mathematics

MDPI

Article

Totally Goldie*-Supplemented Modules

Ayşe Tuğba Güroğlu

Department of Mathematics, Faculty of Arts and Sciences, Manisa Celal Bayar University, Manisa 45140, Turkey; tugba.guroglu@cbu.edu.tr

Abstract: In this paper, we first consider the properties of the Goldie*-supplemented modules, and we study the properties of totally Goldie*-supplemented modules as a version of the Goldie*-supplemented modules. A module M is called Goldie*-supplemented module if, for every submodule U of M, there exists a supplement submodule S of M such that $U\beta^*S$. A module M is called a totally Goldie*-supplemented module if, for every submodule A of M, A is a Goldie*-supplemented module. We emphasize that if M is totally Goldie*-supplemented, then $\frac{M}{U}$ is totally Goldie*-supplemented for some small submodule U of M. In addition, $M = A \oplus B$ is totally Goldie*-supplemented if A and B are totally Goldie*-supplemented. Furthermore, we mention the connection between totally Goldie*-supplemented, totally supplemented, and Goldie*-supplemented.

Keywords: totally supplemented; Goldie*-supplemented module; totally Goldie*-supplemented module

MSC: 16D10; 16D99

1. Introduction

Let M be a unital left R-module over a unital ring R. $Rad(M)$ and $Jac(R)$ denote the Jacobson radical of M and R, respectively. For any submodule U of M, U is called small in M ($U \ll M$) if $U + A = M$ for every nonzero submodule A of M; then, $A = M$. A submodule U is called a (weak) supplement in M if $U + A = M$ and $U \cap A \ll U$ ($U \cap A \ll M$) for some submodule A of M. A module M is said to be a (weakly) supplemented module if every submodule of M has a (weak) supplement in M. Semisimple and artinian modules are supplemented modules. A module M is called totally supplemented if every submodule of M is supplemented.

Totally supplemented modules were introduced by Smith in [1] as a generalization of supplemented modules. A module M is totally supplemented if every submodule of M is supplemented. After this work, various aspects of totally supplemented modules, such as totally cofinitely supplemented, totally weak supplemented, totally cofinitely weak Rad-supplemented, and totally $\oplus$ generalized *cofinitely supplemented modules, were studied by Bilhan in [2], Top in [3], Eryılmaz and Eren in [4], and Wasan and Dnan in [5], respectively. The other generalization of supplemented modules, so-called Goldie*-supplemented modules, were introduced and characterized in [6,7] as another approach to supplemented modules. A module M is called Goldie*-supplemented (or briefly, $\mathcal{G}$*s) if there is a supplement submodule S of M with $U\beta^*S$ for each submodule U of M. This module structure, described in [6], lies between an amply supplemented module and a supplemented module. Furthermore, in [8], the authors determined the Goldie-Rad-supplemented modules inspired by [6].

Although many authors have studied several variations of totally supplemented modules, totally $\mathcal{G}$*-supplemented modules seem to be unexplored. In order to characterize totally $\mathcal{G}$*-supplemented modules, it is important to examine modules whose submodules are $\mathcal{G}$*-supplemented. We expect that all submodules of any module that are $\mathcal{G}$*-supplemented give an idea about totally $\mathcal{G}$*-supplemented, totally supplemented, and amply supplemented modules, and their relationships to each other. Therefore, we prove

check for updates

Citation: Güroğlu, A.T. Totally Goldie*-Supplemented Modules. *Mathematics* **2023**, *11*, 4427. https://doi.org/10.3390/math11214427

Academic Editor: Tomasz Brzezinski

Received: 1 September 2023
Revised: 20 October 2023
Accepted: 23 October 2023
Published: 25 October 2023

that any direct summand of a totally $\mathcal{G}^*$-supplemented module is a totally $\mathcal{G}^*$-supplemented module. We show that every submodule of a totally $\mathcal{G}^*$-supplemented module over a left V-ring is a direct summand. In addition, the relationships between totally $\mathcal{G}^*$-supplemented, $\mathcal{G}^*$-supplemented, and totally supplemented modules are given under some restrictions.

2. Goldie*-Supplemented Modules

Before we start the main study of this work, we need to examine some properties of the Goldie*-supplemented module which was described in [6] by Birkenmeier et al. via the β^* relation. The β^* relation presented in [6] is defined as $U\beta^* A$ if $\frac{U+A}{U} \ll \frac{M}{U}$ and $\frac{U+A}{A} \ll \frac{M}{A}$. This means that if $\frac{U+A}{U} + \frac{X}{U} = \frac{M}{U}$ for any submodule X of M containing U, then $\frac{X}{U} = \frac{M}{U}$. Similarly, if $\frac{U+A}{A} + \frac{X}{A} = \frac{M}{A}$ for any submodule X of M containing A, then $\frac{X}{A} = \frac{M}{A}$. In ([6], Lemma 2.2), the authors said that β^* is an equivalence relation and $0\beta^* U$ with $U \ll M$.

Definition 1 ([6], Definition 3.1)**.** *A module M is said to be Goldie*-supplemented (in short, $\mathcal{G}^*s$) if each submodule of M is β^* equivalent to a supplement submodule of M; equally, there is a supplement submodule S of M such that $U\beta^* S$ for each submodule U of M.*

Example 1. *Semisimple and linearly compact modules are $\mathcal{G}^*s$ modules.*

Example 2. *Let R be a commutative local ring which has two incomparable ideals, I and J. Let $M = R/I \oplus R/J$. Then, M is $\mathcal{G}^*s$ (see [6], Example 3.9(ii)).*

Recall that a submodule U of M has ample supplements in M if, for every submodule A of M with $U + A = M$, there is a supplement S of U with $S \subseteq A$. If all submodules have ample supplements in M, then M is called amply supplemented ([9]). The $\mathbb{Z}$-modules $\mathbb{Z}_{p^k}$, where p is prime and $k \in \mathbb{N}$, are amply supplemented modules. In particular, an amply supplemented module implies a supplemented module.

With ([6], Theorem 3.6 and Proposition 3.11), we say the following implications:

$$\text{amply supplemented} \Rightarrow \mathcal{G}^*\text{s} \Rightarrow \text{supplemented}$$

The converse of the above implications is true under certain conditions. In particular, in [1], it was shown that the finitely generated supplemented modules are amply supplemented. In this case, the finitely generated supplemented modules are also $\mathcal{G}^*$s.

Theorem 1. *If M is $\mathcal{G}^*s$ over a non-local Dedekind domain R, then M is amply supplemented.*

Proof. Applying ([6], Theorem 3.6), we see that M is supplemented. Hence, ([1], Theorem 1.3) implies that M is amply supplemented. □

Proposition 1. *If M is a $\mathcal{G}^*s$, then $\frac{M}{U}$ is a $\mathcal{G}^*s$ for $U \ll M$.*

Proof. Let us take a submodule $\frac{A}{U}$ of $\frac{M}{U}$ for a submodule A of M. By hypothesis, $A\beta^* S$. Here, S is a supplement submodule in M. Let us consider the small epimorphism $\alpha : M \to \frac{M}{U}$. From ([6], Proposition 2.9(i)), $\alpha(A)\beta^*\alpha(S)$. In this situation, $\frac{A}{U}\beta^*(\frac{S+U}{U})$. Using ([10], Lemma 4), we see that $\frac{S+U}{U}$ is also a supplement submodule in $\frac{M}{U}$. Thus, $\frac{M}{U}$ is $\mathcal{G}^*$s. □

Even if M/U is $\mathcal{G}^*$s, it is evident from the following example that M may not be $\mathcal{G}^*$s.

Example 3 (see [11] , Remark 3.3). *Let* $R = \mathbb{Z}_{p,q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0, p \nmid b, q \nmid b\}$ *be a ring where p and q are prime numbers. Then, R is a commutative uniform semilocal noetherian domain with two maximal ideals. As a result, R is a semilocal ring which is not semiperfect. Thus,* $R/Jac(R)$ *is amply supplemented. Additionally,* $R/Jac(R)$ *is* $\mathcal{G}^*s$ *by ([6], Proposition 3.11). However, the R-module R is not supplemented as stated in ([12], 42.6). On the other hand, the R-module R is not* $\mathcal{G}^*s$*, by ([6], Theorem 3.6).*

To prove that M is $\mathcal{G}^*$s while M/U is $\mathcal{G}^*$s, we need to use a refinable module. Recall that M is called *refinable* if $M = U + A$ for any submodules U, A of M, there is a direct summand V of M with $V \subseteq U$, $M = V + A$ [9].

Proposition 2. *If* $\frac{M}{U}$ *is* $\mathcal{G}^*s$ *with* $U \ll M$ *and M is a refinable, then M is* $\mathcal{G}^*s$*.*

Proof. Suppose A is a submodule of M. From our assumption, $(\frac{A+U}{U})\beta^*\frac{S}{U}$, where $\frac{S}{U}$ is a supplement in $\frac{M}{U}$. Then, we deduce that, for some submodule X of M, $\frac{M}{U} = \frac{S}{U} + \frac{X+U}{U}$, $\frac{S}{U} \cap \frac{X+U}{U} = \frac{(S \cap X)+U}{U} \ll \frac{S}{U}$. Consider an epimorphism $f : M \to \frac{M}{U}$. Notice that $M = S + X$ and $S \cap X \ll M$ from ([9], 2.2(5)). In fact, X is a weak supplement of S in M. According to ([6], Proposition 2.9(ii)), $f^{-1}(\frac{A+U}{U})\beta^* f^{-1}(\frac{S}{U})$. It yields that $(A+U)\beta^* S$. By ([6], Corollary 2.12), $A\beta^* S$ in M. Using ([6], Theorem 2.6(ii)), it can be said that X is also a weak supplement of A in M. It is natural to write as $M = A + X$ and $A \cap X \ll M$. As M is refinable, for the direct summand A' of M, $A' \subseteq A$ and $M = A' + X$. In this case, $A' \cap X \ll M$, by ([12], 19.3(2)). This allows us to say that X is a weak supplement of A' in M. Then, ([6], Corollary 2.7) shows that $A'\beta^* A$, where A' is a supplement in M. Hence, M is $\mathcal{G}^*$s. □

Proposition 3. *If M is a finitely generated module over commutative ring R, then M is* $\mathcal{G}^*s$ *if, and only if,* $\frac{M}{U}$ *is* $\mathcal{G}^*s$ *for a linearly compact submodule U of M.*

Proof. Suppose M is $\mathcal{G}^*$s. We know from ([6], Theorem 3.6) that M is supplemented. As such, $\frac{M}{U}$ is supplemented because of ([1], Theorem 2.8). Since M is finitely generated, we deduce that $\frac{M}{U}$ is amply supplemented by ([1], Corollary 4.6). Hence, $\frac{M}{U}$ is $\mathcal{G}^*$s, by ([6], Proposition 3.11). Conversely, ([6], Theorem 3.6) states that if $\frac{M}{U}$ is $\mathcal{G}^*$s, then $\frac{M}{U}$ is supplemented. ([1], Theorem 2.8) allow us to show that M is supplemented. As M is finitely generated, M is amply supplemented by ([1], Corollary 4.6). As a consequence, M is $\mathcal{G}^*$s by ([6], Proposition 3.11). □

We conclude the following consequence by utilizing Proposition 3.

Corollary 1. *Let R be a commutative ring, M be a finitely generated module, and U be a linearly compact submodule of M. In the following exact sequence,*

$$0 \to U \to M \to \frac{M}{U} \to 0$$

M is $\mathcal{G}^*s$ *if, and only if, U and* $\frac{M}{U}$ *are* $\mathcal{G}^*s$*.*

Proposition 4. *Let M be a quasi-projective module and U be a linearly compact submodule of M. If in the exact sequence,*

$$0 \to U \to M \to \frac{M}{U} \to 0$$

*U and $\frac{M}{U}$ are $\mathcal{G}$*s, M is $\mathcal{G}$*s.*

Proof. Let M be a quasi-projective module and U be a linearly compact submodule of M. Assume $\frac{M}{U}$ is a $\mathcal{G}^*$s. Then, $\frac{M}{U}$ is supplemented by ([6], Theorem 3.6). It follows from ([1], Theorem 2.8) that M is supplemented. Since M is quasi-projective, M is $\mathcal{G}^*$s, by ([6], Proposition 3.12), as desired. □

A module M is called *distributive* if $U \cap (A + B) = (U \cap A) + (U \cap B)$ for all submodules U, A, B of M.

Now, apply the distributive property to get the following result.

Proposition 5. *If $M = A \oplus B$ is a distributive module where A and B are $\mathcal{G}^*$s, then M is $\mathcal{G}^*$s.*

Proof. Let U be a submodule of M. Using the distributive property, we have $U = (U \cap A) \oplus (U \cap B)$. Since A and B are $\mathcal{G}^*$s, $(U \cap A)\beta^* X$ and $(U \cap B)\beta^* Y$, where X and Y are supplements in A and B, respectively. In other words, $A = X + X'$, $X \cap X' \ll X$ for some submodule X' of A, and $B = Y + Y'$, $Y \cap Y' \ll Y$ for some submodule Y' of B. According to ([6], Proposition 2.11), $((U \cap A) + (U \cap B))\beta^*(X + Y)$. In this situation, we write as $U\beta^*(X + Y)$. Now, we will show that $X + Y$ is a supplement in M. We have $M = A + B = (X + X') + (Y + Y') = (X + Y) + (X' + Y')$. Since M is distributive, we obtain that $(X + Y) \cap (X' \cap Y') = (X \cap X') + (Y \cap Y') \ll X + Y$ by ([12], 19.3). This indicates that $X + Y$ is a supplement submodule in M. □

Proposition 6. *Let R be a commutative ring, let each M_i be $\mathcal{G}^*$s for $i = 1, \ldots, n$, and let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ such that $R = ann(M_i) + ann(M_j)$ for all $1 \le i < j \le k$. Thus, M is $\mathcal{G}^*$s.*

Proof. Consider a submodule U of M. By ([1], Lemma 4.1), $U = (U \cap M_1) \oplus \ldots \oplus (U \cap M_k)$. Since M_i is $\mathcal{G}^*$s, then $(U \cap M_i)\beta^* S_i$, where S_i is a supplement submodule of M_i for $i = 1, \ldots, k$. It follows from ([6], Proposition 2.11) that $U\beta^*(S_1 + S_2 + \ldots + S_k)$, where $(S_1 + S_2 + \ldots + S_k)$ is a supplement submodule in M. If every S_i is a supplement submodule in M_i, then $M_i = S_i + X_i$ and $S_i \cap X_i \ll S_i$ for $i = 1, \ldots, k$ for some submodule X_i of M_i. Indeed, we have $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k = (S_1 + S_2 + \ldots + S_k) + (X_1 + X_2 + \ldots + X_k)$ and $(S_1 + S_2 + \ldots + S_k) \cap (X_1 + X_2 + \ldots + X_k) = (S_1 \cap X_1) + \ldots + (S_k \cap X_k) \ll S_1 + S_2 + \ldots + S_k$ by ([12], 19.3). □

Proposition 7. *Every submodule of $\mathcal{G}^*$s module over a left V-ring is a direct summand.*

Proof. Assume that M is a $\mathcal{G}^*$s and U is a submodule of M. Then, $U\beta^* S$ for supplement submodule S of M. This implies that $M = S + X$ and $S \cap X$ is small in S for some submodule X of M. If $S \cap X \subseteq Rad(S) \subseteq Rad(M)$, $S \cap X = 0$, since $Rad(M) = 0$. As a result, $M = S \oplus X$. In particular, it follows from ([6], Theorem 2.6) that X is also a supplement of U in M. Notice that $M = U + X$ and $U \cap X \ll X$. Similarly, $U \cap X \subseteq Rad(X) \subseteq Rad(M) = 0$ and $U \cap X = 0$. Thus, $M = U \oplus X$. □

3. Totally Goldie*-Supplemented Modules

In this part, we are interested in some properties of a totally Goldie*-supplemented module. We mention that every factor module of a totally $\mathcal{G}^*$-supplemented module is totally $\mathcal{G}^*$-supplemented, and the finite direct sum of totally $\mathcal{G}^*$-supplemented modules is totally $\mathcal{G}^*$-supplemented. Finally, we point out in Theorem 3 that totally $\mathcal{G}^*$-supplemented, totally supplemented, and $\mathcal{G}^*$-supplemented modules are equivalent under additional circumstances.

Definition 2. *A module M is called a* totally Goldie*-supplemented module *(in short, t$\mathcal{G}^*$s) if every submodule of M is $\mathcal{G}^*$s.*
It is not hard to see that every t$\mathcal{G}^*$s is $\mathcal{G}^*$s.

Example 4. *Semisimple and linearly compact modules are t$\mathcal{G}^*$s.*

Proposition 8. *Every t$\mathcal{G}^*$s module is totally supplemented.*

Proof. If U is a submodule of M, then U is $\mathcal{G}^*$s by assumption. Then, ([6], Theorem 3.6) applies, allowing the fact that U is supplemented to be obtained. Hence, M is totally supplemented. □

Every $\mathcal{G}^*$s module need not be t$\mathcal{G}^*$s. To show this, we can give the following example.

Example 5 (see [13], p. 482)**.** *Let R be a local Dedekind domain which is not a field domain. Suppose that $M = R^{(\mathbb{N})}$. Then, M is not (amply) supplemented. The group $N = R \times M$ is a ring with the operation $(a,b).(x,y) = (ax, ay + xb)$ for $a, x \in R$ and $b, y \in M$. Then, N is a commutative local ring. Thus, N is amply supplemented. It follows from ([6], Proposition 3.11) that N is $\mathcal{G}^*$s. Consider the ideal $A = \{0\} \times M$ of N. Hence, the submodule A of N is not a supplemented N-module. Therefore, A is not $\mathcal{G}^*$s by ([6], Theorem 3.6) and, so, N is not t$\mathcal{G}^*$s.*

We indicate the following relations:

t$\mathcal{G}^*$s	$\Longrightarrow$	$\mathcal{G}^*$s
$\Downarrow$		$\Downarrow$
totally supplemented	$\Longrightarrow$	supplemented

Theorem 2. *Let M be a refinable module. Then, M is $\mathcal{G}^*$s if, and only if, M is t$\mathcal{G}^*$s.*

Proof. Suppose M is $\mathcal{G}^*$s and U is a submodule of M. It suffices to show that U is $\mathcal{G}^*$s. By means of ([6], Corollary 3.3(i)), we conclude that U is $\mathcal{G}^*$s. Let X be a submodule of U. Since M is $\mathcal{G}^*$s, $X\beta^*Y$, where Y is a supplement in M. Note that $M = Y + K$ and $Y \cap K \ll Y$ for some submodule K of M. We can see that $Y \cap K \ll M$ from ([12], 19.3). This means that K is a weak supplement of Y in M. However, K is also a weak supplement of X in M based on ([6], Theorem 2.6). Therefore, $M = X + K$ and $X \cap K \ll M$. Since M is refinable, there exists a direct summand A of M such that $A \subseteq X$ and $M = A + K$. Since $A \subseteq X$, then $A \cap K \subseteq X \cap K \ll M$ implies $A \cap K \ll M$. In this case, we can consider K as a weak supplement of A in M. If A is a direct summand in M, then $M = A \oplus B$ for some submodule B of M. If A has a weak supplement K in M and $X \cap K \ll M$, then $A\beta^*X$ from ([6], Corollary 2.7). Now, it remains to show that A is a supplement submodule of U. If $M = A \oplus B$, then $U = U \cap M = U \cap (A \oplus B) = A \oplus (U \cap B)$ by modularity. Thus, A is a supplement of $U \cap B$ in U. □

Theorem 3. *Consider the following statements:*

(1) M is t$\mathcal{G}^*$s,
(2) M is totally supplemented,
(3) M is $\mathcal{G}^*$s,
(4) M is supplemented.

Then, $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. If M is refinable, $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ holds.

Proof. (1) $\Rightarrow$ (2) Clear from Proposition 8.
(2) $\Rightarrow$ (3) Recall that every totally supplemented is amply supplemented by ([1], Corollary 1.2). Then, we obtain from ([6], Proposition 3.11) that M is $\mathcal{G}^*$s.
(3) $\Rightarrow$ (4) It is clear from ([6], Theorem 3.6).
(4) $\Rightarrow$ (3) Let M be a refinable supplemented module and U be a submodule of M. We claim that U is β^* equivalent to any supplement submodule of M. From (4), U has a supplement in M, say S. Namely, we can write $M = U + S$ and $U \cap S \ll S$. By ([12], 19.3), $U \cap S \ll M$. The refinable property says that there is a submodule A in M which is a direct summand in M so that $A \subseteq U$ and $M = A + S$. If $U \cap S \ll M$, then $A \cap S \ll M$, again by ([12], 19.3). This verifies that A has a weak supplement S in M. Since $U \cap S \ll M$, $U\beta^*A$ by ([6], Corollary 2.7). Here, A is a supplement submodule in M, since A is a direct summand in M. This proves (3).
(3) $\Rightarrow$ (2) If M is refinable $\mathcal{G}^*$s, then M is t$\mathcal{G}^*$s because of Theorem 2. From Proposition 8, M is totally supplemented.
(2) $\Rightarrow$ (1) Suppose U is a submodule of M and X is a submodule of U. We will show that U is $\mathcal{G}^*$s. M is $\mathcal{G}^*$s from (2), which implies (3). As such, $X\beta^*Y$, where Y is a supplement submodule in M. Then, $M = Y + B$ and $Y \cap B \ll Y$ for some submodule B of M. By ([12], 19.3), $Y \cap B \ll M$. In this case, B is a weak supplement of Y in M. Moreover, ([6], Theorem 2.6) shows that $M = X + B$ and $X \cap B \ll M$. Since M is refinable, for a direct summand A of M, $A \subseteq X$ and $M = A + B$. If $X \cap B \ll M$, $A \cap B \ll M$ from ([12], 19.3). Obviously, A is a weak supplement of B in M. Then, ([6], Corollary 2.7) shows that $A\beta^*X$. If A is a direct summand in M, then $M = A \oplus A'$ for some submodule A' of M. By modularity, $U = U \cap M = U \cap (A \oplus A') = A \oplus (U \cap A')$, that is, A is a supplement of $U \cap A'$ in U. Hence, U is $\mathcal{G}^*$s. □

The following two examples show that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) in Theorem 3 do not hold in general.

Example 6. *(1) (see [6], Example 3.9(iii)). Let K be the quotient field of discrete valuation domain R which is not complete. Let $M = K \oplus K$. Then, M is supplemented, but not $\mathcal{G}^*$s.*

(2) (see [1], Example 1.7). Let R be a commutative ring and M be an R-module. Then, $[R, M]$ will denote the commutative ring of matrices of the form:

$$\begin{pmatrix} s & a \\ 0 & s \end{pmatrix}$$

such that $s \in R$, $a \in M$, with the usual matrix addition and multiplication. Let R be any commutative local domain which is not a field domain, let M be any free R-module of infinite rank, and let $S = [R, M]$ be the commutative ring. Then, the S_S-module is local (that is, amply supplemented). By ([6], Proposition 3.11), the S_S-module is $\mathcal{G}^*$s, but S_S is not totally supplemented. Thus, S_S is not t$\mathcal{G}^*$s.

Theorem 4. *Let R be a non-local Dedekind domain. Then, the following statements are equivalent:*

(1) M is t$\mathcal{G}^*$s,
(2) M is totally supplemented,
(3) M is $\mathcal{G}^*$s,
(4) M is supplemented.

Proof. It is clear from Theorem 3 that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Now, we will prove that (4) $\Rightarrow$ (1). Let U be a submodule of M. We need to prove that U is $\mathcal{G}^*$s. From ([1], Theorem 1.3), M is totally supplemented. Then, we can say that U is supplemented. Again, from ([1], Theorem 1.3), U is amply supplemented. According to ([6], Proposition 3.11), U is $\mathcal{G}^*$s. As a consequence, M is t$\mathcal{G}^*$s. □

The following result can be derived from Theorem 4.

Corollary 2. *Let R be a non-local Dedekind domain. Then, M is $\mathcal{G}^*$s if, and only if, M is amply supplemented.*

Proposition 9. *If M is a t$\mathcal{G}^*$s, then $\frac{M}{U}$ is t$\mathcal{G}^*$s for some submodule U of M.*

Proof. Let $\frac{K}{U}$ be a submodule of $\frac{M}{U}$ for submodule K of M containing U. Our aim is to show that $\frac{K}{U}$ is $\mathcal{G}^*$s. Since K is t$\mathcal{G}^*$s, it follows from Proposition 8 that K is totally supplemented. By ([1], Theorem 2.8), $\frac{K}{U}$ is totally supplemented. Using ([1], Corollary 1.2), we can say that $\frac{K}{U}$ is amply supplemented. Hence, it easy to see from ([6], Proposition 3.11) that $\frac{K}{U}$ is $\mathcal{G}^*$s. □

Proposition 10. *Let $M = A \oplus B$ be a distributive module for some submodules A and B of M. If A and B are t$\mathcal{G}^*$s, then M is t$\mathcal{G}^*$s.*

Proof. Take a submodule U of M. By the distributive property, $U = (U \cap A) \oplus (U \cap B)$. Since A and B are totally $\mathcal{G}^*$s, $U \cap A$ and $U \cap B$ are $\mathcal{G}^*$s. By Proposition 5, U is $\mathcal{G}^*$s, as required. □

Now we can adapt ([1], Lemma 4.2) to our situation.

Proposition 11. *Let R be a commutative ring, and let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ be a finite direct sum of t$\mathcal{G}^*$s M_i ($1 \le i \le k$) such that $R = ann(M_i) + ann(M_j)$ for all $1 \le i < j \le k$. Then, M is t$\mathcal{G}^*$s.*

Proof. Suppose that M_i is t$\mathcal{G}^*$s for all $1 \le i \le k$. Consider the submodule U of M. Then, $U = (U \cap M_1) \oplus \ldots \oplus (U \cap M_k)$ from ([1], Lemma 4.1). Since $U \cap M_i \subseteq M_i$ for $i = 1, \ldots, k$ and M_i's are t$\mathcal{G}^*$s, $U \cap M_i$ is $\mathcal{G}^*$s. Proposition 6 states that U is $\mathcal{G}^*$s. Therefore, M is t$\mathcal{G}^*$s. □

Proposition 12 is analogous to ([1], Theorem 2.9). However, the distributive property is needed in our result.

Proposition 12. *Let $M = M_1 \oplus M_2$ for submodules M_1 and M_2 of M such that M_2 is semisimple. If M is distributive, then M is t$\mathcal{G}^*$s if, and only if, M_1 is t$\mathcal{G}^*$s.*

Proof. ($\Rightarrow$) Suppose that M is t$\mathcal{G}^*$s. By Proposition 9, M_1 is t$\mathcal{G}^*$s, since $\frac{M}{M_2} \cong M_1$ ($\Leftarrow$) Let U be a submodule of M. Since M_2 is semisimple, $M_2 = (U \cap M_2) \oplus K$ for some submodule K of M_2. Here, $U \cap M_2 \cap K = U \cap K = 0$. Then, $M = M_1 \oplus M_2 = M_1 \oplus (U \cap M_2) \oplus K$. By modularity, $U = (U \cap M_2) \oplus (U \cap (M_1 \oplus K))$. Let $H = U \cap (M_1 \oplus K)$, that is, $U = (U \cap M_2) \oplus H$. Then, $H \cap K = U \cap (M_1 \oplus K) \cap K = U \cap K = 0$. Hence, H embeds in M_1. By hypothesis, H is $\mathcal{G}^*$s. Since M_2 is semisimple, $U \cap M_2$ is also semisimple. By Example 1, $U \cap M_2$ is $\mathcal{G}^*$s. Thus, U is $\mathcal{G}^*$s, based on Proposition 5. As a result, M is t$\mathcal{G}^*$s. □

Proposition 13. *Let M be a refinable t$\mathcal{G}^*$s. Then, for each submodule U of M, $U = S + A$ such that S is a supplement submodule of M and $A \ll M$.*

Proof. Let U be a submodule of M. From assumption, we can say that M is $\mathcal{G}^*$s. Then, $U\beta^*S$ for a supplement submodule S of M. Namely, $M = S + X$ and $S \cap X \ll S$ for some submodule X of M. Following ([12], 19.3), we can say that $S \cap X \ll M$. Consequently, X is a weak supplement of S in M. Moreover, ([6], Theorem 2.6) states that X is a weak supplement of U in M. This implies that $M = U + X$ and $U \cap X \ll M$. To see the result, we follow the refinable property. Then, for a direct summand A' of M, $A' \subseteq U$ and also $M = A' + X$. It turns out that $U = U \cap (A' + X) = A' + (U \cap X)$ by modular law. □

Proposition 14. *Let M be t$\mathcal{G}$*s over a left V-ring and U be a submodule of M. Then, U is a direct summand in M.*

Proof. Since every t$\mathcal{G}$*s is $\mathcal{G}$*s, the proof is proved by Proposition 7. □

4. Conclusions

In this work, we have investigated the modules with submodules that are Goldie*-supplemented ($\mathcal{G}$*s) modules. The interesting results we obtained allowed us to characterize totally Goldie*-supplemented (t$\mathcal{G}$*s) modules. One of the interesting results is that the factor module of a t$\mathcal{G}$*s module is also t$\mathcal{G}$*s. We have shown that the finite direct sum of $\mathcal{G}$*s (t$\mathcal{G}$*s) is also $\mathcal{G}$*s (t$\mathcal{G}$*s) under the distributive property. Furthermore, we have proved that M is t$\mathcal{G}$*s if, and only if, the direct summand of M is t$\mathcal{G}$*s by using the distributive property. Moreover, every submodule of $\mathcal{G}$*s (t$\mathcal{G}$*s) over a left V-ring is a direct summand. Specifically, we have indicated the connection between $\mathcal{G}$*s, t$\mathcal{G}$*s, totally supplemented, and supplemented modules, provided that these modules coincide under the refinable condition. The theory of supplemented modules, totally supplemented modules, and their variations has been actively studied and is still being studied. There is potential for future work in the study of rings whose modules are t$\mathcal{G}$*s. In addition, it may be intriguing to examine totally cofinitely Goldie*-supplemented modules.

Funding: There is no funding for this work.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Smith, P.F. Finitely generated supplemented modules are amply supplemented. *Arab. Sci. Eng.* **2000**, *25*, 69–79.
2. Bilhan, G. Totally cofinitely supplemented modules. *Int. Electron. J. Algebra* **2007**, *2*, 106–113.
3. Top, S. Totally Weak Supplemented Modules. Master's Thesis, İzmir Institute of Technology, İzmir, Turkey, 2007.
4. Eryılmaz, F.Y.; Eren, Ş. Totally cofinitely weak Rad-supplemented modules. *Int. Pure Appl. Math.* **2012**, *80*, 683–692.
5. Hassan, W.K.; Dnan, R.H.A. Totally ⊕generalized *cofinitely supplemented modules. *Iraqi Sci.* **2015**, *56*, 2038–2042.
6. Birkenmeier, G.F.; Mutlu, F.T.; Nebiyev, C.; Sökmez, N.; Tercan, A. Goldie*-supplemented modules. *Glasg. Math. J.* **2010**, *52*, 41–52. [CrossRef]
7. Sökmez, N. Goldie*-Tümlenmiş ve Goldie*-Radikal Tümlenmiş Modüller. Ph.D. Thesis, Ondokuz Mayıs University, Samsun, Turkey, 2011.
8. Talebi, Y.; Hamzekolaee, A.R.M.; Tercan, A. Goldie-Rad-supplemented modules. *An. Şt. Univ. Ovidius Constanta* **2014**, *22*, 205–218. [CrossRef]
9. Clark, J.; Lomp, C.; Vanaja, N.; Wisbauer, R. *Lifting Modules: Frontiers in Mathematics*; Birkhäuser-Verlag: Boston, MA, USA, 2006.
10. Acar, U.; Harmancı, A. Principally supplemented modules. *Albanian J. Math.* **2010**, *4*, 79–88. [CrossRef]
11. Lomp, C. On semilocal modules and rings. *Comm. Algebra* **1999**, 27, 1921–1935. [CrossRef]
12. Wisbauer, R. *Foundations of Modules and Rings*; Gordon and Breach: New York, NY, USA, 1991.
13. Kaynar, E.; Çalışıcı, H.; Türkmen, E. SS-Supplemented modules. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2020**, *69*, 473–485. [CrossRef]