account the oscillatory behavior of

$$\left(r(s)\left(z'(s)\right)^{\gamma}\right)' + \sum_{j=1}^{\ell} q_j(s)x^{\beta}\left(\sigma_j(s)\right) = 0, \tag{5}$$

where β is a quotient of odd positive integers and $\ell \in \mathbb{Z}^+$. As an improvement to (3), they offered the following relationships:

$$x(s) > z(s)\sum_{m=1}^{n/2}\frac{1}{p^{2m-1}}\left(1-\frac{1}{p}\frac{\pi\left(\tau^{-2m}(s)\right)}{\pi\left(\tau^{-(2m-1)}(s)\right)}\right), \text{for } p > 1 \text{ and } n \in \mathbb{Z}^+ \text{ is even,}$$

and

$$x(s) > z(s)(1-p)\sum_{m=0}^{(n-1)/2} p^{2m}\frac{\pi\left(\tau^{2m+1}(s)\right)}{\pi(s)}, \text{for } p < 1, \text{ and } n \in \mathbb{Z}^+ \text{ is odd,}$$

where $\tau^{[h]}(s) = \tau\left(\tau^{[h-1]}(s)\right)$, for $h = 1, 2, \ldots, 2m$.

Hassan et al. [24] investigated the oscillatory properties of

$$(r(s)(z'(s))^{\gamma})' + q(s)x^{\gamma}(\sigma(s)) = 0,$$

and they offered

$$x(s) > z(s)\sum_{\kappa=0}^{(n-1)/2} p^{2\kappa}\left(1 - p\frac{\pi\left(\tau^{[2\kappa+1]}(s)\right)}{\pi(\tau^{2\kappa}(s))}\right),$$

which is an improvement to (4), where $n \in \mathbb{Z}^+$ is odd.

In their study of the equation

$$(r(s)(z'(s))^{\gamma})' + q(s)x^{\gamma}(\sigma(s)) = 0,$$

Bohner et al. [25] created the new relation

$$x(s) > z(s)(1 - p(s))(1 + H_k(s)),$$

where

$$H_k(s) = \begin{cases} 0 & \text{for } k = 0, \\ \sum_{i=1}^{k}\prod_{j=0}^{2i-1} p\left(\tau^j(s)\right) & \text{for } \tau(s) \leq s \text{ and } k \in \mathbb{N}, \\ \sum_{i=1}^{k}\frac{\pi\left(\tau^{2i}(s)\right)}{\pi(s)}\prod_{j=0}^{2i-1} p\left(\tau^j(s)\right) & \text{for } \tau(s) \geq s \text{ and } k \in \mathbb{N}, \end{cases}$$

where $\tau^{[0]}(s) = s$ and $\tau^{[j]}(s) = \tau\left(\tau^{j-1}(s)\right)$ for all $j \in \mathbb{N}$, which is an improvement of (4). Also, they added additional oscillation criteria that, in essence, improve a number of pertinent criteria from the literature.

In order to oscillate for the solutions to neutral nonlinear even-order DEs with variable coefficients of the form

$$z^{(n)}(s) + q(s)f(x(\sigma(s))) = 0, \tag{6}$$

where $f(x)$ is a continuous function, several sufficient conditions are found by Zhang et al. [26].

Agarwal et al. [27] studied the oscillatory behavior of the equation

$$z^{(n)}(s) + q(s)x(\sigma(s)) = 0, \tag{7}$$

and created criteria that improve the results published in the literature.

Moaaz et al. [28] tested the oscillation of

$$(r(\mathrm{s})(z''(\mathrm{s}))^{\gamma})' + q(\mathrm{s})x^{\gamma}(\sigma(\mathrm{s})) = 0.$$

Using an iterative method, they were able to develop a new criterion for the nonexistence of the so-called Kneser solutions. Also, they employed a variety of techniques to find various criteria. Using the relation

$$x(\mathrm{s}) > z(\mathrm{s})(1-p)\sum_{\kappa=0}^{(n-1)/2} p^{2\kappa}\left(\frac{\tau^{[2\kappa+1]}(\mathrm{s})-\mathrm{s}_1}{\mathrm{s}-\mathrm{s}_1}\right)^2,$$

they improved (3).

By using some inequalities and the Riccati transformation method, Muhib et al. [29] established some improved criteria for the equation

$$(r(\mathrm{s})(z'''(\mathrm{s}))^{\gamma})' + \sum_{j=1}^{\ell} q_j(\mathrm{s})x^{\beta}(\sigma_j(\mathrm{s})) = 0 \tag{8}$$

without necessitating the existence of unknown functions; where β is a quotient of odd positive integers.

Theorem 1 ([29]). *Let $\sigma_j \in \mathbf{C}^1([\mathrm{s}_0,\infty),(0,\infty))$, $\sigma_j' > 0$ and $\sigma_j(\mathrm{s}) < \sigma(\mathrm{s})$. Assume there exists a λ, $\mu \in \mathbf{C}^1([\mathrm{s}_0,\infty),(0,\infty))$ for all $\mathrm{s}_1 \geq \mathrm{s}_0$, there is a $\mathrm{s}_2 > \mathrm{s}_1$ such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left(\ell\lambda(\mathrm{v})\frac{Q(\mathrm{v})}{2^{\beta-1}}M^{\beta-\gamma} - \left(1+\frac{p_0}{\tau_0}\right)\frac{(\lambda_+'(\mathrm{v}))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\lambda(\mathrm{v})\pi_1(\mathrm{v})\sigma'(\mathrm{v}))^{\gamma}}\right)\mathrm{dv} = \infty \tag{9}$$

and

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left(\mu(\mathrm{v})\left(\frac{\tau_0}{\tau_0+p_0}\right)M^{(\beta/\gamma)-1}\int_{\mathrm{v}}^{\infty}\left(\frac{1}{r(u)}\Phi(u)\right)^{1/\gamma}\mathrm{d}u - \frac{(\mu_+'(\mathrm{v}))^2}{4\mu(\mathrm{v})}\right)\mathrm{dv} = \infty, \tag{10}$$

where

$$\Phi = \int_{\sigma^{-1}(u)}^{\infty}\frac{Q(\mathrm{v})}{2^{\beta-1}}\sum_{j=1}^{\ell}\left(\frac{\sigma_j(\mathrm{v})}{\mathrm{v}}\right)^{\beta/\epsilon}\mathrm{dv},$$

$$Q := \min\{q_j(\mathrm{s}) \text{ for } j = 1,2,\ldots,\ell\},$$

$\lambda_+'(\mathrm{s}) = \max\{0,\lambda'(\mathrm{s})\}$, $\mu_+'(\mathrm{s}) = \max\{0,\mu'(\mathrm{s})\}$ and $\epsilon \in (0,1)$. Then, (8) is oscillatory.

Nabih et al. [30] focused on investigating the oscillation of

$$(r(\mathrm{s})(z'''(\mathrm{s}))^{\gamma})' + q(\mathrm{s})x^{\gamma}(\sigma(\mathrm{s})) = 0.$$

They found new properties that enable them to use more effective terms. To obtain criteria that excluded the positive decreasing solutions, they used the general form of Riccati and the comparison approach.

Lemma 1 ([31]). *Assume that $\phi \in \mathbf{C}^n([\mathrm{s}_0,\infty),(0,\infty))$. If the derivative $\phi^{(n)}(\mathrm{s})$ is eventually of one sign for all large s, then there exists an s_x such that $\mathrm{s}_x \geq \mathrm{s}_0$ and an integer l, $0 \leq l \leq n$, with $n+l$ even for $\phi^{(n)}(\mathrm{s}) \geq 0$, or $n+l$ odd for $\phi^{(n)}(\mathrm{s}) \leq 0$ such that*

$$l > 0 \text{ implies } \phi^{(\kappa)}(\mathrm{s}) > 0 \text{ for } \mathrm{s} \geq \mathrm{s}_x,\ \kappa = 0,1,\ldots,l-1,$$

and

$$l \leq n-1 \text{ implies } (-1)^{l+\kappa}\phi^{\kappa}(\mathrm{s}) > 0 \text{ for } \mathrm{s} \geq \mathrm{s}_x,\ \kappa = l,l+1,\ldots,n-1.$$

Lemma 2 ([32]). *Let γ be a ratio of two odd positive integers. Then*

$$Lv^{(\gamma+1)/\gamma} - Mv \geq -\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{M^{\gamma+1}}{L^{\gamma}},\ L > 0 \tag{11}$$

and

$$A^{(\gamma+1)/\gamma} - (A-B)^{(\gamma+1)/\gamma} \leq \frac{1}{\gamma}B^{1/\gamma}[(1+\gamma)A - B],\quad \gamma \geq 1,\ AB \geq 0. \tag{12}$$

For multi-delay functional DEs, we refer to some qualitative aspects of solutions. We begin by deriving a set of optimized relations and inequalities that connect the solution $x(\mathrm{s})$ to its corresponding function $z(\mathrm{s})$ and its derivatives using a modified methodology. Once the positive solutions have been categorized, we use the Riccati technique and the comparison, where fourth-order DEs are compared with first-order DEs to create criteria that exclude the positive solutions. Then, we provide new oscillation conditions. The new results add to and complete the previous relevant results. In addition to the practical value of neutral DEs, this study has theoretical value in that it uncovers some novel relationships that help advance the oscillation theory of higher-order equations.

2. Main Results

For clarity, let

$$\pi_i(\mathrm{s}) = \int_{\mathrm{s}_0}^{\mathrm{s}} \pi_{i-1}(\mathrm{v})\mathrm{dv} \text{ for } i = 1,2,$$

$$p_1(\mathrm{s},m) := \sum_{i=0}^{\kappa}\left(\prod_{h=0}^{2i} p\left(\tau^{[h]}(\mathrm{s})\right)\right)\left[\frac{1}{p(\tau^{[2i]}(\mathrm{s}))} - 1\right]\frac{\pi_2\left(\tau^{[2i]}(\mathrm{s})\right)}{\pi_2(\mathrm{s})},$$

$$p_2(\mathrm{s},m) := \sum_{i=1}^{\kappa}\left(\prod_{h=0}^{2i-1}\frac{1}{p(\tau^{[-h]}(\mathrm{s}))}\right)\left[\frac{\pi_2\left(\tau^{[-2i+1]}(\mathrm{s})\right)}{\pi_2(\tau^{[-2i]}(\mathrm{s}))} - \frac{1}{p(\tau^{[-2i]}(\mathrm{s}))}\right]$$

and

$$\widehat{p}(\mathrm{s},m) := \begin{cases} 1 & \text{for } p_0 = 0; \\ p_1(\mathrm{s},m) & \text{for } p_0 < 1; \\ p_2(\mathrm{s},m) & \text{for } p_0 > \pi_2(\mathrm{s})/\pi_2(\tau(\mathrm{s})), \end{cases}$$

for any integer $m \geq 0$. We will require the following notations for presenting the results:

$$F^{[0]}(\mathrm{s}) := \mathrm{s},\ F^{[h]}(\mathrm{s}) := F\left(F^{[h-1]}(\mathrm{s})\right) \text{ and } F^{[-h]}(\mathrm{s}) := F^{-1}\left(F^{[-h+1]}(\mathrm{s})\right),$$

for $h = 1,2,\ldots,$

$$\sigma(\mathrm{s}) = \min\{\sigma_j(\mathrm{s}) \text{ for } j = 1,2,\ldots,\ell\}$$

and

$$q(\mathrm{s}) = \min\{q_j(\mathrm{s}) \text{ for } j = 1,2,\ldots,\ell\}.$$

Lemma 3. *Let $x(\mathrm{s})$ be a positive solution of (1). Then, $(r(\mathrm{s})(z'''(\mathrm{s}))^{\gamma})' \leq 0$ and there are ultimately two possible cases:*

$$\begin{array}{llllll} \text{Case}(1) & z(\mathrm{s}) > 0 & z'(\mathrm{s}) > 0 & z''(\mathrm{s}) > 0 & z'''(\mathrm{s}) > 0 & z^{(4)}(\mathrm{s}) < 0; \\ \text{Case}(2) & z(\mathrm{s}) > 0 & z'(\mathrm{s}) > 0 & z''(\mathrm{s}) < 0 & z'''(\mathrm{s}) > 0. & \end{array}$$

Proof. Suppose x(s) is a positive solution of (1). We obtain $(r(\mathrm{s})(z'''(\mathrm{s}))^{\gamma})' \leq 0$ from (1). In order to obtain the cases (1) and (2) for the function $z(\mathrm{s})$ and its derivatives, we use Lemma 2.2.1 in [33]. □

Remark 1. *By using the notation $\mathcal{B}_1$, we can identify the class of all eventually positive solutions whose corresponding functions satisfy* Case (1)*.*

Lemma 4 ([34], Lemma 2.1)**.** *Assume that $x(\mathrm{s})$ is an eventually positive solution of (1). Then, eventually,*

$$x(\mathrm{s}) > \sum_{i=0}^{\kappa}\left(\prod_{h=0}^{2i} p\left(\tau^{(h)}(\mathrm{s})\right)\right)\left[\frac{z\left(\tau^{(2i)}(\mathrm{s})\right)}{p\left(\tau^{(2i)}(\mathrm{s})\right)} - z\left(\tau^{(2i+1)}(\mathrm{s})\right)\right], \tag{13}$$

for any integer $\kappa \geq 0$.

Lemma 5. *Assume that $x \in \mathcal{B}_1$. Then, eventually,*

$$\left(\frac{z''(\mathrm{s})}{\pi_0(\mathrm{s})}\right)' \leq 0 \tag{14}$$

and

$$\left(\frac{z(\mathrm{s})}{\pi_2(\mathrm{s})}\right)' \leq 0. \tag{15}$$

Proof. Assume that $x \in \mathcal{B}_1$. We obtain

$$z''(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} r^{-1/\gamma}(\mathrm{v}) r^{1/\gamma}(\mathrm{v}) z'''(\mathrm{v}) d\mathrm{v} \geq r^{1/\gamma}(\mathrm{s}) z'''(\mathrm{s}) \pi_0(\mathrm{s}); \tag{16}$$

hence,

$$r^{1/\gamma}(\mathrm{s}) z'''(\mathrm{s}) \pi_0(\mathrm{s}) - z''(\mathrm{s}) \leq 0.$$

This implies

$$\begin{aligned}\left(\frac{z''(\mathrm{s})}{\pi_0(\mathrm{s})}\right)' &= \frac{\pi_0(\mathrm{s}) r^{1/\gamma}(\mathrm{s}) z'''(\mathrm{s}) - z''(\mathrm{s})}{r^{1/\gamma}(\mathrm{s}) \pi_0^2(\mathrm{s})} \\ &= \frac{1}{r^{1/\gamma}(\mathrm{s}) \pi_0^2(\mathrm{s})}\left[\pi_0(\mathrm{s}) r^{1/\gamma}(\mathrm{s}) z'''(\mathrm{s}) - z''(\mathrm{s})\right] \\ &\leq 0.\end{aligned}$$

Applying this information, we determine that

$$z'(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} \pi_0(\mathrm{v}) \frac{z''(\mathrm{v})}{\pi_0(\mathrm{v})} d\mathrm{v} \geq \frac{z''(\mathrm{s})}{\pi_0(\mathrm{s})} \pi_1(\mathrm{s}) \geq r^{1/\gamma}(\mathrm{s}) z'''(\mathrm{s}) \pi_1(\mathrm{s}) \tag{17}$$

yields

$$\begin{aligned}\left(\frac{z'(\mathrm{s})}{\pi_1(\mathrm{s})}\right)' &= \frac{\pi_1(\mathrm{s}) z''(\mathrm{s}) - \pi_0(\mathrm{s}) z'(\mathrm{s})}{\pi_1^2(\mathrm{s})} \\ &= \frac{1}{\pi_1^2(\mathrm{s})}\left[\pi_1(\mathrm{s}) z''(\mathrm{s}) - \pi_0(\mathrm{s}) z'(\mathrm{s})\right] \leq 0.\end{aligned}$$

Hence,

$$z(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} \pi_1(\mathrm{v}) \frac{z'(\mathrm{v})}{\pi_1(\mathrm{v})} d\mathrm{v} \geq \frac{z'(\mathrm{s})}{\pi_1(\mathrm{s})} \pi_2(\mathrm{s}) \geq r^{1/\gamma}(\mathrm{s}) z'''(\mathrm{s}) \pi_2(\mathrm{s}) \tag{18}$$

yields

$$\begin{aligned}\left(\frac{z(s)}{\pi_2(s)}\right)' &= \frac{\pi_2(s)z'(s) - \pi_1(s)z'(s)}{\pi_2^2(s)} \\ &= \frac{1}{\pi_2^2(s)}\left[\pi_2(s)z'(s) - \pi_1(s)z'(s)\right] \leq 0.\end{aligned}$$

This completes the proof. □

Lemma 6. *Assume that $x \in \mathcal{B}_1$. Eventually,*

$$(r(s)(z'''(s))^{\gamma})' + \sum_{j=1}^{\ell} q_j(s)\widehat{p}^{\gamma}(\sigma_j(s), m)z^{\gamma}(\sigma_j(s)) \leq 0 \tag{19}$$

is reached by Equation (1), when

$$x(s) > \widehat{p}(s, m)z(s).$$

Proof. Assume that $x \in \mathcal{B}_1$ and that $p_0 < 1$. We determine that

$$z\left(\tau^{(2i)}(s)\right) \geq z\left(\tau^{(2i+1)}(s)\right)$$

and

$$z\left(\tau^{(2i)}(s)\right) \geq \frac{\pi_2\left(\tau^{(2i)}(s)\right)}{\pi_2(s)}z(s),\ i = 0, 1, \ldots$$

based on the information that $\tau^{(2i+1)}(s) \leq \tau^{(2i)}(s) \leq s$, $z'(s) > 0$ and $(z(s)/\pi_2(s))' \leq 0$. As a result, (13) becomes

$$x(s) > z(s)\sum_{i=0}^{\kappa}\left(\prod_{h=0}^{2i} p\left(\tau^{(h)}(s)\right)\right)\left[\frac{1}{p\left(\tau^{(2i)}(s)\right)} - 1\right]\frac{\pi_2\left(\tau^{(2i)}(s)\right)}{\pi_2(s)},$$

which, when combined with (1), yields (19). Conversely, suppose that $p_0 > 1$. The definition of $z(s)$ implies that

$$\begin{aligned}p\left(\tau^{-1}(s)\right)x(s) &= z\left(\tau^{-1}(s)\right) - x\left(\tau^{-1}(s)\right) \\ &= z\left(\tau^{-1}(s)\right) - \frac{1}{p(\tau^{-2}(s))}\left[z\left(\tau^{-2}(s)\right) - x\left(\tau^{-2}(s)\right)\right];\end{aligned}$$

hence,

$$\begin{aligned}p\left(\tau^{-1}(s)\right)x(s) &= z\left(\tau^{-1}(s)\right) - z\left(\tau^{-2}(s)\right)\prod_{i=2}^{2}\frac{1}{p(\tau^{[-i]}(s))} \\ &\quad + \left[z\left(\tau^{[-3]}(s)\right) - x\left(\tau^{[-3]}(s)\right)\right]\prod_{i=2}^{3}\frac{1}{p(\tau^{[-i]}(s))},\end{aligned}$$

and so on. As a result, we arrive at

$$x(s) > \sum_{i=1}^{\kappa}\left(\prod_{h=0}^{2i-1}\frac{1}{p(\tau^{[-h]}(s))}\right)\left[z\left(\tau^{[-2i+1]}(s)\right) - \frac{1}{p(\tau^{[-2i]}(s))}z\left(\tau^{[-2i]}(s)\right)\right]. \tag{20}$$

Thus,

$$z\left(\tau^{[-2i+1]}(s)\right) \geq \frac{\pi_2\left(\tau^{[-2i+1]}(s)\right)}{\pi_2(\tau^{[-2i]}(s))}z\left(\tau^{[-2i]}(s)\right)$$

and
$$z\left(\tau^{[-2i]}(s)\right) \geq z(s)$$
are the result of the fact that $s \leq \tau^{[-2i+1]}(s)$, $z'(s) > 0$ and $(z(s)/\pi_2(s))' \leq 0$. Inequality (20) then changes to
$$x(s) > z(s)\sum_{i=1}^{\kappa}\left(\prod_{h=0}^{2i-1}\frac{1}{p\left(\tau^{[-h]}(s)\right)}\right)\left[\frac{\pi_2\left(\tau^{[-2i+1]}(s)\right)}{\pi_2\left(\tau^{[-2i]}(s)\right)} - \frac{1}{p\left(\tau^{[-2i]}(s)\right)}\right],$$
as a result, which, when combined with (1), yields (19). This completes the proof. □

Now, using the Riccati approach, we obtain the following theorem:

Theorem 2. *Assume that there is a $\lambda \in \mathbf{C}^1([s_0,\infty),(0,\infty))$ such that*
$$\limsup_{s\to\infty}\int_{s_0}^{s}\left\{\lambda(v)q(v)\frac{\pi_2^{\gamma}(\sigma(v))}{\pi_2^{\gamma}(v)}\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(v),m) - \frac{(\lambda'(v))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\lambda(v))^{\gamma}\pi_1^{\gamma}(v)}\right\}dv = \infty, \quad (21)$$
then $\mathcal{B}_1 = \varnothing$.

Proof. Assume that $x \in \mathcal{B}_1$. Through Lemma 5, we guarantee (16)–(18). Define the function
$$\Phi(s) = \lambda(s)\frac{r(s)(z'''(s))^{\gamma}}{z^{\gamma}(s)}, \quad (22)$$
where $\Phi(s) > 0$. By differentiating (22) and using (17) and (19), we obtain
$$\begin{aligned}\Phi'(s) &= \frac{\lambda'(s)}{\lambda(s)}\Phi(s) + \lambda(s)\frac{(r(s)(z'''(s))^{\gamma})'}{z^{\gamma}(s)} - \gamma\lambda(s)r(s)\frac{(z'''(s))^{\gamma}z^{\gamma-1}(s)z'(s)}{z^{2\gamma}(s)}\\ &\leq \frac{\lambda'(s)}{\lambda(s)}\Phi(s) - \lambda(s)q(s)\left(\frac{z(\sigma(s))}{z(s)}\right)^{\gamma}\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(s),m) - \gamma\lambda(s)r(s)\frac{(z'''(s))^{\gamma}z'(s)}{z^{\gamma+1}(s)}\\ &\leq \frac{\lambda'(s)}{\lambda(s)}\Phi(s) - \lambda(s)q(s)\left(\frac{z(\sigma(s))}{z(s)}\right)^{\gamma}\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(s),m)\\ &\quad -\gamma\lambda(s)r^{1+1/\gamma}(s)\pi_1(s)\left(\frac{z'''(s)}{z(s)}\right)^{\gamma+1},\end{aligned}$$
which, when combined with the fact that $(z(s)/\pi_2(s))' \leq 0$, results in
$$\Phi'(s) \leq \frac{\lambda'(s)}{\lambda(s)}\Phi(s) - \lambda(s)q(s)\frac{\pi_2^{\gamma}(\sigma(s))}{\pi_2^{\gamma}(s)}\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(s),m) - \gamma\frac{\pi_1(s)}{\lambda^{1/\gamma}(s)}\Phi^{1+1/\gamma}(s).$$

Using inequality (11) with $M = \lambda'(s)/\lambda(s)$ and $L = \gamma\pi_1(s)/\lambda^{1/\gamma}(s)$, we obtain
$$\Phi'(s) \leq -\lambda(s)\frac{\pi_2^{\gamma}(\sigma(s))}{\pi_2^{\gamma}(s)}\sum_{j=1}^{\ell}q_j(s)\widehat{p}^{\gamma}(\sigma_j(s),m) + \frac{(\lambda'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\lambda(s))^{\gamma}\pi_1^{\gamma}(s)}.$$

Integrating the aforementioned inequality from s_0 to s, we obtain
$$\Phi(s_0) \geq \int_{s_0}^{s}\left\{\lambda(v)q(v)\frac{\pi_2^{\gamma}(\sigma(v))}{\pi_2^{\gamma}(v)}\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(v),m) - \frac{(\lambda'(v))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\lambda(v))^{\gamma}\pi_1^{\gamma}(v)}\right\}dv.$$

Using Equation (21), we encounter a contradiction. This completes the proof. □

In the results that follow, the monotonic features of the solutions in class $\mathcal{B}_1$ are enhanced, and better criteria are then reached to support the claim that $\mathcal{B}_1 = \varnothing$, for which the following notations will be used:

$$\delta_0(\mathrm{s}) = \pi_0(\mathrm{s}) + \frac{1}{\gamma}\int_{\mathrm{s}_0}^{\mathrm{s}} q(\mathrm{v})\pi_0(\mathrm{v})\pi_2^{\gamma}(\sigma(\mathrm{v}))\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(\mathrm{v}),m)\mathrm{dv},$$

$$\delta_i(\mathrm{s}) = \int_{\mathrm{s}_0}^{\mathrm{s}} \delta_{i-1}(\mathrm{v})\mathrm{dv} \text{ for } i = 1,2,$$

$$\vartheta_0(\mathrm{s}) = \exp\left(\int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\delta_0(\mathrm{v})r^{1/\gamma}(\mathrm{v})}\mathrm{dv}\right)$$

and

$$\vartheta_i(\mathrm{s}) = \int_{\mathrm{s}_0}^{\mathrm{s}} \vartheta_{i-1}(\mathrm{v})\mathrm{dv} \text{ for } i = 1,2.$$

Lemma 7. *Assume that $x \in \mathcal{B}_1$. Then, eventually,*

$$\left(\frac{z''(\mathrm{s})}{\vartheta_0(\mathrm{s})}\right)' \leq 0, \tag{23}$$

$$\left(\frac{z(\mathrm{s})}{\vartheta_2(\mathrm{s})}\right)' \leq 0 \tag{24}$$

and

$$(r(\mathrm{s})(z'''(\mathrm{s}))^{\gamma})' + \sum_{j=1}^{\ell} q_j(\mathrm{s})\widehat{p}^{\gamma}(\sigma_j(\mathrm{s}),m)z^{\gamma}(\sigma_j(\mathrm{s})) \leq 0. \tag{25}$$

Proof. Assume that $x \in \mathcal{B}_1$. From (19), we obtain

$$\begin{aligned}
\left[z''(\mathrm{s}) - \pi_0(\mathrm{s})\omega(\mathrm{s})\right]' &= -\pi_0(\mathrm{s})\omega'(\mathrm{s}) \\
&= -\pi_0(\mathrm{s})\left((\omega^{\gamma}(\mathrm{s}))^{1/\gamma}\right)' \\
&= -\frac{1}{\gamma}\pi_0(\mathrm{s})\omega^{1-\gamma}(\mathrm{s})(\omega^{\gamma}(\mathrm{s}))' \\
&\geq \frac{1}{\gamma}q(\mathrm{s})\pi_0(\mathrm{s})\omega^{1-\gamma}(\mathrm{s})z^{\gamma}(\sigma(\mathrm{s}))\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(\mathrm{s}),m),
\end{aligned}$$

where $\omega(\mathrm{s}) = r^{1/\gamma}(\mathrm{s})z'''(\mathrm{s})$. By integrating the aforementioned inequality from s_0 to s, we obtain

$$z''(\mathrm{s}) \geq \pi_0(\mathrm{s})\omega(\mathrm{s}) + \frac{1}{\gamma}\int_{\mathrm{s}_0}^{\mathrm{s}} \pi_0(\mathrm{v})\omega^{1-\gamma}(\mathrm{v})q(\mathrm{v})z^{\gamma}(\sigma(\mathrm{v}))\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(\mathrm{v}),m)\mathrm{dv}. \tag{26}$$

From (18), we obtain

$$z(\sigma(\mathrm{s})) \geq r^{1/\gamma}(\sigma(\mathrm{s}))z'''(\sigma(\mathrm{s}))\pi_2(\sigma(\mathrm{s})) \geq \omega(\sigma(\mathrm{s}))\pi_2(\sigma(\mathrm{s})). \tag{27}$$

Combining (26) and (27), we obtain

$$\begin{aligned} z''(\mathrm{s}) &\geq \pi_0(\mathrm{s})\omega(\mathrm{s}) + \frac{1}{\gamma}\omega(\mathrm{s})\int_{\mathrm{s}_0}^{\mathrm{s}} \pi_0(\mathrm{v})\pi_2^{\gamma}(\sigma(\mathrm{v}))q(\mathrm{v})\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(\mathrm{v}),m)\mathrm{dv} \\ &\geq \omega(\mathrm{s})\left[\pi_0(\mathrm{s}) + \frac{1}{\gamma}\int_{\mathrm{s}_0}^{\mathrm{s}} \pi_0(\mathrm{v})\pi_2^{\gamma}(\sigma(\mathrm{v}))q(\mathrm{v})\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(\mathrm{v}),m)\mathrm{dv}\right] \\ &= \delta_0(\mathrm{s})\omega(\mathrm{s}). \end{aligned}$$

Multiplying this inequality by

$$\exp\left(\int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\delta_0(\mathrm{v})r^{1/\gamma}(\mathrm{v})}\mathrm{dv}\right),$$

we obtain

$$\left(\frac{z''(\mathrm{s})}{\vartheta_0(\mathrm{s})}\right)' \leq 0.$$

Using this fact, we obtain

$$z'(\mathrm{s}) \geq \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{z''(\mathrm{v})}{\vartheta_0(\mathrm{v})}\vartheta_0(\mathrm{v})\mathrm{dv} \geq \frac{z''(\mathrm{s})}{\vartheta_0(\mathrm{s})}\vartheta_1(\mathrm{s}).$$

This implies

$$\begin{aligned} \left(\frac{z'(\mathrm{s})}{\vartheta_1(\mathrm{s})}\right)' &= \frac{\vartheta_1(\mathrm{s})z''(\mathrm{s}) - \vartheta_0(\mathrm{s})z'(\mathrm{s})}{\vartheta_1^2(\mathrm{s})} \\ &= \frac{1}{\vartheta_1^2(\mathrm{s})}\left[\widehat{\vartheta}_1(\mathrm{s})z''(\mathrm{s}) - \vartheta_0(\mathrm{s})z'(\mathrm{s})\right] \\ &\leq 0. \end{aligned} \tag{28}$$

Hence,

$$z(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} \vartheta_1(\mathrm{v})\frac{z'(\mathrm{v})}{\vartheta_1(\mathrm{v})}\mathrm{dv} \geq \frac{z'(\mathrm{s})}{\vartheta_1(\mathrm{s})}\vartheta_2(\mathrm{s})$$

yields

$$\begin{aligned} \left(\frac{z(\mathrm{s})}{\vartheta_2(\mathrm{s})}\right)' &= \frac{\vartheta_2(\mathrm{s})z'(\mathrm{s}) - \vartheta_1(\mathrm{s})z'(\mathrm{s})}{\vartheta_2^2(\mathrm{s})} \\ &= \frac{1}{\vartheta_2^2(\mathrm{s})}\left[\vartheta_2(\mathrm{s})z'(\mathrm{s}) - \vartheta_1(\mathrm{s})z'(\mathrm{s})\right] \leq 0. \end{aligned}$$

Now, the connection (13) becomes

$$x(\mathrm{s}) > \widehat{p}(\sigma(\mathrm{s}),m)z(\mathrm{s}).$$

The proof is therefore complete. □

By utilizing (23) and (24) instead of (14) and (15), we may quickly obtain the following theorem:

Theorem 3. *Suppose that there is a* $\lambda \in \mathbf{C}^1([\mathrm{s}_0,\infty),(0,\infty))$ *such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left\{\lambda(\mathrm{v})q(\mathrm{v})\frac{\vartheta_2^{\gamma}(\sigma(\mathrm{v}))}{\vartheta_2^{\gamma}(\mathrm{v})}\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(\mathrm{v}),m) - \frac{(\lambda'(\mathrm{v}))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\lambda(\mathrm{v}))^{\gamma}\pi_1^{\gamma}(\mathrm{v})}\right\}\mathrm{dv} = \infty, \tag{29}$$

for any $\ell, m \geq 0$*, then* $\mathcal{B}_1 = \varnothing$.

Now, using a comparison principle, we obtain the following theorem:

Theorem 4. *Suppose that*

$$\liminf_{s\to\infty}\int_{\sigma(s)}^{s} q(v)\delta_2^{\gamma}(\sigma(s))\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(s),m)\mathrm{d}v > \frac{1}{e}, \tag{30}$$

for any $m \geq 0$*, then* $\mathcal{B}_1 = \varnothing$*.*

Proof. Assume that $x \in \mathcal{B}_1$. From Lemma 7, we arrive at

$$z''(s) \geq \delta_0(s) r^{1/\gamma}(s) z'''(s), \tag{31}$$

where $\omega(s) = r^{1/\gamma}(s)z'''(s)$, just as we had in the proof of Lemma 7. By integrating (31) twice from s_0 to s, we obtain

$$z(s) \geq \delta_2(s) r^{1/\gamma}(s) z'''(s). \tag{32}$$

By changing (32) into (25), we arrive to the conclusion that

$$(r(s)(z'''(s))^{\gamma})' \leq -q(s)\delta_2^{\gamma}(\sigma(s))\left(r^{1/\gamma}(\sigma(s))z'''(\sigma(s))\right)^{\gamma}\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(s),m).$$

By setting $\Omega = r(s)(z'''(s))^{\gamma}$, we can show that Ω is a positive solution to the inequality

$$\Omega' + q(s)\delta_2^{\gamma}(\sigma(s))\Omega(\sigma_j(s))\sum_{j=1}^{\ell}\widehat{p}^{\gamma}(\sigma_j(s),m) \leq 0. \tag{33}$$

However, condition (30) confirms the oscillation of all solutions to (33), which is in disagreement with [35] (Theorem 2.1.1). The proof is therefore complete. □

Application in Oscillation Theory and Discussion

Finding conditions that individually rule out each case of the derivatives of the solution is necessary to determine the oscillation criterion. In this theorem, the criterion for testing oscillation for (1) will be formed by combining the conditions that are obtained to rule out the existence of solutions that satisfy Case (1) with the conditions that are acquired in the literature to rule out Case (2) of the derivatives of the solution.

Theorem 5. *Assume that one of the conditions (21), (29) or (30) is satisfied. If (10) holds, then (1) is oscillatory.*

Proof. Assume that $x(s)$ is to be an eventually positive solution to (1). Lemma 3 has a solution that satisfies either of the possibilities of Case (1) or Case (2). $\mathcal{B}_1 = \varnothing$ is obtained by applying Theorems 2–4. Case (2) is thus valid. We finally come to a contradiction with (10) in the exact same way as [29] (Theorem 2). This completes the proof. □

Example 1. *Take into consideration the neutral delay DE*

$$(x(s) + p_0 x(\theta s))'''' + \frac{q_0}{s^4}x(\rho s) = 0, \tag{34}$$

where $s > 0$*,* $p_0 \geq 0$*,* $q_0 > 0$ *and* $\theta, \rho \in (0,1)$*. We immediately obtain*

$$\pi_0(s) = s,\ \pi_1(s) = s^2/2 \text{ and } \pi_2(s) = s^3/6.$$

We define

$$\widehat{p} := (1 - p_0) \sum_{i=0}^{\kappa} p_0^{2i} \theta^{6i} \textit{ for } p_0 < 1$$

and

$$\varpi = 1 + \frac{1}{6} q_0 \rho^3 \phi.$$

It is easy to confirm that $\widehat{p}(\sigma(\mathrm{s}), m) = \phi$. *Also,*

$$\begin{aligned}
\delta_0(\mathrm{s}) &= \pi_0(\mathrm{s}) + \int_{\mathrm{s}_0}^{\mathrm{s}} q(\mathrm{v}) \widehat{p}(\sigma(\mathrm{v}), m) \pi_0(\mathrm{v}) \pi_2^{\gamma}(\sigma(\mathrm{v})) \mathrm{dv} \\
&= \left(1 + \frac{1}{6} q_0 \phi \rho^3\right) \mathrm{s} \\
&= \varpi \mathrm{s},
\end{aligned}$$

$$\delta_1(\mathrm{s}) = \int_{\mathrm{s}_0}^{\mathrm{s}} \varpi \mathrm{v} \mathrm{dv} = \frac{1}{2} \varpi \mathrm{s}^2,$$

$$\delta_2(\mathrm{s}) = \frac{1}{2} \varpi \int_{\mathrm{s}_0}^{\mathrm{s}} \mathrm{v}^2 \mathrm{dv} = \frac{1}{6} \varpi \mathrm{s}^3,$$

$$\begin{aligned}
\vartheta_0(\mathrm{s}) &= \exp\left(-\int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\left(1 + \frac{1}{6} q_0 \phi(\mathrm{v}) \rho^3\right) \mathrm{v}} \mathrm{dv}\right) \\
&= \mathrm{s}^{1/\varpi},
\end{aligned}$$

$$\begin{aligned}
\vartheta_1(\mathrm{s}) &= \int_{\mathrm{s}_0}^{\mathrm{s}} \mathrm{v}^{1/\varpi} \mathrm{dv} \\
&= \frac{\mathrm{v}^{(1/\varpi)+1}}{(1/\varpi) + 1} \\
&= \frac{1}{(1/\varpi) + 1} \mathrm{s}^{(1/\varpi)+1}
\end{aligned}$$

and

$$\begin{aligned}
\vartheta_2(\mathrm{s}) &= \frac{1}{(1/\varpi) + 1} \int_{\mathrm{s}_0}^{\mathrm{s}} \mathrm{v}^{(1/\varpi)+1} \mathrm{dv} \\
&= \frac{1}{((1/\varpi) + 1)((1/\varpi) + 2)} \mathrm{s}^{(1/\varpi)+2}.
\end{aligned}$$

Using Theorem 2 and choosing $\lambda(\mathrm{s}) = \mathrm{s}^4$, *we have*

$$q_0 > \frac{8}{\phi \ell \rho^3}. \tag{35}$$

Then, $\mathcal{B}_1 = \varnothing$ *if Equation (35) is satisfied. Once again, applying Theorem 3, we find that*

$$q_0 > \frac{8}{\phi \ell (\rho)^{(1/\varpi)+2}}. \tag{36}$$

Then, $\mathcal{B}_1 = \varnothing$ *if Equation (36) is satisfied. In addition to Theorem 4, we find that* $\mathcal{B}_1 = \varnothing$ *if*

$$q_0 > \frac{6}{\phi \ell \varpi \rho^3 \ln\left(\frac{1}{\rho}\right) \mathrm{e}}. \tag{37}$$

Finally, by applying condition (10) of Theorem 1, we find that

$$q_0 > \frac{6}{\rho^4\left(\frac{\theta}{\theta+p_0}\right)}. \tag{38}$$

Thus, if the conditions (35) and (38) are satisfied, then (34) is oscillatory.

Remark 2. *For $p_0 = 0.5$, $\theta = 0.5$ and $\rho = 0.5$, conditions (35) and (36) are satisfied if $q_0 > 127.5$ and $q_0 > 90.9$, respectively. Thus, we notice that the criterion of Theorem 2 improves the criterion of Theorem 3.*

Remark 3. *Consider the particular instance of the above example of the form*

$$(x(s) + 0.9x(0.9s))'''' + \frac{q_0}{s^4}x(0.9s) = 0. \tag{39}$$

Note that conditions (35) and (38) reduce to $q_0 > 62.5$ and $q_0 > 18.28$, respectively. Then, Equation (39) is oscillatory if $q_0 > 62.5$.

On the other hand, using Corollary 2.1 in [36], Equation (39) is oscillatory if $q_0 > 109.74$. Therefore, our results provide a better criterion for oscillation. With regard to previous relevant results, our results are an improvement and a complement to them.

3. Conclusions

The classification of positive solutions according to the sign of their derivatives always comes first when examining oscillations for neutral delay DEs. The constraints that disallow each case of derivatives of the solution determine the oscillation criterion. In the oscillation theory of neutral DEs, the relationships between the solution and the corresponding function are crucial. By using the modified monotonic features of positive solutions, we strengthen these relationships. We then developed criteria to demonstrate that Category $\mathcal{B}_1$ has no solutions based on these relationships. Then, to create a set of oscillation criteria, we brought together results from previous studies that had been published in the literature with new relationships and features. Finally, we provided an example and a comparison with previous work to emphasize the importance of the results. This comparison showed how our findings enhance and add to those in [36]. Recent scientific work has focused heavily on the characteristics of the solution to fractional DEs. Applying our results to fractional DEs might thus be interesting.

Author Contributions: Conceptualization, A.N., O.M., S.S.A., A.M.A. and E.M.E.; Methodology, A.N., O.M., S.S.A., A.M.A. and E.M.E.; Investigation, A.N., O.M., S.S.A., A.M.A. and E.M.E.; Writing—original draft, A.N., O.M. and S.S.A.; Writing—review & editing, A.N., A.M.A. and E.M.E. All authors have read and agreed to the published version of the manuscript.

Funding: This project is funded by King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Not available.

Acknowledgments: The authors present their appreciation to King Saud University for funding the publication of this research through the Researchers Supporting Program (RSPD2023R533), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Braun, M.; Golubitsky, M. *Differential Equations and Their Applications*; Springer: New York, NY, USA, 1983.
2. Altawallbeh, Z.; Az-Zo'bi, E.; Alleddawi, A.O.; Şenol, M.; Akinyemi, L. Novel liquid crystals model and its nematicons. *Opt. Quantum Electron.* **2022**, *54*, 861. [CrossRef]
3. Gao, L.; Guo, C.; Guo, Y.; Li, D. Exact Solutions and Non-Traveling Wave Solutions of the (2 + 1)-Dimensional Boussinesq Equation. *Mathematics* **2022**, *10*, 2522. [CrossRef]
4. Nemytskii, V.V. *Qualitative Theory of Differential Equations*; Princeton University: Princeton, NJ, USA, 2015; Volume 2083.

5. Oguztoreli, M.N.; Stein, R.B.; An analysis of oscillations in neuro-muscular systems. *J. Math. Biol.* **1975**, *2*, 87–105. [CrossRef]
6. Gyori, I.; Ladas, G.E. *Oscillation Theory of Delay Differential Equations: With Applications*; Clarendon Press: Oxford, UK, 1991.
7. Ladas, G.; Lakshmikantham, V.; Papadakis, J.S. Oscillations of higher-order retarded differential equations generated by the retarded argument. In *Delay and Functional Differential Equations and Their Applications*; Academic Press: Cambridge, MA, USA, 1972; pp. 219–231.
8. Jadlovska, I. New criteria for sharp oscillation of second-order neutral delay differential equations. *Mathematics* **2021**, *9*, 2089. [CrossRef]
9. Nabih, A.; Cesarano, C.; Moaaz, O.; Anis, M.; Elabbasy, E.M. Non-Canonical Functional Differential Equation of Fourth-Order: New Monotonic Properties and Their Applications in Oscillation Theory. *Axioms* **2022**, *11*, 636. [CrossRef]
10. Dzurina, J.; Grace, S.R.; Jadlovska, I.; Li, T. Oscillation criteria for second-order Emden–Fowler delay differential equations with a sublinear neutral term. *Math. Nachrichten* **2020**, *293*, 910–922. [CrossRef]
11. Grace, S.R.; Džurina, J.; Jadlovska, I.; Li, T. On the oscillation of fourth-order delay differential equations. *Adv. Differ. Equ.* **2019**, *1*, 118. [CrossRef]
12. Santra, S.S.; Ghosh, A.; Dassios, I. Second-order impulsive differential systems with mixed delays: Oscillation theorems. *Math. Methods Appl. Sci.* **2022**, *45*, 12184–12195. [CrossRef]
13. Santra, S.S.; Khedher, K.M.; Yao, S.W. New aspects for oscillation of differential systems with mixed delays and impulses. *Symmetry* **2021**, *13*, 780. [CrossRef]
14. Moaaz, O.; Nabih, A.; Alotaibi, H.; Hamed, Y.S. Second-Order Non-Canonical Neutral Differential Equations with Mixed Type: Oscillatory Behavior. *Symmetry* **2021**, *13*, 318. [CrossRef]
15. Li, T.; Baculkova, B.; Džurina, J. Oscillation results for second-order neutral differential equations of mixed type. *Tatra Mt. Math. Publ.* **2011**, *48*, 101–116. [CrossRef]
16. Tunc, E.; Özdemir, O. Comparison theorems on the oscillation of even order nonlinear mixed neutral differential equations. *Math. Methods Appl. Sci.* **2023**, *46*, 631–640. [CrossRef]
17. Santra, S.S. Necessary and Sufficient Conditions for Oscillation of Solutions to Second-Order Neutral Differential Equations with Impulses. *Tatra Mt. Math. Publ.* **2020**, *76*, 157–170. [CrossRef]
18. Elabbasy, E.M.; Nabih, A.; Nofal, T.A.; Alharbi, W.R.; Moaaz, O. Neutral differential equations with noncanonical operator: Oscillation behavior of solutions. *Aims Math.* **2021**, *6*, 3272–3287. [CrossRef]
19. Hasanbulli, M.; Rogovchenko, Y.V. Oscillation criteria for second order nonlinear neutral differential equations. *Appl. Math. Comput.* **2010**, *215*, 4392–4399. [CrossRef]
20. Xing, G.; Li, T.; Zhang, C. Oscillation of higher-order quasi-linear neutral differential equations. *Adv. Differ. Equ.* **2011**, *2011*, 45. [CrossRef]
21. Agarwal, R.P.; Zhang, C.; Li, T. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.* **2016**, *274*, 178–181. [CrossRef]
22. Bohner, M.; Grace, S.; Jadlovská, I. Oscillation criteria for second-order neutral delay differential equations. *Electron. J. Qual. Theory* **2017**, *60*, 1–12. [CrossRef]
23. Moaaz, O.; Muhib, A.; Owyed, S.; Mahmoud, E.E.; Abdelnaser, A. Second-order neutral differential equations: Improved criteria for testing the oscillation. *Jpn. J. Math.* **2021**, *2021*, 6665103. [CrossRef]
24. Hassan, T.S.; Moaaz, O.; Nabih, A.; Mesmouli, M.B.; El-Sayed, A. New sufficient conditions for oscillation of second-order neutral delay differential equations. *Axioms* **2021**, *10*, 281. [CrossRef]
25. Bohner, M.; Grace, S.R.; Jadlovská, I. Sharp results for oscillation of second-order neutral delay differential equations. *Electron. J. Qual. Theory* **2023**, *4*, 1–23. [CrossRef]
26. Zhang, Q.; Yan, J.; Gao, L. Oscillation behavior of even-order nonlinear neutral differential equations with variable coefficients. *Comput. Math. Appl.* **2010**, *59*, 426–430. [CrossRef]
27. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. A new approach in the study of oscillatory behavior of even-order neutral delay differential equations. *Appl. Math. Comput.* **2013**, *225*, 787–794. [CrossRef]
28. Moaaz, O.; Mahmoud, E.E.; Alharbi, W.R. Third-order neutral delay differential equations: New iterative criteria for oscillation. *J. Funct. Space* **2020**, *2020*, 6666061. [CrossRef]
29. Muhib, A.; Moaaz, O.; Cesarano, C.; Askar, S.S. New conditions for testing the oscillation of fourth-order differential equations with several delays. *Symmetry* **2022**, *14*, 1068. [CrossRef]
30. Nabih, A.; Moaaz, O.; AlNemer, G.; Elabbasy, E.M. New Conditions for Testing the Asymptotic and Oscillatory Behavior of Solutions of Neutral Differential Equations of the Fourth Order. *Axioms* **2023**, *12*, 219. [CrossRef]
31. Philos, C.G. A new criterion for the oscillatory and asymptotic behavior of delay differential equations. *Bull. Acad. Polon. Sci. Ser. Sci. Math.* **1981**, *39*, 367–370.
32. Zhang, C.; Li, T.; Sun, B.; Thandapani, E. On the oscillation of higher-order half-linear delay differential equations. *Appl. Math. Lett.* **2011**, *24*, 1618–1621. [CrossRef]
33. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Kluwer Academic: Dordrecht, The Netherlands, 2000.
34. Moaaz, O.; Cesarano, C.; Almarri, B. An improved relationship between the solution and its corresponding function in neutral fourth-order differential equations and its applications. *Mathematics* **2023**, *11*, 1708. [CrossRef]

35. Ladde, G.S.; Lakshmikantham, V.; Zhang, B.G. *Oscillation Theory of Differential Equations with Deviating Arguments*; Marcel Dekker: New York, NY, USA, 1987.
36. Bazighifan, O.; Ruggieri, M.; Scapellato, A. An improved criterion for the oscillation of fourth-order differential equations. *Mathematics* **2020**, *8*, 610. [CrossRef]