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On Bishop–Phelps and Krein–Milman Properties

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Abstract: A real topological vector space is said to have the Krein–Milman property if every bounded, closed, convex subset has an extreme point. In the case of every bounded, closed, convex subset is the closed convex hull of its extreme points, then we say that the topological vector space satisfies the strong Krein–Milman property. The strong Krein–Milman property trivially implies the Krein–Milman property. We provide a sufficient condition for these two properties to be equivalent in the class of Hausdorff locally convex real topological vector spaces. This sufficient condition is the Bishop–Phelps property, which we introduce for real topological vector spaces by means of uniform convergence linear topologies. We study the inheritance of the Bishop–Phelps property. Nontrivial examples of topological vector spaces failing the Krein–Milman property are also given, providing us with necessary conditions to assure that the Krein–Milman property is satisfied. Finally, a sufficient condition to assure the Krein–Milman property is discussed.

Keywords: topological vector space; Krein–Milman; Bishop–Phelps; extreme point; convex set

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1. Introduction

Banach's famous classic book "Théorie des opérations linéaries" [1] was a groundbreaking monograph that triggered the strong development of modern analysis and topology. Famous results of that book, such as Banach's contraction principle, boosted the study of metric spaces and linear spaces, which flourished as a result of the category of topological linear spaces. This is then the birth of "topology for analysts", which separated from the other branch of topology: algebraic topology. In the successive decades, analysts developed the study of topological rings and modules as main objects of work, focusing, in particular, on rings of polynomials and of continuous functions, which have plenty of applications in abstract measure theory. Later on, Grothendieck promoted the re-encounter of topology for analysts and algebraic topology by relating functional analysis with algebraic geometry [2–6].

The famous and groundbreaking Krein–Milman theorem [7] asserts that every compact convex subset of a Hausdorff locally convex topological vector space can be recovered via the closed convex hull of its extreme points. Plenty of applications of this result have been provided ever since, not only in other fields of pure mathematics than functional analysis, but also in applied mathematics, statistics, and operation research [8]. The Krein–Milman theorem led to the Krein–Milman property [9–11]. This geometric property has been shown to have strong connections with measure theory through the famous Radon–Nikodym property [12–15]. In fact, there is still an open problem consisting of the equivalence of the Krein–Milman property and the Radon–Nikodym property, although partial solutions have been already found [16,17].

Here, the Krein–Milman property is considered on real topological vector spaces in two forms (weak and strong), which are "essentially" equivalent for Hausdorff locally convex spaces (see Theorem 3 and Corollary 1).

Property 1 (Krein–Milman)**.** *Let X be a real topological vector space. We say that X enjoys the Krein–Milman property provided that every bounded, closed, convex subset of X has an extreme point.*

In [10] (Corollary 2.1), it is proved that if a real topological vector space enjoys the Krein–Milman property, then it must be Hausdorff. Here, we considerably improve this result by using a different technique (see Theorem 4).

Property 2 (Strong Krein–Milman)**.** *Let X be a real topological vector space. We say that X enjoys the strong Krein–Milman property provided that every bounded, closed, convex subset of X is the closed convex hull of its extreme points.*

Trivially, the strong Krein–Milman property implies the Krein–Milman property. A sufficient condition is provided for these two properties to become equivalent in the category of Hausdorff locally convex real topological vector spaces (see Theorem 3 and Corollary 1). This sufficient condition is nothing else but the Bishop–Phelps property (see Property 3), which we present here for real topological vector spaces by means of uniform convergence linear topologies. The hereditariness of the Bishop–Phelps property is deeply studied and discussed (see Theorems 1 and 2, and Section 5). Nontrivial examples of topological vector spaces failing the Krein–Milman property are also given (see Theorems 4 and 5, and Corollary 2), providing us with the necessary conditions to assure that the Krein–Milman property is satisfied. A sufficient condition to guarantee the Krein–Milman property is discussed as well (see Theorem 6).

As we will see, the Bishop–Phelps property plays a fundamental role in this setting. This property is conceived after the appearance of the famous Bishop–Phelps theorem [18,19]. This theorem is not only famous for its numerous applications in functional analysis and approximation theory, but also because of the generalizations that came afterwards [20], always in the Banach-space setting. One of those famous generalizations is the well-known Bishop–Phelps–Bollobás theorem [21], which provides a double approximation, in the dual and the pre-dual spaces (refer to Appendix B for the statement of the Bishop–Phelps–Bollobás theorem). This theorem gave birth to a version of it for operators [22] and the corresponding Bishop–Phelps–Bollobás property [23,24], which has been an intensive and prolific line of research ever since [25–29]. The hereditariness of the Bishop–Phelps–Bollobás property has also been deeply studied since it provides examples of other Banach spaces satisfying it (and, in some cases, it also provides counter-examples). This is why we pay attention and focus on the hereditariness of the Bishop–Phelps property in Theorems 1 and 2, and Section 5.

2. Materials and Methods

All vector spaces considered throughout this manuscript will be over the reals and will also be assumed to be nonzero by default. If X is a topological space and $x \in X$, then by $\mathcal{N}_x(X)$ we intend to denote the filter of neighborhoods of x.

If X is a topological vector space and I is a non-empty set, then every vector subspace F of X^I can be endowed with a vector topology called "uniform convergence linear topology". Indeed, take $\mathcal{G} \subseteq \mathcal{P}(I)$ upward directed (like, for instance, a bornology on I), satisfying that $f(G)$ is bounded in X for all $G \in \mathcal{G}$ and all $f \in F$. For every $G \in \mathcal{G}$ and every 0-neighborhood $U \subseteq X$, the sets $\mathcal{U}(G, U) := \{f \in F : f(G) \subseteq U\}$ form a base of 0-neighborhoods for a vector topology on F called "uniform convergence linear topology generated by $\mathcal{G}$" or "linear topology of uniform convergence on elements of $\mathcal{G}$". Observe that if $\mathcal{G}$ is the set of finite subsets of I, then we obtain the pointwise convergence topology on F or, equivalently, the inherited product topology on F from X^I. The dual space of X, X^*, can be endowed with a uniform convergence linear topology since $X^* \subseteq \mathbb{R}^X$. If we take $\mathcal{G}$ as the set of finite subsets of X, then we obtain the pointwise convergence topology on X^*, which is the w^*-topology. If we take $\mathcal{G} := \mathcal{BCC}_X := \{A \subseteq X : A \text{ is bounded, closed, convex}\}$, then we obtain a stronger uniform convergence linear topology on X^* called the $\mathcal{BCC}_X$-topology, which is the linear topology of uniform convergence on bounded, closed, convex subsets of X (this topology coincides with the dual norm topology if X is a normed space).

Remark 1. *Let X be a topological vector space, I a non-empty set, $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{P}(I)$ upward directed, and $E \subseteq F \subseteq X^I$ vector subspaces such that $f(G)$ is bounded in X for every $f \in F$ and every $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. Then:*

- *The inherited $\mathcal{G}_1$-uniform convergence linear topology of E from F is precisely the $\mathcal{G}_1$-uniform convergence linear topology of E in view of the fact that $\mathcal{U}_E(G,U) = \mathcal{U}_F(G,U) \cap E$ for all $G \in \mathcal{G}_1$ and all $U \in \mathcal{N}_0(M)$.*
- *If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then the uniform convergence linear topology on F generated by $\mathcal{G}_1$ is clearly coarser than the uniform convergence linear topology generated by $\mathcal{G}_2$.*
- *If for every $G_2 \in \mathcal{G}_2$ there exists $G_1 \in \mathcal{G}_1$ such that $G_2 \subseteq G_1$, then the uniform convergence linear topology on F generated by $\mathcal{G}_1$ is finer than the uniform convergence linear topology generated by $\mathcal{G}_2$.*

As a direct consequence of the previous remark, the $\mathcal{BCC}_X$-topology is coarser than the $\mathcal{B}_X$-topology, where $\mathcal{B}_X$ is the family of all bounded subsets of X, for X a topological vector space.

Lemma 1. *If X is a Hausdorff locally convex topological vector space, then $\overline{\mathrm{co}}(A)$ is bounded for every bounded subset A of X.*

Proof. Fix an arbitrary convex, balanced and absorbing 0-neighborhood $U \subseteq X$. Since every topological vector space is regular, we may assume that U is closed. There exists $\alpha > 0$ such that $A \subseteq \alpha U$. Then, $\overline{\mathrm{co}}(A) \subseteq \alpha U$. □

According to Lemma 1 together with Remark 1, the $\mathcal{BCC}_X$-topology and the $\mathcal{B}_X$-topology coincide on Hausdorff locally convex topological vector spaces.

3. Results

We will begin this section with the definition of the Bishop–Phelps property, which is obviously motivated by the famous Bishop–Phelps theorem [18,19].

Property 3 (Bishop–Phelps)**.** *A topological vector space X is said to have the Bishop–Phelps property if for every bounded, closed, convex subset B of X, the set $\mathsf{SA}(B) := \{f \in X^* : \sup f(B)$ is attained on $B\}$, of all functionals of X^* that attain their supremum on B, is dense in X^* for the $\mathcal{BCC}_X$-topology.*

In accordance with the famous Bishop–Phelps theorem [18], every real Banach space satisfies the Bishop–Phelps property. Trivially, both the Krein–Milman property and the strong Krein–Milman property are hereditary to closed subspaces. The next theorem shows that the Bishop–Phelps property is hereditary to closed complemented subspaces (keep in mind that in non-Hausdorff topological vector spaces, the range of a continuous linear projection does not need to be closed).

Theorem 1. *Let X be a topological vector space. Let Y be a closed complemented subspace of X. If X has the Bishop–Phelps property, then so does Y.*

Proof. Take any bounded, closed, and convex subset B of Y. We will show that $\mathsf{SA}_{Y^*}(B)$ is dense in Y^* for the $\mathcal{BCC}_Y$-topology. Indeed, fix an arbitrary bounded, closed, and convex subset A of Y, an arbitrary $\varepsilon > 0$, and an arbitrary $y^* \in Y^*$. We will prove that $\mathsf{SA}_{Y^*}(B) \cap [y^* + \mathcal{U}_{Y^*}(A, (-\varepsilon, \varepsilon))] \neq \varnothing$. Note that A and B are both bounded, closed, and convex in X, thus by hypothesis, $\mathsf{SA}_{X^*}(B)$ is dense in X^* for the $\mathcal{BCC}_X$-topology, that is, $\mathsf{SA}_{X^*}(B) \cap [x^* + \mathcal{U}_{X^*}(A, (-\varepsilon, \varepsilon))] \neq \varnothing$, where $x^* := P^*(y^*) = y^* \circ P$ and $P : X \to Y$

is a continuous linear projection of X onto Y. Take $f \in \mathsf{SA}_{X^*}(B) \cap [x^* + \mathcal{U}_{X^*}(A, (-\varepsilon, \varepsilon))]$. Observe that $|f|_Y(a) - y^*(a)| = |f(a) - x^*(a)| < \varepsilon$ for every $a \in A$, meaning that $f|_Y \in y^* + \mathcal{U}_{Y^*}(A, (-\varepsilon, \varepsilon))$. It only remains to show that $f|_Y \in \mathsf{SA}_{Y^*}(B)$, which is immediate since $f \in \mathsf{SA}_{X^*}(B)$. $\square$

In [10] (Theorem 2.1), it was proved that $O_X := \bigcap_{V \in \mathcal{N}_0(X)} V$ is a bounded and closed vector subspace of X which is topologically complemented with any of its algebraic complements (in the sense that any projection on X whose range is O_X must be continuous since the inherited topology of O_X is the trivial topology). Note that if Y is an algebraic complement of O_X in X, then Y is Hausdorff and dense in X in view of [10] (Theorem 2.1).

Lemma 2. *Let X, Y be topological vector spaces. If $T : X \to Y$ is linear and continuous and Y is Hausdorff, then $O_X \subseteq T^{-1}(O_Y)$. In particular, if Y is Hausdorff, then $O_X \subseteq \ker(T)$.*

Proof. Notice that

$$O_X = \bigcap_{V \in \mathcal{N}_0(X)} V \subseteq \bigcap_{U \in \mathcal{N}_0(Y)} T^{-1}(U) = T^{-1}\left(\bigcap_{U \in \mathcal{N}_0(Y)} U\right) = T^{-1}(O_Y).$$

Finally, if Y is Hausdorff, then $O_Y = \{0\}$, meaning that $O_X \subseteq T^{-1}(O_Y) = T^{-1}(\{0\}) = \ker(T)$. $\square$

Since $\mathbb{R}$ is Hausdorff, Lemma 2 assures that every $f \in X^*$ verifies that $O_X \subseteq \ker(f)$. As a consequence, $\mathsf{SA}(O_X) = X^*$.

Lemma 3. *Let X be a topological vector space. Let Y be an algebraic complement of O_X in X. Let $B \subseteq X$. If $z \in \mathrm{cl}(B)$, then $z - P(z) \in \mathrm{cl}(B)$, where $P : X \to O_X$ is the continuous linear projection of X onto O_X along Y. As a consequence, if B is closed in X, then $(I - P)(B)$ is closed in Y.*

Proof. Let us show first that $z - P(z) \in \mathrm{cl}(B)$. Fix any arbitrary 0-neighborhood V of X. There exists another 0-neighborhood W of X such that $W + W \subseteq V$. Observe that $z + W \subseteq (z - P(z)) + V$. Indeed, if $w \in W$, then $z + w = (z - P(z)) + P(z) + w \in (z - P(z)) + W + W \subseteq (z - P(z)) + V$. At this point, it is sufficient to realize that $(z + W) \cap B \neq \varnothing$ by hypothesis, meaning that $((z - P(z)) + V) \cap B \neq \varnothing$. This shows that $z - P(z) \in \mathrm{cl}(B)$. Next, assume that B is closed in X. For every $b \in B$, $b - P(b) \in \mathrm{cl}(B) = B$, that is, $(I - P)(B) \subseteq B$, meaning that $(I - P)(B) = B \cap Y$, hence $(I - P)(B)$ is closed in Y. $\square$

Theorem 2. *Let X be a topological vector space. Let Y be any algebraic complement of O_X. Then, X has the Bishop–Phelps property if and only if Y does too.*

Proof. Suppose first that X satisfies the Bishop–Phelps property. Take any bounded, closed, and convex subset B of Y. We will show that $\mathsf{SA}_{Y^*}(B)$ is dense in Y^* for the $\mathcal{BCC}_Y$-topology. Indeed, fix an arbitrary bounded, closed, and convex subset A of Y, an arbitrary $\varepsilon > 0$, and an arbitrary $y^* \in Y^*$. We will prove that $\mathsf{SA}_{Y^*}(B) \cap [y^* + \mathcal{U}_{Y^*}(A, (-\varepsilon, \varepsilon))] \neq \varnothing$. Note that $\mathrm{cl}(B)$ and $\mathrm{cl}(A)$ are both bounded, closed, and convex in X, thus, by hypothesis, $\mathsf{SA}_{X^*}(\mathrm{cl}(B))$ is dense in X^* for the $\mathcal{BCC}_X$-topology, that is, at $\mathsf{SA}_{X^*}(\mathrm{cl}(B)) \cap [x^* + \mathcal{U}_{X^*}(\mathrm{cl}(A), (-\varepsilon, \varepsilon))] \neq \varnothing$, where $x^* := y^* \circ (I - P)$ and $P : X \to O_X$ is the continuous linear projection of X onto O_X along Y. Take $f \in \mathsf{SA}_{X^*}(\mathrm{cl}(B)) \cap [x^* + \mathcal{U}_{X^*}(\mathrm{cl}(A), (-\varepsilon, \varepsilon))]$. Let us show first that $\sup f$ is attained on B. Since $f \in \mathsf{SA}_{X^*}(\mathrm{cl}(B))$, $\sup f$ is attained on $\mathrm{cl}(B)$ so there exists $z \in \mathrm{cl}(B)$ such that $f(z) = \sup f(\mathrm{cl}(B))$. According to Lemma 2, $f(P(z)) = 0$, therefore, $f(z - P(z)) = f(z)$. Since $P(z) \in O_X$, Lemma 3 allows one to deduce that $z - P(z) \in \mathrm{cl}(B)$, hence $z - P(z) \in \mathrm{cl}(B) \cap Y = B$. This shows that f attains its sup on B at $z - P(z)$, that is, $f|_Y \in \mathsf{SA}_{Y^*}(B)$. Finally, $|f|_Y(a) - y^*(a)| = |f(a) - x^*(a)| < \varepsilon$ for every $a \in A$, meaning that $f|_Y \in y^* + \mathcal{U}_{Y^*}(A, (-\varepsilon, \varepsilon))$. Conversely, assume that Y

verifies the Bishop–Phelps property. Take any bounded, closed, and convex subset B of X. We will show that $\mathsf{SA}_{X^*}(B)$ is dense in X^* for the $\mathcal{BCC}_X$-topology. Indeed, fix an arbitrary bounded, closed, and convex subset A of X, an arbitrary $\varepsilon > 0$, and an arbitrary $x^* \in X^*$. We will prove that $\mathsf{SA}_{X^*}(B) \cap [x^* + \mathcal{U}_{X^*}(A, (-\varepsilon, \varepsilon))] \neq \varnothing$. Note that $(I - P)(B)$ is bounded because $I - P$ is continuous, hence $(I - P)(B)$ is bounded, closed, and convex in Y in view of Lemma 3. For the same reasons, $(I - P)(A)$ is bounded, closed, and convex in Y. By hypothesis, $\mathsf{SA}_{Y^*}((I - P)(B))$ is dense in Y^* for the $\mathcal{BCC}_Y$-topology, that is, $\mathsf{SA}_{Y^*}((I - P)(B)) \cap [x^*|_Y + \mathcal{U}_{Y^*}((I - P)(A), (-\varepsilon, \varepsilon))] \neq \varnothing$. Take $f \in \mathsf{SA}_{Y^*}((I - P)(B)) \cap [x^*|_Y + \mathcal{U}_{Y^*}((I - P)(A), (-\varepsilon, \varepsilon))]$. Let us show first $f \circ (I - P)$ attains its sup on B. Since $f \in \mathsf{SA}_{Y^*}((I - P)(B))$, $\sup f$ is attained on $(I - P)(B)$, so there exists $y \in (I - P)(B)$ such that $f(y) = \sup f((I - P)(B))$. Take $b \in B$ such that $y = b - P(b)$. Then, $f(y) = f(b)$ by Lemma 2, meaning that $f \circ (I - P)$ attains its sup on B at b, that is, $f \circ (I - P) \in \mathsf{SA}_{X^*}(B)$. Finally, by Lemma 2

$$|(f \circ (I - P))(a) - x^*(a)| = |f((I - P)(a)) - x^*|_Y((I - P)(a)))| < \varepsilon$$

for every $a \in A$, reaching the conclusion that $f \circ (I - P) \in x^* + \mathcal{U}_{X^*}(A, (-\varepsilon, \varepsilon))$. □

Our next efforts are aimed at proving that the Krein–Milman property and the strong Krein–Milman property are equivalent in the class of Hausdorff locally convex topological vector spaces satisfying the Bishop–Phelps property.

Lemma 4. *Let X be a topological vector space satisfying the Bishop–Phelps property. Let $A \subseteq X$ be a bounded, closed, convex subset of X. For every $f \in X^*$ and every $\varepsilon > 0$ there exists $g \in \mathsf{SA}(A)$ such that $|f(a) - g(a)| < \varepsilon$ for all $a \in A$.*

Proof. Simply observe that there exists $g \in \mathsf{SA}(A) \cap [f + \mathcal{U}(A, (-\varepsilon, \varepsilon))]$, meaning that $|f(a) - g(a)| < \varepsilon$ for all $a \in A$. □

The following is a technical lemma.

Lemma 5. *Let X be a topological vector space. Let $A \subseteq X$ be a bounded, closed, convex subset of X. Let $f, g \in X^*$ and $B \subseteq A$. Then,*

$$|\sup f(B) - \sup g(B)| \leq \sup_{a \in A} |f(a) - g(a)|.$$

Proof. Notice that

$$f(b) = f(b) - g(b) + g(b) \leq |f(b) - g(b)| + g(b) \leq \sup_{a \in A} |f(a) - g(a)| + \sup g(B)$$

for every $b \in B$. As a consequence, $\sup f(B) \leq \sup_{a \in A} |f(a) - g(a)| + \sup g(B)$, that is, $\sup f(B) - \sup g(B) \leq \sup_{a \in A} |f(a) - g(a)|$. In a similar way, we obtain that $\sup g(B) - \sup f(B) \leq \sup_{a \in A} |f(a) - g(a)|$, obtaining the desired result. □

The following theorem establishes the equivalence of the Krein–Milman property and the strong Krein–Milman property in the class of Hausdorff locally convex topological vector spaces satisfying the Bishop–Phelps property.

Theorem 3. *Let X be a Hausdorff locally convex topological vector space satisfying the Bishop–Phelps property. If $\operatorname{ext}(A) \neq \varnothing$ for all bounded, closed, convex subset A of X, then $A = \overline{\operatorname{co}}(\operatorname{ext}(A))$ for all bounded, closed, convex subset A of X.*

Proof. Suppose on the contrary that there exists $x_0 \in A \setminus \overline{\operatorname{co}}(\operatorname{ext}(A))$. According to the Hahn–Banach separation theorem, we can find $f \in X^*$ such that $f(x_0) > \sup f(\overline{\operatorname{co}}(\operatorname{ext}(A)))$. Let $\delta := f(x_0) - \sup f(\overline{\operatorname{co}}(\operatorname{ext}(A)))$. In view of Lemma 4, there exists $g \in X^*$ such that g

attains its sup at A and $|f(a) - g(a)| < \frac{1}{4}\delta$ for all $a \in A$. By bearing in mind Lemma 5, we have that

$$|\sup f(\overline{\text{co}}(\text{ext}(A))) - \sup g(\overline{\text{co}}(\text{ext}(A)))| \leq \frac{1}{4}\delta.$$

On the other hand,

$$f(x_0) - \sup f(\overline{\text{co}}(\text{ext}(A))) = \delta > \frac{1}{2}\delta = \frac{1}{4}\delta + \frac{1}{4}\delta,$$

meaning that

$$f(x_0) - \frac{1}{4}\delta > \sup f(\overline{\text{co}}(\text{ext}(A))) + \frac{1}{4}\delta.$$

Then,

$$g(x_0) > f(x_0) - \frac{1}{4}\delta > \sup f(\overline{\text{co}}(\text{ext}(A))) + \frac{1}{4}\delta \geq \sup g(\overline{\text{co}}(\text{ext}(A))).$$

Finally, $\text{ext}(\{a \in A : g(a) = \sup g(A)\}) \neq \varnothing$, in fact, $\text{ext}(\{a \in A : g(a) = \sup g(A)\}) \subseteq \text{ext}(A)$, which is impossible since $\sup g(A) \geq g(x_0) > \sup g(\overline{\text{co}}(\text{ext}(A)))$. □

Corollary 1. *Let X be a Banach space. If $\text{ext}(A) \neq \varnothing$ for all bounded, closed, convex subset A of X, then $A = \overline{\text{co}}(\text{ext}(A))$ for all bounded, closed, convex subset A of X.*

Proof. It only suffices to bear in mind Theorem 3 together with the Bishop–Phelps theorem. □

The idea of searching for spaces failing the Krein–Milman property is to find necessary conditions for satisfying such a property. The following is another technical lemma, which in fact also works for general topological modules over topological rings.

Lemma 6. *Let X be a topological vector space. If U is an open neighborhood of 0 in X, then $v + U = U$ for all $v \in O_X$. If U is a regular closed neighborhood of 0 in X, then $v + U = U$ for all $v \in O_X$.*

Proof. First of all, let us assume that U is open. For every $u \in U$, $U - u \in \mathcal{N}_0(X)$, hence $v \in U - u$, meaning that $v + U \subseteq U$. By the same argument, $-v + U \subseteq U$. As a consequence, we obtain the desired result. Next, let us assume that U is regular closed. Since $\text{int}(U)$ is open, we know that $\text{int}(U + v) = \text{int}(U) + v = \text{int}(U)$. By taking closures, we obtain that $U + v = \text{cl}(\text{int}(U)) + v = \text{cl}(\text{int}(U + v)) = \text{cl}(\text{int}(U)) = U$. □

The following result improves considerably on [10] (Corollary 2.1 and Theorem 2.2). Refer to Appendix A for the notion of extremal points (which coincides with that of extreme points in the convex setting).

Theorem 4. *Let X be a topological vector space. If X is not Hausdorff, then every regular closed 0-neighborhood of X is free of extremal points. In particular, O_X is a bounded, closed, and convex subset of X free of extremal points.*

Proof. Let U be any regular closed 0-neighborhood of X. Fix any arbitrary $u \in U$. We will show that u is not an extremal point of U. Take any $v \in O_X$ such that $v \neq 0$. By Lemma 6, $v + u, -v + u \in U$. Finally, $u = \frac{1}{2}(v + u) + \frac{1}{2}(-v + u)$, meaning that $u \notin \text{ext}(U)$. □

We will find more examples of topological vector spaces not enjoying the Krein–Milman property. The following proposition is an improvement of [10] (Remark 3.1).

Proposition 1. *Let X be a locally convex topological vector space endowed with a biorthogonal system $\left(e_i, e_i^*\right)_{i\in I} \subset X \times X^*$ such that I is infinite. For every sequence $\left(e_{i_n}\right)_{n\in\mathbb{N}} \subseteq (e_i)_{i\in I}$ of different terms, the linear operator*

$$\begin{array}{rccl} T: & c_{00} & \to & X \\ & a & \mapsto & T(a) := \displaystyle\sum_{n=1}^{\infty} \frac{a(n)}{2^n} e_{i_n} \end{array} \tag{1}$$

satisfies the following:

1. *$T(\mathsf{B}_{c_{00}})$ is absolutely convex but free of extreme points.*
2. *$T(\mathsf{B}_{c_{00}})$ is closed in* $\operatorname{span}\{e_i : i \in I\}$.
3. *If $\left(e_{i_n}\right)_{n\in\mathbb{N}}$ is bounded, then $T(\mathsf{B}_{c_{00}})$ is bounded, hence T is continuous.*

Proof. First, T is linear and one-to-one, therefore, $T(\mathsf{B}_{c_{00}})$ is absolutely convex and free of extreme points because so is $\mathsf{B}_{c_{00}}$. Next, let us show that $T(\mathsf{B}_{c_{00}})$ is closed in $\operatorname{span}\{e_i : i \in I\}$. Indeed, take a net $(a_\gamma)_{\gamma\in\Gamma} \subseteq \mathsf{B}_{c_{00}}$ such that $(T(a_\gamma))_{\gamma\in\Gamma}$ converges to some $x \in \operatorname{span}\{e_i : i \in I\}$. Since $e_i^* \in X^*$ is a continuous linear functional for each $i \in I$, we have that $\left(e_i^*(T(a_\gamma))\right)_{\gamma\in\Gamma}$ converges to $e_i^*(x)$ for each $i \in I$. In particular, if $i \in I \setminus \{i_n : n \in \mathbb{N}\}$, then $e_i^*(T(a_\gamma)) = 0$ for all $\gamma \in \Gamma$, meaning that $e_i^*(x) = 0$. This shows that $x \in \operatorname{span}\{e_{i_n} : n \in \mathbb{N}\}$. Also, $\left(e_{i_n}^*(T(a_\gamma))\right)_{\gamma\in\Gamma}$ converges to $e_{i_n}^*(x)$ for each $n \in \mathbb{N}$, in other words, $\left(\frac{a_\gamma(n)}{2^n}\right)_{\gamma\in\Gamma}$ converges to $e_{i_n}^*(x)$ for each $n \in \mathbb{N}$. Then, $\left|2^n e_{i_n}^*(x)\right| \leq 1$ for all $n \in \mathbb{N}$ since $|a_\gamma(n)| \leq \|a_\gamma\|_\infty \leq 1$ for all $n \in \mathbb{N}$ and all $\gamma \in \Gamma$. So, consider $b \in \mathsf{B}_{c_{00}}$ defined by $b(n) := 2^n e_{i_n}^*(x)$ for each $n \in \mathbb{N}$. Then,

$$x = \sum_{n=1}^{\infty} e_{i_n}^*(x) e_{i_n} = \sum_{n=1}^{\infty} \frac{b(n)}{2^n} e_{i_n} = T(b) \in T(\mathsf{B}_{c_{00}}).$$

Finally, assume that $\left(e_{i_n}\right)_{n\in\mathbb{N}}$ is bounded. Fix an arbitrary 0-neighborhood $U \subseteq X$. We may assume that U is absolutely convex. There exists $\lambda \neq 0$ such that $e_{i_n} \in \lambda U$ for each $n \in \mathbb{N}$. Since λU is absolutely convex, we have that $T(a) = \sum_{n=1}^{\infty} \frac{a(n)}{2^n} e_{i_n} \in \lambda U$ for all $a \in \mathsf{B}_{c_{00}}$. As a consequence, $T(\mathsf{B}_{c_{00}})$ is bounded and T is continuous. □

Recall that in any topological vector space X, every neighborhood of 0 is absorbing, or equivalently, 0 is an internal point, and if $\mathcal{B}$ is a base of 0-neighborhoods of X, then $\mathcal{B}_0 := \{[-1,1]U : U \in \mathcal{B}\}$ is also a base of neighborhoods of 0 in X. Notice that if U is a balanced neighborhood of 0, then $\alpha U \subseteq \beta U$ whenever $|\alpha| \leq |\beta|$.

Lemma 7. *Let X be a first-countable topological vector space. Let $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ be a countable nested base of 0-neighborhoods of X. If $u_n \in U_n$ for each $n \in \mathbb{N}$, then $\{u_n : n \in \mathbb{N}\}$ is bounded.*

Proof. By considering $\mathcal{B}_0 := \{[-1,1]U : U \in \mathcal{B}\}$ if necessary, we may assume that every element of $\mathcal{B}$ is balanced. Fix any arbitrary 0-neighborhood V of X. Since $\mathcal{B}$ is nested, there exists $n_0 \in \mathbb{N}$ such that $U_n \subseteq V$ for all $n \geq n_0$. For every $n \in \{1, \ldots, n_0 - 1\}$, there exists $\lambda_n \neq 0$ such that $u_n \in \lambda_n V$. Finally, it is enough to take $\lambda := \max\{|\lambda_1|, \ldots, |\lambda_{n_0-1}|, 1\}$. Then, $u_n \in \lambda V$ for all $n \in \mathbb{N}$. □

Lemma 7 does not remain true if $\mathcal{B}$ is not nested. Indeed, consider

$$\mathcal{B} := \left\{\left[-\frac{1}{n}, \frac{1}{n}\right] : n \in \mathbb{N}\right\} \cup \{[-n, n] : n \in \mathbb{N}\}$$

is a countable base of 0-neighborhoods in $\mathbb{R}$, but taking $u_n := n$ for all $n \in \mathbb{N}$ we conclude that $\{u_n : n \in \mathbb{N}\}$ is unbounded.

Theorem 5. *Let X be a first-countable locally convex topological vector space endowed with a biorthogonal system $(e_i, e_i^*)_{i\in I} \subset X \times X^*$ such that I is infinite. Then,* $\text{span}\{e_i : i \in I\}$ *fails the Krein–Milman property.*

Proof. Extract a sequence $(e_{i_n})_{n\in\mathbb{N}} \subseteq (e_i)_{i\in I}$ of different terms. Let $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ be a countable nested base of absolutely convex and absorbing 0-neighborhoods of X. For every $n \in \mathbb{N}$, take $\lambda_n \neq 0$ such that $\lambda_n e_{i_n} \in U_n$. According to Lemma 7, $\{\lambda_n e_{i_n} : n \in \mathbb{N}\}$ is bounded. Notice that in $(e_i, e_i^*)_{i\in I}$, we can replace $\left(e_{i_n}, e_{i_n}^*\right)$ by $\left(\lambda_n e_{i_n}, \frac{1}{\lambda_n} e_{i_n}^*\right)$ for each $n \in \mathbb{N}$. So, we may assume that $\{e_{i_n} : n \in \mathbb{N}\}$ is bounded. Now, we can call on Proposition 1 to conclude that $T(\mathsf{B}_{c_{00}})$ is a bounded, closed, convex subset of $\text{span}\{e_i : i \in I\}$ free of extreme points, where T is the continuous linear operator given in (1). As a consequence, $\text{span}\{e_i : i \in I\}$ fails the Krein–Milman property. □

Our next goal is to take advantage of the previous results to keep exploring more spaces failing the Krein–Milman property. The following technical lemma [10] (Theorem 3.1) will be employed later on, whose proof is included here for the sake of completeness.

Lemma 8. *Let X be a topological vector space. If $\dim(X) = \aleph_0$, then X is separable. If, in addition, X is Hausdorff and locally convex, then there exists a countably infinite biorthogonal system $(e_i, e_i^*)_{i\in\mathbb{N}} \subset X \times X^*$ such that $X = \text{span}\{e_i : i \in \mathbb{N}\}$ and $X^* = \overline{\text{span}}^{\,\omega^*}\{e_i^* : i \in \mathbb{N}\}$.*

Proof. We will prove first that X is separable. Let $(u_n)_{n\in\mathbb{N}}$ be a Hamel basis for X. We will prove that $S := \{\sum_{i=1}^n q_i u_i : n \in \mathbb{N}, q_1, \dots, q_n \in \mathbb{Q}\}$ is dense in X. Indeed, fix an arbitrary non-empty open subset $U \subseteq X$. Take any $u \in U$. We can write $u = \sum_{i=1}^n t_i u_i$ with $t_1, \dots, t_n \in \mathbb{R}$. There exists a 0-neighborhood V of X such that $u + V \subseteq U$. There also exists a 0-neighborhood $W \subseteq X$ satisfying that $W + \overset{n}{\cdots} + W \subseteq V$. For every $i \in \{1, \dots, n\}$, we can find $\varepsilon_i > 0$ satisfying that $(-\varepsilon_i, \varepsilon_i)u_i \subseteq W$. For every $i \in \{1, \dots, n\}$, let $q_i \in \mathbb{Q} \cap (t_i + (-\varepsilon_i, \varepsilon_i))$. Then,

$$\sum_{i=1}^n q_i u_i = u + \sum_{i=1}^n (q_i - t_i)u_i \in u + \sum_{i=1}^n (-\varepsilon_i, \varepsilon_i)u_i \subseteq u + \left(W + \overset{n}{\cdots} + W\right) \subseteq u + V \subseteq U.$$

Finally, assume that X is Hausdorff and locally convex. We will construct a countably infinite biorthogonal system $(e_i, e_i^*)_{i\in\mathbb{N}} \subset X \times X^*$ such that $X = \text{span}\{e_i : i \in \mathbb{N}\}$. Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a Hamel basis for X. We will construct the biorthogonal system inductively. Take $e_1 := u_1$. Obviously, $\text{span}\{e_1\} = \text{span}\{u_1\}$. The Hahn–Banach theorem allows us to find $e_1^* \in X^*$ such that $e_1^*(e_1) = 1$. Take $e_2 := u_2 - e_1^*(u_2)u_1$. Note that $\text{span}\{e_1, e_2\} = \text{span}\{u_1, u_2\}$. The Hahn–Banach theorem allows us to find $e_2^* \in X^*$ such that $1 = e_2^*(e_2) > \sup e_2^*(\mathbb{R}e_1)$. Therefore, $e_2^*(e_1) = 0$. Take $e_3 := u_3 - e_1^*(u_3)e_1 - e_2^*(u_3)e_2$. Observe that $\text{span}\{e_1, e_2, e_3\} = \text{span}\{u_1, u_2, u_3\}$. The Hahn–Banach theorem allows us to find $e_3^* \in X^*$ such that $1 = e_3^*(e_3) > \sup e_3^*(\mathbb{R}e_1 \oplus \mathbb{R}e_2)$. Therefore, $e_3^*(e_1) = e_3^*(e_2) = 0$. And so on. To see that $X^* = \overline{\text{span}}^{\,\omega^*}\{e_i^* : i \in \mathbb{N}\}$, it suffices to realize that $\text{span}\{e_i^* : i \in \mathbb{N}\}$ separates points of X together with the fact that X is Hausdorff and locally convex. □

The next corollary improves on [10] (Corollary 3.2).

Corollary 2. *Let X be a first-countable Hausdorff locally convex topological vector space. If* $\dim(X) = \aleph_0$, *then X fails the Krein–Milman property.*

Proof. According to Lemma 8, X is separable and there exists a countably infinite biorthogonal system $(e_i, e_i^*)_{i\in\mathbb{N}} \subset X \times X^*$ such that $X = \text{span}\{e_i : i \in \mathbb{N}\}$. Now, we can call on Theorem 5 to assure that $X = \text{span}\{e_i : i \in \mathbb{N}\}$ fails the Krein–Milman property. □

The previous results have been providing necessary conditions to assure that the Krein–Milman property is satisfied. We will finish this section and the manuscript with a sufficient condition. First, the following technical lemma is needed.

Lemma 9. *Let X be a vector space. If A and B are convex subsets of X, then* $\mathrm{ext}(\mathrm{co}(A \cup B)) \subseteq \mathrm{ext}(A) \cup \mathrm{ext}(B)$.

Proof. Let $e \in \mathrm{ext}(\mathrm{co}(A \cup B))$. There exist $t \in [0,1]$, $a \in A$, and $b \in B$ such that $e = ta + (1-t)b$. By hypothesis, either $a = b$ or $t \in \{0,1\}$. In either case, $e \in A \cup B$, hence $e \in \mathrm{ext}(A) \cup \mathrm{ext}(B)$. □

Before stating and proving our sufficient condition to assure the Krein–Milman property, we need to introduce the notion of an absolutely convex prism.

Definition 1 (Absolutely convex prism)**.** *Let X be a topological vector space X. An absolutely convex prism of X is defined as a set of the form* $\mathrm{co}(M \cup -M)$ *for M, a bounded, closed, and convex subset of X.*

Theorem 6. *Let X be a topological vector space. If every absolutely convex prism of X has extreme points, then X has the Krein–Milman property.*

Proof. Fix an arbitrary closed, bounded, convex subset M of X. By hypothesis, there exists an extreme point $e \in \mathrm{ext}(\mathrm{co}(M \cup -M))$. In accordance with Lemma 9, $e \in \mathrm{ext}(M) \cup \mathrm{ext}(-M) = \mathrm{ext}(M) \cup -\mathrm{ext}(M)$. □

4. Nontrivial Examples

As mentioned earlier, every Banach space satisfies the Bishop–Phelps property in virtue of the famous Bishop–Phelps theorem [18]. However, not every Banach space satisfies the Krein–Milman property. Indeed, observe that the unit ball of c_0 is free of extreme points, hence c_0 does not verify the Krein–Milman property. However, c_0 satisfies the Bishop–Phelps property in view of the famous Bishop–Phelps theorem [18] since it is a Banach space. As a consequence, we have an example of a topological vector space satisfying the Bishop–Phelps property, but not the Krein–Milman property. Next, another example of a locally convex topological vector space satisfying the Bishop–Phelps property but failing the Krein–Milman property is given.

Example 1. *If X is a topological vector space endowed with the trivial topology, then X is clearly locally convex and satisfies the Bishop–Phelps property. Indeed, $X^* = \{0\}$ and $\mathcal{BCC}_X = X$. If, in addition, $X \neq \{0\}$, then X is not Hausdorff, since $O_X = \bigcap_{V \in \mathcal{N}_0(X)} V = X \neq \{0\}$, meaning that X fails the Krein–Milman property in view of [10] (Corollary 2.1). Finally, notice that the trivial topology is always a vector topology in any vector space.*

Our next efforts are aimed at showing better examples of non-Hausdorff locally convex topological vector spaces satisfying the Bishop–Phelps property than just topological vector spaces endowed with the trivial topology.

Example 2. *Let Y be an infinite-dimensional Banach space and Z an infinite-dimensional vector space. Take $X := Y \times Z$ endowed with the product topology by previously endowing Z with the trivial topology. Note that $O_X = Z$ and Y is an algebraic complement for O_X in X, meaning that X is not Hausdorff since $O_X = Z \neq \{0\}$. As a consequence, Theorem 2 guarantees that X enjoys the Bishop–Phelps property since Y does because it is a Banach space.*

The following is an example of a Hausdorff locally convex topological vector space failing the Bishop–Phelps property.

Example 3. *In accordance with [30] (Example), there exists a normed space X which is not subreflexive, therefore,* $\mathsf{SA}(\mathsf{B}_X)$ *is not dense in the norm-topology of X^*, which is precisely the $\mathcal{BCC}_X$-topology of X^*. As a consequence, X does not satisfy the Bishop–Phelps property.*

An interesting question is to show whether [30] (Example) satisfies the Krein–Milman property.

Finally, another interesting question is to study how vector spaces endowed with the finest locally convex vector topology behave with respect to these properties (refer to Appendix C for the construction of the finest locally convex vector topology in any real or complex vector space). Our final results are aimed at showing that every vector space endowed with the finest locally convex vector topology enjoys the Bishop–Phelps property.

Lemma 10. *Let X be a vector space endowed with the finest locally convex vector topology. Every linear functional f on X is continuous.*

Proof. Indeed, for every $\varepsilon > 0$, $f^{-1}((-\varepsilon,\varepsilon))$ is clearly absolutely convex and absorbing, hence it is a neighborhood of 0 in X for the finest locally convex vector topology in view of Theorem A5. □

Lemma 11. *Let X be a vector space. If $A \subseteq X$ is bounded for the finest locally convex vector topology, then* $\mathrm{span}(A)$ *is finite dimensional.*

Proof. Assume on the contrary that $\mathrm{span}(A)$ is infinite dimensional. Then, a linearly independent sequence $(a_n)_{n\in\mathbb{N}}$ of infinite terms can be found in A. Then, a linear functional f on X can be constructed by defining $f(a_n) = n$ for all $n \in \mathbb{N}$ and by extending by linearity to the whole of X (after completing $(a_n)_{n\in\mathbb{N}}$ to a Hamel basis). Next, by applying Lemma 10, f is continuous on X, that is, $f^{-1}((-1,1))$ is absolutely convex and absorbing, so it is a neighborhood of 0 in X for the finest locally convex vector topology in view of Theorem A5. By hypothesis, A is bounded, meaning that there exists $\varepsilon > 0$ such that $A \subseteq \varepsilon f^{-1}((-1,1)) = f^{-1}((-\varepsilon,\varepsilon))$, which contradicts the fact that $f(a_n) = n$ for all $n \in \mathbb{N}$. □

Theorem 7. *Let X be a vector space. If X is endowed with the finest locally convex vector topology, then X enjoys the Bishop–Phelps property.*

Proof. Fix an arbitrary bounded, closed, and convex subset $A \subseteq X$. By Lemma 11, $\mathrm{span}(A)$ is finite dimensional. Notice that the inherited topology of $\mathrm{span}(A)$ from X is the Euclidean topology. As a consequence, A is compact and convex in $\mathrm{span}(A)$, meaning that any linear functional on X attains its supremum on A, that is, $\mathsf{SA}_X(A) = X^*$, which is trivially dense in X^* for the $\mathcal{BCC}_X$-topology, □

5. Discussion

In the context of Banach spaces, the Krein–Milman property and strong Krein–Milman property are equivalent. Hence, what keeps the interest of researchers in this field is the proving or disproving of the equivalence between the Krein–Milman property and the Radon–Nikodym property. As far as we know, this problem still remains open.

With respect to the Bishop–Phelps–Bollobás property, according to the literature, this property is hereditary to certain complemented subspaces. We have accomplished the hereditariness of the Bishop–Phelps property to closed complemented subspaces in Theorem 1. Now, the next step is to provide a converse result to Theorem 1, in other words, to recover the Bishop–Phelps property from two closed complemented subspaces enjoying it. We will discuss next that such a converse seems not to work.

Let X be a topological vector space. Let $P : X \to X$ be a continuous linear projection. Suppose that both $P(X)$ and $(I - P)(X)$ both enjoy the Bishop–Phelps property. Let us discuss whether X satisfies the Bishop–Phelps property. Take any bounded, closed, and convex subset B of X. We should show that $\mathsf{SA}_{X^*}(B)$ is dense in X^* for the $\mathcal{BCC}_X$-topology. Indeed, fix an arbitrary bounded, closed, and convex subset A of X, an arbitrary $\varepsilon > 0$, and an arbitrary $x^* \in X^*$. We should prove that $\mathsf{SA}_{X^*}(B) \cap [x^* + \mathcal{U}_{X^*}(A, (-\varepsilon, \varepsilon))] \neq \varnothing$. Note that $P(A)$, $P(B)$ and $(I-P)(A)$, $(I-P)(B)$ are all bounded and convex in X because P and $I-P$ are linear and continuous, but they might not be closed in $P(X)$ and $(I-P)(X)$, respectively. This is the first obstacle. But still, let us assume that $P(A)$, $P(B)$ and $(I-P)(A)$, $(I-P)(B)$ are closed in $P(X)$ and $(I-P)(X)$, respectively. Thus, by hypothesis, $\mathsf{SA}_{P(X)^*}(P(B))$ and $\mathsf{SA}_{(I-P)(X)^*}((I-P)(B))$ are dense in $P(X)^*$ and in $(I-P)(X)^*$ for the $\mathcal{BCC}_{P(X)}$-topology and the $\mathcal{BCC}_{(I-P)(X)}$-topology, respectively. This means that

$$\mathsf{SA}_{P(X)^*}(P(B)) \cap \left[x^*|_{P(X)} + \mathcal{U}_{P(X)^*}\left(P(A), \left(\frac{-\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right) \right] \neq \varnothing$$

and

$$\mathsf{SA}_{(I-P)(X)^*}((I-P)(B)) \cap \left[x^*|_{(I-P)(X)} + \mathcal{U}_{(I-P)(X)^*}\left((I-P)(A), \left(\frac{-\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right) \right] \neq \varnothing.$$

Therefore, we might be able to choose elements

$$y^* \in \mathsf{SA}_{P(X)^*}(P(B)) \cap \left[x^*|_{P(X)} + \mathcal{U}_{P(X)^*}\left(P(A), \left(\frac{-\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right) \right]$$

and

$$z^* \in \mathsf{SA}_{(I-P)(X)^*}((I-P)(B)) \cap \left[x^*|_{(I-P)(X)} + \mathcal{U}_{(I-P)(X)^*}\left((I-P)(A), \left(\frac{-\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right) \right].$$

If $f := y^* \circ P + z^* \circ (I-P) \in X^*$, then

$$|f(a) - x^*(a)| \leq |y^*(P(a)) - x^*(P(a))| + |z^*((I-P)(a)) - x^*((I-P))(a)| < \varepsilon,$$

for all $a \in A$, meaning that $f \in x^* + \mathcal{U}_{X^*}(A, (-\varepsilon, \varepsilon))$. It only remains to show that $f \in \mathsf{SA}_{X^*}(B)$. Indeed, there are $y \in P(B)$ and $z \in (I-P)(B)$ such that $y^*(y) \geq y^*(P(b))$ and $z^*(z) \geq z^*((I-P)(b))$ for all $b \in B$. Note that

$$\begin{aligned} f(y+z) &= y^*(P(y+z)) + z^*((I-P)(y+z)) \\ &= y^*(y) + z^*(z) \\ &\geq y^*(P(b)) + z^*((I-P)(b)) \\ &= f(b) \end{aligned}$$

for each $b \in B$. However, there is no guarantee that $y + z \in B$.

6. Conclusions

According to Theorem 1, the Bishop–Phelps property is hereditary to closed complemented subspaces. In view of Theorem 3, both Krein–Milman and strong Krein–Milman properties are equivalent in the class of Hausdorff locally convex topological vector spaces satisfying the Bishop–Phelps property. Theorem 4 forces topological vector spaces with the Krein–Milman property to be Hausdorff. Another necessary condition is given by Corollary 2, which states that if a first-countable Hausdorff locally convex topological vector space satisfies the Krein–Milman property, then it must have uncountable dimension. Finally, Theorem 6 provides a sufficient condition for a topological vector space to enjoy the Krein–Milman property, which consists in checking only whether the absolutely convex prisms have extreme points.

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Appendix A. Extremal Structure

The following notions are well known among the Banach-space geometers and belong to the folklore of the classic literature of Banach-space theory. For further reading on these topics, we refer to the classical texts [31–33].

Property A1 (Extremal)**.** *Let X be a vector space. Let $E \subseteq F \subseteq X$. Then, E is said to enjoy the extremal condition with respect to F whenever the following property is satisfied:*

$$\forall x, y \in F, \quad \forall t \in (0,1): \quad tx + (1-t)y \in E \Rightarrow x, y \in E. \tag{A1}$$

Under this situation, we say that E is extremal in F.

Extremal subsets which are singletons are called extremal points. The set of extremal points of F is denoted by $\mathrm{ext}(F)$ (when F is convex, then extremal points are often called extreme points). The non-empty intersection of any arbitrary family of extremal subsets is extremal as well. Also, if E is extremal in F and D is extremal in E, then D is extremal in F. As a consequence, if E is extremal in F, then $\mathrm{ext}(E) \subseteq \mathrm{ext}(F)$.

Example A1 (Supporting hyperplane)**.** *Let A be a non-empty subset of a vector space X and consider $f \in X^*$. The supporting hyperplane relative to f in A,*

$$\mathsf{F}(f, A) := \{x \in A : f(x) = \max f(A)\},$$

is an extremal subset of A, provided that $\mathsf{F}(f, A) \neq \varnothing$, that is, $f \in \mathsf{SA}(A)$.

A face of a convex set is a extremal convex subset. Extremal points of convex sets are usually called extreme points.

Appendix B. Bishop–Phelps and Krein–Milman Theorems

The original statement of the Krein–Milman theorem [7] is the following:

Theorem A1 (Krein–Milman Theorem)**.** *Let C be a non-empty compact convex subset of a Hausdorff locally convex topological vector space. Then, C is the closed convex hull of its set of extreme points.*

The proof of the Krein–Milman theorem relies on two key facts: every non-empty compact subset of a Hausdorff locally convex topological vector space has extremal points; and supporting hyperplanes are extremal subsets.

The original statement of the Bishop–Phelps theorem [18,19] is the following:

Theorem A2 (Bishop–Phelps Theorem)**.** *If C is a non-empty bounded closed convex subset of a Banach space X, then the support functionals for C are dense in X^*.*

The set of support functionals for C is by definition $\mathsf{SA}(C)$. The completeness hypothesis is crucial in the proof of the Bishop–Phelps theorem.

The original statement of the Bishop–Phelps–Bollobás theorem [21] is the following:

Theorem A3 (Bishop–Phelps–Bollobás Theorem)**.** *Let X be a Banach space. For every $\varepsilon \in (0,1)$, there exists $\delta(\varepsilon) := \varepsilon^2/4$ such that for all $x^* \in \mathsf{S}_{X^*}$ and all $x \in \mathsf{S}_X$ with $x^*(x) > 1 - \delta(\varepsilon)$ there exist $u \in \mathsf{S}_X$ and $y^* \in \mathsf{S}_{X^*}$ with $y^*(u) = 1$, $\|u - x\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.*

Appendix C. Finest Locally Convex Vector Topology

Let X be a vector space over the real or complex field $\mathbb{K}$. Let $A \subseteq X$. The set of internal points [34] of A is defined as

$$\operatorname{inter}(A) := \{a \in X : \forall x \in X \quad \exists \varepsilon_x > 0 \quad a + \varepsilon_x \mathsf{B}_{\mathbb{K}}(x - a) \subseteq A\},$$

where $\mathsf{B}_{\mathbb{K}}$ is the unit ball of $\mathbb{K}$ for its usual absolute value. The set A is said to be balanced provided that $\mathsf{B}_{\mathbb{K}} A \subseteq A$, and is said to be absorbing provided that $0 \in \operatorname{inter}(A)$.

The following theorem is a characterization of module topology that can be found in [35] (Theorem 3.6) and also in [36–38].

Theorem A4. *If M is a topological module over a topological ring R and $\mathcal{B}$ is a base of neighborhoods of 0 in M, then it is verified that:*

1. *For every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V + V \subseteq U$.*
2. *For every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $-V \subseteq U$.*
3. *For every $U \in \mathcal{B}$ there exist $V \in \mathcal{B}$ and $W \in \mathcal{N}_0(R)$ such that $WV \subseteq U$.*
4. *For every $U \in \mathcal{B}$ and every $r \in R$, there exists $V \in \mathcal{B}$ such that $rV \subseteq U$.*
5. *For every $U \in \mathcal{B}$ and every $m \in M$, there exists $W \in \mathcal{N}_0(R)$ such that $Wm \subseteq U$.*

Conversely, for any filter base on a module over a topological ring verifying all five properties above there exists a unique module topology on the module such that the filter base is a base of neighborhoods of zero.

Remark A1. *Let X be a topological vector space. Then:*

1. *Every neighborhood of 0 is absorbing in view of Theorem A4(5).*
2. *If $\mathcal{B}$ is a base of neighborhoods of 0, then $\mathcal{B}_0 := \{\mathsf{B}_{\mathbb{K}} U : U \in \mathcal{B}\}$ is also base of neighborhoods of 0 and all elements of $\mathcal{B}_0$ are balanced.*

As a consequence, every topological vector space has a base of zero-neighborhoods which are balanced and absorbing. In this sense, a topological vector space is said to be locally convex provided that there exists a base of zero-neighborhoods which are convex.

Remark A2. *Let X be a locally convex topological vector space. If $\mathcal{B}$ is a base of neighborhoods of 0 which are convex, then $\mathcal{B}_0 := \{\mathsf{B}_{\mathbb{K}} U : U \in \mathcal{B}\}$ is also base of neighborhoods of 0 and all elements of $\mathcal{B}_0$ are balanced and convex.*

As a consequence, every locally convex topological vector space has a base of zero-neighborhoods which are convex, balanced, and absorbing.

Remark A3. *Let X be a topological vector space. Let $A \subseteq X$ convex. If $a \in \operatorname{int}(A)$ and $b \in \operatorname{cl}(A)$, then $[a, b) \subseteq \operatorname{int}(A)$.*

Lemma A1. *Let X be a topological vector space and A a non-empty subset of X. Then:*

1. *$\operatorname{int}(A) \subseteq \operatorname{inter}(A) \subseteq A$.*
2. *If A is open, then $A = \operatorname{int}(A) = \operatorname{inter}(A)$.*
3. *If A is convex and $\operatorname{int}(A) \neq \varnothing$, then $\operatorname{int}(A) = \operatorname{inter}(A)$.*
4. *If A is convex, $\operatorname{int}(A) \neq \varnothing$ and $A = \operatorname{inter}(A)$, then A is open.*

Proof.

1. Fix an arbitrary $a \in \mathrm{int}(A)$. Let $x \in X \setminus \{a\}$. Let V be a balanced and absorbing neighborhood of 0 such that $a + V \subseteq A$. Since V is absorbing, we can find $\varepsilon_x > 0$ such that $\varepsilon_x \mathsf{B}_{\mathbb{K}}(x - a) \subseteq V$. Then, $a + \varepsilon_x \mathsf{B}_{\mathbb{K}}(x - a) \subseteq a + V \subseteq A$, which means that $a \in \mathrm{inter}(A)$.
2. By (1), $A = \mathrm{int}(A) \subseteq \mathrm{inter}(A) \subseteq A$.
3. By (1), we know that $\mathrm{int}(A) \subseteq \mathrm{inter}(A)$. Fix $a \in \mathrm{int}(A)$. Let $b \in \mathrm{inter}(A) \setminus \{a\}$. There exists $\varepsilon_a > 0$ such that $b + \varepsilon_a \mathsf{B}_{\mathbb{K}}(a - b) \subseteq A$. In particular,

$$b = \frac{\varepsilon_a}{1+\varepsilon_a} a + \frac{1}{1+\varepsilon_a}(b - \varepsilon_a(a-b)) \in [a, b - \varepsilon_a(a-b)) \subseteq \mathrm{int}(A)$$

in view of Remark A3.
4. By (4), $A = \mathrm{inter}(A) = \mathrm{int}(A)$.

□

See [39] for related topics on the previous Lemma. The next theorem shows the modern construction of the finest locally convex vector topology.

Theorem A5. *Let X be a vector space. Let $\mathcal{B}_X := \{U \subseteq X : U$ is convex, balanced, and absorbing$\}$. Then:*

1. *$\mathcal{B}_X$ is a base of neighborhoods of 0 for a locally convex vector topology τ_X on X.*
2. *If τ is a locally convex vector topology on X, then $\tau \subseteq \tau_X$.*
3. *If $A \subseteq X$ is convex and absorbing, then A is a neighborhood of 0 in τ_X.*
4. *If $A \subseteq X$ is convex and $\mathrm{inter}(A) \neq \varnothing$, then $\mathrm{inter}(A)$ coincides with the interior of A in τ_X.*

Proof.

1. We will check that $\mathcal{B}_X$ satisfies all five conditions given in Theorem A4:
    - For every $U \in \mathcal{B}_X$ there exists $V \in \mathcal{B}_X$ such that $V + V \subseteq U$. Indeed, since U is convex, it suffices to consider $V := \frac{1}{2}U \in \mathcal{B}_X$.
    - For every $U \in \mathcal{B}_X$ there exists $V \in \mathcal{B}_X$ such that $-V \subseteq U$. Indeed, since U is balanced, it suffices to consider $V := U \in \mathcal{B}_X$.
    - For every $U \in \mathcal{B}_X$ there exist $V \in \mathcal{B}_X$ and $W \in \mathcal{N}_0(\mathbb{K})$ such that $WV \subseteq U$. Indeed, since U is balanced, it suffices to consider $V := U \in \mathcal{B}_X$ and $W := \mathsf{B}_{\mathbb{K}} \in \mathcal{N}_0(\mathbb{K})$.
    - For every $U \in \mathcal{B}_X$ and every $\lambda \in \mathbb{K}$ there exists $V \in \mathcal{B}_X$ such that $rV \subseteq U$. Indeed, it suffices to take $V := \lambda^{-1}U \in \mathcal{B}_X$ if $\lambda \neq 0$, and $V := U \in \mathcal{B}_X$ if $\lambda = 0$.
    - For every $U \in \mathcal{B}_X$ and every $x \in X$, there exists $W \in \mathcal{N}_0(\mathbb{K})$ such that $Wx \subseteq U$. Indeed, since U is absorbing, then $0 \in \mathrm{inter}(A)$, so there exists $\varepsilon_x > 0$ such that $\varepsilon_x \mathsf{B}_{\mathbb{K}} x \subseteq U$, hence it only suffices to take $W := \varepsilon_x \mathsf{B}_{\mathbb{K}} \in \mathcal{N}_0(\mathbb{K})$.

    According to Theorem A4, there exists a unique vector topology τ_X on X for which $\mathcal{B}_X$ is a base of neighborhoods of 0. This topology is by construction locally convex.
2. In accordance with Remark A2, there exists a base $\mathcal{B}$ of neighborhoods of 0, whose elements are convex, balanced, and absorbing for the topology τ. Then, $\mathcal{B} \subseteq \mathcal{B}_X$, which implies that $\tau \subseteq \tau_X$.
3. Let $\mathcal{L} := \{B \subseteq A : B$ is absolutely convex$\}$. Clearly $\mathcal{L}$ is non-empty because A is absorbing. Notice that $\mathcal{L}$ can be partially ordered by the inclusion. If $(B_i)_{i \in I}$ is a chain of $\mathcal{L}$, then it is clear that $\bigcup_{i \in I} B_i \in \mathcal{L}$. Using Zorn's lemma, there exists a maximal element B_0. If B_0 is not absorbing, then there exists $x \neq 0$ and $\varepsilon_x > 0$ such that $\varepsilon_x \mathsf{B}_{\mathbb{K}} x \subseteq A$ but $\varepsilon_x \mathsf{B}_{\mathbb{K}} \nsubseteq B_0$. Then, $C := \mathrm{co}(\varepsilon_x \mathsf{B}_{\mathbb{K}} x \cup B_0) \subseteq A$ is absolutely convex and contains B_0 strictly. This is a contradiction, so B_0 is absorbing. As a consequence, $B_0 \in \mathcal{B}_X$, so A is a neighborhood of 0 in τ_X.
4. Suppose now that A is convex and $\mathrm{inter}(A) \neq \varnothing$. By Lemma A1(1), the interior of A in τ_X is contained in $\mathrm{inter}(A)$. Fix an arbitrary $a \in \mathrm{inter}(A)$. Now, $0 \in \mathrm{inter}(A - a)$,

so $A - a$ is absorbing, and it is also convex, therefore, by (3), $A - a$ is a neighborhood of 0 in τ_X. As a consequence, a is an interior point of A in τ_X.

□

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