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On the Fractional Derivative Duality in Some Transforms

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Abstract: Duality is one of the most interesting properties of the Laplace and Fourier transforms associated with the integer-order derivative. Here, we will generalize it for fractional derivatives and extend the results to the Mellin, Z and discrete-time Fourier transforms. The scale and nabla derivatives are used. Some consequences are described.

Keywords: Liouville derivative; scale derivative; Hadamard derivative; Laplace transform; Mellin transform; Z transform; Fourier transform

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1. Introduction

Transform is a term that originated a long time ago, probably in the work of Euler. In fact, the Laplace transform (LT) was introduced by him in 1737 [1]. However, it is used in many applications, not only in mathematics, but in all sciences and engineering. We therefore need to restrict our definition so that it is in line with our objectives. We will use the designation *transform* for operators defined in $\mathbb{R}$ or $\mathbb{Z}$ that generate functions defined in $\mathbb{C}$ and that take on integral or series format. The classic integral transforms, namely the Fourier, Laplace, Mellin, Hankel and Stieltjes transforms [2–5], and some variants are the most useful in practical applications. In engineering, other transforms have been introduced, such as the fractional Fourier transform (FT) [6,7], the Wavelet transform [8,9] and the Radon transform [4]. In recent years, many transforms have been introduced, without their usefulness being clear [10]. Most are modified one-sided Laplace transforms. Although many transforms are defined over continuous domains, there are some discrete transforms such as the Z transform [11], also called characteristic function, the discrete-time Fourier transform [12,13], the discrete Mellin transform [14] and the Fermat number transform [15].

Here, we will consider those transforms that have direct relations with linear systems:

1. the continuous-time (two-sided) Laplace and the Fourier transforms that are associated with the linear systems described by shift-invariant convolution [11,12,16,17];
2. the continuous-scale Mellin transform that is tied with the linear systems described by the Mellin convolution [17,18];
3. the discrete-time Z and Fourier transforms connected with the discrete shift-invariant convolution [11,12,16].

We have not considered the discrete Mellin transform here, as it poses some difficulties that have not been sufficiently studied [19].

These transforms have a duality property that can be expressed as a reversibility characteristic that can be stated as follows: *if a given transform has a particular property, the inverse transform has a similar property*. The extreme expression of this property is found in

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the Fourier transform (FT). To understand this idea, we consider the FT of an absolutely integrable function, $f(t)$,

$$F(i\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t,$$

and its inverse

$$f(t) = \mathcal{F}^{-1}[F(i\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}\mathrm{d}\omega.$$

Duality says that

$$\mathcal{F}[F(it)] = 2\pi f(-\omega),$$

which is very useful in the computation of some transforms.

In this paper, we are interested in the derivative duality. Let D represent the usual derivative and $n \in \mathbb{N}$. As $D^n_\omega e^{-i\omega t} = (-it)^n e^{-i\omega t}$, we have

$$F^{(n)}(i\omega) = \int_{-\infty}^{\infty} (-it)^n f(t)e^{-i\omega t}\mathrm{d}t, \tag{1}$$

and, similarly, $D^n_t e^{i\omega t} = (-i\omega)^n e^{i\omega t}$,

$$f^{(n)}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^n F(i\omega)e^{i\omega t}\mathrm{d}\omega. \tag{2}$$

These relations clearly express the duality property of FT and the associated usual derivative: *deriving t (ω) in one domain corresponds to multiplying ω (t) in the other domain.* This property is useful in transforming some variable coefficient ordinary differential equations.

In the current literature, only Fourier transform cases are dealt with. The fractional case is not considered anywhere. Here, we want to generalize the derivative duality to fractional orders and to other transforms. We will consider Laplace, Mellin (MT), Z (ZT) and discrete-time Fourier transforms (DTFT). To achieve this, we need to introduce the necessary fractional derivatives: Liouville, Hadamard and discrete, nabla and bilinear. The best-known fractional derivatives are not useful for our purposes, as we shall see.

The paper is outlined as follows. In Section 2, we introduce the required properties for the fractional derivatives to use. The integer order case of duality is dealt with in Section 3, and the fractional cases are studied in Section 4. In Section 5, we describe some consequences of the results we obtained in the previous sections. Finally, we present some conclusions.

2. Suitable Derivatives

2.1. Derivative Requirements

The most important transforms have exponentials or powers as kernels due to their relations with shift-invariant or scale-invariant systems [17,18]. Therefore, if we want to compute derivatives of the corresponding transforms and continue having a transform of the same type, we must have derivatives verifying (Liouville requirement)

$$D^\alpha_z e^{az} = a^\alpha e^{az}, \tag{3}$$

or (Hadamard requirement)

$$\mathfrak{D}^\beta_z z^b = b^\beta z^b, \tag{4}$$

for suitable orders $\alpha, \beta \in \mathbb{R}$ and complex variable $z \in \mathbb{C}$. For simplicity matters and later utility, we will assume that $a, b \in \mathbb{R}$ also.

Remark 1. *It is important to note that the usual Riemann–Liouville and Caputo derivatives are not useful in this case since they do not check relationships (3) and (4).*

2.2. Liouville-Type Derivatives

The Liouville-type derivatives can be expressed in a general unified way [20,21] assuming both summation and integral formulations.

Definition 1. *We will consider the Grünwald–Letnikov (GL) derivatives that we define by:*

$$D^{\alpha}_{\pm} f(z) := \lim_{h\to 0^+} (\pm h)^{-\alpha} \sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} f(z \mp nh), \tag{5}$$

where we denoted by $(a)_n,\ n = 1, 2, \cdots$ *the Pochhammer symbol for the rising factorial*

$$(a)_0 = 1,\ (a)_n = \prod_{k=0}^{n-1} (a+k).$$

To avoid confusion of symbols for different derivatives, we will write frequently $D^{\alpha}_{\pm} f(t) = f^{(\alpha)}_{\pm}(t)$. If necessary, we will put the independent variable in the subscript.

Theorem 1. *Let* $f(z) = e^{az}$. *Then, [21,22]*

$$D^{\alpha}_{\pm} e^{az} = a^{\alpha} e^{az}, \tag{6}$$

if $\pm Re(a) > 0$, *while both diverge with* $\pm Re(a) < 0$, *unless* $\alpha \in \mathbb{Z}$.

For applications to the Laplace transform, we need to compute derivatives of $e^{st}, t \in \mathbb{R}$, $s \in \mathbb{C}$. We obtain:

- Derivative in t
$$D^{\alpha}_{t\pm} e^{st} = s^{\alpha} e^{st}, \quad \pm Re(s) > 0; \tag{7}$$
it is useful for the derivative computation of the inverse LT.
- Derivative in s
$$D^{\alpha}_{s\pm} e^{-st} = (-t)^{\alpha} e^{-st}, \quad \mp t > 0; \tag{8}$$
this derivative allows us to express the derivative of the direct LT.

2.3. Hadamard-Type Derivatives

Definition 2. *The Hadamard-type derivatives are scale-invariant and verify the above requirement (4). Similarly to the Liouville's type, we define the stretching (+) and shrinking (−) GL-type derivatives by [18]*

$$\mathfrak{D}^{\alpha}_{v\pm} x(v) = \lim_{q\to 1^+} \ln^{-\alpha}(q^{\pm}1) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(vq^{\mp n}), \tag{9}$$

where $q > 1$.

Theorem 2.

$$\mathfrak{D}^{\alpha}_{v\pm} v^b = b^{\alpha} v^b, \tag{10}$$

provided that $\pm Re(v) > 0$. *If* $\pm Re(v) < 0$, *they diverge.*

Similarly to the LT case, we need to compute derivatives of $\tau^v, \tau \in \mathbb{R}^+, v \in \mathbb{C}$. We obtain:

- Derivative in τ
$$\mathfrak{D}^{\alpha}_{\tau\pm} \tau^v = v^{\alpha} \tau^v, \quad \pm Re(v) > 0; \tag{11}$$
it is useful for the derivative computation of the inverse MT.
- Derivative in v
$$D^{\alpha}_{v\pm} \tau^{-v} = (-\ln \tau)^{\alpha} \tau^{-v}, \quad \tau^{\mp 1} > 1; \tag{12}$$

this derivative allows us to express the derivative of the direct MT.

2.4. Discrete-Time Derivatives

2.4.1. Fractional Nabla and Delta Derivatives

In the following, we consider that our domain is the *time scale* or *time sequence*

$$\mathbb{T}_h = (h\mathbb{Z}) = \{\ldots, -nh \ldots, -2h, -h, 0, h, 2h, \ldots, nh, \ldots\},$$

with $h \in \mathbb{R}^+$, which is called the graininess or sampling interval [23,24].

Definition 3. *Let $f(t)$ be a function defined on $\mathbb{T}$. Set $t = nh$. We define the nabla derivative by:*

$$\nabla f(t) = \frac{f(t) - f(t-h)}{h} \tag{13}$$

and the delta derivative by

$$\Delta f(t) = \frac{f(t+h) - f(t)}{h}. \tag{14}$$

The corresponding fractional derivatives read [24]:
nabla

$$\nabla^\alpha f(t) = \frac{\sum\limits_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t-nh)}{h^\alpha} \tag{15}$$

and delta

$$\Delta^\alpha f(t) = \frac{\sum\limits_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(t+nh)}{h^\alpha}. \tag{16}$$

Theorem 3. *The eigenvalue of these derivatives is s^α, and the corresponding eigenfunctions are the nabla and delta exponentials given by [24]*

$$e_\nabla(kh, s) = \frac{1}{(1-sh)^k} \tag{17}$$

and

$$e_\Delta(kh, s) = (1+sh)^k. \tag{18}$$

These results were generalized for irregular time sequences in [24]. In the following, we will continue with the nabla derivative.

Definition 4. *With the nabla exponential, we can define the nabla Laplace transform [24] through*

$$F_\nabla(s) = h \sum_{k=-\infty}^{+\infty} f(kh) e_\nabla(-kh, s), \tag{19}$$

with its inverse transform being given by

$$f(kh) = -\frac{1}{2\pi j} \oint_\gamma F_\nabla(s) e_\nabla((k+1)h, s) ds, \tag{20}$$

where the integration path, γ, is any simple closed contour in a region of analyticity of the integrand that includes the point $s = \frac{1}{h}$. The simplest path is a circle with center at $s = \frac{1}{h}$.

Corollary 1. *Let $z^{-1} = 1 - sh$. Then*

$$\nabla^\alpha z^n = \left[\frac{1-z^{-1}}{h}\right]^\alpha z^n. \tag{21}$$

The proof is immediate.

With this change, we entered in the framework of the Z transform.

2.4.2. Forward and Backward Derivatives Based on the Bilinear Transformation

The Tustin transformation is usually expressed by [11,13]

$$s = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}, \tag{22}$$

where s is the derivative operator associated with the (continuous-time) LT and z^{-1} the delay operator tied to the Z transform.

Definition 5. *Let $x(nh)$ be a discrete-time function. We define the order 1 forward or nabla bilinear derivative $\nabla_b x(nh)$ of $x(nh)$ as the solution of the difference equation*

$$\nabla_b x(nh) + \nabla_b x(nh-h) = \frac{2}{h}[x(nh) - x(nh-h)]. \tag{23}$$

Similarly, we define the order 1 backward or delta bilinear derivative $\Delta_b x(nh)$ of $x(nh)$ as the solution of

$$\Delta_b x(nh+h) + \Delta_b x(nh) = \frac{2}{h}[x(nh+h) - x(nh)]. \tag{24}$$

Definition 6. *We define the nabla bilinear derivative (∇_b) as an elementary DT system such that*

$$\nabla_b z^n = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} z^n. \tag{25}$$

The transfer function of such derivative, $H_b(z)$, is defined by

$$H_\nabla(z) = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}, \qquad |z| > 1. \tag{26}$$

For the backward bilinear derivative, a transfer function is defined similarly.

Let $\alpha \in \mathbb{R}$. The α-order nabla bilinear fractional derivative is a discrete-time linear system with transfer function

$$H_\nabla(z) = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha, \qquad |z| > 1, \tag{27}$$

such that

$$\nabla_b^\alpha z^n = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha z^n, \qquad |z| > 1. \tag{28}$$

With this formulation, we entered again in the context of the Z transform.

3. Main Transforms and Integer-Order Derivatives

3.1. Continuous-Time Laplace and Fourier Transforms

Definition 7. *The direct bilateral LT (BLT) is given by [25]*

$$\mathcal{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,\mathrm{d}t,\ s \in \mathbb{C} \cap \mathcal{R}_c, \tag{29}$$

while the inverse LT (synthesis equation) reads

$$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,\mathrm{d}s,\ t \in \mathbb{R}, \tag{30}$$

where $a \in \mathbb{R}$ must be located inside the region of convergence, $\mathcal{R}_c$, (ROC) of $F(s)$. The right-hand side represents the Bromwich integral. In the following, we will denote by γ the integration path.

We can obtain existence conditions for the BLT from those of the FT [12,26,27]. Let $f(t)$ be a function

- piecewise continuous,
- with bounded variation,
- locally integrable (in the sense that the function is absolutely integrable in any real interval $[a,b]$, so that $\int_a^b |f(t)|\,\mathrm{d}t < \infty$),
- of exponential order,

then there exists the BLT of $f(t)$.

Remark 2. *Loosely speaking, a function of exponential order is the one that does not "grow faster" than given exponentials as $t \to \pm\infty$. This means two things. First, that there are real constants A, $a > 0$ such that $|f(t)| < A \cdot e^{at}$, when t is large and negative (say, for $t < t_1 \in \mathbb{R}$). Second, that there are real constants B, $b > 0$ such that $|f(t)| < B \cdot e^{bt}$, when t is large (say, for $t > t_2 \in \mathbb{R}$). It also has to be true that $b < a$. We are interested in dealing with functions for which $b < 0$ and $a > 0$ so that the function has Fourier transform.*

Under the conditions indicated, the integral in (29) converges absolutely and uniformly in a vertical band in the complex plane defined by $b < Re(s) < a$, where $F(s)$ is analytic. This band is called the *region of convergence* (ROC), and the values of the constants a and b are the abscissae of convergence. It can be shown that:

1. If $f(t)$ is absolutely integrable and of finite duration, then the ROC is the entire s-plane since the Laplace transform is finite and $F(s)$ exists for any s.
2. If $f(t)$ is right-handed (i.e., it exists with $t \geq t_0 \in \mathbb{R}$) and $Re(s) = a \in \mathcal{R}_c$, then any s to the right of a is also in $\mathcal{R}_c$.
3. If $f(t)$ is left-handed (i.e., exists with $t \leq t_0 \in \mathbb{R}$) and $Re(s) = a \in \mathcal{R}_c$, then any s to the left of a is also in $\mathcal{R}_c$.
4. A function $f(t)$ is absolutely integrable (satisfying the Dirichlet conditions and having the Fourier transform) if and only if the ROC of the corresponding Laplace transform $F(s)$ includes the imaginary axis since $Re(s) = 0$ and $s = i\omega$.
5. A given complex variable function only can define univocally an LT if it has attached a suitable ROC.
6. If $F(s) = \mathcal{L}[f(t)]$, then $\mathcal{L}[f(-t)] = F(-s)$.
7. If the region of convergence of $F(s)$ includes the frontiers $b \leq Re(s) \leq a$, then $F(s)$ is completely defined in that region by the values at the lines $F(a + i\tau)$ and $F(b + i\tau)$, $\tau \in \mathbb{R}$.
8. $F(s)$ is bounded in the strip $a + \epsilon \leq Re(s) \leq b - \epsilon$, with $\epsilon > 0$.

As the integer-order GL derivative of an exponential exists for any value, provided that $s \neq 0$, we obtain easily

Theorem 4.

$$F^{(n)}(s) = \int_{-\infty}^{\infty} (-t)^n f(t) e^{-st}\,\mathrm{d}t,\ s \in \mathbb{C} \tag{31}$$

and

$$f^{(n)}(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} s^n F(s) e^{st}\,\mathrm{d}s,\ t \in \mathbb{R}. \tag{32}$$

From these relations, we obtain the corresponding properties that we introduced above for the FT.

3.2. The Mellin Transform

Definition 8. *Let us define the Mellin transform by*

$$G(v) = \int_0^\infty g(u) u^{-v-1} du, \tag{33}$$

which is a modified version of the usual Mellin transform. This has a parameter sign change $-v \to v$ relative to the current [14]. The idea is to keep the parallelism with the LT so that the properties related to the transform domain are essentially the same. In fact, it results from the change $e^t \to u$. The inverse Mellin transform related to (10) is

$$g(\tau) = \mathcal{M}^{-1}[X(v)] = \frac{1}{2\pi i} \int_\gamma G(v) \tau^v dv, \quad \tau \in \mathbb{R}^+,$$

where γ is a vertical straight line in the ROC of the transform.

As the MT results from the LT through an exponential variable change, the convergence properties are easily deduced. In particular, the integral in (33) converges absolutely and uniformly in a vertical strip in the complex plane defined by $b < Re(v) < a$, where $G(v)$ is analytic.

Theorem 5. *The duality in the MT is expressed by*

$$\mathfrak{D}^n_{\pm} g(\tau) = \frac{1}{2\pi i} \int_\gamma (\pm v)^n G(v) \tau^v dv, \quad \tau \in \mathbb{R}^+, \tag{34}$$

and

$$G^{(n)}(v) = \int_0^\infty (-\ln(u))^n g(u) u^{-v-1}\, \mathrm{d}u,\ v \in \mathbb{C}. \tag{35}$$

The proof is immediate using the scale-derivative in the inverse and the classical derivative in the direct MT. Although this result may seem strange, it expresses the effect of the exponential transformation mentioned above.

3.3. On the Z and Discrete-Time Fourier Transforms

Definition 9. *Let $x(n)$ denote any function defined on $\mathbb{T}$, leaving implicit the graininess, unless it is convenient to display it. The Z transform (ZT) is defined by*

$$X(z) = \mathcal{Z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n},\ \ z \in \mathbb{C}. \tag{36}$$

The inverse ZT can be obtained by the integral defined by

$$x(n) = \frac{1}{2\pi i} \oint_\gamma X(z) z^{n-1} \mathrm{d}z, \tag{37}$$

where γ is a circle centered at the origin, located in the ROC of the transform and taken in a counterclockwise direction.

In some scientific fields, such as geophysics, z is used instead of z^{-1}, and sometimes the ZT is called the "generating function" or "characteristic function". The existence conditions of the ZT are similar to those of the bilateral LT [11,13,16]. They can be stated as follows.

Definition 10. *A discrete-time signal $x(n)$ is called an* exponential order signal *if there exist integers n_1 and n_2 and positive real numbers $a, b, A,$ and B such that $A\,a^{n_1} < |x(n)| < B\,b^{n_2}$ for $n_1 < n < n_2$.*

For these signals, the ZT exists, and the ROC is an annulus centered at the origin, generally delimited by two circles of radius r_- and r_+ such that $r_- < |z| < r_+$. However, there are some cases where the annulus can become infinite:

- If the signal is right (i.e., $x(n) = 0,\ n < n_0 \in \mathbb{Z}$), then the ROC is the exterior of a circle centered at the origin ($r_+ = \infty$): $|z| > r_-$.
- If the signal is left (i.e., $x(n) = 0,\ n > n_0 \in \mathbb{Z}$), then the ROC is the interior of a circle centered at the origin ($r_- = 0$): $|z| < r_+$.
- If the signal is a pulse (i.e., non null only on a finite set), then the ROC is the whole complex plane, possibly with the exception of the origin. In the ROC, the ZT defines an analytical function.

If the ROC contains the unit circle, then by making $z = e^{i\omega},\ |\omega| < \pi,\ i = \sqrt{-1}$, we obtain the discrete-time Fourier transform, which we will shortly call Fourier transform. This means that not all signals with ZT have FT. The signals with ZT and FT are those for which the ROC is non-degenerate and contains the unit circle ($r_- < 1,\ r_+ > 1$).

In this situation, the integral in (37) converges uniformly. The calculation uses Cauchy's theorem for functions of complex variable [16].

To treat the duality, we must note the importance of the unit circle that suggests the use of the scale-derivative $\mathfrak{D}_\pm$ according to the ROC: (+) for $|z| > 1$ and (-) for $|z| < 1$.

Theorem 6. *Using the stretching and shrinking derivatives, we obtain*

$$\mathfrak{D}^n_{z\pm} X(z) = \sum_{k=-\infty}^{\infty} (-k)^n x(kh)\, z^{-k}, \tag{38}$$

for $|z|^{\pm 1} > 1$. Concerning the inverse ZT, we use the nabla derivative to obtain

$$\nabla^n x(kh) = \frac{1}{2\pi i} \oint_\gamma X(z) \left[\frac{1 - z^{-1}}{h}\right]^n z^{k-1} \mathrm{d}z \tag{39}$$

For both, the proofs are immediate.

Definition 11. *For functions that have an ROC including the unit circle or for functions having a degenerate ROC, as it is the case of the periodic signals, it is preferable to work with the discrete-time Fourier transform that can be obtained from the ZT through the transformation $z = e^{i\omega h},\ |\omega h| < \pi$*

$$X(e^{i\omega}) = \sum_{k=-\infty}^{\infty} x(kh) e^{-i\omega hk}, \tag{40}$$

with the inversion integral

$$x(k) = \frac{1}{2\pi h} \int_{-\pi/h}^{\pi/h} X(e^{i\omega}) e^{i\omega hk} \mathrm{d}\omega. \tag{41}$$

To obtain the duality, we must note that we made an exponential transformation to pass from (36) to (40). Therefore, the Liouville derivative must be used. We can state:

Theorem 7.

$$X^{(n)}(e^{i\omega}) = \sum_{k=-\infty}^{\infty} (-ikh)^n x(kh) e^{-i\omega hk} \tag{42}$$

and

$$\nabla^n x(kh) = \frac{1}{2\pi h} \int_{-\pi/h}^{\pi/h} \left[\frac{1 - e^{-i\omega h}}{h} \right]^n X(e^{i\omega}) e^{i\omega hk} \mathrm{d}\omega. \tag{43}$$

Remark 3. *The change from the nabla derivative stated in (13) to the corresponding bilinear implies a change of the factor* $\left[\frac{1-z^{-1}}{h}\right]$ *by* $\left[\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right]$ *so that we obtain alternative derivative properties in (39) and (43).*

4. Main Transforms and Non Integer-Order Derivatives

4.1. Laplace Transform

We reproduce here the results stated in Theorem 4

$$F^{(n)}(s) = \int_{-\infty}^{\infty} (-t)^n f(t) e^{-st} \,\mathrm{d}t,\ s \in \mathbb{C}$$

and

$$h^{(n)}(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) s^n e^{st} \,\mathrm{d}s,\ t \in \mathbb{R}.$$

As it is clear, the substitution $n \to \alpha$ creates problems, since the complex variable expression s^α is no longer a function, it involves setting a branchcut line. The simplest way is to choose the negative or positive real semi-axis. The results in Section 2 allow us to write:

1. Right function case ($Re(s) > 0$)

$$F_{-}^{(\alpha)}(s) = \int_{0}^{\infty} (-t)^\alpha f(t) e^{-st} \,\mathrm{d}t, \tag{44}$$

and

$$f_{+}^{(\alpha)}(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) s^\alpha e^{st} \,\mathrm{d}s,\ a > 0. \tag{45}$$

2. Left function case ($Re(s) < 0$)

$$F_{+}^{(\alpha)}(s) = \int_{-\infty}^{0} (-t)^\alpha f(t) e^{-st} \,\mathrm{d}t, \tag{46}$$

and

$$f_{-}^{(\alpha)}(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) s^\alpha e^{st} \,\mathrm{d}s,\ a < 0. \tag{47}$$

3. Two-sided function case ($|Re(s)| < b$)
   The above relations suggest we introduce the two-sided fractional derivative [28]:

$$D_\theta^\alpha f(t) := \lim_{h \to 0^+} h^{-\alpha} \sum_{n=-\infty}^{+\infty} (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha+\theta}{2} - n + 1\right)\Gamma\left(\frac{\alpha-\theta}{2} + n + 1\right)} f(t - nh), \tag{48}$$

   for which

$$D_\theta^\alpha e^{\pm i\omega t} = |\omega|^\alpha e^{\mp i\theta \frac{\pi}{2} \operatorname{sgn}(\omega)} e^{\pm i\omega t}.$$

   Assuming that $|\theta| \neq |\alpha|$, then

$$D_\theta^\alpha F(i\omega) = \int_{-\infty}^{\infty} |t|^\alpha e^{-i\theta\frac{\pi}{2}\operatorname{sgn}(t)} f(t) e^{-i\omega t}\,\mathrm{d}t, \tag{49}$$

and

$$D_\theta^\alpha f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^\alpha e^{i\theta\frac{\pi}{2}\operatorname{sgn}(\omega)} F(i\omega) e^{i\omega t}\,\mathrm{d}\omega. \tag{50}$$

In particular, we obtain

$$D_0^\alpha F(i\omega) = \int_{-\infty}^{\infty} |t|^\alpha f(t) e^{-i\omega t}\,\mathrm{d}t, \tag{51}$$

and

$$D_0^\alpha f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^\alpha F(i\omega) e^{i\omega t}\,\mathrm{d}\omega. \tag{52}$$

4.2. Mellin Transform

The MT case is very similar to the LT case. The relationships are easily obtained from those in the previous subsection, taking into account the situations in which the Liouville derivative is replaced by the Hadamard derivative. The results in Section 2 allow us to write:

1. Stretching case ($Re(v) > 0$)

$$G_-^{(\alpha)}(v) = \int_1^\infty (-\ln\tau)^\alpha g(\tau)\tau^{-v-1}\,\mathrm{d}\tau, \tag{53}$$

and

$$\mathfrak{D}_{\tau+}^\alpha g(\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} v^\alpha G(v)\tau^v\,\mathrm{d}s,\ a > 0. \tag{54}$$

2. Shrinking case ($Re(v) < 0$)

$$G_+^{(\alpha)}(v) = \int_0^1 (\ln\tau)^\alpha g(\tau)\tau^{-v-1}\,\mathrm{d}\tau, \tag{55}$$

and

$$\mathfrak{D}_{\tau-}^\alpha g(\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} v^\alpha G(v)\tau^v\,\mathrm{d}v,\ a < 0. \tag{56}$$

3. Bilateral scale case ($|Re(v)| < b$)
The above relations suggest we introduce the two-sided scale derivative by:

$$\mathfrak{D}_{\tau,\theta}^\alpha g(\tau) = \lim_{q\to 1^+} \ln(q)^{-\alpha}\sum_{n=-\infty}^{+\infty}(-1)^n \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha+\theta}{2}-n+1\right)\Gamma\left(\frac{\alpha-\theta}{2}+n+1\right)} g(\tau q^{-n}), \tag{57}$$

for which

$$\mathfrak{D}_{\tau,\theta}^\alpha \tau^{i\omega} = |\omega|^\alpha e^{-i\theta\frac{\pi}{2}\operatorname{sgn}(\omega)}\tau^{i\omega}.$$

Assuming that $|\theta| \neq |\alpha|$, then

$$D_\theta^\alpha G(i\omega) = \int_0^\infty |\ln\tau|^\alpha e^{i\theta\frac{\pi}{2}\operatorname{sgn}(\ln\tau)} g(\tau)\tau^{-i\omega-1}\,\mathrm{d}\tau, \tag{58}$$

and

$$\mathfrak{D}^{\alpha}_{\tau,\theta} g(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^{\alpha} e^{i\theta \frac{\pi}{2} \operatorname{sgn}(\omega)} G(i\omega) \tau^{i\omega t} \, \mathrm{d}\omega. \tag{59}$$

In particular, we obtain

$$D^{\alpha}_{0} G(i\omega) = \int_{0}^{\infty} |\ln \tau|^{\alpha} g(\tau) \tau^{-i\omega - 1} \, \mathrm{d}\tau, \tag{60}$$

and

$$\mathfrak{D}^{\alpha}_{0} g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^{\alpha} G(i\omega) e^{i\omega t} \, \mathrm{d}\omega. \tag{61}$$

4.3. Z and Discrete-Time Fourier Transforms

Theorem 6 expresses the duality of the ZT for the integer-order case. The situation here is similar to the one we found in the LT and MT cases, having to consider separately the three cases, corresponding to the exterior of the unit circle, $|z| > 1$, the unity disk, $|z| < 1$, and the annulus containing the unit circle.

1. Right sequence case ($|z| > 1$)

$$\mathfrak{D}^{\alpha}_{z+} X(z) = \sum_{k=0}^{\infty} (-k)^{\alpha} x(kh) \, z^{-k} \tag{62}$$

and

$$\nabla^{\alpha} x(kh) = \frac{1}{2\pi i} \oint_{\gamma} X(z) \left[\frac{1 - z^{-1}}{h} \right]^{\alpha} z^{k-1} \mathrm{d}z, \tag{63}$$

where γ is a closed path lying outside the unity circle.

2. Left sequence case ($|z| < 1$)

$$\mathfrak{D}^{\alpha}_{z-} X(z) = \sum_{k=-\infty}^{-1} (-k)^{\alpha} x(kh) \, z^{-k} \tag{64}$$

and

$$\nabla^{\alpha} x(kh) = \frac{1}{2\pi i} \oint_{\gamma} X(z) \left[\frac{1 - z^{-1}}{h} \right]^{\alpha} z^{k-1} \mathrm{d}z, \tag{65}$$

where γ is now inside the unity circle.

3. Two-sided function case ($r_- < |z| < r_+$)

Definition 12. *As above, attending to the relation [29,30]:*

$$(1 - z)^{a} (1 - z^{-1})^{b} = \sum_{n=-\infty}^{+\infty} (-1)^{n} \frac{\Gamma(a + b + 1)}{\Gamma(a - n + 1)\Gamma(b + n + 1)} z^{-n}, \tag{66}$$

and considering again two real parameters α, the derivative order, and θ, the asymmetry parameter, such that $\alpha > -1$ if $\theta \neq \pm\alpha$, or $\alpha \in \mathbb{R}$ if $\theta = \pm\alpha$, we define a discrete-time two-sided derivative by

$$\Theta^{\alpha}_{\theta} x(kh) := h^{-\alpha} \sum_{n=-\infty}^{+\infty} (-1)^{n} \frac{\Gamma(\alpha + 1)}{\Gamma\left(\frac{\alpha+\theta}{2} - n + 1\right)\Gamma\left(\frac{\alpha-\theta}{2} + n + 1\right)} x(kh - nh). \tag{67}$$

Theorem 8. *Let $x(kh) = e^{i\omega hk}$. Then, [29]*

$$\Theta^{\alpha}_{\theta} e^{i\omega hk} = |2 \sin(\omega h/2)|^{\alpha} e^{i\theta \frac{\pi}{2} \operatorname{sgn}(\omega)} e^{i\omega hk}. \tag{68}$$

The proof comes from the left side in (66) by letting $z = e^{i\omega hk}$.
Assuming that $|\theta| \neq |\alpha|$, then

Theorem 9.
$$D_\theta^\alpha X(e^{i\omega}) = \sum_{k=-\infty}^{\infty} |kh|^\alpha e^{i\theta\frac{\pi}{2}\operatorname{sgn}(k)} x(kh) e^{-i\omega hk} \tag{69}$$
and
$$\Theta_\theta^\alpha x(kh) = \frac{1}{2\pi h} \int_{-\pi/h}^{\pi/h} |2\sin(\omega h/2)|^\alpha e^{i\theta\frac{\pi}{2}\operatorname{sgn}(\omega)} X(e^{i\omega}) e^{i\omega hk} \mathrm{d}\omega. \tag{70}$$

If h is very small, $2\sin(\omega h/2) \approx \omega h$, making (70) more similar to (69).
In particular, we obtain
$$D_0^\alpha X(e^{i\omega}) = \sum_{k=-\infty}^{\infty} |kh|^\alpha x(kh) e^{-i\omega hk} \tag{71}$$
and
$$\Theta_0^\alpha x(kh) = \frac{1}{2\pi h} \int_{-\pi/h}^{\pi/h} |2\sin(\omega h/2)|^\alpha e^{i\omega hk} \mathrm{d}\omega. \tag{72}$$

Remark 4. *As stated in the previous section, the change from the nabla derivative stated in (13) to the corresponding bilinear implies to substitute the factor* $\left[\frac{1-z^{-1}}{h}\right]$ *by* $\left[\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right]$ *so that we obtain alternative derivative properties in (63) and (70). Relative to (70), the substitution consists of* $2\sin(\omega h/2) \to \tan(\omega h/2)$.

5. Some Consequences

The above results allow us to obtain some consequences that express a full coherence with known theory. We start by considering the case of the fractional derivative in the complex plane [30]. Return back to (44) and (45), where we treat the right-handed functions; the corresponding left case is similar, so we will not deal with it here.
With simple substitutions, we obtain:

1. Substitute the inverse LT (30) in (44) and note that the LT converges uniformly in the ROC to commute the integrations. Using the definition of the gamma function [31], we obtain
$$F_-^{(\alpha)}(s) = \frac{(-1)^\alpha \Gamma(\alpha+1)}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(u)(s-u)^{-\alpha-1} \mathrm{d}u, \quad Re(s > 0). \tag{73}$$
This last formula is another way of expressing the left derivative in the complex plane, suitable for dealing with LT, but in agreement with previous formulations [30,32–34]. It is important to highlight an interesting fact: (73) is defined only in the right-handed complex plane. In general, we do not know what happens in the left half plane since it is out of the ROC. We profit on this fact to define there a branchcut line implicit in the definition. For this reason, we choose $(-1)^\alpha = e^{i\alpha\pi}$.
2. Let $f(t) = \varepsilon(t)$ be the Heaviside unit step. Insert it into (44) and use again the definition of gamma function. Assume that $\alpha > 0$. We arrive at
$$F_-^{(\alpha)}(s) = \frac{(-1)^\alpha \Gamma(\alpha+1)}{s^{\alpha+1}}, \quad Re(s) > 0. \tag{74}$$
From this relation, we conclude that
$$\mathcal{L}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\varepsilon(t)\right] = \frac{1}{s^{\alpha+1}}, \quad Re(s) > 0, \tag{75}$$

and, as $\mathcal{L}[\varepsilon(t)] = 1/s$,

$$D_{-}^{\alpha}\frac{1}{s} = \frac{(-1)^{\alpha}\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad Re(s) > 0,$$

from where we deduce the interesting result [34]

$$D_{-}^{\alpha}s^{-\beta} = \frac{(-1)^{\alpha}\Gamma(\alpha+\beta)}{\Gamma(\beta)}s^{-\beta-\alpha}, \quad Re(s) > 0, \tag{76}$$

where $\beta > 0$.

We established a correspondence between two sequences of powers, positive in time ($\alpha > 0$), negative in transform domain:

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)}\varepsilon(t) \Longleftrightarrow s^{-\alpha-1}, \quad Re(s) > 0, \tag{77}$$

and we ask ourselves what happens when $\alpha < 0$. The answer can be searched in (45). However, we must be careful since we are going to enter in the field of the generalized function [35,36]. In fact, letting $F(s) = 1$, we obtain the formal relation

$$D_{+}^{\alpha}\delta(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} s^{\alpha}e^{st}\,\mathrm{d}s,\ a > 0. \tag{78}$$

Let us try another way [37] by calculating the derivative of the Heaviside function using the Grünwald–Letnikov definition (5). Starting from ([20]):

$$\sum_{k=0}^{n}\binom{\alpha}{k}(-1)^{k} = \binom{\alpha-1}{n}(-1)^{n} = \frac{1}{\Gamma(1-\alpha)}\frac{\Gamma(-\alpha+n+1)}{\Gamma(n+1)},$$

and setting $n = t/h$, with $h \to 0^{+}$, we can show that

$$D^{\alpha}\varepsilon(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\varepsilon(t). \tag{79}$$

Remark 5. *An important fact must be highlighted: The Caputo derivative of the unit step is null. This implies the same for the derivatives we are going to compute in the following.*

As $\delta(t) = D\varepsilon(t)$, we deduce using (79) that

$$D^{\alpha}\delta(t) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha+1)}\varepsilon(t). \tag{80}$$

Attending to (78), we conclude that

$$\mathcal{L}\left[\frac{t^{-\alpha-1}}{\Gamma(\alpha+1)}\varepsilon(t)\right] = s^{\alpha}, \tag{81}$$

which suggests the inversion of the exponent roles relative to (77)

$$\frac{t^{-\alpha-1}}{\Gamma(-\alpha+1)}\varepsilon(t) \Longleftrightarrow s^{\alpha}, \quad Re(s) > 0. \tag{82}$$

To test the coherence of this result and the agreement with (78), we set $\alpha = n,\ n \in \mathbb{N}$. As it is easy to show,

$$\mathcal{L}[D^{n}\delta(t)] = s^{n}.$$

Therefore, we must have

$$\frac{t^{-n-1}}{\Gamma(-n+1)} = D^{n}\delta(t).$$

As strange as this relationship may seem, it was deduced by Gelf and Shilov [35].

With (79), it is possible to obtain the derivative of any order of the continuous function $p(t) = t^\beta u(t)$, with $\beta > 0$. The *LT* of $p(t)$ yields $P(s) = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$, for $Re(s) > 0$ and the FD of order α is given by $s^\alpha \frac{\Gamma(\beta+1)}{s^{\beta+1}}$. Therefore, the expression

$$D_+^\alpha t^\beta \varepsilon(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}\varepsilon(t) \tag{83}$$

is valid for any $\alpha \in \mathrm{R}$ [38].

These results are readily reproduced for the Mellin transform. We only have to remember that we pass from the LT to the MT with the substitution $e^t = \tau$. Therefore, it is enough to perform such substitution and set $v \to s$.

The above presented application to the LT can be repeated with the Z transform. We will consider the $|z| > 1$, case, since the other is similar.

Take relation (62) and substitute there $x(kh)$ by its expression in terms of the inverse ZT and compute the summation and integral to obtain

$$\mathfrak{D}_{z+}^\alpha X(z) = \frac{1}{2\pi i}\oint_\gamma X(u)\sum_{k=0}^{\infty}(-k)^\alpha (z/u)^{-k}\frac{\mathrm{d}u}{u}. \tag{84}$$

If we introduce the kernel (a discrete Mellin transform),

$$K(u) = \sum_{k=0}^{\infty}(-k)^\alpha u^{-k},$$

we can write

$$\mathfrak{D}_{z+}^\alpha X(z) = \frac{1}{2\pi i}\oint_\gamma X(u)K(z/u)\frac{\mathrm{d}u}{u}. \tag{85}$$

This expression represents a new scale-derivative defined in the complex plane. However, it has one major difficulty: the lack of a closed form for $K(z)$. The same is true in most situations that we can obtain from (62) or (63), which prevents us from obtaining simple expressions similar to those obtained in the case of LT. In any case, we can fall back on the Fourier transform and implement the expressions using the fast Fourier transform.

6. Conclusions

The duality property of the Laplace and Fourier transforms associated with the integer-order derivative was reviewed. We generalized it for fractional derivatives and extended the results to the Mellin, Z and discrete-time Fourier transforms. To do it, we used the scale and discrete derivatives. Some consequences of the proposed theory are described.

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