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Solving Integral Equation and Homotopy Result via Fixed Point Method

Badriah Alamri

Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia; baalamri@uj.edu.sa

Abstract: The aim of the present research article is to investigate the existence and uniqueness of a solution to the integral equation and homotopy result. To achieve our objective, we introduce the notion of (α, η, ψ)-contraction in the framework of $\mathfrak{F}$-bipolar metric space and prove some fixed point results for covariant and contravariant mappings. Some coupled fixed point results in $\mathfrak{F}$-bipolar metric space are derived as outcomes of our principal theorems. A non-trivial example is also provided to validate the authenticity of the established results.

Keywords: fixed point; generalized contractions; $\mathfrak{F}$-bipolar metric space; integral equation; homotopy

MSC: 47H10; 46S40; 54H25

1. Introduction

In pure mathematics, one of the most well-known and classical theories is fixed point theory, which has vast applications in various fields. The fundamental and inaugural result in the aforementioned theory is the Banach fixed point theorem [1], which is an attractive and effective tool in investigating existence problems. Over the years, it has been generalized in different directions by several mathematicians. Recently, Samet et al. [2] initiated the conception of α-admissibility and α-ψ-contractions in complete metric spaces and presented some fixed point problems for the aforementioned mappings. Subsequently, Salimi et al. [3] modified these ideas of α-admissibility and α-ψ-contractions and established new fixed point theorems for such mappings in complete metric space.

In all the above outcomes, the idea of metric space represents a crucial and significant aspect, which was introduced by Frechet [4] in 1906. Later on, various researchers extended the notion of metric space by considering the metric postulates or changing its range and domain (see [5–8]). Jleli et al. [9] introduced a fascinating generalization of classical metric space, b-metric space and Branciari metric space, which is well known as an $\mathfrak{F}$-metric space. Subsequetly, Hussain et al. [10] employed the idea of $\mathfrak{F}$-metric space ($\mathfrak{F}$-MS) and demonstrated a number of results for (β, ψ)-contractions.

We take the distance between members of only one set in all these generalizations of metric space. Thus, a question arises: how can the distance between members of two different sets be analyzed? Such questions of computing the distance can be considered in different fields. Mutlu et al. [11] presented the idea of bipolar metric space (bip MS) to address such matters. Moreover, this up-to-date conception of bip MS leads to the evolution and advancement of fixed point theorems. In due course, Mutlu et al. [12] established coupled fixed point results in the framework of bip MS. Kishore et al. [13] extended the concept of coupled fixed point to common coupled fixed point and presented an application of it. Rao et al. [14] proved common coupled fixed point results for Geraghty-type contractions and applied their result to homotopy theory. Gürdal et al. [15] utilized the notion of bip MS to obtain fixed point theorems for (α, ψ)-contractions. A significant task relates to the existence of fixed points in the setting of bip MS (see [16–20]). Rawat et al. [21]

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unified the above two important notions, specifically $\mathfrak{F}$-MS and bip MS, and introduced the notion of $\mathfrak{F}$-bipolar metric space ($\mathfrak{F}$-bip MS) and presented some results.

In the present research article, we introduce the notion of (α, η, ψ)-contraction against the background of $\mathfrak{F}$-bipolar metric space and establish fixed point results for covariant and contravariant mappings. As a consequence, we derive some coupled fixed point results in $\mathfrak{F}$-bipolar metric spaces. An integral equation is explored as an application of our principal result.

2. Preliminaries

The conventional Banach fixed point theorem [1] is given in the following way.

Theorem 1 ([1]). *Let $(\mathcal{S}, \mathfrak{d})$ be a complete metric space (CMS) and let $\mathcal{B} : \mathcal{S} \to \mathcal{S}$. If there exists $\lambda \in [0, 1)$ such that*

$$\mathfrak{d}(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar) \leq \lambda \mathfrak{d}(\mathfrak{w}, \hbar),$$

for all $\mathfrak{w}, \hbar \in \mathcal{S}$, then $\mathcal{B}$ has a unique fixed point.

Samet et al. [2] initiated the following concepts.

Definition 1. *Let Ψ be a family of mappings $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:*
(ψ_1) ψ is nondecreasing,
$(\psi_2)_{\imath=1}^{\infty} \psi^{\imath}(t) < +\infty$, for all $t > 0$, where $\psi^{\imath}$ is the $\imath$-th iterate of ψ.

Lemma 1. *If $\psi \in \Psi$, then, for each $t > 0$, $\psi(t) < t$ and $\psi(0) = 0$.*

Definition 2 ([2]). *Let $\alpha : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ be any function. A mapping $\mathcal{B} : \mathcal{S} \to \mathcal{S}$ is said to be an α-admissible if*

$$\alpha(\mathfrak{w}, \hbar) \geq 1 \implies \alpha(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar) \geq 1,$$

for all $\mathfrak{w}, \hbar \in \mathcal{S}$.

Definition 3 ([2]). *Let $(\mathcal{S}, \mathfrak{d})$ be a metric space. A mapping $\mathcal{B} : \mathcal{S} \to \mathcal{S}$ is said to be (α, ψ)-contraction if there exist some $\alpha : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(\mathfrak{w}, \hbar)\mathfrak{d}(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar) \leq \psi(\mathfrak{d}(\mathfrak{w}, \hbar)),$$

for all $\mathfrak{w}, \hbar \in \mathcal{S}$.

Jleli et al. [9] presented an impressive extension of MS as follows.

Let $\mathfrak{F}$ be the class of mappings $f : (0, +\infty) \to \mathbb{R}$ fulfilling the following assertions:

$(\mathcal{F}_1)$ $f(t) < f(s)$, for $t < s$,
$(\mathcal{F}_2)$ for each sequence $\{t_\imath\} \subseteq \mathbb{R}^+$, $\lim_{\imath \to \infty} t_\imath = 0 \iff \lim_{\imath \to \infty} f(t_\imath) = -\infty$.

Definition 4 ([9]). *Let $\mathcal{S} \neq \emptyset$ and let $\mathfrak{d} : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$. Assume that there exist $(f, \kappa) \in \mathfrak{F} \times [0, +\infty)$ such that for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{S}$,*

(i) $\mathfrak{d}(\mathfrak{w}, \hbar) = 0 \iff \mathfrak{w} = \hbar$,
(ii) $\mathfrak{d}(\mathfrak{w}, \hbar) = \mathfrak{d}(\hbar, \mathfrak{w})$,
(iii) *for every $(u_\imath)_{\imath=1}^{p} \subset \mathcal{S}$ with $(u_1, u_p) = (\mathfrak{w}, \hbar)$, we have*

$$\mathfrak{d}(\mathfrak{w}, \hbar) > 0 \Rightarrow f(\mathfrak{d}(\mathfrak{w}, \hbar)) \leq f\left(\sum_{\imath=1}^{p-1} \mathfrak{d}(u_\imath, u_{\imath+1})\right) + \kappa,$$

for $p \geq 2$ and $p \in \mathbb{N}$. Then, $\mathfrak{d}$ is said to be an $\mathfrak{F}$- metric on $\mathcal{S}$ and $(\mathcal{S}, \mathfrak{d})$ is said to be an $\mathfrak{F}$-MS.

Example 1 ([9]). *Let* $\mathcal{S}=\mathbb{R}$, $f(t) = \ln(t)$ *and* $\kappa = \ln(3)$. *Define* $\mathfrak{d} : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ *by*

$$\mathfrak{d}(\mathfrak{w}, \hbar) = \begin{cases} (\mathfrak{w} - \hbar)^2 & \text{if } (\mathfrak{w}, \hbar) \in [0, 3] \times [0, 3] \\ |\mathfrak{w} - \hbar| & \text{if } (\mathfrak{w}, \hbar) \notin [0, 3] \times [0, 3] \end{cases},$$

and $(\mathcal{S}, \mathfrak{d})$ *is an* $\mathfrak{F}$*-MS.*

Mutlu et al. [11] introduced the idea of bipolar metric space (bip MS) in the following manner.

Definition 5 ([11]). *Let* $\mathcal{S} \neq \emptyset$ *and* $\mathcal{T} \neq \emptyset$ *and let* $\mathfrak{d} : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ *satisfy*

(bi$_1$*)* $\mathfrak{d}(\mathfrak{w}, \hbar) = 0 \Longleftrightarrow \mathfrak{w} = \hbar$,
(bi$_2$*)* $\mathfrak{d}(\mathfrak{w}, \hbar) = \mathfrak{d}(\hbar, \mathfrak{w})$, *if* $\mathfrak{w}, \hbar \in \mathcal{S} \cap \mathcal{T}$,
(bi$_3$*)* $\mathfrak{d}(\mathfrak{w}, \hbar) \leq \mathfrak{d}(\mathfrak{w}, \hbar^/) + \mathfrak{d}(\mathfrak{w}^/, \hbar^/) + \mathfrak{d}(\mathfrak{w}^/, \hbar)$,

for all $(\mathfrak{w}, \hbar), (\mathfrak{w}^/, \hbar^/) \in \mathcal{S} \times \mathcal{T}$. *Then, the triple* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *is called a bip MS.*

Example 2 ([11]). *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be the set of all compact and singleton subsets of* $\mathbb{R}$ *independently. Define* $\mathfrak{d} : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ *by*

$$\mathfrak{d}(\mathfrak{w}, \Xi) = |\mathfrak{w} - \inf(\Xi)| + |\mathfrak{w} - \sup(\Xi)|,$$

for $\{\mathfrak{w}\} \subseteq \mathcal{S}$ *and* $\Xi \subseteq \mathcal{T}$, *and then* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *is a complete bip MS.*

Definition 6. *Let* $(\mathcal{S}_1, \mathcal{T}_1, \mathfrak{d}_1)$ *and* $(\mathcal{S}_2, \mathcal{T}_2, \mathfrak{d}_2)$ *be two bip MSs. A mapping* $\mathcal{B} : \mathcal{S}_1 \cup \mathcal{T}_1 \rightrightarrows \mathcal{S}_2 \cup \mathcal{T}_2$ *is said to be a covariant mapping, if* $\mathcal{B}(\mathcal{S}_1) \subseteq \mathcal{S}_2$ *and* $\mathcal{B}(\mathcal{T}_1) \subseteq \mathcal{T}_2$. *Similarly, a mapping* $\mathcal{B} : \mathcal{S}_1 \cup \mathcal{T}_1 \rightrightarrows \mathcal{S}_2 \cup \mathcal{T}_2$ *is called a contravariant mapping, if* $\mathcal{B}(\mathcal{S}_1) \subseteq \mathcal{T}_2$ *and* $\mathcal{B}(\mathcal{S}_2) \subseteq \mathcal{T}_1$.

We will symbolize the covariant mapping as $\mathcal{B} : (\mathcal{S}_1, \mathcal{T}_1) \rightrightarrows (\mathcal{S}_2, \mathcal{T}_2)$ and the contravariant mapping as $\mathcal{B} : (\mathcal{S}_1, \mathcal{T}_1) \rightleftarrows (\mathcal{S}_2, \mathcal{T}_2)$.

Rawat et al. [21] unified the above two novel notions, $\mathfrak{F}$-MS and bip MS, and introduced the notion of $\mathfrak{F}$-bipolar metric space ($\mathfrak{F}$-bip MS) in the following way.

Definition 7 ([21]). *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be nonempty sets and let* $\mathfrak{d} : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$. *Suppose that there exist* $(f, \kappa) \in \mathfrak{F} \times [0, +\infty)$ *such that, for all* $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$,

(D$_1$*)* $\mathfrak{d}(\mathfrak{w}, \hbar) = 0 \Longleftrightarrow \mathfrak{w} = \hbar$,
(D$_2$*)* $\mathfrak{d}(\mathfrak{w}, \hbar) = \mathfrak{d}(\hbar, \mathfrak{w})$, *if* $\mathfrak{w}, \hbar \in \mathcal{S} \cap \mathcal{T}$,
(D$_3$*) for every* $(u_\imath)_{\imath=1}^p \subset \mathcal{S}$ *and* $(v_\imath)_{\imath=1}^p \subset \mathcal{T}$ *with* $(u_1, v_p) = (\mathfrak{w}, \hbar)$, *we have*

$$\mathfrak{d}(\mathfrak{w}, \hbar) > 0 \Rightarrow f(\mathfrak{d}(\mathfrak{w}, \hbar)) \leq f\left(\sum_{\imath=1}^{p-1} \mathfrak{d}(u_{\imath+1}, v_\imath) + \sum_{\imath=1}^{p} \mathfrak{d}(u_\imath, v_\imath)\right) + \kappa,$$

for $p \geq 2$ *and* $p \in \mathbb{N}$. *Then,* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *is called an* $\mathfrak{F}$*-bip MS.*

Example 3. *Let* $\mathcal{S} = \{1, 2\}$ *and* $\mathcal{T} = \{2, 7\}$. *Define* $\mathfrak{d} : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ *by*

$$\mathfrak{d}(1, 2) = 6,\ \mathfrak{d}(1, 7) = 10, \mathfrak{d}(2, 7) = 2,\ \mathfrak{d}(2, 2) = 0,$$

and then $\mathfrak{d}$ *satisfies all the conditions of an* $\mathfrak{F}$*-bip metric with* $\kappa = 0$ *and* $f(\mathfrak{t}) = \ln t$, *for* $t > 0$. *Thus,* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *is an* $\mathfrak{F}$*-bip MS but not a bip MS.*

Remark 1 ([21]). *Taking* $\mathcal{T} = \mathcal{S}$, $p = 2\imath$, $u_\jmath = u_{2\jmath-1}$ *and* $v_\jmath = u_{2\jmath}$ *in the above definition (7), we obtain a sequence* $(u_\jmath)_{\jmath=1}^{2\imath} \in \mathcal{S}$ *with* $(u_1, u_{2\imath}) = (\mathfrak{w}, \hbar)$ *such that condition (iii) of Definition 4 holds. Thus, every* $\mathfrak{F}$*-MS is an* $\mathfrak{F}$*-bip MS but the converse is not true in general.*

Definition 8 ([21]). *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be an $\mathfrak{F}$-bip MS.*

(i) *An element $\mathfrak{w} \in \mathcal{S} \cup \mathcal{T}$ is called a left point if $\mathfrak{w} \in \mathcal{S}$ and $\mathfrak{w} \in \mathcal{S} \cup \mathcal{T}$ is called a right point if $\mathfrak{w} \in \mathcal{T}$. Moreover, $\mathfrak{w}$ is called a central point if it is both a left and right point.*

(ii) *A sequence $(\mathfrak{w}_\imath)$ on $\mathcal{S}$ is said to be a left sequence and $(\hbar_\imath)$ on $\mathcal{T}$ is called a right sequence. A left sequence or a right sequence is called a sequence in an $\mathfrak{F}$-bip MS.*

(iii) *The sequence $(\mathfrak{w}_\imath)$ converges to a point $\mathfrak{w}$, if and only if $(\mathfrak{w}_\imath)$ is a left sequence, $\mathfrak{w}$ is a right point and $\lim_{\imath \to \infty} \mathfrak{d}(\mathfrak{w}_\imath, \mathfrak{w}) = 0$ or $(\mathfrak{w}_\imath)$ is a right sequence, $\mathfrak{w}$ is a left point and $\lim_{\imath \to \infty} \mathfrak{d}(\mathfrak{w}, \mathfrak{w}_\imath) = 0$. A bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ on $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is a sequence on the set $\mathcal{S} \times \mathcal{T}$. If $(\mathfrak{w}_\imath)$ and $(\hbar_\imath)$ are convergent, then the bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ is also convergent, and if $(\mathfrak{w}_\imath)$ and $(\hbar_\imath)$ converge to a common element, then the bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ is said to be biconvergent.*

(iv) *A bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ in an $\mathfrak{F}$-bip MS $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is called a Cauchy bisequence if, for each $\epsilon > 0$, there exists $\imath_0 \in \mathbb{N}$, such that $\mathfrak{d}(\mathfrak{w}_\imath, \hbar_p) < \epsilon$, for all $\imath, p \geq \imath_0$.*

Definition 9 ([21]). *An $\mathfrak{F}$-bip MS $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is said to be complete, if every Cauchy bisequence in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is convergent.*

3. Fixed Point Results for Covariant Mappings

Definition 10. *Let $\alpha : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ be any function. A mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is said to be covariant α-admissible if*

$$\alpha(\mathfrak{w}, \hbar) \geq 1 \implies \alpha(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar) \geq 1, \tag{1}$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Example 4. *Let $\mathcal{S} = [0, +\infty)$ and $\mathcal{T} = (-\infty, 0]$ and $\alpha : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ is defined as*

$$\alpha(\mathfrak{w}, \hbar) = \begin{cases} 1, & \text{if } \mathfrak{w} \neq \hbar, \\ 0, & \text{if } \mathfrak{w} = \hbar. \end{cases}$$

A covariant mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ defined by $\mathcal{B}(\mathfrak{w}) = \mathfrak{w}$ is covariant α-admissible.

Definition 11. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be an $\mathfrak{F}$-bip MS and $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightleftarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is a covariant mapping. A mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is said to be covariant α-admissible with respect to η if there exist the functions $\alpha, \eta : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ such that*

$$\alpha(\mathfrak{w}, \hbar) \geq \eta(\mathfrak{w}, \hbar) \ \text{ implies } \ \alpha(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar) \geq \eta(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar),$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Remark 2. *If we take $\eta(\mathfrak{w}, \hbar) = 1$, then this Definition 14 reduces to Definition 13. Moreover, if we take $\alpha(\mathfrak{w}, \hbar) = 1$, then we can say that $\mathcal{B}$ is an η-subadmissible mapping.*

Definition 12. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be an $\mathfrak{F}$-bip MS. A mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is said to be a covariant (α, η, ψ)-contraction if $\mathcal{B}$ is covariant and there exist two functions $\alpha, \eta : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(\mathfrak{w}, \hbar) \geq \eta(\mathfrak{w}, \hbar) \implies \mathfrak{d}(\mathcal{B}\hbar, \mathcal{B}\mathfrak{w}) \leq \psi(\mathfrak{d}(\mathfrak{w}, \hbar)), \tag{2}$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Remark 3. *A mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ satisfying the Banach contraction in $\mathfrak{F}$-bipolar metric space $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is a covariant (α, η, ψ)-contraction with*

$$\alpha(\mathfrak{w}, \hbar) = \eta(\mathfrak{w}, \hbar) = 1,$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$ and $\psi(t) = kt$, for some $k \in [0, 1)$ and for $t \geq 1$.

(P) there exists $z \in \mathcal{S} \cap \mathcal{T}$ such that $\alpha(\mathfrak{w}, z) \geq 1$ and $\alpha(z, \hbar) \geq 1$ for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Theorem 2. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be a covariant (α, η, ψ)-contraction. Assume that the following assertions hold:*

(i) $\mathcal{B}$ is covariant α-admissible with respect to η,
(ii) there exists $\mathfrak{w}_0 \in \mathcal{S}, \hbar_0 \in \mathcal{T}$ such that $\alpha(\mathfrak{w}_0, \hbar_0) \geq \eta(\mathfrak{w}_0, \hbar_0)$ and $\alpha(\mathfrak{w}_0, \mathcal{B}\hbar_0) \geq \eta(\mathfrak{w}_0, \mathcal{B}\hbar_0)$,
(iii) $\mathcal{B}$ is continuous or, if $(\mathfrak{w}_\imath, \hbar_\imath)$ is a bisequence in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ such that $\alpha(\mathfrak{w}_\imath, \hbar_\imath) \geq \eta(\mathfrak{w}_\imath, \hbar_\imath)$, for all $\imath \in \mathbb{N}$ with $\mathfrak{w}_\imath \to \omega$ and $\hbar_\imath \to \omega$, as $\imath \to \infty$ for $\omega \in \mathcal{S} \cap \mathcal{T}$, then $\alpha(\omega, \hbar_\imath) \geq \eta(\omega, \hbar_\imath)$, for all $\imath \in \mathbb{N}$.

Then, the mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ has a fixed point. Furthermore, if the property (P) holds, then the fixed point is unique.

Proof. Let $\mathfrak{w}_0$ and $\hbar_0$ be arbitrary points in $\mathcal{S}$ and $\mathcal{T}$, respectively, and suppose that $\alpha(\mathfrak{w}_0, \hbar_0) \geq \eta(\mathfrak{w}_0, \hbar_0)$ and $\alpha(\mathfrak{w}_0, \mathcal{B}\hbar_0) \geq \eta(\mathfrak{w}_0, \mathcal{B}\hbar_0)$. Define the bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ by

$$\mathfrak{w}_{\imath+1} = \mathcal{B}\mathfrak{w}_\imath \text{ and } \hbar_{\imath+1} = \mathcal{B}\hbar_\imath,$$

for all $\imath \in \mathbb{N}$. As $\mathcal{B}$ is a covariant α-admissible mapping with respect to η, we have

$$\alpha(\mathfrak{w}_0, \hbar_0) \geq \eta(\mathfrak{w}_0, \hbar_0),$$

which implies

$$\alpha(\mathfrak{w}_1, \hbar_1) = \alpha(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_0) \geq \eta(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_0) = \eta(\mathfrak{w}_1, \hbar_1).$$

and

$$\alpha(\mathfrak{w}_0, \hbar_1) = \alpha(\mathfrak{w}_0, \mathcal{B}\hbar_0) \geq \eta(\mathfrak{w}_0, \mathcal{B}\hbar_0) = \eta(\mathfrak{w}_0, \hbar_1),$$

which implies

$$\alpha(\mathfrak{w}_1, \hbar_2) = \alpha(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_1) \geq \eta(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_1) = \eta(\mathfrak{w}_1, \hbar_2).$$

Similarly,

$$\alpha(\mathfrak{w}_1, \hbar_1) = \alpha(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_0) \geq \eta(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_0) = \eta(\mathfrak{w}_1, \hbar_1),$$

which implies

$$\alpha(\mathfrak{w}_2, \hbar_2) = \alpha(\mathcal{B}\mathfrak{w}_1, \mathcal{B}\hbar_1) \geq \eta(\mathcal{B}\mathfrak{w}_1, \mathcal{B}\hbar_1) = \eta(\mathfrak{w}_2, \hbar_2),$$

and

$$\alpha(\mathfrak{w}_1, \hbar_2) = \alpha(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_1) \geq \eta(\mathcal{B}\mathfrak{w}_0, \mathcal{B}\hbar_1) = \eta(\mathfrak{w}_1, \hbar_2),$$

which implies

$$\alpha(\mathfrak{w}_2, \hbar_3) = \alpha(\mathcal{B}\mathfrak{w}_1, \mathcal{B}\hbar_2) \geq \eta(\mathcal{B}\mathfrak{w}_1, \mathcal{B}\hbar_2) = \eta(\mathfrak{w}_2, \hbar_3).$$

Likewise,

$$\alpha(\mathfrak{w}_2, \hbar_2) = \alpha(\mathcal{B}\mathfrak{w}_1, \mathcal{B}\hbar_1) \geq \eta(\mathcal{B}\mathfrak{w}_1, \mathcal{B}\hbar_1) = \eta(\mathfrak{w}_2, \hbar_2),$$

which implies

$$\alpha(\mathfrak{w}_3, \hbar_3) = \alpha(\mathcal{B}\mathfrak{w}_2, \mathcal{B}\hbar_2) \geq \eta(\mathcal{B}\mathfrak{w}_2, \mathcal{B}\hbar_2) = \eta(\mathfrak{w}_3, \hbar_3).$$

Continuing in this way, we have

$$\alpha(\mathfrak{w}_{\imath+1}, \hbar_\imath) \geq \eta(\mathfrak{w}_{\imath+1}, \hbar_\imath) \text{ and } \alpha(\mathfrak{w}_{\imath+1}, \hbar_{\imath+1}) \geq \eta(\mathfrak{w}_{\imath+1}, \hbar_{\imath+1}), \quad (3)$$

for all $\imath \in \mathbb{N}$. Now, by (2) and (3), we have

$$\mathfrak{d}(\mathfrak{w}_\imath, \hbar_{\imath+1}) = \mathfrak{d}(\mathcal{B}\mathfrak{w}_{\imath-1}, \mathcal{B}\hbar_\imath) \leq \mathfrak{d}(\mathcal{B}\mathfrak{w}_{\imath-1}, \mathcal{B}\hbar_\imath) \leq \psi(\mathfrak{d}(\mathfrak{w}_{\imath-1}, \hbar_\imath)) \tag{4}$$

for all $\imath \in \mathbb{N}$. Additionally,

$$\mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_{\imath+1}) = \mathfrak{d}(\mathcal{B}\mathfrak{w}_\imath, \mathcal{B}\hbar_\imath) \leq \mathfrak{d}(\mathcal{B}\mathfrak{w}_\imath, \mathcal{B}\hbar_\imath) \leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath)), \tag{5}$$

for all $\imath \in \mathbb{N}$. By (4) and mathematical induction, we obtain

$$\mathfrak{d}(\mathfrak{w}_\imath, \hbar_{\imath+1}) \leq \psi(\mathfrak{d}(\mathfrak{w}_{\imath-1}, \hbar_\imath)) \leq \psi(\psi(\mathfrak{d}(\mathfrak{w}_{\imath-2}, \hbar_{\imath-1}))) \leq \ldots \leq \psi^\imath(\mathfrak{d}(\mathfrak{w}_0, \hbar_1)). \tag{6}$$

Similarly, by (5) and mathematical induction, we obtain

$$\mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_{\imath+1}) \leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath)) \leq \psi(\psi(\mathfrak{d}(\mathfrak{w}_{\imath-1}, \hbar_{\imath-1}))) \leq \ldots \leq \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)), \tag{7}$$

for all $\imath \in \mathbb{N}$. Let $(f, \kappa) \in \mathfrak{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \kappa. \tag{8}$$

Let there exist $\epsilon > 0$ and $\imath(\epsilon) \in \mathbb{N}$ such that

$$\sum_{\imath \geq \imath(\epsilon)} \psi^\imath(\mathfrak{d}(\mathfrak{w}_0, \hbar_1)) < \frac{\epsilon}{2},$$

and

$$\sum_{\imath \geq \imath(\epsilon)} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) < \frac{\epsilon}{2}.$$

Now, for $p > \imath \geq \imath(\epsilon)$, by applying (D_3), we have that $\mathfrak{d}\big(\mathfrak{w}_\imath, \hbar_p\big) > 0$ implies

$$\begin{aligned}
f\big(\mathfrak{d}\big(\mathfrak{w}_\imath, \hbar_p\big)\big) &\leq f\left(\begin{array}{c} \mathfrak{d}(\mathfrak{w}_\imath, \hbar_{\imath+1}) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_{\imath+1}) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_{\imath+2}) + \\ \ldots + \mathfrak{d}\big(\mathfrak{w}_{p-1}, \hbar_{p-1}\big) + \mathfrak{d}\big(\mathfrak{w}_{p-1}, \hbar_p\big) \end{array}\right) + \kappa \\
&\leq f\left(\sum_{j=\imath}^{p-1} \mathfrak{d}\big(\mathfrak{w}_j, \hbar_{j+1}\big) + \sum_{j=\imath}^{p-2} \mathfrak{d}\big(\mathfrak{w}_{j+1}, \hbar_{j+1}\big)\right) + \kappa \\
&\leq f\left(\sum_{j=\imath}^{p-1} \psi^j(\mathfrak{d}(\mathfrak{w}_0, \hbar_1)) + \sum_{j=\imath}^{p-2} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0))\right) + \kappa \\
&\leq f\left(\sum_{\imath \geq \imath(\epsilon)} \psi^\imath(\mathfrak{d}(\mathfrak{w}_0, \hbar_1)) + \sum_{\imath \geq \imath(\epsilon)} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0))\right) + \kappa \\
&< f(\epsilon).
\end{aligned}$$

for all $j \in \mathbb{N}$. Similarly, for $\imath > p \geq \imath(\epsilon)$, by applying (D_3), we have that $\mathfrak{d}\left(\mathfrak{w}_\imath, \hbar_p\right) > 0$ implies

$$
\begin{aligned}
f\left(\mathfrak{d}\left(\mathfrak{w}_\imath, \hbar_p\right)\right) &\leq f\left(\begin{array}{c} \mathfrak{d}\left(\mathfrak{w}_p, \hbar_p\right) + \mathfrak{d}\left(\mathfrak{w}_p, \hbar_{p+1}\right) + \mathfrak{d}\left(\mathfrak{w}_{p+1}, \hbar_{p+1}\right) + \\ \ldots + \mathfrak{d}\left(\mathfrak{w}_\imath, \hbar_{\imath+1}\right) + \mathfrak{d}\left(\mathfrak{w}_\imath, \hbar_\imath\right) \end{array}\right) + \kappa \\
&\leq f\left(\sum_{j=p}^{\imath} \mathfrak{d}\left(\mathfrak{w}_j, \hbar_j\right) + \sum_{j=\imath}^{\imath} \mathfrak{d}\left(\mathfrak{w}_j, \hbar_{j+1}\right)\right) + \kappa \\
&\leq f\left(\sum_{j=p}^{\imath} \psi^j(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) + \sum_{j=p}^{\imath} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_1))\right) + \kappa \\
&\leq f\left(\sum_{\imath \geq \imath(\epsilon)} \psi^\imath(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) + \sum_{\imath \geq \imath(\epsilon)} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_1))\right) + \kappa, \\
&< f(\epsilon)
\end{aligned}
$$

for all $j \in \mathbb{N}$. Then, by $(\mathfrak{F}_1)$, $\mathfrak{d}\left(\mathfrak{w}_\imath, \hbar_p\right) < \epsilon$, for all $p, \imath \geq \imath_0$. Thus, $(\mathfrak{w}_\imath, \hbar_\imath)$ is a Cauchy bisequence in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$. As $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is complete, $(\mathfrak{w}_\imath, \hbar_\imath)$ biconverges to a point $\omega \in \mathcal{S} \cap \mathcal{T}$. Thus, $(\mathfrak{w}_\imath) \to \omega$, $(\hbar_\imath) \to \omega$. Moreover, as $\mathcal{B}$ is continuous, we obtain

$$(\mathfrak{w}_\imath) \to \omega \implies (\mathfrak{w}_{\imath+1}) = (\mathcal{B}\mathfrak{w}_\imath) \to \mathcal{B}\omega.$$

Additionally, since $(\hbar_\imath)$ has a limit ω in $\mathcal{S} \cap \mathcal{T}$. Since the limit is unique in $\mathfrak{F}$-bip MS, $\mathcal{B}\omega = \omega$. Thus, $\mathcal{B}$ has a fixed point.

As a bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is such that $\alpha(\mathfrak{w}_\imath, \hbar_\imath) \geq \eta(\mathfrak{w}_\imath, \hbar_\imath)$, for all $\imath \in \mathbb{N}$ with $\mathfrak{w}_\imath \to \omega$ and $\hbar_\imath \to \omega$, as $\imath \to \infty$ for $\omega \in \mathcal{S} \cap \mathcal{T}$, then, by hypothesis (iii), we have $\alpha(\omega, \hbar_\imath) \geq \alpha(\omega, \hbar_\imath)$, for all $\imath \in \mathbb{N}$. Now, by (19), we have

$$
\begin{aligned}
f(\mathfrak{d}(\mathcal{B}\omega, \omega)) &\leq f(\mathfrak{d}(\mathcal{B}\omega, \mathcal{B}\hbar_\imath) + \mathfrak{d}(\mathcal{B}\mathfrak{w}_\imath, \mathcal{B}\hbar_\imath) + \mathfrak{d}(\mathcal{B}\mathfrak{w}_\imath, \omega)) + \kappa \\
&\leq f(\alpha(\omega, \hbar_\imath)\mathfrak{d}(\mathcal{B}\omega, \mathcal{B}\hbar_\imath) + (\mathcal{B}\mathfrak{w}_\imath, \mathcal{B}\hbar_\imath) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \omega)) + \kappa \\
&\leq f(\psi(\mathfrak{d}(\omega, \hbar_\imath)) + \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath)) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \omega)) + \kappa \\
&\leq f\left(\begin{array}{c} \psi(\mathfrak{d}(\omega, \hbar_\imath)) \\ +\psi\left(\begin{array}{c} \mathfrak{d}(\mathfrak{w}_\imath, \omega) \\ +\mathfrak{d}(\omega, \omega) + \mathfrak{d}(\omega, \hbar_\imath) \end{array}\right) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \omega) \end{array}\right) + \kappa.
\end{aligned}
$$

Taking the limit as $\imath \to \infty$ and using the continuity of f and ψ at $t = 0$, we have $\mathfrak{d}(\mathcal{B}\omega, \omega) = 0$. Thus, $\mathcal{B}\omega = \omega$. Hence, $\mathcal{B}$ has a fixed point.

Now, if ϖ is another fixed point of $\mathcal{B}$, then $\mathcal{B}\varpi = \varpi$ implies that $\varpi \in \mathcal{S} \cap \mathcal{T}$ such that $\omega \neq \varpi$. Then, by the property (P), there exists $z \in \mathcal{S} \cap \mathcal{T}$ such that

$$\alpha(\omega, z) \geq \eta(\omega, z) \text{ and } \alpha(z, \varpi) \geq \eta(z, \varpi). \tag{9}$$

Since $\mathcal{B}$ is a covariant α-admissible mapping with respect to η, by (9), we have

$$\alpha(\omega, \mathcal{B}^\imath z) \geq \eta(\omega, \mathcal{B}^\imath z) \text{ and } \alpha(\mathcal{B}^\imath z, \varpi) \geq \eta(\mathcal{B}^\imath z, \varpi), \tag{10}$$

for all $\imath \in \mathbb{N}$. Now, by $(\mathfrak{F}_1)$ and (2), we have

$$
\begin{aligned}
f(\mathfrak{d}(\omega, \mathcal{B}^\imath z)) &\leq f\left(\mathfrak{d}\left(\mathcal{B}\omega, \mathcal{B}\left(\mathcal{B}^{\imath-1} z\right)\right)\right) \\
&\leq f\left(\mathfrak{d}\left(\mathcal{B}\omega, \mathcal{B}\mathcal{B}^{\imath-1} z\right)\right) \\
&\leq f\left(\psi\left(\mathfrak{d}\left(\omega, \mathcal{B}^{\imath-1} z\right)\right)\right) \\
&\leq \ldots \leq f(\psi^\imath(\mathfrak{d}(\omega, z))).
\end{aligned} \tag{11}
$$

Similarly, we have

$$\begin{aligned} f(\mathfrak{d}(\mathcal{B}^{\imath}z,\varpi)) &\leq f\left(\mathfrak{d}\left(\mathcal{B}\left(\mathcal{B}^{\imath-1}z\right),\mathcal{B}\varpi\right)\right) \\ &\leq f\left(\mathfrak{d}\left(\mathcal{B}\left(\mathcal{B}^{\imath-1}z\right),\mathcal{B}\varpi\right)\right) \\ &\leq f\left(\psi\left(\mathfrak{d}\left(\mathcal{B}^{\imath-1}z,\varpi\right)\right)\right) \\ &\leq \ldots \leq f\left(\psi^{\imath}\left(\mathfrak{d}\left(\mathcal{B}^{\imath-1}z,\varpi\right)\right)\right). \end{aligned} \tag{12}$$

Letting $\imath \to +\infty$ in (11) and (12) and using the continuity of f and ψ, we have

$$\lim_{\imath\to\infty} f(\mathfrak{d}(\omega,\mathcal{B}^{\imath}z)) = -\infty, \tag{13}$$

and

$$\lim_{\imath\to\infty} f(\mathfrak{d}(\mathcal{B}^{\imath}z,\varpi)) = -\infty. \tag{14}$$

Thus, from (13) and (14) by $(\mathfrak{F}_2)$, we have

$$\mathcal{B}^{\imath}z \to \omega \text{ and } \mathcal{B}^{\imath}z \to \varpi,$$

which is a contradiction because the limit is unique. Hence, $\omega = \varpi \in \mathcal{S} \cap \mathcal{T}$. □

Example 5. *Let* $\mathcal{S}=\{9,10,18,20\}$ *and* $\mathcal{T}=\{3,5,11,18\}$*. Define the usual metric* $\mathfrak{d} : \mathcal{S}\times\mathcal{T}\to [0,\infty)$ *by*

$$\mathfrak{d}(\mathfrak{w},\hbar) = 2^{|\mathfrak{w}-\hbar|}.$$

Then, $(\mathcal{S},\mathcal{T},\mathfrak{d})$ *is a complete* $\mathfrak{F}$*-bip MS. Define the covariant mapping* $\mathcal{B} : \mathcal{S}\cup\mathcal{T} \to \mathcal{S}\cup\mathcal{T}$ *by*

$$\mathcal{B}(\mathfrak{w}) = \begin{cases} 18, & \text{if } \mathfrak{w} \in \mathcal{S}\cup\{11\} \\ 9, & \text{otherwise.} \end{cases}$$

Then, all the conditions of Theorem 2 are satisfied with $\psi(\mathfrak{t}) = \frac{3}{4}\mathfrak{t}$*. Hence, by Theorem 2,* $\mathcal{B}$ *must have a unique fixed point, which is* $18 \in \mathcal{S}\cap\mathcal{T}$*.*

By taking $\eta(\mathfrak{w},\hbar) = 1$ in Theorem 2, we have the following result.

Corollary 1. *Let* $(\mathcal{S},\mathcal{T},\mathfrak{d})$ *be a complete* $\mathfrak{F}$*-bip MS and let* $\mathcal{B} : (\mathcal{S},\mathcal{T},\mathfrak{d}) \rightrightarrows (\mathcal{S},\mathcal{T},\mathfrak{d})$ *be a covariant mapping. Assume that there exists* $\psi \in \Psi$ *and* $\alpha : \mathcal{S}\times\mathcal{S}\to [0,+\infty)$ *such that*

$$\alpha(\mathfrak{w},\hbar) \geq 1 \Longrightarrow \mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar) \leq \psi(\mathfrak{d}(\mathfrak{w},\hbar)),$$

for all $(\mathfrak{w},\hbar) \in \mathcal{S}\times\mathcal{T}$*.*

Moreover, suppose that the following postulations hold:

(i) $\mathcal{B}$ *is covariant* α*-admissible,*
(ii) there exists $\mathfrak{w}_0 \in \mathcal{S}$, $\hbar_0 \in \mathcal{T}$ *such that* $\alpha(\mathfrak{w}_0,\hbar_0) \geq 1$ *and* $\alpha(\mathfrak{w}_0,\mathcal{B}\hbar_0) \geq 1$,
(iii) $\mathcal{B}$ *is continuous or, if* $(\mathfrak{w}_\imath,\hbar_\imath)$ *is a bisequence in* $(\mathcal{S},\mathcal{T},\mathfrak{d})$ *such that* $\alpha(\mathfrak{w}_\imath,\hbar_\imath) \geq 1$*, for* $\imath \in \mathbb{N}$ *with* $\mathfrak{w}_\imath \to \omega$ *and* $\hbar_\imath \to \omega$*, as* $\imath \to \infty$ *for* $\omega \in \mathcal{S}\cap\mathcal{T}$*, then* $\alpha(\omega,\hbar_\imath) \geq 1$*, for* $\imath \in \mathbb{N}$.

Then, the mapping $\mathcal{B} : (\mathcal{S},\mathcal{T},\mathfrak{d}) \rightrightarrows (\mathcal{S},\mathcal{T},\mathfrak{d})$ *has a fixed point.*

By taking $\alpha(\mathfrak{w},\hbar) = 1$ in Theorem 2, we have the following result.

Corollary 2. *Let* $(\mathcal{S},\mathcal{T},\mathfrak{d})$ *be a complete* $\mathfrak{F}$*-bip MS and let* $\mathcal{B} : (\mathcal{S},\mathcal{T},\mathfrak{d}) \rightrightarrows (\mathcal{S},\mathcal{T},\mathfrak{d})$ *be a covariant mapping. Assume that there exists* $\psi \in \Psi$ *and* $\eta : \mathcal{S}\times\mathcal{S}\to [0,+\infty)$ *such that*

$$\eta(\mathfrak{w},\hbar) \leq 1 \Longrightarrow \mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar) \leq \psi(\mathfrak{d}(\mathfrak{w},\hbar)),$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Moreover, suppose that the following postulations hold:

(i) $\mathcal{B}$ *is covariant* η*-subadmissible,*
(ii) *there exists* $\mathfrak{w}_0 \in \mathcal{S}$, $\hbar_0 \in \mathcal{T}$ *such that* $\eta(\mathfrak{w}_0, \hbar_0) \leq 1$ *and* $\eta(\mathfrak{w}_0, \mathcal{B}\hbar_0) \leq 1$,
(iii) $\mathcal{B}$ *is continuous or, if* $(\mathfrak{w}_\imath, \hbar_\imath)$ *is a bisequence in* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *such that* $\eta(\mathfrak{w}_\imath, \hbar_\imath) \leq 1$, *for all* $\imath \in \mathbb{N}$ *with* $\mathfrak{w}_\imath \to \omega$ *and* $\hbar_\imath \to \omega$, *as* $\imath \to \infty$ *for* $\omega \in \mathcal{S} \cap \mathcal{T}$, *then* $\eta(\omega, \hbar_\imath) \leq 1$, *for all* $\imath \in \mathbb{N}$.

Then, the mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *has a fixed point.*

The following result is a direct consequence of Corollary 1.

Corollary 3. *Let* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *be a complete* $\mathfrak{F}$*-bip MS and let* $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *be a covariant mapping. Assume that there exists* $\psi \in \Psi$ *and* $\alpha : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ *such that*

$$\alpha(\mathfrak{w}, \hbar)\mathfrak{d}(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar) \leq \psi(\mathfrak{d}(\mathfrak{w}, \hbar)),$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Moreover, suppose that the following postulations hold:

(i) $\mathcal{B}$ *is covariant* α*-admissible,*
(ii) *there exists* $\mathfrak{w}_0 \in \mathcal{S}$, $\hbar_0 \in \mathcal{T}$ *such that* $\alpha(\mathfrak{w}_0, \hbar_0) \geq 1$ *and* $\alpha(\mathfrak{w}_0, \mathcal{B}\hbar_0) \geq 1$,
(ii) $\mathcal{B}$ *is continuous or, if* $(\mathfrak{w}_\imath, \hbar_\imath)$ *is a bisequence in* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *such that* $\alpha(\mathfrak{w}_\imath, \hbar_\imath) \geq 1$, *for* $\imath \in \mathbb{N}$ *with* $\mathfrak{w}_\imath \to \omega$ *and* $\hbar_\imath \to \omega$, *as* $\imath \to \infty$ *for* $\omega \in \mathcal{S} \cap \mathcal{T}$, *then* $\alpha(\omega, \hbar_\imath) \geq 1$, *for all* $\imath \in \mathbb{N}$.

Then, the mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *has a fixed point.*

Remark 4. *If we define* $\alpha : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ *by* $\alpha(\mathfrak{w}, \hbar) = 1$ *and* $\psi(\mathfrak{t}) = k\mathfrak{t}$, *where* $0 < k < 1$ *in Corollary 3, then we deduce the principal result of Rawat et al. [21].*

Remark 5. *Taking* $f(t) = \ln(t)$, *for* $t > 0$ *and* $\kappa = 0$ *in Definition 7, then* $\mathfrak{F}$*-bip MS is reduced to bip MS. Thus, the main result of Gürdal et al. [15] is a direct consequence of the above result.*

Remark 6. *If we take* $\mathcal{S} = \mathcal{T}$ *in Definition 7, then the* $\mathfrak{F}$*-bip MS is reduced to* $\mathfrak{F}$*-MS and we derive the leading result of Hussain et al. [10] from the above corollary.*

Corollary 4. *Let* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *be a complete* $\mathfrak{F}$*-bip MS and let* $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$. *Assume that there exist* $\psi \in \Psi$, $\alpha : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ *and* $\ell > 0$ *such that*

$$(\alpha(\mathfrak{w}, \hbar) + \ell)^{\mathfrak{d}(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar)} \leq (1 + \ell)^{\psi(\mathfrak{d}(\mathfrak{w}, \hbar))}, \tag{15}$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Moreover, suppose that the following postulations hold:

(i) $\mathcal{B}$ *is covariant* α*-admissible,*
(ii) *there exists* $\mathfrak{w}_0 \in \mathcal{S}$, $\hbar_0 \in \mathcal{T}$ *such that* $\alpha(\mathfrak{w}_0, \hbar_0) \geq 1$ *and* $\alpha(\mathfrak{w}_0, \mathcal{B}\hbar_0) \geq 1$,
(iii) $\mathcal{B}$ *is continuous or, if* $(\mathfrak{w}_\imath, \hbar_\imath)$ *is a bisequence in* $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *such that* $\alpha(\mathfrak{w}_\imath, \hbar_\imath) \geq 1$, *for all* $\imath \in \mathbb{N}$ *with* $\mathfrak{w}_\imath \to \omega$ *and* $\hbar_\imath \to \omega$, *as* $\imath \to \infty$ *for* $\omega \in \mathcal{S} \cap \mathcal{T}$, *then* $\alpha(\omega, \hbar_\imath) \geq 1$, *for* $\imath \in \mathbb{N}$.

Then, the mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ *has a fixed point.*

Proof. Let $\alpha(\mathfrak{w}, \hbar) \geq 1$. Then, by (15), we have

$$(1 + \ell)^{\mathfrak{d}(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar)} \leq (\alpha(\mathfrak{w}, \hbar) + \ell)^{\mathfrak{d}(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar)} \leq (1 + \ell)^{\psi(\mathfrak{d}(\mathfrak{w}, \hbar))},$$

which implies $\mathfrak{d}(\mathcal{B}\mathfrak{w}, \mathcal{B}\hbar) \leq \psi(\mathfrak{d}(\mathfrak{w}, \hbar))$, and all the conditions of Corollary 1 are satisfied and $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightrightarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ has a fixed point. □

Similarly, we have the following corollary.

Corollary 5. *Let $(\mathcal{S},\mathcal{T},\mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightrightarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$. Assume that there exist $\psi\in\Psi$, $\alpha:\mathcal{S}\times\mathcal{S}\rightarrow[0,+\infty)$ and $\ell>0$ such that*

$$(\mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar)+\ell)^{\alpha(\mathfrak{w},\hbar)}\leq\psi(\mathfrak{d}(\mathfrak{w},\hbar))+\ell, \tag{16}$$

for all $(\mathfrak{w},\hbar)\in\mathcal{S}\times\mathcal{T}$.

Moreover, suppose that the following postulations hold:

(i) *$\mathcal{B}$ is covariant α-admissible,*
(ii) *there exists $\mathfrak{w}_0\in\mathcal{S}$, $\hbar_0\in\mathcal{T}$ such that $\alpha(\mathfrak{w}_0,\hbar_0)\geq1$ and $\alpha(\mathfrak{w}_0,\mathcal{B}\hbar_0)\geq1$,*
(iii) *$\mathcal{B}$ is continuous or, if $(\mathfrak{w}_\imath,\hbar_\imath)$ is a bisequence in $(\mathcal{S},\mathcal{T},\mathfrak{d})$ such that $\alpha(\mathfrak{w}_\imath,\hbar_\imath)\geq1$, for $\imath\in\mathbb{N}$ with $\mathfrak{w}_\imath\rightarrow\omega$ and $\hbar_\imath\rightarrow\omega$, as $\imath\rightarrow\infty$ for $\omega\in\mathcal{S}\cap\mathcal{T}$, then $\alpha(\omega,\hbar_\imath)\geq1$, for $\imath\in\mathbb{N}$.*

Then, the mapping $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightrightarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ has a fixed point.

Proof. Let $\alpha(\mathfrak{w},\hbar)\geq1$. Then, by (16), we have

$$(\mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar)+\ell)\leq(\mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar)+\ell)^{\alpha(\mathfrak{w},\hbar)}\leq\psi(\mathfrak{d}(\mathfrak{w},\hbar))+\ell,$$

which implies $\mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar)\leq\psi(\mathfrak{d}(\mathfrak{w},\hbar))$, and all the conditions of Corollary 1 are satisfied and $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightrightarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ has a fixed point. □

Corollary 6. *Let $(\mathcal{S},\mathcal{T},\mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightrightarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ be a covariant and continuous mapping. Assume that there exists $\psi\in\Psi$ such that*

$$\mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar)\leq\psi(\mathfrak{d}(\mathfrak{w},\hbar)),$$

for all $(\mathfrak{w},\hbar)\in\mathcal{S}\times\mathcal{T}$.

Then, the mapping $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightrightarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ has a unique fixed point.

Proof. Take $\alpha,\eta:\mathcal{S}\times\mathcal{T}\rightarrow[0,+\infty)$ by $\alpha(\mathfrak{w},\hbar)=\eta(\mathfrak{w},\hbar)=1$, for $\mathfrak{w}\in\mathcal{S}$ and $\hbar\in\mathcal{T}$ in Theorem 2. □

Corollary 7. *Let $(\mathcal{S},\mathcal{T},\mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightrightarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ be a convariant and continuous mapping. Assume that there exists $0<k<1$ such that*

$$\mathfrak{d}(\mathcal{B}\mathfrak{w},\mathcal{B}\hbar)\leq k\mathfrak{d}(\mathfrak{w},\hbar),$$

for all $(\mathfrak{w},\hbar)\in\mathcal{S}\times\mathcal{T}$.

Then, the mapping $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightrightarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ has a unique fixed point.

Proof. Define $\psi:[0,+\infty)\rightarrow[0,+\infty)$ by $\psi(t)=kt$, where $0<k<1$ and $\alpha,\eta:\mathcal{S}\times\mathcal{T}\rightarrow[0,+\infty)$ by $\alpha(\mathfrak{w},\hbar)=\eta(\mathfrak{w},\hbar)=1$, for $\mathfrak{w}\in\mathcal{S}$ and $\hbar\in\mathcal{T}$ in Theorem 2. □

4. Fixed Point Results for Contravariant Mappings

Definition 13. *Let $(\mathcal{S},\mathcal{T},\mathfrak{d})$ be an $\mathfrak{F}$-bip MS and $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightleftarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ is a contravariant mapping. A mapping $\mathcal{B}:(\mathcal{S},\mathcal{T},\mathfrak{d})\rightleftarrows(\mathcal{S},\mathcal{T},\mathfrak{d})$ is said to be contravariant α-admissible if there exists a function $\alpha:\mathcal{S}\times\mathcal{T}\rightarrow[0,+\infty)$ such that*

$$\alpha(\mathfrak{w},\hbar)\geq1\implies\alpha(\mathcal{B}\hbar,\mathcal{B}\mathfrak{w})\geq1, \tag{17}$$

for all $(\mathfrak{w},\hbar)\in\mathcal{S}\times\mathcal{T}$.

Example 6. *Let $\mathcal{S}=[0,+\infty)$ and $\mathcal{T}=(-\infty,0]$ and $\alpha:\mathcal{S}\times\mathcal{T}\rightarrow[0,+\infty)$ is defined as*

$$\alpha(\mathfrak{w},\hbar)=\begin{cases}1, & \text{if } \mathfrak{w}\neq\hbar,\\ 0, & \text{if } \mathfrak{w}=\hbar.\end{cases}$$

A contravariant mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightleftarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ defined by $\mathcal{B}(\mathfrak{w}) = -\mathfrak{w}$ is contravariant α-admissible.

Definition 14. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be an $\mathfrak{F}$-bip MS and $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightleftarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is a contravariant mapping. A mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightleftarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is said to be contravariant α-admissible with respect to η if there exist two functions $\alpha, \eta : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ such that*

$$\alpha(\mathfrak{w}, \hbar) \geq \eta(\mathfrak{w}, \hbar) \implies \alpha(\mathcal{B}\hbar, \mathcal{B}\mathfrak{w}) \geq \eta(\mathcal{B}\hbar, \mathcal{B}\mathfrak{w}), \tag{18}$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Definition 15. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be an $\mathfrak{F}$-bip MS. A mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightleftarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is said to be a contravariant (α, η, ψ)-contraction if $\mathcal{B}$ is contravariant and there exist some $\alpha, \eta : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(\mathfrak{w}, \hbar)\mathfrak{d}(\mathcal{B}\hbar, \mathcal{B}\mathfrak{w}) \leq \psi(\mathfrak{d}(\mathfrak{w}, \hbar)), \tag{19}$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$.

Remark 7. *A mapping $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightleftarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ satisfying the Banach contraction in a $\mathfrak{F}$-bip MS $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is a contravariant (α, η, ψ)-contraction with*

$$\alpha(\mathfrak{w}, \hbar) = \eta(\mathfrak{w}, \hbar) = 1,$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$ and $\psi(t) = kt$, for some $k \in [0, 1)$ and for $t \geq 1$.

Theorem 3. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let $\mathcal{B} : (\mathcal{S}, \mathcal{T}, \mathfrak{d}) \rightleftarrows (\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be a contravariant (α, η, ψ)-contraction. Assume that the following postulations hold:*

(i) $\mathcal{B}$ is contravariant α-admissible with respect to η,
(ii) there exists $\mathfrak{w}_0 \in \mathcal{S}$ such that $\alpha(\mathfrak{w}_0, \mathcal{B}\mathfrak{w}_0) \geq \eta(\mathfrak{w}_0, \mathcal{B}\mathfrak{w}_0)$,
(iii) $\mathcal{B}$ is continuous or, if $(\mathfrak{w}_\imath, \hbar_\imath)$ is a bisequence in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ such that $\alpha(\mathfrak{w}_\imath, \hbar_\imath) \geq \eta(\mathfrak{w}_\imath, \hbar_\imath)$, for $\imath \in \mathbb{N}$ with $\mathfrak{w}_\imath \to \omega$ and $\hbar_\imath \to \omega$, as $\imath \to \infty$ for $\omega \in \mathcal{S} \cap \mathcal{T}$, then $\alpha(\mathfrak{w}_\imath, \omega) \geq \eta(\mathfrak{w}_\imath, \omega)$, for $\imath \in \mathbb{N}$.

Then, the mapping $\mathcal{B} : \mathcal{S} \cup \mathcal{T} \to \mathcal{S} \cup \mathcal{T}$ has a fixed point. Furthermore, if the property (P) holds, then the fixed point is unique.

Proof. Let $\mathfrak{w}_0$ and $\hbar_0$ be arbitrary points in $\mathcal{S}$ and $\mathcal{T}$, respectively, and suppose that $\alpha(\mathfrak{w}_0, \mathcal{B}\mathfrak{w}_0) \geq \eta(\mathfrak{w}_0, \mathcal{B}\mathfrak{w}_0)$. Define the bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ by

$$\hbar_\imath = \mathcal{B}\mathfrak{w}_\imath \text{ and } \mathfrak{w}_{\imath+1} = \mathcal{B}\hbar_\imath$$

for all $\imath \in \mathbb{N}$. As $\mathcal{B}$ is a contravariant α-admissible mapping with respect to η, we have

$$\alpha(\mathfrak{w}_0, \hbar_0) = \alpha(\mathfrak{w}_0, \mathcal{B}\mathfrak{w}_0) \geq \eta(\mathfrak{w}_0, \mathcal{B}\mathfrak{w}_0) = \eta(\mathfrak{w}_0, \hbar_0),$$

which implies

$$\alpha(\mathfrak{w}_1, \hbar_0) = \alpha(\mathcal{B}\hbar_0, \mathcal{B}\mathfrak{w}_0) \geq \eta(\mathcal{B}\hbar_0, \mathcal{B}\mathfrak{w}_0) = \eta(\mathfrak{w}_1, \hbar_0),$$

and $\alpha(\mathfrak{w}_1, \hbar_0) \geq \eta(\mathfrak{w}_1, \hbar_0)$ implies

$$\alpha(\mathfrak{w}_1, \hbar_1) = \alpha(\mathcal{B}\hbar_0, \mathcal{B}\mathfrak{w}_1) \geq \eta(\mathcal{B}\hbar_0, \mathcal{B}\mathfrak{w}_1) = \eta(\mathfrak{w}_1, \hbar_1).$$

Similarly, $\alpha(\mathfrak{w}_1, \hbar_1) \geq \eta(\mathfrak{w}_1, \hbar_1)$ implies

$$\alpha(\mathfrak{w}_2, \hbar_1) = \alpha(\mathcal{B}\hbar_1, \mathcal{B}\mathfrak{w}_1) \geq \eta(\mathcal{B}\hbar_1, \mathcal{B}\mathfrak{w}_1) = \eta(\mathfrak{w}_2, \hbar_1),$$

and $\alpha(\mathfrak{w}_2, \hbar_1) \geq \eta(\mathfrak{w}_2, \hbar_1)$ implies

$$\alpha(\mathfrak{w}_2, \hbar_2) = \alpha(\mathcal{B}\hbar_1, \mathcal{B}\mathfrak{w}_2) \geq \eta(\mathcal{B}\hbar_1, \mathcal{B}\mathfrak{w}_2) = \eta(\mathfrak{w}_2, \hbar_2).$$

Continuing in this way, we have

$$\alpha(\mathfrak{w}_\imath, \hbar_\imath) \geq \eta(\mathfrak{w}_\imath, \hbar_\imath) \text{ and } \alpha(\mathfrak{w}_{\imath+1}, \hbar_\imath) \geq \eta(\mathfrak{w}_{\imath+1}, \hbar_\imath), \tag{20}$$

for all $\imath \in \mathbb{N}$. Now, by (19) and (20), we have

$$\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath) = \mathfrak{d}(\mathcal{B}\hbar_{\imath-1}, \mathcal{B}\mathfrak{w}_\imath) \leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_{\imath-1})), \tag{21}$$

for all $\imath \in \mathbb{N}$. Moreover,

$$\mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_\imath) = \mathfrak{d}(\mathcal{B}\hbar_\imath, \mathcal{B}\mathfrak{w}_\imath) \leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath)), \tag{22}$$

for all $\imath \in \mathbb{N}$. By (21) and mathematical induction, we obtain

$$\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath) \leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_{\imath-1})) \leq \psi(\psi(\mathfrak{d}(\mathfrak{w}_{\imath-1}, \hbar_{\imath-2}))) \leq \ldots \leq \psi^\imath(\mathfrak{d}(\mathfrak{w}_1, \hbar_0)). \tag{23}$$

Similarly, by (22) and mathematical induction, we obtain

$$\mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_\imath) \leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath)) \leq \psi(\psi(\mathfrak{d}(\mathfrak{w}_{\imath-1}, \hbar_{\imath-1}))) \leq \ldots \leq \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)), \tag{24}$$

for all $\imath \in \mathbb{N}$. Let $(f, \kappa) \in \mathfrak{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \kappa. \tag{25}$$

Let there exist $\epsilon > 0$ and $\imath(\epsilon) \in \mathbb{N}$ such that

$$\sum_{\imath \geq \imath(\epsilon)} \psi^\imath(\mathfrak{d}(\mathfrak{w}_1, \hbar_0)) < \frac{\epsilon}{2},$$

and

$$\sum_{\imath \geq \imath(\epsilon)} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) < \frac{\epsilon}{2}.$$

Now, for $p > \imath \geq \imath(\epsilon)$, by applying (D_3), we have that $\mathfrak{d}(\mathfrak{w}_\imath, \hbar_p) > 0$ implies

$$\begin{aligned} f(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_p)) &\leq f\left(\begin{array}{c} \mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_\imath) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_{\imath+1}) + \\ \ldots + \mathfrak{d}(\mathfrak{w}_p, \hbar_{p-1}) + \mathfrak{d}(\mathfrak{w}_p, \hbar_p) \end{array} \right) + \kappa \\ &\leq f\left(\sum_{j=\imath}^{p} \mathfrak{d}(\mathfrak{w}_j, \hbar_j) + \sum_{j=\imath}^{p-1} \mathfrak{d}(\mathfrak{w}_{j+1}, \hbar_j) \right) + \kappa \\ &\leq f\left(\sum_{j=\imath}^{p} \psi^j(\mathfrak{d}(\mathfrak{w}_1, \hbar_0)) + \sum_{j=\imath}^{p-1} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) \right) + \kappa \\ &\leq f\left(\sum_{\imath \geq \imath(\epsilon)} \psi^\imath(\mathfrak{d}(\mathfrak{w}_1, \hbar_0)) + \sum_{\imath \geq \imath(\epsilon)} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) \right) + \kappa \\ &< f(\epsilon), \end{aligned}$$

for all $j \in \mathbb{N}$. Similarly, for $\imath > p \geq \imath(\epsilon)$, by applying (D_3), we have that $\mathfrak{d}(\mathfrak{w}_\imath, \hbar_p) > 0$ implies

$$\begin{aligned} f(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_p)) &\leq f\begin{pmatrix} \mathfrak{d}(\mathfrak{w}_\imath, \hbar_{\imath-1}) + \mathfrak{d}(\mathfrak{w}_{\imath-1}, \hbar_{\imath-1}) + \mathfrak{d}(\mathfrak{w}_{\imath-1}, \hbar_{\imath-2}) + \\ \ldots + \mathfrak{d}(\mathfrak{w}_p, \hbar_{p-1}) + \mathfrak{d}(\mathfrak{w}_p, \hbar_p) \end{pmatrix} + \kappa \\ &\leq f\left(\sum_{j=p}^{\imath-1} \mathfrak{d}(\mathfrak{w}_j, \hbar_j) + \sum_{j=\imath}^{\imath} \mathfrak{d}(\mathfrak{w}_j, \hbar_{j-1}) \right) + \kappa \\ &\leq f\left(\sum_{j=p}^{\imath-1} \psi^j(\mathfrak{d}(\mathfrak{w}_1, \hbar_0)) + \sum_{j=p}^{\imath} \psi^{\imath+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) \right) + \kappa \\ &\leq f\left(\sum_{\imath \geq \imath(\epsilon)} \psi^{j+1}(\mathfrak{d}(\mathfrak{w}_0, \hbar_0)) + \sum_{\imath \geq \imath(\epsilon)} \psi^{\imath}(\mathfrak{d}(\mathfrak{w}_1, \hbar_0)) \right) + \kappa \\ &< f(\epsilon), \end{aligned}$$

for all $j \in \mathbb{N}$. Then, by $(\mathfrak{F}_1)$, $\mathfrak{d}(\mathfrak{w}_\imath, \hbar_p) < \epsilon$, for all $p, \imath \geq \imath_0$. Thus, $(\mathfrak{w}_\imath, \hbar_\imath)$ is a Cauchy bisequence in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$. As $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is complete, $(\mathfrak{w}_\imath, \hbar_\imath)$ biconverges to a point $\omega \in \mathcal{S} \cap \mathcal{T}$. Thus, $(\mathfrak{w}_\imath) \to \omega$, $(\hbar_\imath) \to \omega$. Additionally, since $\mathcal{B}$ is continuous, we obtain

$$(\mathfrak{w}_\imath) \to \omega \implies (\hbar_\imath) = (\mathcal{B}\mathfrak{w}_\imath) \to \mathcal{B}\omega.$$

Moreover, since $(\hbar_\imath)$ has a limit ω in $\mathcal{S} \cap \mathcal{T}$ and the limit is unique, $\mathcal{B}\omega = \omega$. Thus, $\mathcal{B}$ has a fixed point. Now, since a bisequence $(\mathfrak{w}_\imath, \hbar_\imath)$ in $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ is such that $\alpha(\mathfrak{w}_\imath, \hbar_\imath) \geq \eta(\mathfrak{w}_\imath, \hbar_\imath)$, for all $\imath \in \mathbb{N}$ with $\mathfrak{w}_\imath \to \omega$ and $\hbar_\imath \to \omega$, as $\imath \to \infty$ for $\omega \in \mathcal{S} \cap \mathcal{T}$, then, by hypothesis (iii), we have $\alpha(\mathfrak{w}_\imath, \omega) \geq \eta(\mathfrak{w}_\imath, \omega)$, for $\imath \in \mathbb{N}$. Now, by (19), we have

$$\begin{aligned} f(\mathfrak{d}(\mathcal{B}\omega, \omega)) &\leq f(\mathfrak{d}(\mathcal{B}\omega, \mathcal{B}\mathfrak{w}_\imath) + \mathfrak{d}(\mathcal{B}\hbar_\imath, \mathcal{B}\mathfrak{w}_\imath) + \mathfrak{d}(\mathcal{B}\hbar_\imath, \omega)) + \kappa \\ &\leq f(\psi(\mathfrak{d}(\mathfrak{w}_\imath, \omega)) + \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath)) + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \omega)) + \kappa \\ &\leq f\left(\begin{array}{c} \psi(\mathfrak{d}(\mathfrak{w}_\imath, \omega)) \\ +\psi\begin{pmatrix} \mathfrak{d}(\mathfrak{w}_\imath, \omega) + \mathfrak{d}(\omega, \omega) \\ +\mathfrak{d}(\omega, \hbar_\imath) \end{pmatrix} + \mathfrak{d}(\mathfrak{w}_{\imath+1}, \omega) \end{array} \right) + \kappa. \end{aligned}$$

Taking the limit as $\imath \to \infty$ and using the continuity of f and ψ at $t = 0$, we have $\mathfrak{d}(\mathcal{B}\omega, \omega) = 0$. Thus, $\mathcal{B}\omega = \omega$. Hence, $\mathcal{B}$ has a fixed point. □

The uniqueness of the fixed point is the same as given in Theorem 2.

5. Coupled Fixed Point Theorems

In the present section, we obtain coupled fixed point results from our established results.

Definition 16. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let $\mathcal{F} : (\mathcal{S} \times \mathcal{T}, \mathcal{T} \times \mathcal{S}) \rightrightarrows (\mathcal{S}, \mathcal{T})$ be a convariant mapping. A point $(a, b) \in \mathcal{S} \times \mathcal{T}$ is alleged to be a coupled fixed point of $\mathcal{F}$ if*

$$\mathcal{F}(a, b) = a \text{ and } \mathcal{F}(b, a) = b.$$

Lemma 2. *Let $\mathcal{F} : (\mathcal{S} \times \mathcal{T}, \mathcal{T} \times \mathcal{S}) \rightrightarrows (\mathcal{S}, \mathcal{T})$ be a convariant mapping. If we define a convariant mapping $\aleph : (\mathcal{S} \times \mathcal{T}, \mathcal{T} \times \mathcal{S}) \rightrightarrows (\mathcal{S} \times \mathcal{T}, \mathcal{T} \times \mathcal{S})$ by*

$$\aleph(\mathfrak{w}, \hbar) = (\mathcal{F}(\mathfrak{w}, \hbar), \mathcal{F}(\hbar, \mathfrak{w})),$$

for all $(\mathfrak{w}, \hbar) \in \mathcal{S} \times \mathcal{T}$, then $(\mathfrak{w}, \hbar)$ is a coupled fixed point of $\mathcal{F}$ if only if $(\mathfrak{w}, \hbar)$ is a fixed point of $\aleph$.

We state a property (P') that is required in our result.

(P') there exists $(z_1, z_2) \in (\mathcal{S} \times \mathcal{T}) \cap (\mathcal{T} \times \mathcal{S})$ such that

$$\alpha((\mathfrak{w}, \hbar), (z_1, z_2)) \geq 1,\ \alpha((z_2, z_1), (\hbar, \mathfrak{w})) \geq 1,$$

and

$$\alpha((u,v),(z_1,z_2)) \geq 1, \ \alpha((z_2,z_1),(u,v)) \geq 1,$$

for all $(\mathfrak{w},\hbar) \in \mathcal{S}\times\mathcal{T}$ and $(u,v) \in \mathcal{T}\times\mathcal{S}$.

Theorem 4. *Let $(\mathcal{S},\mathcal{T},\mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let $\mathcal{F} : (\mathcal{S}\times\mathcal{T},\mathcal{T}\times\mathcal{S}) \rightrightarrows (\mathcal{S},\mathcal{T})$ be a convariant mapping. Assume that there exist $\alpha : (\mathcal{S}\times\mathcal{T})\times(\mathcal{T}\times\mathcal{S}) \to [0,+\infty)$ and $\psi \in \Psi$ such that*

$$\alpha((\mathfrak{w},\hbar),(u,v))\mathfrak{d}(\mathcal{F}(\mathfrak{w},\hbar),\mathcal{F}(u,v)) \leq \psi\left(\frac{\mathfrak{d}(\mathfrak{w},u)+\mathfrak{d}(v,\hbar)}{2}\right), \tag{26}$$

for all $(\mathfrak{w},\hbar),(u,v) \in \mathcal{S}\times\mathcal{T}$, and the following hypotheses also hold:

(i) $\alpha((\mathfrak{w},\hbar),(u,v)) \geq 1$ *implies* $\alpha((\mathcal{F}(\mathfrak{w},\hbar),\mathcal{F}(\hbar,\mathfrak{w})),(\mathcal{F}(u,v),\mathcal{F}(v,u))) \geq 1$,

(ii) *there exists $(\mathfrak{w}_0,\hbar_0) \in \mathcal{S}\times\mathcal{T}$ such that*

$$\alpha((\mathfrak{w}_0,\hbar_0),(\mathcal{F}(\hbar_0,\mathfrak{w}_0),\mathcal{F}(\mathfrak{w}_0,\hbar_0))) \geq 1,$$

and

$$\alpha((\mathcal{F}(\mathfrak{w}_0,\hbar_0),\mathcal{F}(\hbar_0,\mathfrak{w}_0)),(\mathfrak{w}_0,\hbar_0)) \geq 1,$$

(iii) *$\mathcal{F}$ is continuous or, if $(\mathfrak{w}_\imath,\hbar_\imath)$ is a bisequence in $(\mathcal{S},\mathcal{T},\mathfrak{d})$ such that $\alpha((\mathfrak{w}_\imath,\hbar_\imath),(\hbar_{\imath+1},\mathfrak{w}_{\imath+1})) \geq 1$ and $\alpha((\hbar_{\imath+1},\mathfrak{w}_{\imath+1}),(\mathfrak{w}_\imath,\hbar_\imath)) \geq 1$, for all $\imath \in \mathbb{N}$ with $\mathfrak{w}_\imath \to \mathfrak{w}$ and $\hbar_\imath \to \hbar$, as $\imath \to \infty$ for $(\mathfrak{w},\hbar) \in \mathcal{S}\cap\mathcal{T}$, then*

$$\alpha((\mathfrak{w}_\imath,\hbar_\imath),(\mathfrak{w},\hbar)) \geq 1 \text{ and } \alpha((\mathfrak{w},\hbar),(\mathfrak{w}_\imath,\hbar_\imath)) \geq 1,$$

for all $\imath \in \mathbb{N}$.

Then, $\mathcal{F}$ has a coupled fixed point. Furthermore, if the property (P') holds, then the coupled fixed point is unique.

Proof. Let $L = \mathcal{S}\times\mathcal{T}$ and $H = \mathcal{T}\times\mathcal{S}$ and

$$\delta((\mathfrak{w},\hbar),(u,v)) = \mathfrak{d}(\mathfrak{w},u)+\mathfrak{d}(v,\hbar),$$

for all $(\mathfrak{w},\hbar) \in L$ and $(u,v) \in H$. Then, (L,H,δ) is a complete $\mathfrak{F}$-bipolar metric space. By (26), we have

$$\alpha((\mathfrak{w},\hbar),(u,v))\mathfrak{d}(\mathcal{F}(\mathfrak{w},\hbar),\mathcal{F}(u,v)) \leq \psi\left(\frac{\mathfrak{d}(\mathfrak{w},u)+\mathfrak{d}(v,\hbar)}{2}\right), \tag{27}$$

and

$$\alpha((\mathfrak{w},\hbar),(u,v))\mathfrak{d}(\mathcal{F}(\mathfrak{w},\hbar),\mathcal{F}(u,v)) \leq \psi\left(\frac{\mathfrak{d}(\mathfrak{w},u)+\mathfrak{d}(v,\hbar)}{2}\right). \tag{28}$$

Combining (27) and (28), we obtain

$$\beta(\varkappa,\varrho)\mathfrak{d}(\mathcal{F}\varkappa,\mathcal{F}\varrho) \leq \psi(\delta(\varkappa,\varrho)),$$

for all $\varkappa = (\varkappa_1,\varkappa_2) \in L$ and $\varrho = (\varrho_1,\varrho_2) \in H$. Moreover, the function $\beta : L\times H \to [0,+\infty)$ is defined as

$$\beta(\varkappa,\varrho) = \min\{\alpha((\varkappa_1,\varkappa_2),(\varrho_1,\varrho_2)),\alpha((\varrho_2,\varrho_1),(\varkappa_2,\varkappa_1))\},$$

and $\aleph : (L,H) \rightrightarrows (L,H)$ is defined by

$$\aleph(\mathfrak{w},\hbar) = (\mathcal{F}(\mathfrak{w},\hbar),\mathcal{F}(\hbar,\mathfrak{w})).$$

Then, $\aleph$ is a continuous and covariant (β, ψ)-contraction. Now, we suppose that $\beta(\varkappa, \varrho) \geq 1$. Then, by (i), we have $\beta(\aleph\varkappa, \aleph\varrho) \geq 1$. By condition (ii), there exists $(\mathfrak{w}_0, \hbar_0) \in L$ (or $(\hbar_0, \mathfrak{w}_0) \in H$) such that

$$\beta((\mathfrak{w}_0, \hbar_0), \aleph(\mathfrak{w}_0, \hbar_0)) \geq 1,$$

(or $\beta(\aleph(\mathfrak{w}_0, \hbar_0), (\hbar_0, \mathfrak{w}_0,)) \geq 1$). Since $\aleph$ is continuous, $\aleph$ has a fixed point. Now, if $(\mathfrak{w}_\imath, \hbar_\imath)$ is a bisequence in $L = \mathcal{S} \times \mathcal{T}$ and $(\hbar_\imath, \mathfrak{w}_\imath)$ is a bisequence in $H = \mathcal{T} \times \mathcal{S}$ such that $\alpha((\mathfrak{w}_\imath, \hbar_\imath), (\hbar_{\imath+1}, \mathfrak{w}_{\imath+1})) \geq 1$ and $(\mathfrak{w}_\imath, \hbar_\imath) \to (\mathfrak{w}, \hbar)$ as $n \to \infty$. Then, by (iii), we have $\alpha((\mathfrak{w}_\imath, \hbar_\imath), (\hbar, \mathfrak{w})) \geq 1$. Thus, all the conditions of Corollary 3 are satisfied and $\aleph$ has a fixed point. Hence, by Lemma 2, $\mathcal{F}$ has a coupled fixed point. Now, since the property ($P^/$) holds, $\mathcal{F}$ has a unique coupled fixed point. □

Remark 8. *Taking* $\alpha((\mathfrak{w}, \hbar), (u, v)) = 1$ *and* $\psi(t) = kt$, *where* $0 < k < 1$ *in Theorem 4, we can obtain the leading result of Mutlu et al. [12].*

6. Application

6.1. Integral Equations

Fixed point theory is a valuable tool used to solve differential and integral equations, which are used to investigate the solutions of various mathematical models, as well as in game theory, dynamical systems, physics, engineering, computer science, neural networks and many other domains (see [22–24]). In the present section, we discuss the uniqueness and existence of an integral equation.

$$\varphi(\mathfrak{w}) = g(\mathfrak{w}) + \int_{\mathcal{S}\cup\mathcal{T}} K(\mathfrak{w}, \hbar, \varphi(\mathfrak{w}))\partial\hbar, \tag{29}$$

where $\mathcal{S} \cup \mathcal{T}$ is a Lebesgue measurable set and g is real-valued continuous function.

Theorem 5. *Suppose that the following conditions hold:*

(i) $K : (\mathcal{S}^2 \cup \mathcal{T}^2) \times [0, \infty) \to [0, \infty)$ *and* $f \in \mathcal{L}^\infty(\mathcal{S}) \cup \mathcal{L}^\infty(\mathcal{T})$,

(ii) *there exists a continuous function* $\Upsilon : \mathcal{S}^2 \cup \mathcal{T}^2 \to [0, \infty)$ *such that*

$$|K(\mathfrak{w}, \hbar, \varphi(\hbar)) - K(\mathfrak{w}, \hbar, \phi(\hbar))| \leq \frac{1}{2}\Upsilon(\mathfrak{w}, \hbar)|\phi(\hbar) - \varphi(\hbar)|,$$

for all $\mathfrak{w}, \hbar \in (\mathcal{S}^2 \cup \mathcal{T}^2)$,

(iii) $\left\| \int_{\mathcal{S}\cup\mathcal{T}} \Upsilon(\mathfrak{w}, \hbar)\partial\hbar \right\| \leq 1$, *that is,* $\sup_{\mathfrak{w}\in\mathcal{S}\cup\mathcal{T}} \int_{\mathcal{S}\cup\mathcal{T}} |\Upsilon(\mathfrak{w}, \hbar)|\partial\hbar \leq 1$.

Then, the integral Equation (29) has a unique solution in $\mathcal{L}^\infty(\mathcal{S}) \cup \mathcal{L}^\infty(\mathcal{T})$.

Proof. Let $\Xi = \mathcal{L}^\infty(\mathcal{S})$ and $\Theta = \mathcal{L}^\infty(\mathcal{T})$ be two normed linear spaces, where $\mathcal{S}$ and $\mathcal{T}$ are Lebesgue measurable sets and $m(\mathcal{S} \cup \mathcal{T}) < \infty$. Consider $\mathfrak{d} : \Xi \times \Theta \to [0, \infty)$ to be defined by

$$\mathfrak{d}(\xi, \zeta) = \|\xi - \zeta\|_\infty$$

for all $\xi, \zeta \in \Xi \times \Theta$. Then, $(\Xi, \Theta, \mathfrak{d})$ is a complete $\mathfrak{F}$-bip MS. Define the mapping $I : \Xi \cup \Theta \to \Xi \cup \Theta$ by

$$I(\varphi(\mathfrak{w})) = g(\mathfrak{w}) + \int_{\mathcal{S}\cup\mathcal{T}} K(\mathfrak{w}, \hbar, \varphi(\mathfrak{w}))\partial\hbar,$$

for $\mathfrak{w} \in \mathcal{S} \cup \mathcal{T}$ and $\alpha, \eta : \Xi \times \Theta \to [0, +\infty)$ by

$$\alpha(\varphi(\mathfrak{w}), \phi(\mathfrak{w})) = \eta(\varphi(\mathfrak{w}), \phi(\mathfrak{w})) = 1.$$

Now, we have

$$\begin{aligned}\mathfrak{d}(I(\varphi(\mathfrak{w})), I(\phi(\mathfrak{w}))) &= \|I(\varphi(\mathfrak{w})) - I(\phi(\mathfrak{w}))\| \\ &= \left| \int_{\mathcal{S}\cup\mathcal{T}} K(\mathfrak{w}, \hbar, \varphi(\mathfrak{w}))\partial\hbar - \int_{\mathcal{S}\cup\mathcal{T}} K(\mathfrak{w}, \hbar, \phi(\mathfrak{w}))\partial\hbar \right| \\ &\leq \int_{\mathcal{S}\cup\mathcal{T}} |K(\mathfrak{w}, \hbar, \varphi(\mathfrak{w})) - K(\mathfrak{w}, \hbar, \phi(\mathfrak{w}))|\partial\hbar \\ &\leq \int_{\mathcal{S}\cup\mathcal{T}} \frac{1}{2}\Upsilon(\mathfrak{w}, \hbar)|\phi(\hbar) - \varphi(\hbar)|\partial\hbar \\ &\leq \frac{1}{2}\|\phi(\hbar) - \varphi(\hbar)\| \int_{\mathcal{S}\cup\mathcal{T}} |\Upsilon(\mathfrak{w}, \hbar)|\partial\hbar \\ &\leq \frac{1}{2}\|\phi - \varphi\| \sup_{\mathfrak{w}\in\mathcal{S}\cup\mathcal{T}} \int_{\mathcal{S}\cup\mathcal{T}} |\Upsilon(\mathfrak{w}, \hbar)|\partial\hbar \\ &\leq \frac{1}{2}\|\phi - \varphi\| \\ &= \psi(\mathfrak{d}(\phi, \varphi)).\end{aligned}$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{1}{2}t$, for $t > 0$. Thus, by result 2, I has a unique fixed point in $\Xi \cup \Theta$. □

6.2. Homotopy Result

Theorem 6. *Let $(\mathcal{S}, \mathcal{T}, \mathfrak{d})$ be a complete $\mathfrak{F}$-bip MS and let (Ξ, Θ) be an open subset of $(\mathcal{S}, \mathcal{T})$ and $(\overline{\Xi}, \overline{\Theta})$ be a closed subset of $(\mathcal{S}, \mathcal{T})$ and $(\Xi, \Theta) \subseteq (\overline{\Xi}, \overline{\Theta})$. Suppose that $\mathcal{L} : (\overline{\Xi} \cup \overline{\Theta}) \times [0, 1] \to \mathcal{S} \cup \mathcal{T}$ satisfies the following conditions:*

(hom1) $\mathfrak{w} \neq \mathcal{L}(\mathfrak{w}, q)$ for each $\mathfrak{w} \in \partial\Xi \cup \partial\Theta$ and $q \in [0, 1]$,

(hom2) for all $\mathfrak{w} \in \overline{\Xi}$, $\hbar \in \overline{\Theta}$ and $q \in [0, 1]$

$$\mathfrak{d}(\mathcal{L}(\hbar, q), \mathcal{L}(\mathfrak{w}, q)) \leq \psi(\mathfrak{d}(\mathfrak{w}, \hbar)),$$

where $\psi \in \Psi$,

(hom3) there exists $M \geq 0$ such that

$$\mathfrak{d}(\mathcal{L}(\mathfrak{w}, r), \mathcal{L}(\hbar, o)) \leq M|r - o|,$$

for all $\mathfrak{w} \in \overline{\Xi}$, $\hbar \in \overline{\Theta}$ and $r, o \in [0, 1]$.

Then, the mapping $\mathcal{L}(\cdot, 0)$ has a fixed point if and only if $\mathcal{L}(\cdot, 1)$ has a fixed point.

Proof. Let

$$\Re_1 = \{\tau \in [0, 1] : \mathfrak{w} = \mathcal{L}(\mathfrak{w}, \tau), \mathfrak{w} \in \Xi\}$$

and

$$\Re_1 = \{o \in [0, 1] : \hbar = \mathcal{L}(\hbar, o), \hbar \in \Theta\}.$$

Since $\mathcal{L}(\cdot, 0)$ has a fixed point in $\Xi \cup \Theta$, then we get $0 \in \Re_1 \cap \Re_2$. Thus $\Re_1 \cap \Re_2 \neq \emptyset$. Now, we shall prove that $\Re_1 \cap \Re_2$ is both open and closed in $[0, 1]$ and so, by connetedness, $\Re_1 = \Re_2 = [0, 1]$. Let $\left(\{\tau_\imath\}_{\imath=1}^{\infty}\right), \left(\{o_\imath\}_{\imath=1}^{\infty}\right) \subseteq (\Re_1, \Re_2)$ with $(\tau_\imath, o_\imath) \to (\rho, \rho) \in [0, 1]$ as $\imath \to \infty$. We also claim that $\rho \in \Re_1 \cap \Re_2$. Since $(\tau_\imath, o_\imath) \in \Re_1 \cap \Re_2$, for $\imath \in \mathbb{N} \cup \{0\}$. Hence there exists a bisequence $(\mathfrak{w}_\imath, \hbar_\imath) \in (\Xi, \Theta)$ such that $\hbar_\imath = \mathcal{L}(\mathfrak{w}_\imath, \tau_\imath)$ and $\mathfrak{w}_{\imath+1} = \mathcal{L}(\hbar_\imath, o_\imath)$. Also, we get

$$\begin{aligned}\mathfrak{d}(\mathfrak{w}_{\imath+1}, \hbar_\imath) &= \mathfrak{d}(\mathcal{L}(\hbar_\imath, o_\imath), \mathcal{L}(\mathfrak{w}_\imath, \tau_\imath)) \\ &\leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath)).\end{aligned}$$

And,

$$\begin{aligned} \mathfrak{d}(\mathfrak{w}_\imath, \hbar_\imath) &= \mathfrak{d}(\mathcal{L}(\hbar_{\imath-1}, o_{\imath-1}), \mathcal{L}(\mathfrak{w}_\imath, \tau_\imath)) \\ &\leq \psi(\mathfrak{d}(\mathfrak{w}_\imath, \hbar_{\imath-1})). \end{aligned}$$

Following the proof of Theorem 2, one can easily show that $(\mathfrak{w}_\imath, \hbar_\imath)$ is a Cauchy bisequence in (Ξ, Θ). Since (Ξ, Θ) is complete, so there exists $\rho_1 \in \Xi \cap \Theta$ such that $\lim_{\imath\to\infty}(\mathfrak{w}_\imath) = \lim_{\imath\to\infty}(\hbar_\imath) = \rho_1$. Now, we have

$$\begin{aligned} f(\mathfrak{d}(\mathcal{L}(\rho_1, o), \hbar_\imath)) &= f(\mathfrak{d}(\mathcal{L}(\rho_1, o), \mathcal{L}(\mathfrak{w}_\imath, \tau_\imath))) \\ &\leq f(\psi(\mathfrak{d}(\mathfrak{w}_\imath, \rho_1))) = -\infty, \end{aligned}$$

whenever $\imath \to \infty$. Hence by $(\mathfrak{F}_2)$, we get $\mathfrak{d}(\mathcal{L}(\rho_1, o), \rho_1) = 0$, which implies that $\mathcal{L}(\rho_1, o) = \rho_1$. Similarly, $\mathcal{L}(\rho_1, \tau) = v_1$. Thus $\tau = o \in \Re_1 \cap \Re_2$, and evidently $\Re_1 \cap \Re_2$ is closed set in $[0,1]$. Next, we have to prove that $\Re_1 \cap \Re_2$ is open in $[0,1]$. Suppose $(\tau_0, o_0) \in (\Re_1, \Re_2)$, then there is a bisequence $(\mathfrak{w}_0, \hbar_0)$ so that

$$\mathfrak{w}_0 = \mathcal{L}(\mathfrak{w}_0, \tau_0), \hbar_0 = \mathcal{L}(\hbar_0, o_0).$$

Since $\Xi \cup \Theta$ is open, so there exists $r > 0$ so that $B_{\mathfrak{d}}(\mathfrak{w}_0, r) \subseteq \Xi \cup \Theta$ and $B_{\mathfrak{d}}(r, \hbar_0) \subseteq \Xi \cup \Theta$. Choose $\tau \in (o_0 - \epsilon, o_0 + \epsilon)$ and $o \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$ such that

$$|\tau - o_0| \leq \frac{1}{M^\imath} < \frac{\epsilon}{2}$$

$$|o - \tau_0| \leq \frac{1}{M^\imath} < \frac{\epsilon}{2},$$

and

$$|\tau_0 - o_0| \leq \frac{1}{M^\imath} < \frac{\epsilon}{2}.$$

Hence, we have

$$\hbar \in \overline{B_{\Re_1 \cup \Re_2}(\mathfrak{w}_0, r)} = \left\{ \begin{array}{c} \hbar : \hbar_0 \in \Theta : \\ \mathfrak{d}(\mathfrak{w}_0, \hbar) \leq r + \mathfrak{d}(\mathfrak{w}_0, \hbar_0) \end{array} \right\}$$

and

$$\mathfrak{w} \in \overline{B_{\Re_1 \cup \Re_2}(r, \hbar_0)} = \left\{ \begin{array}{c} \mathfrak{w} : \mathfrak{w}_0 \in \Xi : \\ \mathfrak{d}(\mathfrak{w}, \hbar_0) \leq r + \mathfrak{d}(\mathfrak{w}_0, \hbar_0) \end{array} \right\}.$$

Moreover, we have

$$\begin{aligned} \mathfrak{d}(\mathcal{L}(\mathfrak{w}, \tau), \hbar_0) &= \mathfrak{d}(\mathcal{L}(\mathfrak{w}, \tau), \mathcal{L}(\hbar_0, o_0)) \\ &\leq \mathfrak{d}(\mathcal{L}(\mathfrak{w}, \tau), \mathcal{L}(\hbar, o_0)) \\ &\quad + \mathfrak{d}(\mathcal{L}(\mathfrak{w}_0, \tau), \mathcal{L}(\hbar, o_0)) \\ &\quad + \mathfrak{d}(\mathcal{L}(\mathfrak{w}_0, \tau), \mathcal{L}(\hbar_0, o_0)) \\ &\leq 2M|\tau - o_0| + \mathfrak{d}(\mathcal{L}(\mathfrak{w}_0, \tau), \mathcal{L}(\hbar, o_0)) \\ &\leq \frac{2}{M^\imath - 1} + \psi(\mathfrak{d}(\mathfrak{w}_0, \hbar)) \\ &\leq \frac{2}{M^\imath - 1} + \mathfrak{d}(\mathfrak{w}_0, \hbar). \end{aligned}$$

Letting $\imath \to \infty$, we get

$$\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \tau), \hbar_0) \leq \mathfrak{d}(\mathfrak{w}_0, \hbar) \leq r + \mathfrak{d}(\mathfrak{w}_0, \hbar_0).$$

By corresponding fashion, we get

$$\mathfrak{d}(\mathfrak{w}_0, \mathcal{L}(\hbar, o)) \leq \mathfrak{d}(\mathfrak{w}, \hbar_0) \leq r + \mathfrak{d}(\mathfrak{w}_0, \hbar_0).$$

But

$$\begin{aligned} \mathfrak{d}(\mathfrak{w}_0, \hbar_0) &= \mathfrak{d}(\mathcal{L}(\mathfrak{w}_0, \tau_0), \mathcal{L}(\hbar_0, o_0)) \\ &\leq M|\tau_0 - o_0| \leq \frac{1}{M^{\iota-1}} \to 0, \end{aligned}$$

as $\iota \to \infty$, which yields that $\mathfrak{w}_0 = \hbar_0$. As a result, $o = \tau \in (o_0 - \epsilon, o_0 + \epsilon)$ for each fixed o and $\mathcal{L}(\cdot, \tau) : \overline{B_{\Re_1 \cup \Re_2}(\mathfrak{w}_0, r)} \to \overline{B_{\Re_1 \cup \Re_2}(\mathfrak{w}_0, r)}$. Since all the conditions of Corollary 3 hold, $\mathcal{L}(\cdot, \tau)$ has a fixed point in $\overline{\Xi} \cap \overline{\Theta}$, which certainly exists in $\Xi \cap \Theta$. Then $\tau = o \in \Re_1 \cap \Re_2$ for each $o \in (o_0 - \epsilon, o_0 + \epsilon)$. Hence $(o_0 - \epsilon, o_0 + \epsilon) \in \Re_1 \cap \Re_2$ which gives $\Re_1 \cap \Re_2$ is open in $[0, 1]$. Similarly, we can prove the converse of it. □

7. Conclusions

In this research article, we have defined (α, η, ψ)-contractions against the background of $\mathfrak{F}$-bip MS and established fixed point results. Some coupled fixed point results in $\mathfrak{F}$-bip MS are also derived as a result of our main theorems. An important example is also provided to validate the authenticity of the established theorems. We have explored the existence and uniqueness of a solution of an integral equation by applying our main result. Additionally, we have explored the unique solution of the homotopy result.

The given results in this research work can be extended to some multivalued mappings and fuzzy mappings in the framework of $\mathfrak{F}$-bip MS. In addition, a number of common fixed point results for these contractions can be obtained. As applications of these outcomes against the background of $\mathfrak{F}$-bip MS, some differential and integral inclusions can be explored.

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