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Article

On the Concept of Equilibrium in Sanctions and Countersanctions in a Differential Game

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Abstract: This paper develops the methodology for modeling decision processes in complex controlled dynamic systems. The idea of balancing such systems (driving them to equilibrium) is implemented, and a new mechanism for the equilibria's stability is proposed. Such an approach involves economic–mathematical modeling jointly with systems analysis methods, economics, law, sociology, game theory, management, and performance measurement. A linear-quadratic positional differential game of several players is considered. Coefficient criteria under which the game has an equilibrium in sanctions and countersanctions and, simultaneously, no Nash equilibrium are derived. The economic and legal model of active equilibrium is studied through the legal concept of sanctions, which enlarges the practical application of this class of problems.

Keywords: sanctions; countersanctions; balance of sanctions and countersanctions; active equilibriums; stability; efficiency; Pareto maximum

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1. Introduction

A sanction (from Latin "sanction"—strictest decree) is a measure of influence applied to an economic agent (an individual or a legal entity, industry, or the state) and entailing certain consequences. Sanctions may be of an economic, legal, or social nature. In terms of form, a sanction may be expressed as a prohibition, restriction of operations, fines, and more. Depending on the nature of violated rights, it is customary to distinguish such legal sanctions as criminal, administrative, property, and international legal sanctions. At the same time, according to the consequences' nature, legal norms' sanctions can be both negative and positive. The first one implies the application of penalties, and the second one—actions of encouragement. Game Theory is actively used as a tool for modeling socio-economic processes in general and sanctions processes in particular, so, for example, in the research [1] based on a game-theoretic approach, the concept of a model of criminal sanctions and a model of criminal punishment system, corresponding to fundamental requirements of game theory, taking into account Nash's equilibrium is offered. The author concludes that "the game-theoretic approach to the formation of both individual criminal law sanctions and the system of criminal law sanctions is quite applicable and complements the formal legal logic of construction of criminal law sanctions and their system, allows to avoid defects in the construction of criminal law sanctions and helps to eliminate legal uncertainty in sentencing".

It is well known that social sanctions are commonly understood as measures of encouragement and punishment that stimulate an individual to comply with social norms. Sanctions can be formal (medals, diplomas, scholarships, fines, and others) and informal (praise, ridicule, boycott, and others), positive and negative. The research on social sanctions is beyond the scope of this work. Further, we will consider economic sanctions in detail, as much as possible, within the framework of this section.

The first economic sanctions recorded in written sources were imposed by the Athenian Maritime Union (Delos Symmachus) on the city of Megara (part of the Peloponnesian Union) in 432 BC in order to stop the practice of receiving runaway Athenian enslaved people in that city and plowing the sacred border territories. They are known as "Megarian psephism". The effectiveness of the sanctions was not obvious. On the one hand, the Megarian merchants suffered significant losses, but on the other hand, they were forced to turn to their allies (primarily Sparta) for military support. As a result of the Peloponnesian War, Athens suffered a crushing defeat, and the Athenian alliance was destroyed. Studying the above fact and using it as an example, G. Tsebelis, in his article "Are Sanctions Effective? A Game-Theoretic Analysis" [2], reasoned that "Although economic sanctions have been quite frequent in the twentieth century, a close examination of the low success rate (33 out of 83 cases) indicates that sender countries are unable to select the appropriate cases. Moreover, analysts sometimes offer contradictory advice for such selection". This research provides a game-theoretic explanation of these phenomena. Six different game-theoretic scenarios lead to the same equilibrium outcome. It is a mixed strategy equilibrium. The success ratio is the outcome of the selection of mixed strategies by sender and receiver countries. Under a wide range of (specified) circumstances, the size of the sanction has no impact on the behavior of the target country. Finally, some empirical implications of the game-theoretic analysis are compared to existing empirical generalizations, and further implications for empirical research are discussed.

In the Middle Ages in Europe, economic sanctions were primarily local and short-lived because of the constantly changing configuration of trade and military alliances and the changing interests of individual rulers and influential individuals. In the 19th century, the primary tool of economic sanctions was naval blockades—measures to prevent a country's maritime trade with other countries without a declaration of war. Between 1827 (the first known naval blockade) and 1914, 21 blockades were recorded against Turkey, Portugal, the Netherlands, Colombia, Panama, Mexico, Argentina, and El Salvador. The organizers of the blockades were mainly Great Britain (12 times) and France (11 times), but also Italy and Germany (three times each), Russia and Austria (twice each), and Chile [3].

Economic sanctions became widespread in the twentieth century with the development of international trade relations. Before World War II, Yugoslavia, Greece, Bolivia, Paraguay, and Italy were subject to collective economic sanctions. During the Cold War, sanctions were largely ineffective because they were not supported by either the Western or Eastern blocs of countries (with the United States and the Soviet Union as leaders, respectively). Unified sanctions, supported by both blocs, were imposed only twice: on Rhodesia and South Africa, and according to international relations experts, in both cases, were not effective enough.

One of the best-known examples of collective economic sanctions at the time was the restriction of deliveries of "strategic" goods and technologies, primarily military and computer technology, to socialist countries. The Coordination Committee (CoCom) was created in 1949 specifically to control exports to the Eastern Bloc countries, which included 17 states, while another six countries cooperated with the committee without formally being part of it. The committee ceased its activities in 1994. The most famous example of long-term unilateral sanctions is the U.S. embargo against Cuba, which began in 1960–1962 and continues today. U.S. companies are prohibited from any economic contact with Cuba without special permission, including in third countries. According to Cuban authorities, direct damage from the embargo amounted to about USD 1 trillion in current prices. Nevertheless, the goal of U.S. economic sanctions—establishing democracy in Cuba—has not been achieved. Since 1990, the UN has made greater use of international economic sanctions against various states. They have been subjected to Iraq (since 1990), Yugoslavia (1991–2001), Somalia (since 1992), Libya (1992–2003), Liberia (since 1992), Angola (1993–2002), Haiti (1993–1994), Rwanda (1994–2008), Sierra Leone (since 1997), Afghanistan (since

1999), Eritrea and Ethiopia (since 2000), DR Congo (since 2003), Cŏte d'Ivoire (since 2004), Sudan (since 2004), Lebanon (since 2005), Iran (since 2006), and DPRK (since 2006). The sanctions are mostly partial and restrict weapons and military equipment supplies to these countries. In some cases, foreign assets are frozen [4,5].

Global economic sanctions have been repeatedly imposed on several states. Sanctions, whether international or unilateral, currently apply to 24 states worldwide. However, experience has shown that states subject to sanctions have almost always found ways to minimize their damage or use them to their advantage [6].

Many publications are devoted to economic sanctions and the application of the game-theoretic apparatus in their study. Let us note the research of Marc V. Simon "When sanctions can work: Economic sanctions and the theory of moves" [7]; Shidiqi, Khalifany Ash and Pradiptyo, Rimawan "A Game Theoretical Analysis of Economic Sanctions" [8]; Karimi, Mohammad and Maleki, Abbas and Haieri Yazdi, Asieh "How the Possibility of a Fight-Back Strategy Affects the Consequences of a Sanction's Regime" [9]; Onder, Mehmet "The Impact of Decision-Makers on Economic Sanctions: A Game Theoretical Perspective" [10].

In contrast to other research, this article offers an economical and legal substantiation of theoretical and game constructions modeling the process of application of sanctions and countersanctions, using in general terms the idea of systemic balance of three macrosystems: economic, legal, and social [11]. In a particular case, the economic-legal substantiation of the proposed theoretical-game model of the balance of sanctions and countersanctions is based on the use of the legal concept of sanctions as a component of the definition of legal responsibility of subjects, i.e., in practical terms, the regulator implements the principle of inevitability of legal responsibility, in particular, sanctions for offense and crime (can be a departure from the established in the legal prescriptions rules, rules of social and economic relations), which is manifested in the application in the description of real socio-economic processes, various concepts of static and active equilibriums.

The current national market system is based on the neoliberal economic doctrine. In differentiated forms, it covers all public relations spheres and appears in decision processes at all complex control and controlled systems levels. Various concepts of static [12] and active equilibria [13] are adopted to balance controlled systems [14,15] in game-theoretic economic–mathematical modeling. If an analytically constructed differential game describes decision-making in a complex system, then, according to leading researchers, equilibrium as an acceptable solution of a differential game should have the property of stability [16–18]. In a practical interpretation, stability means no player will increase their payoff by any unilateral deviation from equilibrium.

Within the neoliberal economic doctrine, the well-known solution proposed by J. Nash [19–22] meets this requirement in many situations. (In 1994, J. Nash, J. Harsanyi and R. Selten were awarded the Nobel Prize in Economic Sciences "for the pioneering analysis of equilibria in the theory of non-cooperative games"). Note that this equilibrium does not always exist under certain conditions and (or) has several negative properties. For example, Nash equilibria can be internally and externally unstable. The mathematical problem of ensuring the stability of equilibria can be solved using active equilibria, e.g., the classical equilibrium in threats and counterthreats or the equilibrium in objections and counter-objections [23,24], simultaneously with requiring efficiency (Pareto maximality) [25–27].

The economic and legal justification of game-theoretic models generally uses the idea of system balancedness: complex controlled systems correlate through interaction, and balancedness means that the legal order of public relations corresponds to the laws and trends of economic development. In a particular case, the economic and legal justification of the equilibrium in sanctions and countersanctions is based on the legal concept of sanctions defining the legal responsibility of subjects (agents). In practical terms, the adjuster implements the principle of inevitable legal liability (particularly sanctions) for an offense and crime (any deviation from the behavioral rules established by legal regulations, e.g., any deviation from the equilibrium mentioned above).

2. Methods

The theoretical and methodological basis of the research is the theory of systems analysis, economics, theory of state and law, sociology, synthesis of the provisions and principles of economic and mathematical modeling, game theories, and management and developed on their basis the theoretical and game models of decision-making in complex systems with uncertainty.

Let us proceed to the conceptual framework used below. At the micro level, the concept of threat contains both a real action by a market participant and its possibility to compel the latter to comply with the previously established rules; at the macro level, the matter concerns "the threat of coercion", i.e., a regulatory definition of the legal inevitability of punishment. In the legal literature, the threat of coercion means a sanction and refers to measures of impact applied in regulation [28,29]. In mathematical game theory, the terms "threat" and "sanction" can be used as a synonym for the word "objection" to mitigate their "aggressive" nature. The economic content of sanctions in legal sources includes measures of compulsory economic impact for violating the established procedure for activities and often have preventive, compensatory, and repressive functions. Accordingly, in the macro- and micromodels of decision-making, the player resisting coercion and using counter methods of impact is determined through counter objection, counter-threat, and countersanction.

Like other equilibria, the equilibrium in sanctions and countersanctions can yield stable solutions, but each concept uses different mechanisms. In the case of Nash equilibrium, all players, except for the deviating one, continue using the same strategies as before. In the case of equilibrium in sanctions and countersanctions, the players pass to the legally admissible actions that force the deviating player to follow the equilibrium: the players implement countersanctions. Hence, the inevitability of punishment is a good reason for the players to stay within such an equilibrium. From the legislative viewpoint, their actions rest on the concept of legal responsibility for changing the content of a current obligation and future negative consequences due to its violation [29].

As mentioned above, the classical concept of threats and counterthreats is not widespread in mathematical game theory [24] and is restricted to static or differential two-player games [30–37]. The approach presented below is novel for the theory of differential games:

1. We augment the equilibrium in sanctions and countersanctions with the property of Pareto maximality.
2. We identify a relatively large class of differential games of players in which there is an equilibrium in sanctions and countersanctions, and, at the same time, no Nash equilibrium exists.
3. We propose an algorithm for constructing the equilibrium in sanctions and countersanctions.

3. Results

To construct a game-theoretic model of equilibrium in sanctions and countersanctions, we consider a noncooperative linear-quadratic differential N-player game in normal form described by an ordered quadruple

$$\Gamma = \langle \mathbb{N}, \Sigma, \{\mathcal{U}_i\}_{i\in\mathbb{N}}, \{J_i(U, t_0, x_0\}_{i\in\mathbb{N}} \rangle.$$

In the game Γ, the set of players (e.g., market participants) is $\mathbb{N} = \{1, \dots N\}$, where $N \geq 2$. The dynamics of the controlled system Σ (the interacting subjects of market activity, also called agents) obey the vector linear differential equation

$$\Sigma \div \dot{x} = A(t)x + u_1 + \dots + u_N, x(t_0) = x_0, \quad (1)$$

with the following notations: $x \in \mathbb{R}^n$ is the n-dimensional state vector of the system Σ $t \in [t_0, \vartheta]$ is a finite time interval of the game with a fixed terminal time instant $\vartheta = const$; $u_i \in \mathbb{R}^n$ is the control action of player i ($i \in \mathbb{N}$); $(t, x) \in [0, \vartheta] \times \mathbb{R}^n$ is a pair determining a current position in the game Γ; finally, (t_0, x_0) as an initial position, where $0 \leq t_0 < \vartheta$. The matrix $A(t)$ of dimensions $n \times n$ is assumed to have continuous elements on $[0, \vartheta]$, which is denoted by $A(\cdot) \in C_{n\times n}[0, \vartheta]$.

The strategy U_i of player i will be identified with an n-dimensional vector function $u_i(t, x)$, and this fact will be denoted by $U_i \div u_i(t, x)$. Then, the strategy set of player i can be written as

$$\mathcal{U}_i = \{U_i \div u_i(t, x), u_i(t, x) = Q_i(t)x | \forall Q_i(\cdot) \in C_{n\times n}[0, \vartheta]\}.$$

Thus, player i chooses their (strategy by specifying a matrix $Q_i(t)$ $(i \in \mathbb{N})$ of dimensions $n \times n$ from the space $C_{n\times n}[0, \vartheta]$.

The game evolves over time under market competition in the following way. Without forming coalitions with other players, each player i chooses a particular strategy $U_i \div Q_i(t)x$, which yields *a strategy profile* $U = (U_1, \ldots, U_N) \in \mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_N$ of the game. Next, each player finds the solution $x(t)$, $t_0 \leq t \leq \vartheta$, of System (1) with $u_i = Q_i(t)x$ $(i \in \mathbb{N})$, i.e.,

$$\dot{x}(t) = [A(t) + Q_1(t) + \ldots + Q_N(t)]x(t), x(t_0) = x_0. \tag{2}$$

The system of linear homogeneous differential Equation (2) with continuous coefficients on $[t_0, \vartheta]$ has a continuous solution $x(t)$ that is extendable to $[t_0, \vartheta]$ $\forall t_0 \in [0, \vartheta)$. Then, each player constructs the realization of their strategy $u_i[t] = Q_i(t)x(t)$ $(i \in \mathbb{N})$ and the corresponding realization of the strategy profile $u[t] = (u_1[t], \ldots, u_N[t])$, which consists of the N continuous n-dimensional vectors $u_1[t], \ldots, u_N[t]$ on $[t_0, \vartheta]$. The payoff function of player i is a quadratic functional

$$J_i(U_1, \ldots U_N, t_0, x_0) = x'(\vartheta)C_i x(\vartheta) + \int_{t_0}^{\vartheta} \sum_{j\in\mathbb{N}} u'_j[t]D_{ij}u_j[t]dt (i \in \mathbb{N}) \tag{3}$$

defined on the continuous pairs $(x(t), u[t])$, $t \in [t_0, \vartheta]$. Without loss of generality, let the constants matrices C_i and D_{ij} of dimensions $n \times n$ be symmetric. The prime denotes transposition: x' is a row vector. The value of the functional (3) is called the payoff of player i. The neoliberal economic doctrine assumes that each player in the game Γ seeks to maximize their payoff only.

This research aims to find a rather general class of noncooperative linear-quadratic differential N-player games in normal form Γ that have no Nash equilibrium but simultaneously have an equilibrium in sanctions and countersanctions. To this effect, we will associate with the game Γ the N-criteria dynamic choice problem

$$\Gamma_v = \langle \Sigma, \mathcal{U}, \{J_i(U, t_0, x_0)\}_{i\in\mathbb{N}} \rangle.$$

Here, the controlled dynamic system Σ coincides with (1); the set of alternatives $\mathcal{U}$ coincides with the set of strategy profiles $\mathcal{U} = \prod_{i=1}^{N} \mathcal{U}_i$ of the game Γ; the N criteria $J_i(U, t_0, x_0)$ $(i \in \mathbb{N})$ are given by (3). The DM's goal in the problem Γ_v is to choose an alternative $U^P \in \mathcal{U}$ for which the N criteria (3) will take the maximum possible values. V. Pareto proposed a conventional approach to such problems in 1909; see [38,39].

Note two results, which are immediate from Definition 1.

Definition 1. *An alternative $U^P = (U_1^P, \ldots, U_N^P) \in \mathcal{U}$ is said to be Pareto-maximal in the problem Γ_v if $\forall U \in \mathcal{U}$ and $\forall (t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n$, $x_0 \neq 0_n$, the system of inequalities*

$$J_i(U, t_0, x_0) \geq J_i(U^P, t_0, x_0) (i \in \mathbb{N}),$$

with at least one strict inequality, is inconsistent. In this case, the vector $J^P = J^P[t_0, x_0] = (J_1(U^P, t_0, x_0), \ldots, J_N(U^P, t_0, x_0))$ is called a Pareto maximum in the problem Γ_v.

Note two results, which are immediate from Definition 1.

Property 1.

$$[J_i(\hat{U}, t_0, x_0) > J_i(U^P, t_0, x_0)] \Rightarrow [J_j(\hat{U}, t_0, x_0) < J_j(U^P, t_0, x_0)]$$

for at least one number $j \in \mathbb{N}, j \neq i$ *and* $\hat{U} \in \mathcal{U}$.

Property 2. *If the condition*

$$\max_{U \in \mathcal{U}} \{\sum_{i \in \mathbb{N}} \alpha_i J_i(U, t_0, x_0)\} = Idem\{U \to U^P\} \tag{4}$$

holds for constants $\alpha_i > 0$ $(i \in \mathbb{N})$, *then the alternative* U^P *is Pareto-maximal in the problem* Γ_v. *Here,* $Idem\{U \to U^P\}$ *indicates the bracketed expression from (4) with replaced* U *by* U^P.

Consider two concepts of equilibrium for the game Γ, where $J = (J_1, \dots, J_N) \in \mathbb{R}^N$.

Definition 2. *A pair* $(U^e, J^e = J(U^e, t_0, x_0)) \in \mathcal{U} \times \mathbb{R}^N$ *is called a Nash equilibrium of the game* Γ *if*

$$\begin{cases} \max\limits_{U_1 \in \mathcal{U}_1} J_1(U_1, U_2^e, \dots, U_N^e, t_0, x_0) = J_1(U_1^e, U_2^e, \dots, U_N^e, t_0, x_0) = J_1^e, \\ \max\limits_{U_2 \in \mathcal{U}_2} J_2(U_1^e, U_2, \dots, U_N^e, t_0, x_0) = J_2(U_1^e, U_2^e, \dots, U_N^e, t_0, x_0) = J_2^e, \\ \dots \\ \max\limits_{U_N \in \mathcal{U}_N} J_N(U_1^e, U_2^e, \dots, U_{N-1}^e, U_N, t_0, x_0) = \\ = J_N(U_1^e, \dots, U_{N-1}^e, U_N^e, t_0, x_0) = J_N^e \end{cases}$$

for any $(t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n, x_0 \neq 0_n$ *(*0_n *denotes a zero vector of dimension* n*).*

Now we construct the equilibrium in sanctions and countersanctions.

Let $U = (U_1, U_2, \dots, U_N)$ be some fixed strategy profile of the game Γ. Player 1 is said to impose a sanction to the strategy profile U if there exists their strategy $U_1^T \in \mathcal{U}_1$ such that

$$J_1(U_1^T, U_2, \dots, U_N, t_0, x_0) > J_1(U_1, U_2, \dots, U_N, t_0, x_0). \tag{5}$$

An existing sanction is not necessarily implemented: it means the threat of coercion. Recall that the role of a sanction is revealed through the legal responsibility of players: sanctions make them refrain from violating the established game rules and are implemented in case of "frustration". In terms of game theory, implementing the sanction is beneficial to Player 1: according to (5), their individual payoff will increase compared to the previous strategy profile U.

The complex of punitive measures taken by one party against the other in response to sanctions is manifested in countersanctions. Player 2 is said to impose *an incomplete countersanction* to a sanction U_1^T of Player 1 if there exists a strategy $U_2^C \in \mathcal{U}_2$ such that

$$J_1(U_1^T, U_2^C, \dots, U_N, t_0, x_0) \leq J_1(U_1, U_2, \dots, U_N, t_0, x_0). \tag{6}$$

Player 2 is said to impose *a complete countersanction* to U_1^T if there exists a strategy $U_2^C \in \mathcal{U}_2$ such that Inequality (6) is satisfied simultaneously with

$$J_2(U_1^T, U_2^C, \dots, U_N, t_0, x_0) > J_2(U_1^T, U_2, \dots, U_N, t_0, x_0). \tag{7}$$

Incomplete and complete countersanctions of other players to a sanction U_i^T are formalized by analogy.

In the presence of an incomplete countersanction, Player 2 can choose their strategy U_2^C for making the payoff of Player 1 (who imposes an original sanction) equal to a value *not exceeding* their original payoff in the strategy profile U; see (6). (Note that he (it) may even reduce the payoff of Player 1!) Therefore, the presence of an incomplete countersanction negates the implementation of a sanction. In addition, a complete countersanction motivates Player 2 to choose U_2^C because their payoff in the resulting strategy prole $(U_1^T, U_2^C, \ldots, U_N)$ yielded by implementing the sanction and countersanction will increase compared to the strategy prole $(U_1^T, U_2, \ldots, U_N)$ yielded by implementing the sanction U_1^T. A sanction U_i^T of player i to a strategy profile U and a (complete) countersanction of one of the other players are defined by analogy.

If at least one of the other players has a countersanction to each sanction imposed by any player to U, then it makes no sense for him to implement the sanction: due to the countersanction of another player, their payoff will not increase (but it may even decrease!).

Definition 3. *A strategy profile* $U^P = (U_1^P, U_2^P, \ldots, U_N^P) \in \mathcal{U}$ *is called an active equilibrium* [13] *of the game* Γ *if, for any initial position* $(t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n, x_0 \neq 0_n$*:*

1. *The alternative* U^P *is Pareto-maximal in the N-criteria dynamic choice problem* Γ_v*.*
2. *At least one of the other players has an incomplete countersanction to each sanction* $U_i^T \in \mathcal{U}_i$ *of any player.*

Definition 4. *A pair* $(U^P, J^P) \in \mathcal{U} \times \mathbb{R}^N$ *is called an equilibrium in sanctions and countersanctions in the differential N-player game* Γ *if, for any initial position* $(t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n, x_0 \neq 0_n$*:*

1. *The alternative* U^P *is Pareto-maximal in the N-criteria dynamic choice problem* Γ_v*.*
2. *At least one of the other players has an complete countersanction to each sanction* $U_i^T \in \mathcal{U}_i$ *of any player.*

As before, $J^P = (J_1^P, J_2^P, \ldots, J_N^P)$ and $J_i^P = J_i(U^P, t_0, x_0) (i \in \mathbb{N})$. From Definitions 3 and 4, it follows that any equilibrium in sanctions and countersanctions is simultaneously an active equilibrium. Active equilibria and equilibria in sanctions and countersanctions are based on threats and counterthreats, well known in game theory [24]. They have all the positive properties of Nash equilibria [18]. More specifically:

1. They are stable against the deviations of an individual player.
2. They satisfy individual rationality.
3. They coincide with the saddle point in the case of zero-sum two-player games.

At the same time, these equilibria are free from the following disadvantages of Nash equilibrium [18]:

- They exist in several cases when there is no Nash equilibrium (e.g., in the game Γ).
- Unlike Nash equilibrium, they are unimprovable and internally stable due to Pareto maximality.
- The presence of a Nash equilibrium in the game implies the existence of certain types of unimprovable equilibria in which the payoffs of all players are no smaller than in the Nash equilibrium.
- The best Nash equilibria (in the sense of Pareto maximality) are equilibria in sanctions and countersanctions.

Let us emphasize again: the requirement of efficiency (Pareto maximality) has been incorporated into Definitions 3 and 4 to eliminate some negative properties of Nash equilibrium, such as the internal and external instability of the set of Nash equilibria.

N.N. Krasovskii [40] formalized the concepts of players' strategies and the motions of a dynamic system induced by them for a two-player zero-sum positional differential game. The constructions underlying the positive properties above are valid for a more general class—-noncooperative positional differential games [41].

Note that under economic sanctions [42,43], the methodology for constructing active equilibria, particularly the concept of equilibrium in sanctions and countersanctions, is of utmost importance for economic–mathematical modeling of decision processes and applications.

Hereinafter, the notation $D < 0(> 0)$ means that a quadratic form $x'Dx$ is negative definite (positive definite, respectively).

Consider the auxiliary N-criteria static problem

$$\Gamma_N = \left\langle \mathbb{R}^{N_n}, \{f_i(u) = u'_1 D_{i1} u_1 + \ldots + u'_N D_{iN} u_N\}_{i \in \mathbb{N}} \right\rangle, \quad (8)$$

in which the DM chooses an alternative $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^{Nn}$ for simultaneously maximizing all components of a vector criterion $f(u) = (f_1(u), f_2(u), \ldots, f_N(u))$.

For this problem, Definition 1 can be reformulated as follows: an alternative u^P is Pareto-maximal in Γ_N if $\forall u \in \mathbb{R}^{Nn}$ the system of inequalities $f_i(u) \geq f_i(u^P) (i \in \mathbb{N})$, with at least one strict inequality, is inconsistent.

We present some auxiliary properties of the quadratic forms $x'Dx = \sum_{\gamma,\beta=1}^{n} d_{\gamma\beta} x_\gamma x_\beta$ with constant coefficients $d_{\gamma\beta}$ (the elements of a matrix D of dimensions $n \times n$) and the components $x_1, \ldots, x_n$ of an n-dimensional vector $x \in \mathbb{R}^n$.

Lemma 1. *With the change of coefficients $b_{\gamma\beta} = \frac{d_{\gamma\beta} + d_{\beta\gamma}}{2}$, a quadratic form $x'Dx$ is reduced to the form $x'Bx$, where the matrix $B = (b_{ij})$ of dimensions $n \times n$ is symmetric, i.e., $B = B'$.*

Without loss of generality, all quadratic forms below are supposed to have symmetric matrices.

Lemma 2. *If $D_{ii} > 0$, then all n roots of the characteristic equation $\det[D_{ii} - \Lambda E_n] = 0$ are real and positive $(i \in \mathbb{N})$, where E_n denotes an identity matrix of dimensions $n \times n$ [44].*

Let $\Lambda_{ii} > 0$ be the greatest root under consideration. Then

$$u'_i D_{ii} u_i \leq \Lambda_{ii} \|u_i\|^2 = \Lambda_{ii} u'_i u_i \forall u_i \in \mathbb{R}^n. \quad (9)$$

Since $D > 0 \Leftrightarrow -D = (-1)D < 0$, for $D < 0$, all n roots $\lambda_{ij} < 0$ of the characteristic equation $\det[D_{ij} - \lambda E_n] = 0$ are negative $(i, j \in \mathbb{N}, i \neq j)$.

Let $-\lambda_{ij}$ be the greatest (smallest by magnitude) root among them. By analogy with (9), we have

$$u'_j D_{ij} u_i \leq -\lambda_{ij} u'_i u_j \forall u_j \in \mathbb{R}^n. \quad (10)$$

Without loss of generality, consider the N-criteria choice problem Γ_N in which a sanction is imposed by Player 1 and a countersanction by Player 2. (The players are numbered subjectively).

The next result follows from Property 2.

Lemma 3. *Assume that in the problem Γ_N :*

1. The symmetric matrices D_{ij} of dimensions $n \times n$ satisfy the inequalities $D_{ii} > 0, D_{ij} < 0$ $(i, j \in \mathbb{N}, i \neq j)$ and

$$\Lambda_{11} \Lambda_{22} < \lambda_{12} \lambda_{21}. \quad (11)$$

2. The nonzero matrix

$$A = \begin{bmatrix} \Lambda_{11} & -\lambda_{21} & \ldots & -\lambda_{N1} \\ -\lambda_{12} & \Lambda_{22} & \ldots & -\lambda_{N2} \\ \ldots & \ldots & \ldots & \ldots \\ -\lambda_{1N} & -\lambda_{2N} & \ldots & \Lambda_{NN} \end{bmatrix}$$

of dimensions $N \times N$ is singular, i.e., its determinant is $\det A = 0$.

Then, the system of strict homogeneous linear inequalities

$$\begin{cases} +\alpha_1\Lambda_{11} - \alpha_2\lambda_{21} - \alpha_3\lambda_{31} - \ldots - \alpha_N\lambda_{N1} < 0, \\ -\alpha_1\lambda_{12} + \alpha_2\Lambda_{22} - \alpha_3\lambda_{32} - \ldots - \alpha_N\lambda_{N2} < 0, \\ -\alpha_1\lambda_{13} - \alpha_2\lambda_{23} + \alpha_3\Lambda_{33} - \ldots - \alpha_N\lambda_{N3} < 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -\alpha_1\lambda_{1N} - \alpha_2\lambda_{2N} - \alpha_3\lambda_{3N} - \ldots - \alpha_{N-1}\lambda_{N-1N} + \alpha_N\Lambda_{NN} < 0, \end{cases} \tag{12}$$

has a positive solution $\alpha = (1, \alpha_2, \ldots, \alpha_N), \alpha_j > 0, j = 2, \ldots, N$, *where* Λ_{ii} *and* $-\lambda_{ij}(i, j \in \mathbb{N}, i \neq j)$ *are the greatest roots of the characteristic equations* $\det[D_{ii} - \Lambda E_n] = 0$ *and* $\det[D_{ij} - \lambda E_n] = 0$, *respectively* $(i, j \in \mathbb{N}, i \neq j)$.

Proof. To construct a Pareto-maximal strategy profile in the game Γ, we will apply Property 2 and the linear convolution $\sum\limits_{i\in\mathbb{N}} \alpha_i f_i(u)$ of the criteria (8) with positive coefficients $\alpha_i > 0$. For (8), we construct a quadratic form from the components of the Nn-dimensional vector $u = (u_1, \ldots, u_N)$:

$$f(u) = \sum_{i\in\mathbb{N}} \alpha_i f_i(u) = \sum_{i\in\mathbb{N}} \alpha_i [u'_1 D_{i1} u_1 + u'_2 D_{i2} u_2 + \ldots + u'_N D_{iN} u_N].$$

Due to (8)–(10),

$$\begin{aligned} & f(u) = u'_1[\alpha_1\Lambda_{11} - \alpha_2\lambda_{21} - \alpha_3\lambda_{31} - \ldots - \alpha_N\lambda_{N1}]u_1 + \\ & +u'_2[-\alpha_1\lambda_{12} + \alpha_2\Lambda_{22} - \alpha_3\lambda_{32} - \ldots - \alpha_N\lambda_{N2}]u_2 + \ldots + \\ & +u'_{N-1}[-\alpha_1\lambda_{1N-1} - \alpha_2\lambda_{2N-1} - \ldots + \alpha_{N-1}\Lambda_{N-1N-1} - \alpha_N\lambda_{NN-1}]u_{N-1} + \\ & +u'_N[-\alpha_1\lambda_{1N} - \alpha_2\lambda_{2N} - \ldots - \alpha_{N-1}\lambda_{N-1N} + \alpha_N\Lambda_{NN}]u_N = \\ & = d_1\|u_1\|^2 + \ldots + d_N\|u_N\|^2. \end{aligned}$$

The numbers $\alpha_i = const > 0$ $(i \in \mathbb{N})$ can be chosen so that all d_i will become negative and

$$\begin{aligned} & d_1 = [\alpha_1\Lambda_{11} - \alpha_2\lambda_{21} - \alpha_3\lambda_{31} - \ldots - \alpha_N\lambda_{N1}], \\ & d_2 = [-\alpha_1\lambda_{12} + \alpha_2\Lambda_{22} - \alpha_3\lambda_{32} - \ldots - \alpha_N\lambda_{N2}], \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & d_{N-1} = [-\alpha_1\lambda_{1N-1} - \alpha_2\lambda_{2N-1} - \ldots + \alpha_{N-1}\Lambda_{N-1N-1} - \alpha_N\lambda_{NN-1}], \\ & d_N = [-\alpha_1\lambda_{1N} - \alpha_2\lambda_{2N} - \ldots - \alpha_{N-1}\lambda_{N-1N} + \alpha_N\Lambda_{NN}]. \end{aligned} \tag{13}$$

To prove this fact, we associate with (12) the system of strict homogeneous linear inequalities

$$\begin{cases} \alpha_1\Lambda_{11} - \alpha_2\lambda_{21} < 0, \\ -\alpha_1\lambda_{12} + \alpha_2\Lambda_{22} < 0, \\ -\alpha_1\lambda_{13} - \alpha_2\lambda_{23} + \alpha_3\Lambda_{33} < 0, \\ -\alpha_1\lambda_{14} - \alpha_2\lambda_{24} - \alpha_3\lambda_{34} + \alpha_4\Lambda_{44} < 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -\alpha_1\lambda_{1N-1} - \alpha_2\lambda_{2N-1} - \ldots + \alpha_{N-1}\Lambda_{N-1N-1} < 0, \\ -\alpha_1\lambda_{1N} - \alpha_2\lambda_{2N} - \ldots - \alpha_{N-1}\lambda_{N-1N} + \alpha_N\Lambda_{NN} < 0. \end{cases} \tag{14}$$

This system is obtained from (12) by discarding all negative terms, except for $-\alpha_2\lambda_{21}$ in the first inequality and $-\alpha_1\lambda_{12}$ in the second one. Adding negative terms only strengthens strict inequalities. Hence, for $\lambda_{ij} > 0$ and $\alpha_i > 0$, any positive solutions of System (14) are also positive for (12). Then, the quadratic form $f(u)$ will be negative definite with respect to the Nn-dimensional vector $u = (u_1, \ldots, u_N)$ if System (12) (or System (14)) has the positive solution $\alpha = (1, \alpha_1, \ldots, \alpha_N)$, $\alpha_\gamma > 0$ $(\gamma = 2, \ldots, N)$, that is, all numbers. Let us establish this result for System (14). To this end, we find α_2 from the first two strict inequalities of (14):

$$\begin{cases} \alpha_1\Lambda_{11} - \alpha_2\lambda_{21} < 0, \\ -\alpha_1\lambda_{12} + \alpha_2\Lambda_{22} < 0. \end{cases}$$

If $\alpha_1 = 1$, then for $\Lambda_{ii} > 0$ and $-\lambda_{ij} > 0$ $(i, j \in \mathbb{N}, i \neq j)$, there exists a positive number α_2 such that

$$\left\{ \begin{array}{c} \Lambda_{11} - \alpha_2\lambda_{21} < 0 \\ -\lambda_{12} + \alpha_2\Lambda_{22} < 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \frac{\Lambda_{11}}{\lambda_{21}} < \alpha_2 \\ \frac{\lambda_{12}}{\Lambda_{22}} > \alpha_2 \end{array} \right\} \Leftrightarrow \left\{ \frac{\Lambda_{11}}{\lambda_{21}} < \alpha_2 < \frac{\lambda_{12}}{\Lambda_{22}} \right\}.$$

This inequality holds under $\Lambda_{11}\Lambda_{22} < \lambda_{12}\lambda_{21}$, e.g., for

$$\alpha_2 = \frac{1}{2}\left[\frac{\Lambda_{11}}{\lambda_{21}} + \frac{\lambda_{12}}{\Lambda_{22}}\right]. \tag{15}$$

From the third inequality of (14), for $\alpha_1 = 1$ and α_2 (15), we obtain

$$0 < \alpha_3 < \frac{1}{\Lambda_{33}}\left[\lambda_{13} + \frac{\lambda_{23}}{2}\left(\frac{\Lambda_{11}}{\lambda_{21}} + \frac{\lambda_{12}}{\Lambda_{22}}\right)\right],$$

e.g., by letting $\alpha_3 = \frac{1}{2\Lambda_{33}}\left[\lambda_{13} + \frac{\lambda_{23}}{2}\left(\frac{\Lambda_{11}}{\lambda_{31}} + \frac{\lambda_{32}}{\Lambda_{22}}\right)\right]$. Continuing the considerations above, using the subsequent inequalities of (14) and the calculated values $\alpha_1 = 1, \alpha_2, \alpha_3, \ldots,$ we finally arrive at the recurrent formula $\alpha_N = \frac{1}{2\Lambda_{NN}}[\lambda_{1N} + \alpha_2\lambda_{2N} + \ldots + \alpha_{N-1}\lambda_{N-1N}]$. □

Remark 1. *By analogy with Lemma 3, we have the following result. Assume that in the problem Γ_N, the symmetric matrices D_{ij} of dimensions $n \times n$ and positive numbers Λ_{ii} and λ_{ij} are such that $D_{ii} > 0, D_{ij} < 0$ for $i \neq j$ and $\Lambda_{11}\Lambda_{33} < \Lambda_{13}\Lambda_{31}$.*

Then, for

$$\alpha_2 = \frac{1}{2}\left(\frac{\lambda_{13}}{\Lambda_{33}} + \frac{\Lambda_{11}}{\lambda_{31}}\right), \alpha_3 = \frac{1}{2}\left[\frac{\lambda_{12}}{\Lambda_{22}} + \frac{1}{2}\left(\frac{\lambda_{13}}{\Lambda_{33}} + \frac{\Lambda_{11}}{\lambda_{31}}\right)\frac{\lambda_{32}}{\Lambda_{22}}\right], \ldots$$

and $\alpha_\gamma (\gamma = 4, \ldots, N)$ given by Lemma 3, the quadratic form

$$f(u) = f_1(u) + \alpha_2 f_2(u) + \alpha_3 f_3(u) + \ldots + \alpha_N f_N(u) = u'_1 d_1 u_1 + u'_2 d_2 u_2 + u'_3 d_3 u_3 + \ldots + u'_N d_N u_N$$

becomes negative definite; the constants $d_i < 0$ are given by (13).

Really, α_2 and α_3 in this expression satisfy the strict inequalities (12).

Note that in addition to the solutions α_γ (Lemma 3 and Remark 1), the system of strict inequalities (12) has a continuum of other solutions. As demonstrated below, each of them induces a specific equilibrium in sanctions and countersanctions of the differential game Γ under the conditions $D_{ii} > 0, D_{ij} < 0$ $(i, j \in \mathbb{N}, i \neq j)$.

Remark 2. *Positive solutions of (12) can be found using S.N. Chernikov's approach [35]. To avoid cumbersome transformations and notations dictated by this approach, we propose an original method for proving Lemma 3.*

Lemma 4. *The solutions $x(t)$ of the system $\dot{x} = K(t)x$, $x(t_0) = x_0$, where $K(\cdot) \in C_{n\times n}[0, \vartheta]$, satisfy the nontrivial property*

$$x_0 \neq 0_n \Rightarrow x(t) \neq 0_n \forall t \in [t_0, \vartheta];$$

here 0_n denotes a zero vector from the space $\mathbb{R}^n$.

Proof. Assume on the contrary that $\exists t_1 (t_0, \vartheta]$ such that $x(t) = 0_n$. In other words, at the time instant t_1, two different solutions of the system $\dot{x} = K(t)x$ are passing through the position $(t_1, 0_n)$: the trivial one $x^{(1)}(t) = 0_n$ $\forall t \in [0, \vartheta]$ and the nontrivial one $x^{(2)}(t)$ induced by the nonzero initial condition $x_0 \neq 0_n$. This obviously contradicts the existence and uniqueness theorem for the solution of a matrix linear differential equation with continuous coefficients. □

Proposition 1. *Assume that in the differential game* Γ*,*

$$D_{ii} > 0, D_{ij} < 0, C_i < 0 (i, j \in \mathbb{N}, i \neq j) \text{ and } \Lambda_{11}\Lambda_{22} < \Lambda_{12}\Lambda_{21}. \tag{16}$$

Then, a Pareto-maximal alternative U^P *in the N-criteria choice problem* Γ_v *has the form*

$$\begin{aligned} U^P &= (U_1^P, U_2^P, \ldots, U_N^P) \div (u_1^P(t,x), u_2^P(t,x), \ldots, u_N^P(t,x)) = \\ &= u^P(t,x) = (Q_1^P(t)x, Q_2^P(t)x, \ldots, Q_N^P(t)x) = (-d_1^{-1}\Theta^P(t)x, -d_2^{-1}\Theta^P(t)x, \ldots, -d_N^{-1}\Theta^P(t)x), \end{aligned} \tag{17}$$

where

$$\Theta^P(t) = [X^{-1}(t)]'\{C^{-1} + \int_t^{\vartheta} X^{-1}(\tau)[d_1^{-1} + d_2^{-1} + \ldots + d_N^{-1}][X^{-1}(\tau)]'d\tau\}^{-1}X^{-1}(t) \tag{18}$$

is a continuous symmetric matrix of dimensions $n \times n$ *on the time interval* $[0, \vartheta]$ *; the negative constants* d_i *are given by (13);*

$$\alpha_2 = \frac{1}{2}\left[\frac{\Lambda_{11}}{\lambda_{21}} + \frac{\lambda_{12}}{\Lambda_{22}}\right], \alpha_3 = \frac{1}{2}\left[\frac{\lambda_{13}}{\Lambda_{33}} + \frac{1}{2}\left(\frac{\Lambda_{12}}{\lambda_{21}} + \frac{\lambda_{12}}{\Lambda_{22}}\right)\frac{\lambda_{23}}{\Lambda_{33}}\right], \tag{19}$$

and the other numbers $\alpha_\gamma (\gamma = 4, \ldots, N)$ *are calculated by the recurrent formulas*

$$\alpha_m = \frac{1}{2\Lambda_{mm}}[\lambda_{1m} + \alpha_2\lambda_{2m} + \ldots + \alpha_{m-1}\lambda_{m-1m}] (m = 4, \ldots, N), \tag{20}$$

where Λ_{ii} *is the greatest root of the characteristic equation* $\det[D_{ii} - \Lambda E_n] = 0$ $(i \in \mathbb{N})$*, and* $-\lambda_{ij}$ *is the greatest root of the characteristic equation* $\det[D_{ij} - \lambda E_n] = 0$ $(i, j \in \mathbb{N}, i \neq j)$*;* E_n *denotes an identity matrix of dimensions* $n \times n$*; finally,* $X(t)$ *means the fundamental matrix of the system* $\dot{x} = A(t)x, X(\vartheta) = E_n$*.*

Proof. We construct a Pareto-maximal alternative U^P using Lemma 3 (formula (4)) and dynamic programming [16]. Due to Property 2, the application of dynamic programming reduces to two stages as follows.

First stage. For the problem Γ, find $(N-1)$ positive numbers $\alpha_m (m = 2, \ldots, N-1)$, a continuously differentiable scalar function $V(t,x) = x'\Theta(t)x$, $\Theta(t) = \Theta'(t)$ $\forall t \in [0, \vartheta]$, and Nn-dimensional vector functions $u_i(t,x,V)$ $(i \in \mathbb{N})$ such that

$$V(\vartheta, x) = x'Cx, C = C_1 + \alpha_2 C_2 + \ldots + \alpha_N C_N, \forall x \in \mathbb{R}^n. \tag{21}$$

Then, using the scalar function

$$W(t,x,u_1,u_2,\ldots,u_N,V) = \frac{\partial V}{\partial x} + \left[\frac{\partial V}{\partial t}\right]'(A(t)x + u_1 + u_2 + \ldots + u_N) + u'_1 d_1 u_1 + u'_2 d_2 u_2 + \ldots + u'_N d_N u_N,$$

find Nn-dimensional vector functions $u_i(t,x,V)$ $(i \in \mathbb{N})$ from

$$\max_{u_1,u_2,\ldots,u_N} W(t,x,u_1,u_2,\ldots,u_N,V) = Idem\{u_i \to u_i(t,x,V)\} (i \in \mathbb{N}) \tag{22}$$

for any $t \in [0, \vartheta], x \in \mathbb{R}^n$ and $V \in \mathbb{R}$ $\left(\frac{\partial V}{\partial x} = grad_x V\right)$. The functions $u_i(t,x,V)$ in (22) exist under the following sufficient conditions: for all $(t,x) \in [0, \vartheta) \times \mathbb{R}^n$,

$$\begin{aligned} &\left.\frac{\partial V}{\partial x}\right|_{u(t,x,V)} = \frac{\partial V}{\partial x} + 2d_i u_i(t,x,V) = 0_n, (i \in \mathbb{N}), \\ &\frac{\partial^2 W}{\partial u_i^2} = 2d_i E_n < 0, (i \in \mathbb{N}), \end{aligned} \tag{23}$$

where (as before) 0_n denotes an n-dimensional zero vector from the space $\mathbb{R}^n$, and $d_i < 0$ by Lemma 3.

From (23), it follows that

$$u_i(t,x,V) = -\frac{1}{2}d_1^{-1}\frac{\partial V}{\partial x}(i \in \mathbb{N}). \tag{24}$$

Then

$$W(t,x,u(t,x,V),V) = W[t,x,V] = \frac{\partial V}{\partial t} + \left[\frac{\partial V}{\partial x}\right]' A(t)x - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)' [d_1^{-1} + d_2^{-1} + \ldots + d_N^{-1}]\frac{\partial V}{\partial x}.$$

The second stage. Find the solution $V = V^P(t,x) = x'\Theta^P x, \Theta^P = [\Theta^P(t)]$, of the partial differential equation

$$W[t,x,V] = 0$$

with the boundary condition

$$V(\vartheta,x) = x'Cx \forall x \in \mathbb{R}^n,$$

where $C = C_1 + \alpha_2 C_2 + \ldots + \alpha_N C_N$. In other words, for all $t \in [0,\vartheta]$ and all $x \in \mathbb{R}^n$,

$$W[t,x,V(t,x)] = x'\Theta^P x = 0, V(\vartheta,x) = x'Cx \forall x \in \mathbb{R}^n.$$

Consequently, the symmetric matrix $\Theta^P(t)$ of dimensions $n \times n$ satisfies the Riccati matrix differential equation

$$\begin{aligned}
&\dot{\Theta}^P(t) + \Theta^P(t)A(t) + A'(t)\Theta^P(t) - \Theta^P(t)[d_1^{-1} + d_2^{-1} + \ldots + d_N^{-1}]\Theta^P(t) = 0_{n\times n},\\
&\Theta^P(\vartheta) = C = C_1 + \alpha_2 C_2 + \ldots + \alpha_N C_N,
\end{aligned}$$

where $0_{n\times n}$ denotes a zero matrix of dimensions $n \times n$.

As is well known [16], the solution $\Theta^P(t)$ of the Riccati matrix differential equation has the form (18). (Here, the implication

$$C_i < 0 (i \in \mathbb{N}) \Rightarrow C_1 + \alpha_2 C_2 + \ldots + \alpha_N C_N < 0$$

has been taken into account). Formula (18), in combination with another implication

$$[V(t,x) = x'\Theta^P(t)x] = \left[\frac{\partial V(t,x)}{\partial x} = 2\Theta^P(t)x\right],$$

finally yields (17). Thus, a Pareto-maximal alternative U^P in the multicriteria choice problem Γ_v is given by (17) and (18).

Now we construct the Pareto-maximal payoffs

$$J^P = (J_1(U^P,t_0,x_0), J_2(U^P,t_0,x_0), \ldots, J_N(U^P,t_0,x_0)) = (J_1^P, J_2^P, \ldots, J_N^P)$$

using dynamic programming and [44]. □

Proposition 2. *Let Conditions (16) of Proposition 1 be valid. Assume that for the differential game* Γ *, there are N scalar continuously differentiable functions* $V_i(t,x) = x'\Theta_i(t)x (i \in \mathbb{N})$ *such that:*

1. $V_i(\vartheta,x) = x'C_i x \forall x \in \mathbb{R}^n$,
2. *The system of N partial differential equations*

$$\frac{\partial V_i}{\partial t} + \left(\frac{\partial V_i}{\partial x}\right)'(N(t)x + x'\Theta^P(t)M_i(t)\Theta^P(t)x) = 0, V_i(\vartheta,x) = x'C_i x, \forall x \in \mathbb{R}^n (i \in \mathbb{N}) \tag{25}$$

has the solution $V_i(t,x) = x'\Theta_i(t)x$, $[\Theta_i(t)]' = \Theta_i(t) (i \in \mathbb{N})$.

Then, for any initial position $(t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n, x_0 \neq 0_n$,

$$J_i^P = J_i(U^P, t_0, x_0) = x'_0 \Theta_i^P(t_0) x_0 (i \in \mathbb{N}).$$

In (25),

$$N(t) = A(t) - (d_1^{-1} + d_2^{-1} + \ldots + d_N^{-1}) E_n,$$
$$M_i(t) = \Theta^P(t) \Big([d_1^{-1}]^2 D_{i1} + [d_2^{-1}]^2 D_{i2} + \ldots + [d_N^{-1}]^2 D_{iN} \Big) \Theta^P(t)$$
$$(i \in \mathbb{N})$$

are continuous matrices of dimensions $n \times n$ *; the matrix* $\Theta^P(t)$ *is given by (17) and (18); the numbers* d_i *are given by (13);*

$$\Theta_i(t) = \Big[Y^{-1}(t)\Big]' \left\{ C_i - \int_t^{\vartheta} Y'(\tau)\Theta(\tau)M_i(\tau)Y(\tau)d\tau \right\} Y^{-1}(t) \quad (i \in \mathbb{N}) \tag{26}$$

are symmetric matrices of dimensions $n \times n$*; finally,* $Y(t)$ *denotes the fundamental matrix of the homogeneous system* $\dot{y} = N(t)y, Y(\vartheta) = E_n$.

Proof. We construct the N scalar functions

$$\begin{array}{l} W[t, x, V_i] = \frac{\partial V}{\partial t} + \left[\frac{\partial V}{\partial x}\right]' (N(t)x + [u_1^P(t, x)]' D_{i1} u_1^P(t, x) + \\ + [u_2^P(t, x)]' D_{i2} u_2^P(t, x) + \ldots + [u_N^P(t, x)]' D_{iN} u_N^P(t, x) (i \in \mathbb{N}), \end{array} \tag{27}$$

where $u_i^P(t, x)$ are the n-dimensional vector functions given by (17) and (18).

Let us find the solution $V_i(t, x)$ $(i \in \mathbb{N})$ of the system of N partial differential equations

$$W_i[t, x, V_i] = 0, V_i(\vartheta, x) = x' C_i x \tag{28}$$

as the quadratic form $V_i(t, x) = x' \Theta_i(t) x, [\Theta_i(t)]' = \Theta_i(t)$ $(i \in \mathbb{N})$.

We will establish two facts as follows.

First, the solution of Systems (27) and (28) has the property

$$V_i(t_0, x_0) = J_i(U^P, t_0, x_0)(i \in \mathbb{N}), \tag{29}$$

where the strategy profile $U^P = (U_1^P, U_2^P, \ldots, U_N^P)$ has the form (17) and (18). Really, if U^P is a strategy profile from (12)–(14), then by (27) and (28), the solution $x^P(t)$ of the system $\dot{x} = N(t)x$, $x(t_0) = x_0 \neq 0_n$, for $x = x^P(t)$ will be

$$\begin{array}{l} 0 = W_i[t, x^p(t), V_i(t, x^p(t))] = \\ = \frac{\partial V_i(t, x^p(t))}{\partial t} + \left[\frac{\partial V_i(t, x^p(t))}{\partial x}\right]' N(t)x^p(t) + \sum\limits_{j=1}^{N} [u_j^P(t, x^P(t))]' D_{ij} u_j^P(t, x^P(t)) = \overline{W}_i[t] \\ \forall t \in [t_0 \vartheta] (i \in \mathbb{N}). \end{array}$$

Integrating both sides of this identity from t_0 to ϑ subject to the boundary conditions (28) yields

$$\begin{array}{l} 0 = \int\limits_{t_0}^{\vartheta} \overline{W}_i[t] dt = \int\limits_{t_0}^{\vartheta} \frac{dV_i(t, x^P(t))}{dt} dt + \int\limits_{t_0}^{\vartheta} \sum\limits_{j=1}^{N} [u_j^P(t, x^P(t))]' D_{ij} u_j^P(t, x^P(t)) dt = \\ = V_i^P(\vartheta, x^P(\vartheta)) - V_i^P(t_0, x^P(t_0)) + \int\limits_{t_0}^{\vartheta} \sum\limits_{j=1}^{N} [u_j^P(t, x^P(t))]' D_{ij} u_j^P(t, x^P(t)) dt = \\ = x'(\vartheta) C_i x(\vartheta) + \int\limits_{t_0}^{\vartheta} \sum\limits_{j=1}^{N} [u_j^P(t, x^P(t))]' D_{ij} u_j^P(t, x^P(t)) dt - V_i^P(t_0, x^P(t_0)) = \\ = J_i(U^P, t_0, x_0) - V_i^P(t_0, x^P(t_0)) (i \in \mathbb{N}). \end{array}$$

Hence, Property (29) is proved.

Second, the solution of System (29) has the form $V_i(t,x) = x'\Theta_i(t)x$, where a symmetric matrix $\Theta_i(t)$ of dimensions $n \times n$ is given by (26). Really, substituting $V_i(t,x) = x'\Theta_i(t)x$ into (28) leads to (29) if $\Theta_i(t)$ is the solution of the matrix linear inhomogeneous differential equation

$$\Theta_i + \Theta_i N + N'\Theta_i + \Theta^P(t)M_i\Theta^P(t) = 0_{n\times n}, \Theta^P(\vartheta) = C_i (i \in \mathbb{N}).$$

A direct substitution of (26) into equation (29) shows that this symmetric matrix $\Theta_i(t)$ of dimensions $n \times n$ is the desired solution. The proof of Proposition 2 is complete.

Note that

$$\frac{dY^{-1}(t)}{dt} = -Y^{-1}(t)A(t), \frac{d[Y^{-1}(t)]'}{dt} = -A(t)[Y^{-1}(t)]'.$$

□

Remark 3. *Propositions 1 and 2 considered together finally yield the following explicit form of the Pareto-maximal solution* $(U^P, J^P) \in \mathcal{U} \times \mathbb{R}^n$ *of the game* Γ.

Assume that in the differential game Γ*:*

1. *The symmetric constant matrices* D_{ij} *and* C_i *of dimensions* $n \times n$ *are such that*

$$D_{ii} > 0, D_{ij} < 0, C_i < 0 (i, j \in \mathbb{N}, i \neq j).$$

2. $\Lambda_{11}\Lambda_{22} < \lambda_{12}\lambda_{21}$.

Then, for all $(t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n, x_0 \neq 0_n$,

$$U^P \div u^P(t,x) = (-d_1^{-1}\Theta^P(t)x, -d_2^{-1}\Theta^P(t)x, \ldots, -d_N^{-1}\Theta^P(t)x),$$
$$J^P = (J_1^P, J_2^P, \ldots, J_N^P), (J_i^P = x'_0\Theta_i(t_0)x_0 (i \in \mathbb{N}),$$

where

$$\Theta^P(t) = [X^{-1}(t)]'\left\{C^{-1} + \int_t^{\vartheta} X^{-1}(\tau)[d_1^{-1} + d_2^{-1} + \ldots + d_N^{-1}][X^{-1}(\tau)]'d\tau\right\}^{-1} X^{-1}(t),$$
$$\Theta_i(t) = [Y^{-1}(t)]'\left\{C_i - \int_t^{\vartheta} Y'(\tau)\Theta^P M_i(\tau)\Theta^P Y(\tau)d\tau\right\}Y^{-1}(t),$$

are symmetric matrices of dimensions $n \times n$ *; the matrices* $X(t)$ *and* $Y(t)$ *of dimensions* $n \times n$ *are the fundamental matrices of the systems* $\dot{x} = A(t)x$, $X(\vartheta) = E_n$ *and* $\dot{y} = N(t)x$, $Y(\vartheta) = E_n$, *respectively;*

$$C = C_1 + \alpha_2 C_2 + \ldots + \alpha_N C_N,$$
$$N(t) = A(t) - (d_1^{-1} + d_2^{-1} + \ldots + d_N^{-1})\Theta^P(t),$$
$$M_i(t) = \Theta^P(t)\left([d_1^{-1}]^2 D_{i1} + [d_2^{-1}]^2 D_{i2} + \ldots + [d_N^{-1}]^2 D_{iN}\right)\Theta^P(t),$$

the numbers $\alpha_2 = \frac{1}{2}\left(\frac{\lambda_{13}}{\Lambda_{33}} + \frac{\Lambda_{11}}{\lambda_{31}}\right)$, $\alpha_3 = \frac{1}{2}\left[\frac{\lambda_{12}}{\Lambda_{22}} + \frac{1}{2}\left(\frac{\lambda_{13}}{\Lambda_{33}} + \frac{\Lambda_{11}}{\lambda_{31}}\right)\frac{\lambda_{32}}{\Lambda_{22}}\right], \ldots,$ *and* $\alpha_\gamma (\gamma = 4, \ldots N)$ *are given by (20); the negative numbers* d_i *are given by (13);* Λ_{ij} *and* $-\lambda_{ij}$ *are the greatest roots of the characteristic equations* $\det[D_{ii} - \Lambda E_n] = 0$ *and* $\det[D_{ij} - \lambda E_n] = 0$, *respectively* $(i, j \in \mathbb{N}, i \neq j)$.

Lemmas of Majorants

Next, let us proceed to the propositions that

1. Will allow us to reveal the absence of Nash equilibrium in differential games if (11) is satisfied;

2. Realize for the game the concept of sanctions and countersanctions for the differential game Γ.

These items (1) and (2) involve the definiteness of quadratic forms figuring in the integral terms of the payoff functions (3). From this point onwards, assume that Conditions (11) are satisfied. Hence, there exists a Pareto-maximal alternative

$$\begin{aligned} &U^P = (U_1^P, U_2^P, \ldots, U_N^P) \div (u_1^P(t,x), u_2^P(t,x), \ldots, u_N^P(t,x)) = \\ &= u^P(t,x) = (Q_1^P(t)x, Q_2^P(t)x, \ldots, Q_N^P(t)x) = \\ &= (-d_1^{-1}\Theta^P(t)x, -d_2^{-1}\Theta^P(t)x, \ldots, -d_N^{-1}\Theta^P(t)x). \end{aligned}$$

in the N-criteria choice problem Γ_v.

Lemma 5. *Let the payoff function (3) be such that $D_{11} > 0$. Then, for a Pareto-maximal strategy profile alternative U^P in the game Γ, there exists a constant $\alpha^{(1)}(U^P, t_0, x_0) > 0$ such that, for all $\alpha = \alpha^{(1)}(U^P, t_0, x_0) > 0$ and the strategy $U_1^T \div \alpha x$ of Player 1, the inequality*

$$J_1(U_1^T, U_2^P, \ldots, U_N^P, t_0, x_0) > J_1(U_1^P, U_2^P, \ldots, U_N^P, t_0, x_0) \quad (30)$$

will hold for any initial positions $(t_0, x_0) \in [0, \vartheta) \times [\mathbb{R}^n \backslash \{0_n\}]$.

Proof. According to Proposition 2, there exists a Bellman function $V_1(t,x) = x'\Theta_1(t)x$ such that

$$J_1(U^P, t_0, x_0) = V_1(t_0, x_0) = x'_0\Theta_1(t_0)x_0,$$

where the symmetric matrix $\Theta_1(t)$ of dimensions $n \times n$ is continuous on $[0, \vartheta)$ and has the form (26) $(i = 1)$.

Consider the strategy $U_1^T \div u_1^T(t,x) = \alpha x$ of Player 1, in which the numerical parameter $\alpha > 0$ will be determined below. Due to the symmetry of the matrix D_{11} and $D_{11} > 0$,

$$u'_1 D_{11} u_1 \geq \lambda_1 \|u_1\|^2 = \lambda_1 u'_1 u_1 \forall u_1 \in \mathbb{R}^n, \quad (31)$$

where $\|\cdot\|$ denotes the Euclidean norm, and $\lambda_1 > 0$ is the smallest root of the characteristic equation $\det[D_{11} - \lambda E_n]$; see [45].

We take the symmetric matrix $\Theta^P(t)$ of dimensions $n \times n$ from (18) and the strategies $U_2^P \div Q_2^P(t)x, \ldots, U_N^P \div Q_N^P(t)x$ of players $2, \ldots, N$, respectively, from (17). We introduce the scalar function

$$\begin{aligned} &W_1[t,x] = \\ &= [W_1(t, x, u_1^T(t,x) = \alpha x, u_2^T(t,x) = Q_2^P x, \ldots\ldots, u_N^T(t,x) = Q_N^P x, \\ &V(t,x)_1 = x'\Theta_1(t)x] = \\ &= \tfrac{\partial V_1(t,x)}{\partial t} + \left[\tfrac{\partial V_1(t,x)}{\partial x}\right]'(A(t)x + u_1^T(t,x) + u_2^T(t,x) + \ldots + u_N^T(t,x)) + \\ &+ [u_1^T(t,x)]'d_1 u_1^T(t,x) + [u_2^T(t,x)]'d_2 u_2^T(t,x) + \ldots + [u_N^T(t,x)]'d_N u_N^T(t,x) \geq \\ &\geq x'\tfrac{d\Theta_1(t)}{dt}x + 2x'\Theta_1(t)[A(t) + \alpha E_n + Q_2^P(t) + \ldots Q_N^P(t)]x + \\ &+ x'(\lambda_1\alpha^2 E_n)x + x'[Q_2^P(t)]'D_{12}Q_2^P(t)x + \ldots \\ &\ldots + x'[Q_N^P(t)]'D_{1N}Q_N^P(t)x = x'\{\tfrac{d\Theta_1(t)}{dt} + \Theta_1(t)[A(t) + \alpha E_n + Q_2^P(t) + \ldots + Q_N^P(t)] + \\ &+ [A'(t) + \alpha E_n + (Q_2^P(t))' + \ldots + (Q_N^P(t))']\Theta_1(t) + \lambda_1\alpha^2 E_n + \\ &+ [Q_2^P(t)]'D_{12}Q_2^P(t) + \ldots + [Q_N^P(t)]'D_{1N}Q_N^P(t)\}x = \\ &= x'M_1(t,\alpha)x. \end{aligned}$$

The matrix $M_1(t,\alpha)$ in curly brackets is symmetric and has the form

$$M_1(t,\alpha) = \lambda_1\alpha^2 E_n + 2\alpha\Theta_1(t) + K_1(t),$$

where

$$K_1(t) = \dot{\Theta}_1(t) + \Theta_1(t)[A(t) + Q_2^P(t) + \ldots + Q_N^P(t)] + [Q_2^P(t)]' D_{12} Q_2^P(t) + \ldots + [Q_N^P(t)]' D_{1N} Q_N^P(t) + \\ +[A'(t) + (Q_2^P(t))' + \ldots + (Q_N^P(t))']\Theta_1(t)$$

is a symmetric and continuous matrix of dimensions $n \times n$.

The elements of the matrices $\Theta_1(t)$ and $K_1(t)$ are continuous on $[0, \vartheta]$ and hence uniformly bounded on the compact set $[0, \vartheta]$. The factor α^2 enters only the diagonal elements of the matrix $M_1(t, \alpha)$. Recall that $\lambda_1 > 0$ is the *smallest* root of the characteristic equation $\det[D_{11} - \lambda E_n] = 0$, where E_n denotes an identity matrix of dimensions $n \times n$. Therefore, the constant $\alpha = \alpha^{(1)}(U^P, t_0, x_0) > 0$ can be chosen sufficiently great for making all principal minors of the matrix $M_1(t, \alpha)$ positive $\forall t \in [0, \vartheta]$ $\alpha \geq \alpha^{(1)}(U^P, t_0, x_0)$. (This fact will be proved below). According to Lemma 2 and [46], the quadratic form $x' M_1(t, \alpha) x$ is positive definite for all $t \in [0, \vartheta]$ and all constants $\alpha \geq \alpha^{(1)}(U^P, t_0, x_0)$.

Now we show the existence of a constant $\alpha^{(1)}(U^P, t_0, x_0) > 0$ such that, for all $\alpha \geq \alpha^{(1)}(U^P, t_0, x_0)$, the quadratic form $x' M_1(t, \alpha) x$ is positive definite for all $t \in [0, \vartheta]$ and $x \in \mathbb{R}^n$. Note that the matrix $M_1(t, \alpha)$ of dimensions $n \times n$ is symmetric. By Sylvester's criterion, the quadratic form $x' M_1(t, \alpha) x$ is positive definite if all principal minors $\Delta_r (r = 1, \ldots, n)$ of the matrix $M_1(t, \alpha)$ are positive. The minors Δ_r are located in the first r rows and first r columns of the matrix $M_1(t, \alpha)$ $(r = 1, \ldots, n)$:

$$\Delta_r(t, \alpha) = \begin{vmatrix} \lambda_1 \alpha^2 n + \alpha l_{11}(t) + k_{11}(t) & \cdots & \alpha l_{1r}(t) + k_{1r}(t) \\ \vdots & \ddots & \vdots \\ \alpha l_{r1}(t) + k_{r1}(t) & \cdots & \lambda_1 \alpha^2 n + \alpha l_{rr}(t) + k_{rr}(t) \end{vmatrix}.$$

They must be positive $\forall t \in [0, \vartheta]$ and $\forall \alpha \geq \alpha^{(1)}(U^P, t_0, x_0)$. Expanding the determinants $\Delta_r(t, \alpha)$ and rearranging the terms in the descending order of the power of the parameter α, we obtain

$$\Delta_r(t, \alpha) = \alpha_0(t)\alpha^{2r} + \alpha_1(t)\alpha^{2r-1} + \ldots + \alpha_{2r-1}(t)\alpha + \alpha_{2r}(t),$$

where $\alpha_0 = \lambda^r n^r > 0$ (constant), and the other coefficients $\alpha_1(t), \ldots, \alpha_{2r}(t)$ are continuous on the compact set $[0, \vartheta]$, hence being uniformly bounded. This uniform boundedness guarantees the existence of $\Omega_r = const > 0$ such that

$$\max_{0 \leq t \leq \vartheta} \{\alpha_p(t) | p = 0, 1, \ldots, 2r\} < \Omega_r.$$

Let us demonstrate that if

$$\alpha > \frac{\Omega_r}{|\alpha_0|} + 1 = \alpha^{(1)}(U^P, t_0, x_0),$$

then

$$|\alpha_1(t)\,\alpha^{2r-1} + \alpha_2(t)\alpha^{2r-2} + \ldots + \alpha_{2r-1}(t)\alpha + \alpha_{2r}(t)| < \left|\alpha_0 \alpha^{2r}\right|.$$

In other words, for a sufficiently great value $|\alpha|$, the sign of the polynomial $\Delta_r(t, \alpha)$ is determined by the sign of its leading term. Really,

$$|\alpha_1(t)\,\alpha^{2r-1} + \alpha_2(t)\alpha^{2r-2} + \ldots + \alpha_{2r-1}(t)\alpha + \alpha_{2r}(t)| \leq \\ \leq \left|\alpha_1(t)\alpha^{2r-1}\right| + \left|\alpha_2(t)\alpha^{2r-2}\right| + \ldots + |\alpha_{2r-1}(t)\alpha| + |\alpha_{2r}(t)| \leq \\ \leq \Omega_r(\alpha^{2r-1} + \alpha^{2r-2} + \ldots + \alpha + 1) = \Omega_r \tfrac{\alpha^{2r-1}}{\alpha - 1}.$$

In addition,

$$\left[\alpha > \frac{\Omega_r}{|\alpha_0|} + 1\right] \Rightarrow [\Omega_r < \alpha_0(\alpha - 1)].$$

Replacing Ω_r in this inequality by a greater value $\alpha_0(\alpha-1)$ yields

$$|\alpha_1(t)\,\alpha^{2r-1}+\alpha_2(t)\alpha^{2r-2}+\ldots+\alpha_{2r-1}(t)\alpha+\alpha_{2r}(t)|<$$
$$<\alpha_0(\alpha-1)\tfrac{\alpha^{2r-1}}{\alpha-1}=\alpha_0(\alpha-1)<\alpha_0\alpha^{2r}.$$

Thus $\forall\alpha\geq\Omega_r=\alpha^{(r)}(U,t_0,x_0)>0$, and $\forall t\in[0,\vartheta]$, we have

$$|\alpha_1(t)\,\alpha^{2r-1}+\alpha_2(t)\alpha^{2r-2}+\ldots+\alpha_{2r-1}(t)\alpha+\alpha_{2r}(t)|<\alpha_0\alpha^{2r}.$$

Well, for a sufficiently great value α, the sign of the polynomial $\Delta_r(t,\alpha)$ is determined by the sign of its leading term. Finally, for each $r=1,\ldots,n$, we calculate $\Omega_r>0$ and let $\alpha^{(1)}(U^P,t_0,x_0)=\max\limits_{r=1,\ldots,n}\Omega_r$.

Then, for $\alpha^{(1)}(U^P,t_0,x_0)$, it follows that

$$\widetilde{W}_1[t,x]=x'M_1(t,\alpha^{(1)})x>0\forall t\in[0,\vartheta]\forall x\in\mathbb{R}^n\backslash\{0_n\}. \tag{32}$$

We denote by $\widetilde{x}(t)$, $t\in[0,\vartheta]$, the solution of the vector differential equation

$$\dot{x}=A(t)x+\alpha^{(1)}x+Q_2^P(t)x+\ldots+Q_N^P(t)x,x(t_0)=x_0\neq 0_n.$$

Due to Lemma 2, the implication $[x_0\neq 0_n]\Rightarrow(\widetilde{x}(t)\neq 0_n\;\,t\in[0,\vartheta])$ and (32),

$$\widetilde{W}_1[t,\widetilde{x}(t)]>0,\forall t\in[0,\vartheta].$$

Integrating both sides of (32) from t_0 to ϑ subject to the boundary condition $\Theta_1(\vartheta)=C_1$ and $u_1^T[t]=\alpha^{(1)}\widetilde{x}(t)$ gives:

$$\begin{aligned}
0&=\int\limits_{t_0}^{\vartheta}\widetilde{W}_1[t,\widetilde{x}(t)]dt=\\
&=\int\limits_{t_0}^{\vartheta}\left\{\frac{\partial V_1(t,x)}{\partial t}+\left[\frac{\partial V_1(t,x)}{\partial x}\right]'[A[t]x+\alpha^{(1)}E_nx+Q_2^P(t)x+\ldots+Q_N^P(t)x]\right\}_{x=\widetilde{x}(t)}dt=\\
&=\int\limits_{t_0}^{\vartheta}\{(\alpha^{(1)})^2x'D_{11}x+x'[Q_2^P(t)]'D_{12}Q_2^P(t)+\ldots+x'[Q_N^P(t)]'D_{1N}Q_N^P(t)\}_{x=\widetilde{x}(t)}dt=\\
&=\int\limits_{t_0}^{\vartheta}\frac{dV_1(t,\widetilde{x}(t))}{dt}+\int\limits_{t_0}^{\vartheta}\sum_{j=1}^{N}[u_j^T[t]]'D_{1j}u_j^T[t]dt=\\
&=[\widetilde{x}(\vartheta)]'C_1\widetilde{x}(\vartheta)+\int\limits_{t_0}^{\vartheta}\sum_{j=1}^{N}[u_j^T[t]]'D_{1j}u_j^T[t]dt-V_1(t_0,x_0)=\\
&=J_1(U_1^T,U_2^P,\ldots,U_N^P,t_0,x_0)-V_1(t_0,x_0).
\end{aligned}$$

This result, in combination with the equality $J_1(U_1^P,U_2^P,\ldots,U_N^P,t_0,x_0)=V_1(t_0,x_0)$, finally proves Lemma 5. □

Remark 4. *Consider the inner optimization problem in the game Γ: for fixed strategies $U_2=U_2^P\in\mathcal{U}_2,\ldots,U_N=U_N^P\in\mathcal{U}_N$ of players $2,\ldots,N$, respectively, and for any $(t_0,x_0)\in[0,\vartheta)\times[\mathbb{R}^n\backslash\{0_n\}]$, find $\max\limits_{U_1\in\mathcal{U}_1}J_1(U_1,U_2^P,\ldots,U_N^P,t_0,x_0)$ subject to System (1). Lemma 5 claims that for $D_{11}>0$ and $x_0\neq 0_n$, this problem maximization problem has no solution. Really, whatever strategy $U_1\in\mathcal{U}_1$ is chosen by Player 1, there always exists another strategy U_1^T of this player such that*

$$J_1(\widetilde{U}_1^T,U_2^P,\ldots,U_N^P,t_0,x_0)>J_1(U_1,U_2^P,\ldots,U_N^P,t_0,x_0)\forall(t_0,x_0)\in[0,\vartheta)\times[\mathbb{R}^n\backslash\{0_n\}].$$

When choosing an appropriate solution of the game Γ , this result allows directly eliminating those concepts of equilibrium that involve the maximization of the payoff function of Player 1 with *respect to U_1. (For example, if $D_{11} > 0$, the concept of Nash equilibrium should not be used as the solution of the game Γ).*

Thus, under Conditions (11), the differential game Γ has no Nash equilibrium. At the same time, the strategy $U_1^T \div \alpha x \forall \alpha \geq \alpha^{(1)}(U^P, t_0, x_0)$ implements the sanction of Player 1 to the Pareto-maximal (efficient) strategy profile U^P ; see (5). In the lemmas below, the initial position (t_0, x_0) is fixed and coincides with the one from Lemma 5; in addition, the sanction strategy $U_1^T \div \alpha x$ of Player 1 has a constant scalar $\alpha = \alpha^{(1)}$. Recall that Conditions (11) are assumed to hold without special mention. Well, Lemma 5 establishes the following result.

Proposition 3. *Assume that in the game Γ, at least one of the constant symmetric matrices $D_{ii} > 0$ $(i \in \mathbb{N})$ of dimensions $n \times n$ is positive definite. Then, this game has no Nash equilibrium, i.e., there does not exist a strategy $U_i^e \in \mathcal{U}_i$ satisfying Definition 2.*

Note that:

1. *The condition $D_{ii} > 0$ with a fixed number $i \in \mathbb{N}$ breaks only the i th equality of Definition 2. This is enough for the absence of a Nash equilibrium U^e in the game Γ . If $D_{ii} > 0$ for all $i \in \mathbb{N}$, then the N equalities of Definition 2 will be violated.*
2. *The equivalence*

$$D > 0 \Leftrightarrow -D < 0$$

is obvious. (Here,—D means that all elements of the matrix D are multiplied by –1).

Then, Lemma 5 also implies the following.

Lemma 6. *Let the payoff function (3) be such that $D_{12} > 0$. Then, there exists a constant $\alpha^{(2)} = \alpha^{(2)}(U^P, U_1^T, t_0, x_0) > 0$ such that, for all $\forall \alpha \geq \alpha^{(2)}$ and the strategy $U_2^C \div \alpha x$ of Player 2,*

$$J_1(U_1^T, U_2^C, U_3^P, \ldots, U_N^P, t_0, x_0) < J_1(U^P, t_0, x_0). \tag{33}$$

In other words, the strategy $U_2^C \div \alpha x \forall \alpha \geq \alpha^{(2)}$ implements in the game Γ an incomplete countersanction to the sanction U_1^T of Player 1.

Proof. With some obvious modifications, the proof is immediate from Lemma 5.

Let a Bellman function $\widetilde{V}_1(t, x) = x'\widetilde{\Theta}(t)x$, $\widetilde{\Theta}(t) = \Theta_1(t)$, be constructed, satisfying

$$J_1(U_1^T, U_2^P, \ldots, U_N^P, t_0, x_0) = \widetilde{V}_1(t_0, x_0). \tag{34}$$

By analogy with Lemmas 5 and 6, we will establish another important result below. Recall that an initial position (t_0, x_0), a continuous matrix $\Theta^P(t)$ of dimensions $n \times n$, and an incomplete countersanction strategy $U_2^C \div \alpha^{(2)} x$ figuring in Lemmas 5 and 6 are assumed to be fixed, and Conditions (11) are assumed to hold. □

Lemma 7. *The condition $D_{22} > 0$ implies the existence of a value $\alpha^{(3)}(U^P, U_1^T, t_0, x_0) = const > 0$ such that, for all $\alpha > \alpha^{(3)}$ and the strategy $U_2^C \div \alpha x$,*

$$J_2(U_1^T, U_2^C, U_3^P, t_0, x_0) < J_2(U_1^T, U_2^P, U_3^P, t_0, x_0).$$

In other words, the strategy $U_2^C \div (\max\{\alpha^{(2)}, \alpha^{(3)}\})x$ of Player 2 implements a complete countersanction, jointly with $U_2^C \div \alpha^{(2)} x$, to the sanction of Player 1 to U^P.□

The sanctions of another player and countersanctions of the other players are constructed similarly.

Proof of Existence

Theorem 1. *Assume that the game* $\Gamma = \langle \mathbb{N}, \Sigma \div (2), \{\mathcal{U}_i\}_{i\in\mathbb{N}}, \{J_i(U, t_0, x_0) \div (3)\}_{i\in\mathbb{N}} \rangle$ *satisfies Conditions (16). Then, the (N + 1)-tuple*

$$\begin{aligned}(U^P, J_1^P, J_2^P, \ldots, J_N^P) = ((U_1^P, U_2^P, \ldots, U_N^P), J_1(U^P, t_0, x_0), J_2(U^P, t_0, x_0), \ldots \\ \ldots, J_N(U^P, t_0, x_0) = ((-d_1^{-1}\Theta^P(t)x, -d_2^{-1}\Theta^P(t)x, \ldots \\ \ldots, -d_N^{-1}\Theta^P(t)x), x'_0\Theta_1(t_0)x_0, x'_0\Theta_2(t_0)x_0, \ldots, x'_0\Theta_N(t_0)x_0)\end{aligned}$$

is an equilibrium in sanctions and countersanctions of differential game Γ *where:*

$$\Theta^P(t) = [X^{-1}(t)]'\{C^{-1} - \int_t^{\vartheta} X^{-1}(\tau)[d_1^{-1} + d_2^{-1} + \ldots + d_N^{-1}][X^{-1}(\tau)]'dt\}^{-1}X^{-1}(t);$$

the constants d_i *are given by (13);* $C = C_1 + \alpha_2 C_2 + \ldots + \alpha_N C_N$, *where* $\alpha_2 = \frac{1}{2}\left(\frac{\Lambda_{11}}{\lambda_{21}} + \frac{\lambda_{12}}{\Lambda_{22}}\right)$, $\alpha_3 = \frac{1}{2}\left[\frac{\Lambda_{13}}{\lambda_{33}} + \frac{1}{2}\left(\frac{\Lambda_{11}}{\lambda_{21}} + \frac{\lambda_{12}}{\Lambda_{22}}\right)\frac{\lambda_{23}}{\Lambda_{33}}\right]$ *and* $\alpha_m = \frac{1}{2\Lambda_{mn}}[\lambda_{1m} + \alpha_2\lambda_{2m} + \ldots + \alpha_{m-1}\lambda_{m-1m}]$ $(m = 4, \ldots, N)$ *and* Λ_i *and* $-\lambda_{ij}$ *are the smallest and greatest roots of the equations* $\det[D_{ij} - \Lambda E_n] = 0$ *and* $\det[D_{ij} - \lambda E_n] = 0$, *respectively;* $X(t)$ *denotes the fundamental matrix of the system* $\dot{x} = A(t)x$, $X(\vartheta) = E_n$ $(i, j \in \mathbb{N}, i \neq j)$*; finally, the symmetric matrices* $\Theta_i(t) (i \in \mathbb{N})$ *are given by (26).*

Proof. The absence of a Nash equilibrium in the game Γ and the presence of a sanction U_1^T imposed by Player 1 to a Pareto-maximal alternative U^P in the N-criteria choice problem Γ_v immediately follow from $D_{11} > 0$; see Remark 4. The existence of a Pareto-maximal alternative and Pareto-maximal outcomes in Γ_v (including their explicit forms in this case) has been established by Propositions 2 and 3, respectively. The condition $D_{21} > 0$ allows constructing an incomplete countersanction U_2^C of Player 2 to the sanctions of Player 1 (Lemma 6). The condition $D_{22} > 0$ and Lemma 7 enable transforming the incomplete countersanction U_2^C of Player 2 into the complete one U_2^C. The requirement $D_{22} > 0$ simultaneously implies the absence of a Nash equilibrium ($\max_{U_1} J(U_1, U_2^e, U_3^e, \ldots, U_N^e, t_0, x_0)$ is not achieved $\overline{U}_1^C \in \mathcal{U}_1$) and the ability of Player 2 to design analytically a sanction U_2^T to U^P in the game Γ:

$$J_2(U_1^C, U_2^T, U_3^P, \ldots, U_N^P) \leq J_2(U^P t_0, x_0). \quad (35)$$

The condition $D_{22} > 0$ and Lemma 7 guarantee the existence of an incomplete countersanction $\overline{U}_1^C \in \mathcal{U}_1$ of Player 1 to the sanction U_2^T of Player 2:

$$J_2(U_1^C, U_2^T, U_3^P, \ldots, U_N^P) < J_2(U^P t_0, x_0). \quad (36)$$

Finally, the Pareto maximality of U^P and Property 1 lead to

$$J_1(U_1^P, U_2^T, U_3^P, \ldots, U_N^P) < J_1(U^P t_0, x_0). \quad (37)$$

Due to $D_{11} > 0$ and Lemma 5, there exists a $\overline{U}_1^C \in \mathcal{U}_1$ such that

$$J_1(\overline{U}_1^C, U_2^T, U_3^P, \ldots, U_N^P) > J_1(U_1^P, U_2^T, U_3^P, \ldots, U_N^P, t_0, x_0). \quad (38)$$

The countersanction to the sanctions imposed by players $3, \ldots, N$ to U^P is designed by analogy.

Thus, one of the other players in the game Γ always has a complete countersanction to a sanction imposed by any player to the Pareto-maximal strategy profile U^P. The proof of Theorem 1 is complete. □

4. Discussion

Game-theoretic analysis of international legal agreements is based on a qualitative assessment of the interaction mechanisms between their participants. The application of this methodology is also justified by the fact that, unlike national law, international law lacks a superstructure with the legal personality to punish violations of these agreements and to compel the participants to fulfill them. As a rule, the subjects of international law agree with each other on the mechanisms of fulfilling legal prescriptions and responsibility for their violations. However, recently the system of international law is undergoing significant changes, which changes the basis of relations between its subjects and, accordingly, some consequences affect the functioning of national socio-economic systems. At the same time, the application of sanctioning legal regimes concerning individual national economies becomes a demanded multilateral mechanism of economic and political coercion. The so-called "treaty games" are often used to analyze the parties' benefits and the effectiveness of existing and planned international legal regimes. They make it possible to estimate the number of sanctions and compensatory payments to ensure that the signing (implementation) of an agreement benefits all parties and prevents the violation of international legal regimes. As a rule, classical symmetric games with nonzero sums are widely used to model situations arising during the creation and modification of international legal regimes.

In contrast to such studies [47–58], this article presents a new methodology for modeling decision-making processes in complex controlled dynamic systems and forms a mechanism of equilibrium of sanctions and countersanctions, which contributes to solving the problems of stability of equilibria: a linear-quadratic positional differential game of many individuals is considered, coefficient criteria are established if they are met, there is an equilibrium of sanctions and countersanctions in the game, and there is no generally accepted Nash equilibrium.

The substantive meaning of the result obtained in this study is as follows. Both for the subjects using sanction coercive regimes and for the subjects stabilizing the situation by countersanctions, under certain conditions, there is a situation of stable effective active equilibrium with the most tremendous benefits for its participants. Thus, a debatable question is raised: If there is a way to use sanctions for one's benefit, how expedient is their use?

5. Conclusions

This research demonstrates that there is no Nash equilibrium in a linear-quadratic game when Constraints (11) are satisfied. However, there is an equilibrium of sanctions and countersanctions. The novelty of the study of the theory of differential games is that

- The equilibrium of sanctions and countersanctions is simultaneously Pareto-maximal;
- A wide sufficient class of differential games of N persons at $N \geq 2$, in which there is an equilibrium of sanctions and countersanctions and simultaneously there is no Nash equilibrium;
- An algorithm for practical constructing the equilibrium of sanctions and countersanctions is proposed.

The research results allow us to assert that the economic and legal justification of the construction of game-theoretic equilibrium models expands the practical application of the class of problems considered in this paper. At the same time, the authors do not claim the universality of game-theoretic methodology in modeling socio-economic processes. At the same time, they show the urgent need for additional research into the properties used in the analytical construction of various equilibria, including the equilibrium of sanctions and countersanctions, as a mechanism for researching the equilibrium of complex controlled systems.

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