mathematics

Article

The Shape of a Compressible Drop on a Vibrating Solid Plate

Andrey Ivantsov [1], Tatyana Lyubimova [1,2,*], Grigoriy Khilko [1] and Dmitry Lyubimov [2]

[1] Institute of Continuous Media Mechanics of the Ural Branch of the Russian Academy of Sciences Perm, 614013 Perm, Russia
[2] Theoretical Physics Department, Perm State University, 614068 Perm, Russia
* Correspondence: lubimova@icmm.ru

Abstract: The influence of high-frequency vibrations on the shape of a compressible drop placed on an oscillating solid substrate is studied in this paper. Due to the significant difference in characteristic temporal scales, the average and pulsating motions of the drop can be considered separately. For nearly hemispherical drop, the solution to the problem of pulsating motion is found in the form of series in Legendre polynomials. Frequencies of natural sound oscillations of hemispherical axisymmetric drop are obtained. Resonances of the acoustic mode of drop oscillations are found. The problem of forced oscillations of hemispherical drop in the limit of weakly compressible liquid is considered. It is found that drop oscillation amplitude grows with vibration intensity according to quadratic law, which is consistent with the solution of the pulsation problem for finite compressibility assumption. A variational principle for calculation of average drop shape is formulated based on minimization of energy functional for the case, so the compressibility of the liquid should be taken into account. It is shown that the functional (the sum of the kinetic and potential energies of the pulsating flow, the kinetic energy of the averaged flow, and the surface tension energy of the drop) decreases and reaches a minimum value at quasi-equilibrium state, in which the average shape of the drop becomes static. The influence of vibrations on the drop shape is studied for small values of the vibrational parameter. The surface of the drop in the absence of vibrations is assumed to be hemispherical. Calculations showed that under vibrations, drop height decreases, while the area of the base increases.

Keywords: vibrations; resonance; natural frequencies; hemispherical drop; drop shape; Legendre polynomials; variational principle

MSC: 76M45; 76D45; 76N17

check for updates

Citation: Ivantsov, A.; Lyubimova, T.; Khilko, G.; Lyubimov, D. The Shape of a Compressible Drop on a Vibrating Solid Plate. *Mathematics* **2023**, *11*, 4527. https://doi.org/10.3390/math11214527

Academic Editors: Nikolay M. Zubarev, Evgeny A. Kochurin and James M. Buick

Received: 8 September 2023
Revised: 13 October 2023
Accepted: 31 October 2023
Published: 3 November 2023

1. Introduction

Vibrations are one of the promising ways of controlling inhomogeneous hydrodynamic systems. It is known that many technological processes can be significantly improved by proper vibration impact. On the other hand, vibrations are often an intractable accompanying phenomenon of technological processes, which also necessitates studying the effects they lead. Situations where gas and liquid phases exist in the state of liquid droplets are very common in technology; for instance, in the process of phase separation in containers with cryogenic fuel [1], in boiling processes [2], in prilling processes [3], etc.

Problems on a drop or bubble dynamics under high-frequency vibrations can be divided into two classes. In the first case, the vibration frequency is high enough to state that the vibration period is much smaller than viscous dissipation time and the fluid can be considered inviscid, but is comparable to the natural frequencies of the problem. Resonant phenomena of various nature arise in these systems caused by linear, nonlinear or parametric resonances. At vibration frequencies that are much larger compared to the first natural frequency of the system, the main modes of natural oscillations are not excited. At the same time, motion corresponding to high modes is usually very small-scale, and therefore it is suppressed by viscosity. Thus, in this case, the averaged effects play the main role. It is worth noting that for intermediate frequencies, both resonant and averaged effects can be important.

The influence of high-frequency vibrations on the dynamics of interfaces in hydrodynamic system is studied in [4]. It is shown that in all cases, the interface under vibrations is oriented perpendicularly to the vibration axis. On the other hand, vibrations also cause viscous fluid flow generation in inhomogeneous hydrodynamic systems. The generation of flows in a closed container is studied experimentally in [5] for the case of a solid walls and in [6] for the container with a deformable boundary. Vibrations can also suppress instability development at the interface [7].

Vibrations can induce a variety of behaviors in liquid drops and bubbles. At low vibration frequencies or amplitudes, drops remain pinned in place due to contact angle hysteresis. However, as the vibration frequency and amplitude increase, the drops can slide or climb [8], or exhibit subharmonic modes and stick–slip motion [9]. The direction of drop motion depends on the contact angle of the drop [8]. More hydrophilic drops do not climb, only slide [10]. Brunet et al. [8] found that drops on an inclined vibrating surface can move upward against gravity, linking this to drop deformation and symmetry breaking. The effects of micro-texture placed on the inclined plate on drop dynamics are studied experimentally and numerically by Xu et al. [11]. In [12] the effects of a dual-plate enclosure on droplet transport for the anisotropic ratchet conveyor system are studied, in which an asymmetric pattern of hydrophilic rungs is used to move droplets by means of vibration.

Influence of viscosity on the vibration-induced motion of a gas bubble near a solid wall was reported in [13]. It has been observed that with the increase in viscosity, the vibrational attraction to the wall diminishes, subsequently being replaced by repulsion. This phenomenon can be attributed to the influence of viscosity in the boundary layer near the rigid surface, where the average flow becomes less intense.

The interfacial dynamics of a sessile water drop were investigated experimentally in [14]. The drop went through a series of motions during which the force was gradually increased from zero, including small asymmetrical waves at the interface, small waves that were asymmetrical at the contact line, full waves that were both asymmetrical and asymmetrical, lattice waves, a disordered pre-ejection mode, and finally, droplets ejecting from individual wave cells.

An alternating electric field has a similar vibrational effect on a water drop due to the polar properties of the water molecules or an electric charge. The impact of electric charges on the behavior of single sessile drops is investigated experimentally by Löwe et al. [15]. In the work of [16], it is demonstrated that the effect of the electric field can be described by an effective condition at the contact line. The specific geometry of a drop's base also affects its oscillations. Tankovsky et al. [17] found that the boundary conditions at the drop's base depend on the shape and material of the base, impacting the resonant frequency of the drop. However, for hemispherical drops, the influence of the base can be calibrated out, making surface tension measurement possible.

Multiple papers have explored how the shape and material of the substrate under the drop affect its motion. In [18] the influence of the properties of the plate surface on the oscillations of the cramped drop is studied. The forced oscillations of a drop clamped between different surfaces is discussed in [19] for normal vibration and in [20] for translational vibration.

Boughzala et al. [21] found that polyhedral bubbles of different shapes oscillate at frequencies similar to spheres of the same volume. Vos et al. [22] used high-speed imaging to observe that microbubbles deform when oscillating near a wall, changing between oblate and prolate shapes. Geng et al. [23] studied how the aspect ratio of acoustically levitated drops affects their vertical oscillation frequency, and found that the frequency increases with the aspect ratio.

Lyubimov et al. [24] found that the natural oscillations of a hemispherical drop attenuate over time due to energy loss at the contact line between the drop and surface. The drop's oscillations can also be forced by vibrating the surface it is located on, resulting in resonant oscillations at certain frequencies. Shklyaev and Straube [25] studied how hemispherical bubbles oscillate on solid substrates, and identified resonant phenomena and interactions between shape and volume oscillations. They determined when bubble compressibility can be neglected and found that compressible hemispherical bubbles also demonstrate interactions between shape and volume oscillations, as well as a double resonance causing unbounded growth.

This work is devoted to studying the dynamics of a drop placed on a vibrating solid substrate, in the case in which the vibrational frequency is so high that drop compressibility should be taken into account. The influence of the compressibility of the drop on its averaged shape was not considered before, although the fluid compressibility can be important for various applications [26]. The paper is organized as follows. In Section 2, the mathematical statement of the problem is given, and the main assumptions used are formulated. Then, the equations of averaged and pulsating flow are derived. In Section 3, the variational principle is formulated. In Section 4, the solution of the pulsating flow is given for natural and forced oscillations. Finally, in Section 5, the averaged shape of the drop is found, and all results are outlined and discussed in the Conclusion.

2. Problem Statement

2.1. Governing Equations

We consider the behavior of a liquid drop placed on a planar vibrating solid substrate. Vibrations are normal to the substrate plane and are described by the harmonic law: $\vec{r}_s = \vec{r}_0 + a\vec{\gamma}\cos\omega t$, where $\vec{r}_s$ is radius vector of arbitrary point of substrate, ω and a are the vibration frequency and amplitude respectively, t is time, and $\vec{\gamma}$ is a unit vector directed along the z axis (the geometry of the problem and the coordinate axes is shown in Figure 1).

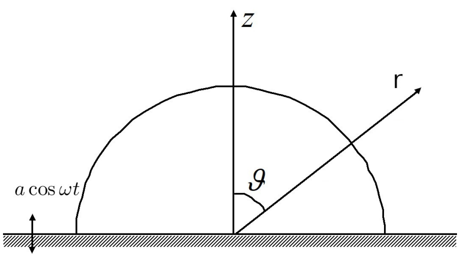

Figure 1. Geometry of the problem.

The vibration amplitude is assumed to be small compared to the average droplet radius. The drop is surrounded by a gaseous medium, the density of which is low; the effect of gravity is neglected. It is supposed that the thickness of the dynamic boundary layer formed near the solid surface is small compared to the equilibrium droplet radius

$$\omega \gg \frac{\nu}{R^2}, \tag{1}$$

where ν is the kinematic viscosity of the liquid and R is the average droplet radius. In addition, we will assume that vibration frequency is comparable to the frequency of acoustic oscillations of the drop (i.e., the length of the sound wave corresponding to the frequency ω is comparable to the size of the drop):

$$\omega \sim \frac{c}{R}. \tag{2}$$

Here, c is the speed of sound in the liquid. In this case, to describe the dynamics of the drop, it is necessary to take the compressibility of the liquid into account.

The governing equations describing oscillations of the drop in the frame of reference associated with the substrate have the following form:

$$\begin{aligned} &\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \vec{v} + \left(\frac{\xi}{\rho} + \frac{\nu}{3} \right) \nabla \mathrm{div} \vec{v} + a\omega^2 \vec{\gamma} \cos(\omega t), \\ &\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0. \end{aligned} \tag{3}$$

where p is the pressure, ρ is the density, $\vec{v}$ is the velocity and ξ is the bulk viscosity coefficient and $\vec{\gamma}$ is the unit vector directed upward. In the present study, we neglect the volume viscosity, assuming that the bulk viscosity coefficient is small.

The shape of the drop is given by

$$F(\vec{r}, t) = 0. \tag{4}$$

The dynamic boundary conditions on the free surface are

$$\begin{aligned} p - \sigma_{nn} &= \alpha K, \\ \sigma_{n\tau} &= 0, \end{aligned} \tag{5}$$

where $\sigma_{nn},\ \sigma_{n\tau}$ are normal and tangential components of the viscous stress tensor, K is the average curvature of the free surface, and α is the surface tension coefficient of the liquid. The pressure in the external gaseous environment is taken as the reference point for pressure. The kinematic boundary condition is also satisfied on the surface:

$$\frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F = 0. \tag{6}$$

On the solid boundary, the no-slip conditions are set

$$z = 0: \quad \vec{v} = 0. \tag{7}$$

Due to significant differences in characteristic dissipative and vibrational temporal scales, one could divide velocity, density, and pressure fields into averaged and fluctuating parts. We will use the method of multiple scales to achieve this.

$$\begin{aligned} p &= p_0(t) + q(\tau), \\ \rho &= \rho_0(t) + g(\tau), \\ \vec{v} &= \vec{u}(t) + \vec{w}(\tau). \end{aligned} \tag{8}$$

Here, $q,\ g,\ \vec{w}$ are the pulsating parts, for which characteristic time is comparable with the period of the substrate vibrations, τ is the fast time; $p,\ \rho_0\ ,\vec{u}$ are the averaged parts, that change slowly over vibration period. The relative changes in the density of the liquid in the drop caused by vibrations of the substrate are small, i.e., $g \ll \rho_0$. The interface can also be given by an equation containing a sum of two functions that vary slowly and rapidly with time

$$F = F_0(t) + f(\tau) = 0, \tag{9}$$

where F_0 describes its average position, f represents the deviation from the average value. Problem (3)–(7) using expressions (8) and (9) is divided into two, describing the pulsating and averaged dynamics of the system.

We will use a spherical coordinate system (r, ϑ, Ψ), where ϑ is the polar angle and r is the radial coordinate, as well as the Cartesian coordinate system (x, y, z), where z is axis normal to the substrate surface, as shown in Figure 1.

2.2. The Problem of Pulsating Motion

Splitting velocity and pressure fields into fluctuating and averaged parts is an effective method for describing the dynamics of an oscillating drop, since there are reasons to neglect nonlinear terms in the equation for the fluctuating component of motion. Indeed, let us compare, for example, the values of the first and second terms in (3). The pulsating velocity of the liquid is determined by the vibration intensity; therefore, its value has the same order of magnitude as $a\omega$. The characteristic time of the pulsation motion is equal to the period of the substrate vibration, i.e., order ω^{-1}. The order of spatial derivatives is R^{-1} outside the boundary layers. As a result, we find that the ratio of the nonlinear term to the first term is of order a/R

$$\left|\frac{\vec{w}\cdot\nabla\vec{w}}{\partial\vec{w}/\partial t}\right| \sim \frac{a}{R} \ll 1 \tag{10}$$

According to condition (1), the characteristic vibration temporal scale is much less than the hydrodynamic time, i.e., the thickness of the viscous skin layer $\sqrt{\nu/\omega}$ is small, which makes it possible to neglect the existence of the boundary layer. As a result, when considering pulsating motion, one can discard the dissipative term in the equation of motion (3) [27].

By removing from (3) all terms that do not depend on the fast time, we obtain the following equations for the pulsating motion:

$$\begin{aligned} &\frac{\partial\vec{w}}{\partial\tau} = -\frac{1}{\rho_0}\nabla q + a\omega^2\vec{\gamma}\cos(\omega t), \\ &\frac{\partial g}{\partial\tau} + \rho_0 \mathrm{div}\vec{w} = 0. \end{aligned} \tag{11}$$

Thus, the vibrational behavior of the drop is described by the equations of linear acoustics with an inhomogeneous term due to the force of inertia.

Boundary conditions (5), (6) for pulsating motion are written as follows:

$$\begin{aligned} F_0 = 0: \ &\frac{\partial f}{\partial\tau} + \vec{w}\cdot\nabla F_0 + \vec{u}\cdot\nabla f = 0, \\ &q = \alpha\tilde{K}. \end{aligned} \tag{12}$$

Here, $\tilde{K}$ is the fluctuating part of the surface curvature. Due to the smallness of the oscillation amplitude, the boundary conditions can be shifted from the moving free surface to the averaged surface of the drop. Obviously, the characteristic average velocity can be viewed as small, compared to the speed of sound, then relation (2) allows us to neglect the term $\vec{u}\cdot\nabla f$ in the kinematic boundary condition.

Since energy dissipation is neglected when solving the pulsating problem, the order of the pulsating motion equation is reduced. Consequently, the condition on the free surface for shear stresses can be discarded, and the no-slip condition (7) must be replaced by the impermeability condition:

$$z = 0: \quad w_n = 0. \tag{13}$$

Here, w_n is the component of the fluctuating part of the velocity normal to the substrate.

The system of Equation (11) is not closed; to solve the pulsating problem, it is necessary to add the equation of state of the system. Since irreversible dissipation processes are insignificant for pulsating motion, the adiabatic law can be used as follows.

$$g = \frac{q}{c^2}. \tag{14}$$

Let us rewrite the problem in dimensionless form, choosing R as a unit of distance, R/c as a unit of time, a as a unit of surface deviations from the mean, $a\omega$ as a unit of velocity, $\rho_0 a/R$ as a unit of density, $\rho_0 ac^2/R$ as a unit of pressure, and $\alpha/R^2 a\omega$ as a unit of drop shape curvature.

By rewriting Equations (11)–(14) and introducing the velocity potential $\vec{w} = \nabla\varphi$, we obtain the dimensionless form of the pulsating motion problem:

$$\begin{aligned} &k\frac{\partial\varphi}{\partial\tau} = -q + k^2 z\cos k\tau, \\ &\frac{\partial q}{\partial\tau} + k\Delta\varphi = 0, \\ &g = q, \end{aligned} \tag{15}$$

where $k = \omega R/c$ is a dimensionless wavenumber. Boundary conditions (12), (13) can be written as

$$\begin{aligned} z = 0 : \frac{\partial\varphi}{\partial z} &= 0, \\ F_0 = 0 : \frac{\partial f}{\partial\tau} &= -\nabla\varphi\cdot\nabla F_0, \\ q &= \beta\tilde{K}. \end{aligned} \tag{16}$$

where $\beta = \alpha/\left(\rho_0 c^2 R\right)$ is a dimensionless parameter characterizing the surface tension of the liquid. Since substrate vibration frequency is assumed to be comparable with the acoustic frequencies (12), for many liquids, the parameter β will be small. Indeed, for a water drop with a radius of 1 cm, $\beta \approx 10^{-5}$. The parameter β can be equal to one only if the radius of the drop is of the order of 10^{-7} m, and in this case, condition (1) is violated, i.e., the viscosity of the liquid cannot be neglected. Thus, we will assume

$$\beta \ll 1. \tag{17}$$

Surface Equation (9) in dimensionless variables has the following form:

$$F = F_0(t) + \varepsilon f(\tau) = 0, \tag{18}$$

where $\varepsilon = a/R$ is a dimensionless parameter characterizing the vibration amplitude of the substrate. As noted above, the vibration amplitude is assumed to be small; hence $\varepsilon \ll 1$.

Let us represent the pulsating pressure, the surface deviation and the velocity potential in the form:

$$q = \tilde{q}(\vec{r})\cos k\tau; \quad f = \tilde{f}(\vec{r})\cos k\tau; \quad \varphi = \tilde{\varphi}(\vec{r})\sin k\tau, \tag{19}$$

where $\tilde{q}(\vec{r})$, $\tilde{f}(\vec{r})$, $\tilde{\varphi}(\vec{r})$ are functions that do not depend on fast time. Substituting (19) into (16), we obtain a system of equations and boundary conditions for the amplitudes:

$$\begin{aligned} k^2\tilde{\varphi} &= -\tilde{q} + k^2 z, \\ -\tilde{q} + \Delta\tilde{\varphi} &= 0, \end{aligned} \tag{20}$$

$$\begin{aligned} z = 0 : \frac{\partial\tilde{\varphi}}{\partial z} &= 0, \\ F_0 = 0 : k^2\tilde{\varphi} - \beta\tilde{K} &= k^2 z, \\ \tilde{f} &= \nabla\tilde{\varphi}\cdot\nabla F_0. \end{aligned} \tag{21}$$

By eliminating the pulsating pressure from these equations, we obtain the wave equation for the velocity potential

$$\Delta\tilde{\varphi} + k^2\tilde{\varphi} = k^2 z. \tag{22}$$

The tilde sign over the amplitudes of pulsating pressure, velocity potential, etc., is omitted from this point forward.

2.3. The Averaged Motion Problem

Describing the slow average dynamics of the system, the compressibility of the fluid can be neglected, while the viscosity and surface tension must be taken into account . Substituting expansions (8) into (3)–(5) and removing terms that depend on fast time, we obtain the equation system describing the averaged dynamics of the drop. The equations of average motion in dimensional form are written as follows:

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \overline{\vec{w} \cdot \nabla \vec{w}} = -\left(\frac{\nabla p}{\rho_0} - \overline{\frac{g}{\rho_0^2} \nabla q} \right) + \nu \Delta \vec{u}, \\ &\mathrm{div} \vec{u} = 0, \end{aligned} \tag{23}$$

where the bar denotes the operation of averaging over period of substrate vibration. Note that neither pulsation motion equations nor averaged motion equations contain terms dependent on bulk viscosity.

Boundary conditions (12), (13) can be written as:

$$\begin{aligned} z = 0 &: \vec{u} = 0, \\ F_0 = 0 &: \frac{\partial F_0}{\partial t} + \vec{u} \cdot \nabla F_0 + \overline{\vec{w} \cdot \nabla f} = 0, \\ &\quad p + \bar{q}|_{F_0 + f} - \sigma_{nn} = \alpha \mathrm{div} \vec{n}, \\ &\quad \sigma_{n\tau} = 0, \end{aligned} \tag{24}$$

where $\vec{n}$ is the unit vector, normalized to the free surface of the drop. The dynamic boundary condition contains a term equal to the average of the fluctuating pressure $\bar{q}|_{F_0+f}$. When averaging this term, it is necessary to take fluctuating deviations of the droplet surface into account. The last term in the kinematic boundary condition vanishes on averaging, as the period average of $\sin 2\tau$.

Let us rewrite the problem (23) and (24) using dimensionless variables. For the average velocity $\vec{u}$, time t and pressure p, the following units are chosen: ν/R, R^2/ν, $\rho_0 a^2 \omega^2$. Thus, the averaged dynamics of the system is governed by

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + CaQ \overline{\vec{w} \cdot \nabla \vec{w}} = -CaQ \left(\frac{\nabla p}{\rho_0} - \overline{q \nabla q} \right) + \Delta \vec{u}, \\ &\mathrm{div} \vec{u} = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} z = 0 &: \vec{u} = 0, \\ F_0 = 0 &: \frac{\partial F_0}{\partial t} + \vec{u} \cdot \nabla F_0 = 0, \\ &\quad Qp + Q\bar{q}|_{F_0+f} - \frac{1}{Ca} \sigma_{nn} = \mathrm{div} \vec{n}, \\ &\quad \sigma_{n\tau} = 0. \end{aligned} \tag{26}$$

In the equation of motion (25), the fluctuating density of the liquid is excluded using the equation of state (14). Problems (25) and (26) contain two dimensionless parameters: $Q = \rho_0 a^2 \omega^2 R/\alpha$ is the vibration parameter, $Ca = \alpha R/\rho_0 \nu^2$ is the capillary number.

The expressions under the averaging sign in (25) can be transformed as follows:

$$\begin{aligned} &\vec{w} \cdot \nabla \vec{w} = \nabla \varphi \cdot \nabla \nabla \varphi = \frac{1}{2} \nabla (\nabla \varphi)^2, \\ &q \cdot \nabla q = \frac{1}{2} \nabla q^2. \end{aligned} \tag{27}$$

Substituting the expressions for the velocity potential and fluctuating pressure introduced in (19) and averaging over the fast time τ, we obtain:

$$\begin{aligned}\overline{\vec{w}\cdot\nabla\vec{w}} &= \frac{1}{2}\nabla(\nabla\tilde{\varphi})^2\overline{(\sin\tau)^2} = \frac{1}{4}\nabla(\nabla\tilde{\varphi})^2,\\ \overline{q\cdot\nabla q} &= \frac{1}{2}\nabla\tilde{q}^2\overline{(\cos\tau)^2} = \frac{1}{4}\nabla\tilde{q}^2.\end{aligned} \tag{28}$$

Let us discuss the averaging of the term $q|_{F_0+f}$ in the dynamic boundary condition. Due to the smallness of the oscillation amplitude $q|_{F_0+f}$ can be represented as an expansion in terms of the pulsation deviation of the surface from the mean position $\delta\vec{r}$. Up to the first order in $\delta\vec{r}$, the pulsating pressure has the form:

$$q|_{F_0+f} = q|_{F_0} + \delta\vec{r}\cdot\nabla q = q|_{F_0} + \delta r_n\nabla_n q, \tag{29}$$

where ∇_n is the gradient component normalized to the droplet surface (the tangent component $\delta\vec{r}$ does not cause the fluid element to leave the free surface and can be set equal to zero).

The function defining the interface can also be represented as an expansion in $\delta\vec{r}$

$$F_0(\vec{r}+\delta\vec{r}) = F_0(\vec{r}) + \delta\vec{r}\cdot\nabla F_0(\vec{r}) = F_0(\vec{r}) + \delta r_n\cdot\nabla_n F_0(\vec{r}). \tag{30}$$

Comparing (30) with Formula (9), we write the normal component of the deviation of the droplet surface from the mean position in the form:

$$\delta r_n = -\frac{f}{\nabla_n F_0}. \tag{31}$$

Then, expansion (29) is rewritten as follows:

$$q|_{F_0+f} = q|_{F_0} - \nabla_n q\frac{f}{\nabla_n F_0}. \tag{32}$$

Substituting the expressions and using the boundary condition for the pulsating deviation of the surface and the equation for the pulsating pressure (20), we obtain:

$$q|_{F_0+f} = q|_{F_0} + (\nabla_n\tilde{\varphi} - \gamma_n)\nabla_n\tilde{\varphi}(\sin\tau)^2, \tag{33}$$

where γ_n is the projection of the vector $\vec{\gamma}$ onto the normal $\vec{n}$ to the mean interface. Therefore, the desired average term is equal to

$$\overline{q|_{F_0+f}} = \frac{1}{2}(\nabla_n\tilde{\varphi} - \gamma_n)\nabla_n\tilde{\varphi}. \tag{34}$$

As a result, we obtain equations describing the averaged dynamics of the system:

$$\begin{aligned}&\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -QCa\nabla\left(p + \frac{1}{4}(\nabla\tilde{\varphi})^2 - \frac{k^2}{4}\tilde{q}^2\right) + \Delta\vec{u},\\ &\mathrm{div}\vec{u} = 0.\end{aligned} \tag{35}$$

The boundary conditions are

$$\begin{aligned}z = 0 &: \vec{u} = 0,\\ F_0 = 0 &: Q\left(p + \frac{1}{2}(\nabla_n\tilde{\varphi} - \gamma_n)\nabla_n\tilde{\varphi}\right) - \frac{1}{Ca}\sigma_{nn} + p_0 = \mathrm{div}\vec{n},\\ &\sigma_{n\tau} = 0,\end{aligned} \tag{36}$$

where p_0 is a constant characterizing the average overpressure in a drop. Equations (35) and (36) contain the pulsation velocity potential and the pulsation pressure; on the other hand, the pulsation fields are determined by the average shape of the drop. Thus, the fluctuating and averaged problems must be solved jointly.

Due to the presence of free surface and vibrations, there is no solution to the problem corresponding to the state of complete equilibrium (the absence of both averaged and pulsating motion). Therefore, it is reasonable to consider stationary solutions to the average motion problem, i.e., state of the system, called in [4] quasi-equilibrium. Quasi-equilibrium is a state in which the average flow velocity is equal to zero, while other average characteristics are stationary. If in (35) and (36), we set the velocity of the averaged motion equal to zero, we obtain a problem describing the state of quasi-equilibrium:

$$p = -\frac{1}{4}(\nabla\tilde{\varphi})^2 + \frac{k^2}{4}\tilde{q}^2, \tag{37}$$

$$F_0 = 0: \quad Q\left(\frac{1}{4}(\nabla_n\tilde{\varphi})^2 - \frac{1}{4}(\nabla_\tau\tilde{\varphi})^2 + \frac{1}{4}k^2\tilde{q}^2 - \frac{1}{2}\gamma_n\nabla_n\tilde{\varphi}\right) + p_0 = \operatorname{div}\vec{n}, \tag{38}$$

where ∇_τ is the gradient component tangential to the droplet surface. Obviously, $(\nabla\varphi)^2 = (\nabla_n\varphi)^2 + (\nabla_\tau\varphi)^2$. The resulting equations do not contain a capillary number, because this parameter characterizes the effect of fluid viscosity, i.e., the importance for the average dynamics. The capillary number determines the rate at which the system reaches the quasi-equilibrium state, and not this state itself.

It is worth noting that the vibrational field can generate averaged flows in viscous boundary layers near solid bodies or free surfaces. In this case, the described state of quasi-equilibrium is not realized. However, taking into account the accepted assumption about the thickness of the skin layer (1), it is necessary in this case to consider such flows as weak, and thus incapable of affecting the behavior of the drop.

3. Variational Principle for Compressible Drop

The quasi-equilibrium state could be studied from the point of view of the total energy of the system. In [27], a variational principle was obtained for the case of both homogeneous and arbitrary vibrations of an incompressible fluid. It is shown that the stable state of quasi-equilibrium corresponds to the minimum of the functional equal to the total energy of the system. Examples of the effective application of the variational principle to various problems of determining the equilibrium shape of a free surface are given. The advantage of this method is that the study of the nature of the energy extremum allows one to automatically reject unstable solutions. In this section, we consider the possibility of using this approach to describe hydrodynamic systems in which fluid compressibility is important.

Multiplying the equations for the averaged motion (35) by the averaged flow velocity $\vec{u}$ and integrating them over the drop volume, we obtain

$$\left\langle \frac{\partial u_j}{\partial t} u_j \right\rangle + \langle u_j u_i \nabla_i u_j \rangle = -QCa\langle u_j \nabla_j \Pi \rangle + \langle u_j \Delta u_j \rangle, \tag{39}$$

where

$$\Pi = p + \frac{1}{4}(\nabla\varphi)^2 - \frac{k^2}{4}q^2. \tag{40}$$

Summation over doubly repeating indices is assumed. The angle brackets show the operation of integration over the droplet volume

$$\langle \ldots \rangle = \int dV. \tag{41}$$

Here, and below, the tildes above the amplitudes of the pulsating pressure and the velocity potential are omitted.

The first term of Equation (39) can be written

$$\int \frac{\partial \vec{u}}{\partial t} \vec{u} dV = \frac{1}{2} \int \frac{\partial}{\partial t} \vec{u}^2 dV. \tag{42}$$

Let us take out the operation of differentiation with respect to time from the integral, taking into account the dependence of the volume element dV near the surface of the drop on time:

$$dV = u_n dt dS, \tag{43}$$

where u_n is the velocity component normal to the droplet surface. Thus, we obtain

$$\int \frac{\partial \vec{u}}{\partial t} \vec{u} dV = \frac{1}{2} \frac{\partial}{\partial t} \int \vec{u}^2 dV - \frac{1}{2} \oint u^2 u_n dS. \tag{44}$$

The first term of expression (44) can be rewritten by introducing the kinetic energy of the mean motion:

$$K = \frac{1}{2} \int u^2 dV. \tag{45}$$

In the second term of Equation (39), the integrand is represented as a divergence from $\vec{u}\ u^2$. When transferring the differentiation operator, no additional terms appear, since the divergence of the mean flow velocity is zero. According to the Ostrogradsky–Gauss formula, the integral over the drop volume is reduced to the integral over the surface:

$$\langle u_j u_i \nabla_i u_j \rangle = \frac{1}{2} \langle u_i \nabla_i u_j u_j \rangle = \frac{1}{2} \left\langle \mathrm{div}(\vec{u}\ u^2) \right\rangle = \frac{1}{2} \oint u^2 u_n dS. \tag{46}$$

When adding the obtained expressions (44), (46), the surface integrals cancel out. As a result, the left side of the equation of motion (39) is equal to the derivative of the kinetic energy of the mean motion:

$$\left\langle \frac{\partial u_j}{\partial t} u_j \right\rangle + \langle u_j u_i \nabla_i u_j \rangle = \frac{dK}{dt}. \tag{47}$$

Let us proceed to the integration of the right side of Equation (39). Consider first the last term of this equation. In this case, a term equal to the divergence from some function can also be separated from the integrand. Then, according to the Ostrogradsky–Gauss formula, the corresponding volume integral is transformed into surface integral. Thus, we obtain:

$$\langle u_j \Delta u_j \rangle = \left\langle u_j \frac{\partial^2}{\partial x_i^2} u_j \right\rangle = \oint u_j \frac{\partial u_j}{\partial x_i} dS_i - \left\langle \left(\frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle. \tag{48}$$

The volume integral in the resulting relation can be written as:

$$\left\langle \left(\frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle = \frac{1}{2} \left\langle {\sigma_{ij}}^2 \right\rangle - \oint u_j \frac{\partial u_i}{\partial x_j} dS_i, \tag{49}$$

where $\sigma_{ij} = \partial u_i / \partial x_j + \partial u_j / \partial x_i$ is the dimensionless viscous stress tensor. Indeed, by extracting the surface integral from $\left\langle {\sigma_{ij}}^2 \right\rangle$, we obtain Formula (49):

$$\begin{aligned} \left\langle {\sigma_{ij}}^2 \right\rangle &= 2\left\langle \left(\frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle + 2\left\langle \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \right\rangle = 2\left\langle \left(\frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle + 2\left\langle \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_i}{\partial x_j} \right) \right\rangle \\ &= 2\left\langle \left(\frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle + 2 \oint u_j \frac{\partial u_i}{\partial x_j} dS_i \end{aligned} \tag{50}$$

By substituting expression (49) into (48) and combining the surface integrals, we obtain the term under consideration:

$$\langle u_j \Delta u_j \rangle = -\frac{1}{2}\left\langle {\sigma_{ij}}^2 \right\rangle + \oint u_j \sigma_{ji} dS_i. \quad (51)$$

On the substrate, the condition of equality to zero of the average velocity component is satisfied; therefore, the integral over the base of the drop is equal to zero, i.e., the surface integral reduces to the integral over the free surface of the drop. By taking the condition of zero shear stresses (36) on the free surface into account, we obtain:

$$\langle u_j \Delta u_j \rangle = -\frac{1}{2}\left\langle {\sigma_{ij}}^2 \right\rangle + \int\limits_{S_{free}} u_n \sigma_{nn} dS \quad (52)$$

Let us calculate the remaining term of Equation (39). Due to the non-compressibility of the liquid and the no-slip condition on the substrate, the volume integral also reduces to the free surface integral:

$$\langle u_j \nabla_j \Pi \rangle = \langle \mathrm{div} \Pi \vec{u} \rangle = \int\limits_{S_{free}} u_n \left(p + \frac{1}{4}(\nabla \varphi)^2 - \frac{k^2}{4} q^2 \right) dS. \quad (53)$$

The average pressure p on the drop surface can be expressed from the boundary condition (36). Eliminating pressure in this way from expression (53), we obtain:

$$\begin{aligned} \langle u_j \nabla_j \Pi \rangle = \int\limits_{S_{free}} u_n \bigg(& \frac{1}{Q} \mathrm{div} \vec{n} + \frac{1}{CaQ} \sigma_{nn} \\ & - \frac{1}{2}\left[(\nabla_n \varphi)^2 - \gamma_n \nabla_n \varphi \right] + \frac{1}{4}\left[(\nabla \varphi)^2 + k^2 q \right] \bigg) dS. \end{aligned} \quad (54)$$

The product $dr_n \mathrm{div} \vec{n}\, dS$ is the area increment of the surface element dS, therefore:

$$\int\limits_{S_{free}} u_n \mathrm{div} \vec{n}\, dS = \int\limits_{S_{free}} \frac{dr_n}{dt} \mathrm{div} \vec{n}\, dS = \frac{dS}{dt}, \quad (55)$$

where S is the free surface area. The second term of expression (54) cancels out with the surface integral obtained in (52) when they are substituted into the original Equation (39). Thus, the right side of this equation can be written as:

$$\begin{aligned} & - CaQ \langle u_j \nabla_j \Pi \rangle + \langle u_j \Delta u_j \rangle \\ & = -\frac{1}{2}\left\langle {\sigma_{ij}}^2 \right\rangle - Ca \frac{dS}{dt} - \frac{CaQ}{4} \int\limits_{S_{free}} u_n \Big((\nabla_\tau \varphi)^2 + 2\gamma_n \nabla_n \varphi - (\nabla_n \varphi)^2 + k^2 q \Big) dS. \end{aligned} \quad (56)$$

Let us introduce the kinetic energy of the pulsating motion, K_p, and the potential energy of the pulsating motion, U_p, as follows:

$$K_p = \frac{1}{4} \int (\nabla \varphi)^2 dV, \quad (57)$$

$$U_p = \frac{1}{4} \int k^2 q^2 dV. \quad (58)$$

Consider the time derivative of the potential energy of pulsations. As noted above, when differentiating the volume integral, it is necessary to take into account the deformation of the droplet surface. As a result, the derivative of the potential energy is equal to the sum of the integral over the volume and the surface integral:

$$\frac{dU_p}{dt} = \frac{1}{4}\frac{d}{dt}\int k^2 q^2 dV = \frac{1}{2}k^2 \int q \frac{\partial q}{\partial t} dV + \frac{1}{4}\int\limits_{S_{free}} k^2 q^2 u_n dS. \quad (59)$$

The resulting integral over the free surface of the drop is equal to the last term in expression (56). Similarly, the derivative of the kinetic energy of the pulsating motion is transformed:

$$\frac{dK_p}{dt} = \frac{1}{4}\frac{d}{dt}\int (\nabla\varphi)^2 dV = \frac{1}{2}\int \nabla\varphi \frac{\partial \nabla\varphi}{\partial t} dV + \frac{1}{4}\int\limits_{S_{free}} u_n (\nabla\varphi)^2 dS. \quad (60)$$

In the expression obtained from the volume integral, using the Ostrogradsky–Gauss formula, one can also extract the surface integral. Taking out the differentiation operation, we obtain:

$$\int \nabla\varphi \frac{\partial \nabla\varphi}{\partial t} dV = \int \mathrm{div}\left(\nabla\varphi \frac{\partial\varphi}{\partial t}\right) dV - \int \Delta\varphi \frac{\partial\varphi}{\partial t} dV. \quad (61)$$

The normal velocity component on the substrate is zero (21); therefore, the first integral of relation (61) is reduced to the integral over the free surface of the drop. The second term on the right side of expression (61) using Formulae (20) is transformed to the integral obtained in (59). Thus, the expression in question will be written in the form:

$$\int \nabla\varphi \frac{\partial \nabla\varphi}{\partial t} dV = \int\limits_{S_{free}} \nabla_n \varphi \frac{\partial\varphi}{\partial t} dS + \int k^2 q \frac{\partial q}{\partial t} dV. \quad (62)$$

Transforming the convective (substantial) derivative, we obtain:

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + u_n \nabla_n \varphi. \quad (63)$$

On the other hand, from the condition on the free surface of the drop for the velocity potential of pulsating motion (20), it follows that

$$\frac{d\varphi}{dt} = \frac{dz}{dt} = \frac{d(\vec{\gamma}\cdot\vec{r})}{dt} = \vec{\gamma}\cdot\frac{d\vec{r}}{dt} = \gamma_n u_n. \quad (64)$$

Thus, the derivative of the velocity potential will be written as:

$$\frac{\partial\varphi}{\partial t} = \gamma_n u_n - u_n \nabla_n \varphi. \quad (65)$$

Substituting relations (65), (62) into (60), we write the derivative of the kinetic energy of pulsation motion as follows:

$$\frac{dK_p}{dt} = \frac{1}{4}\int\limits_{S_{free}} u_n \Big((\nabla_n\varphi)^2 - 2\gamma_n \nabla_n\varphi - (\nabla_\tau\varphi)^2\Big) dS + \frac{1}{2}\int k^2 q \frac{\partial\varphi}{\partial t} dV. \quad (66)$$

Comparing the obtained expressions for potential (60) and kinetic (66) energies with Formula (56), we determine that the surface integral in (56) is equal to the derivative of the energy difference. Therefore, expression (56) will be rewritten in the form:

$$-CaQ\langle u_j \nabla_j \Pi\rangle + \langle u_j \Delta u_j\rangle = -\frac{1}{2}\Big\langle {\sigma_{ij}}^2 \Big\rangle - Ca\frac{dS}{dt} - QCa\frac{d}{dt}(K_p - U_p). \quad (67)$$

Substituting the expressions (47) and (67) into the original Equation (39), we obtain

$$\frac{d}{dt}\left[K + Ca\left(S + Q\left(K_p - U_p\right)\right)\right] = -\frac{1}{2}\left\langle\sigma_{ij}^2\right\rangle. \quad (68)$$

At non-zero average motion velocities, the right side of Equation (68) is negative; therefore, the value $K + Ca(S + Q(K_p - U_p))$ (as can be shown, limited from below) decreases when the system passes to a quasi-equilibrium state. In quasi-equilibrium, the kinetic energy of the mean motion is equal to zero; therefore, in a stable state of quasi-equilibrium, the functional

$$F = S + Q(K_p - U_p) \quad (69)$$

should be minimal. If the vibration frequency is significantly lower than the acoustic frequency, then compressibility of the liquid should be neglected. In this case, the potential energy of the pulsating motion is equal to zero. As a result, we obtain the variational principle formulated in [27]. By choosing such a shape of the average surface, so that this functional is equal to the minimum value, it is possible to determine the averaged shape of a drop in a state of quasi-equilibrium. In this case, to calculate the kinetic and potential energies of "fast" motion, it is necessary to solve a pulsating problem with a given average shape of the droplet surface.

4. Pulsating Motion of a Hemispherical Drop in an Acoustic Field

4.1. Natural Oscillations

Let us analyze the natural oscillations of compressible hemispherical drop, i.e., oscillations occurring in the absence of variable external forces and due to the compressibility of the liquid and surface tension. Discarding the terms in problem (20), (21) that describe the effect of vibrations in the substrate and excluding f, we obtain

$$\Delta\varphi + k^2\varphi = 0, \quad (70)$$

$$\vartheta = \frac{\pi}{2}: \ \frac{\partial\varphi}{\partial\vartheta} = 0, \quad (71)$$

$$r = 1: \ k^2\varphi + \beta(\Delta_\vartheta + 2)\frac{\partial\varphi}{\partial r} = 0, \quad (72)$$

where k is the natural frequency.

An axisymmetric solution of Equation (70) bounded at the origin and satisfying the boundary conditions on a solid surface has the form

$$\varphi = \sum_{n=0}^{\infty} C_n j_{2n}(kr) P_{2n}(\cos\vartheta), \quad (73)$$

where $j_{2n}(kr)$ are spherical Bessel functions of the first kind, $P_{2n}(\cos\vartheta)$ are Legendre polynomials of even order. Note that each term in (73) satisfies condition (71), so an arbitrary term of the series is an eigenfunction of the problem and can be considered separately.

The condition (72) on the free boundary gives the dispersion relation

$$kj_{2n}(k) - (2n-1)(2n+2)\beta j_{2n}'(k) = 0. \quad (74)$$

Here, the dash denotes the derivative with respect to the argument. Let us analyze this condition for small values of β. For finite values of the meridional number n, the second term can be discarded. As a result, we obtain an equation that determines the sequence of natural wavenumbers k_{nl}, and thus the natural frequencies of the acoustic mode,

$$j_{2n}(k_{nl}) = 0. \quad (75)$$

For a solution independent of the angle ϑ ($n = 0$), the Bessel function has the form $j_0(x) = \sin x/x$. The eigenfrequencies corresponding to this mode are equal to the roots of the equation $\sin x = 0$. Solutions to Equation (75) for higher n are given in Table 1. Accounting for surface tension gives corrections of the order of β to natural frequencies determined by (75).

Table 1. The complex wavenumber eigenvalues of sound oscillations, corresponding to the main modes.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
k_{n1}	3.1416	5.7635	8.1826	10.513	12.791
k_{n2}	6.2832	9.0950	11.705	14.2074	16.641
k_{n3}	9.4248	12.323	15.040	17.6480	20.183

The spectrum could also be obtained in another limiting case, corresponding to higher harmonics of the shape oscillations. For $\beta n^3 \sim 1$, the second term in (74) cannot be neglected. For large values of n and finite k, the spherical Bessel function satisfies the power asymptotics:

$$j_{2n}(k) = \frac{k^{2n}}{(4n+1)!!}\Big(1 + O\Big(k^2 n^{-1}\Big)\Big),\quad j'_{2n}(k) \approx 2nk^{-1} j_{2n}(k). \tag{76}$$

Then, Equation (74) gives the natural oscillation frequencies of the shape of a spherical drop of an incompressible liquid (even modes).

$$k^2 = 2n(2n-1)(2n+2)\beta \approx (2n)^3\beta. \tag{77}$$

As shown, for high values of the meridional number, the compressibility of the liquid does not play a significant role. Indeed, in this case, the characteristic length at which the flow changes significantly is R/n, which is much smaller than the sound wavelength (to the order of R for finite values of k).

4.2. The Solution of the Pulsation Motion Problem

Due to the symmetry of the problem, the shape of the oscillating drop can be considered axisymmetric. In this case, the solution of the equation system (20) and (21) has the form:

$$\varphi(r,\vartheta) = \sum_{n=0}^{\infty} \frac{\alpha_n}{k} P_{2n}(\cos\vartheta)[C_n j_{2n}(kr) + w_n(kr)], \tag{78}$$

where $w_n(s)$ is the solution of the inhomogeneous equation

$$\frac{d^2 w_n}{ds^2} + \frac{2}{s}\frac{dw_n}{ds} + \left[1 - \frac{2n(2n+1)}{s^2}\right] w_n = s. \tag{79}$$

In Formula (78) α_n, the expansion coefficients of s in even Legendre polynomials for $0 \le s \le 1$:

$$s = \sum_{n=0}^{\infty} \alpha_n P_{2n}(s), \quad \alpha_n = -\frac{(4n+1)P_{2n}(0)}{(2n-1)(2n+2)}. \tag{80}$$

The solution of Equation (79) limited at the origin can be written as follows:

$$w_n = s + \frac{q_n h_{2n}(s)}{(4n-1)!!} + \sum_{m=0}^{n} \frac{q_m}{s^{2m+1}}, \tag{81}$$

where $h_{2n}(kr)$ are spherical Bessel functions of the second kind. The coefficients q_k are calculated using the formulae

$$q_0 = (2n-1)(2n+2), \quad q_k = q_{k-1}[2n(2n+1) - (2k-2)(2k-1)]. \tag{82}$$

It is easy to see that, up to the factor $-(4n-1)!!/q_n$, the series in (81) gives the main part of the expansion h_{2n}. Based on the condition on the free surface, we have

$$C_n = -\frac{kw_n(k) - \beta(2n-1)(2n+2)w'_n(k) - k^2}{kj_{2n}(k) - \beta(2n-1)(2n+2)j'_{2n}(k)}. \tag{83}$$

According to the well-known solution for the velocity potential, using the boundary condition (21), it is easy to find the fluctuating deviation of the surface based on its average position.

4.3. Weakly Compressible Liquid Approximation

The summation of series (78) was carried out using the mathematical package Maple. In this case, series (78) were replaced by series with a finite but sufficiently large upper limit N. Comparison of the results obtained for different values of N shows that it is necessary to calculate at least 30–40 terms to ensure an accuracy of about 1%. The calculations were carried out at $N = 50$. Test calculations showed that series (78) near the top of the drop $(\vartheta = 0)$ is sign-changing, and, despite the fact that it converges, the absolute values of the terms decrease quite slowly. In view of the need to calculate the spherical Bessel functions of a sufficiently high order, the calculations were carried out with up to sixty significant digits.

To test the series summation algorithm (78), the calculation results were compared with the solution obtained in the limit of small k. For low substrate vibration frequencies, the solution to problem (20), (21) can be sought in the form of power expansions in the small parameter k^2:

$$\varphi = \varphi_0 + k^2\varphi_1 + \ldots \tag{84}$$

In order zero, we obtain the problem of oscillations of an incompressible liquid drop:

$$\Delta\varphi_0 = 0, \tag{85}$$

$$\vartheta = \frac{\pi}{2} : \ \frac{\partial\varphi_0}{\partial\vartheta} = 0, \tag{86}$$

$$r = 1 : \ f_0 = \frac{\partial\varphi_0}{\partial r}, \quad \varphi_0 + \tilde{\beta}(\Delta_\vartheta + 2)\frac{\partial\varphi_0}{\partial r} = \cos\vartheta. \tag{87}$$

Here, a new dimensionless parameter is introduced, which is equal to the square of the ratio of the natural frequency of the shape oscillations to substrate vibration frequency, $\tilde{\beta} = \beta/k^2 = \alpha/\left(\rho_0 R^3 \omega^2\right)$. Obviously, in the case of a weakly compressible fluid, $\tilde{\beta}$ must be finite. The solution of Laplace Equation (85), which satisfies boundary conditions (86) and (87), has form:

$$\varphi_0 = \sum_{n=0}^{\infty} A_n r^{2n} P_{2n}(\cos\vartheta). \tag{88}$$

The expansion coefficients A_n are determined based on the dynamic boundary condition. The solution to the zero order problem has the form

$$\varphi_0 = \sum_{n=0}^{\infty} \alpha_n \frac{P_{2n}(\cos\vartheta)}{(1-\Omega_n^2\tilde{\beta})} r^{2n}, \tag{89}$$

where $\Omega_n^2 = 2n(2n-1)(2n+2)$. A similar result was obtained in [27] for the case of a fixed contact angle.

In the first order, we obtain an inhomogeneous problem:

$$\Delta\varphi_1 = -\varphi_0 - r\cos\vartheta, \tag{90}$$

$$\vartheta = \frac{\pi}{2}: \ \frac{\partial\varphi_1}{\partial\vartheta} = 0, \tag{91}$$

$$r = 1: \ f_1 = \frac{\partial\varphi_1}{\partial r}, \quad \varphi_1 + \tilde{\beta}(\Delta_\vartheta + 2)\frac{\partial\varphi_1}{\partial r} = \cos\vartheta. \tag{92}$$

We write the solution of the first-order problem in the form

$$\varphi_0 = \sum_{n=0}^{\infty} a_n r^{2n} P_{2n}(\cos\vartheta) + \sum_{n=0}^{\infty} b_n r^{2n+2} P_{2n}(\cos\vartheta) + r^3 \sum_{n=0}^{\infty} c_n P_{2n}(\cos\vartheta). \tag{93}$$

Here, the first term is the solution of the homogeneous equation, the second and third are the solutions of the inhomogeneous equations $\Delta\varphi_1 = -\varphi_0$ and $\Delta\varphi_1 = -r\cos\vartheta$. The expansion coefficients, as in the zero-order problem, are determined based on the boundary conditions. Thus, the surface perturbation for a weakly compressible fluid can be calculated using the formula:

$$f = -\sum_{n=0}^{\infty} \zeta_n P_{2n}(\cos\vartheta), \tag{94}$$

where

$$\zeta_n = \frac{\alpha_n}{(1 - \Omega_n^2\tilde{\beta})}\left[2n + k^2\left(\frac{1}{2n+4} - \frac{1}{(4n+3)(1-\Omega_n^2\tilde{\beta})}\right)\right].$$

An increase in the amplitude of oscillations of a weakly compressible drop with an increase in the frequency of vibrations occurs according to quadratic law.

Figure 2 shows the dependence of the drop volume oscillation amplitude on the substrate vibration frequency. It can be seen that Formula (94) describes the surface deviation at $k < 0.6$. Thus, the calculations carried out for a small value of the parameter k are in good agreement with the results obtained in the limit of a weakly compressible fluid.

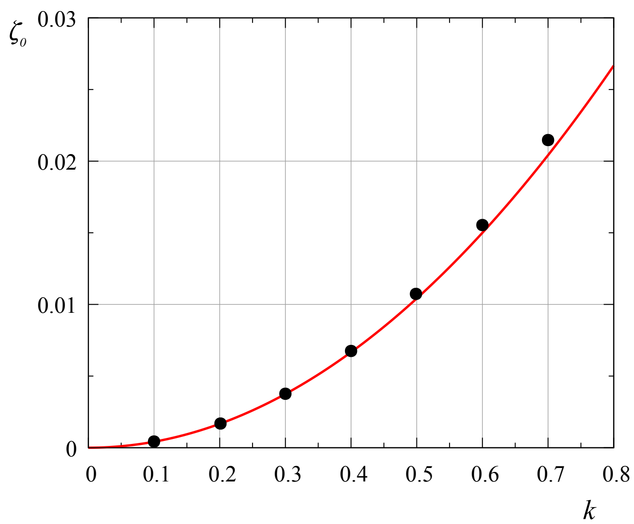

Figure 2. The dependence of the drop volume oscillation amplitude on the frequency at $\tilde{\beta} = 0.001$. The line shows the results obtained in the limit of a weakly compressible fluid (94). The points indicate the solution to finite k (78).

4.4. Drop Oscillations without Capillary Forces

We begin the study of the pulsating motion of a drop by analyzing the situation when the frequency of the substrate vibrations is high compared to the natural oscillation frequencies of the drop shape: $\omega^2 \gg \alpha/(\rho R^3)$ $(\beta \ll 1)$. In this case, the effect of surface tension can be neglected. The condition for the balance of normal stresses (21) on the free surface can be written simply:

$$r = 1: \ \varphi = \cos\vartheta. \tag{95}$$

The solution of the equation system (20), (21), (95) has the form

$$\varphi(r,\vartheta) = \sum_{n=0}^{\infty} \frac{\alpha_n}{k} P_{2n}(\cos\vartheta)\left[\frac{w_n(k)-k}{j_{2n}(k)} j_{2n}(kr) + w_n(kr)\right]. \tag{96}$$

The obtained isolines of the velocity potential are shown in Figure 3. As k approaches the first eigenvalue k_{01}, the radial component of the velocity increases and the drop volume oscillation amplitude increases. The resonance of the acoustic mode is observed: if the substrate vibration frequency coincides with the acoustic oscillation frequencies of the drop, the amplitude of the surface deflection increases indefinitely.

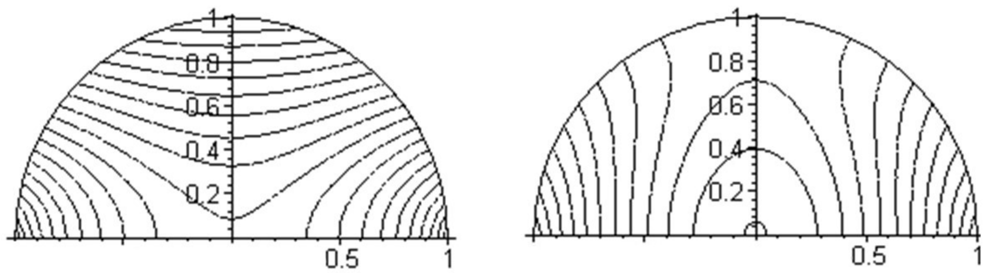

Figure 3. Isolines of the velocity potential at $k = 1$ (**left**), $k = 2.5$ (**right**). The isolines in the left figure are drawn through 0.05. On the right, the step of the isolines is 0.1.

With a further increase of k, resonance occurs with higher acoustic modes. Figure 4 shows resonances at frequencies $k_{01},\ k_{11},\ k_{21}$ (see Table 1). The unlimited growth of the amplitude is explained by the absence of dissipation in the system. It is shown that there are no resonances corresponding to wavenumbers of natural frequencies k_{0s}, where s is an even number.

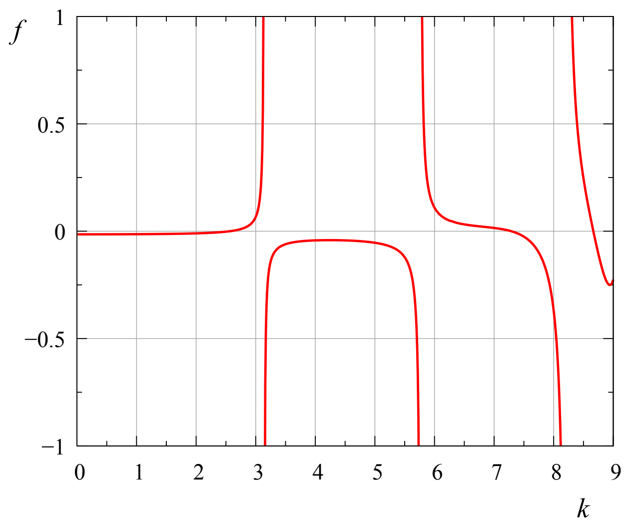

Figure 4. The dependence of the amplitude of oscillations of the top point of the droplet surface (see (19)) on the wavenumber, $\varepsilon = 0.02$.

Figure 5 shows the maximum deviations of the surface in terms of the vibration period at a vibration amplitude equal to 0.1 of the drop radius. The dashed–dotted line corresponds to the moment when the substrate is in the uppermost position; the solid line shows the shape of the droplet half vibration period after that.

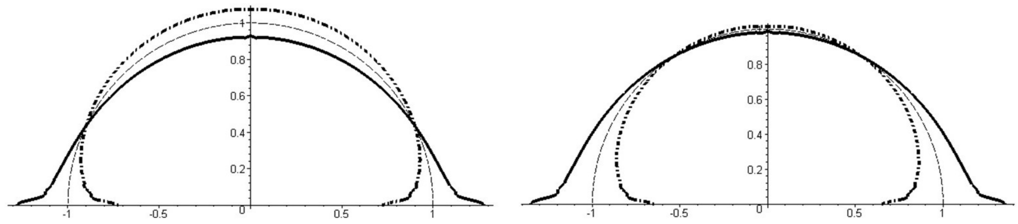

Figure 5. Drop oscillations at $k = 1$ (**left**) and $k = 2.5$ (**right**). The dashed line shows the average (hemispherical) shape of the drop; the solid and dash–dotted lines show the maximum surface deviations. The results are given for dimensionless vibrational amplitude $\varepsilon = 0.1$.

Near the contact line, the deviation of the surface and, accordingly, the normal component of the velocity of the pulsating flow tend to infinity, i.e., series (78) diverges. This result is explained by the contradiction of the boundary conditions: on the free surface, the velocity potential increases linearly with an increase in the coordinate normal to the substrate, and on the basis of the drop, the normal derivative of the potential is equal to zero, i.e., on the contact line $\partial\varphi/\partial\vartheta$ breaks. It can be shown that, in the limit $r \to 1$, $\vartheta \to \pi/2$, the fluctuation deviation of the surface is

$$f = \frac{2}{\pi} \ln\left(\frac{\pi}{2} - \vartheta\right). \tag{97}$$

Thus, near the contact line, the singularity for the deviation of the surface f is logarithmic, i.e., integrable. It should also be noted that the velocity potential always remains finite.

4.5. Forced Oscillations with Surface Tension Effect

In the presence of surface forces, series (78) converges for any ϑ. Figure 6 shows the drop shapes obtained as a result of calculations. Surface deviations correspond to the moments in time when the substrate is in the uppermost and lowermost positions. Accounting for surface tension, as noted above, leads to the appearance of an additional set of eigenfrequencies of oscillations of the drop shape. However, since the parameter β is small, the frequency of vibrations of the substrate is high compared to the first (lowest) frequencies of shape oscillations, i.e., only high oscillation modes of the drop shape can be excited in a resonant way.

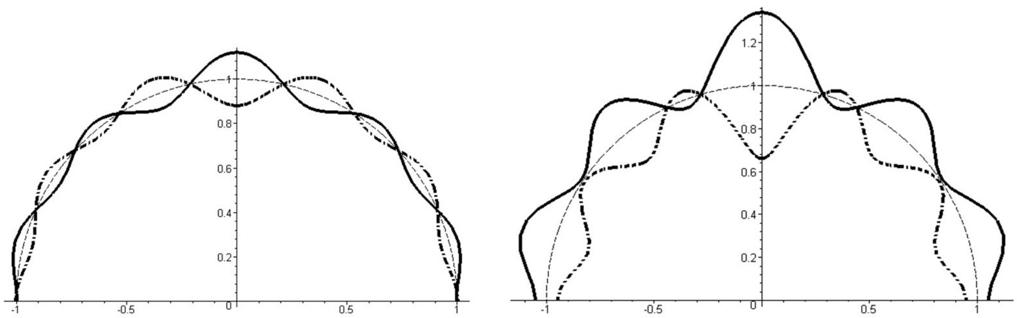

Figure 6. Drop oscillations with surface tension forces for $k = 1$, $\beta = 0.001$ (**left**) and $k = 3$, $\beta = 0.009$ (**right**). The results are given for dimensionless vibrational amplitude $\varepsilon = 0.02$. The dashed line shows the average (semispherical) shape of the drop; the solid and dash–dotted lines show the maximum deviations of the surface.

In this case, at $\beta = 0.001$, the strongest response is observed in the fifth even mode of capillary oscillations, corresponding to P_{10} (see (77)). As a result, the shape of the resulting droplet surface looks much more complicated than the previously considered solution to the problem without taking surface effects into account. The solutions obtained for the velocity potential are shown in Figure 7. As can be seen, the potential fields calculated without taking into account and with taking into account surface tension are qualitatively similar. However, in the latter case, small-scale flows arise near the surface, corresponding to resonant excitation of shape oscillations. As the wavenumber k approaches k_{01}, the radial velocity component also increases.

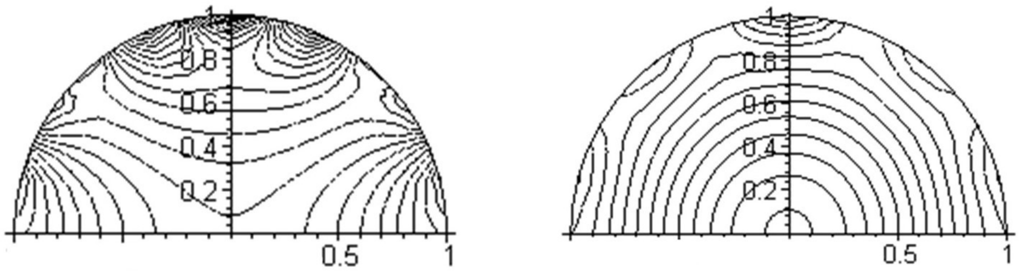

Figure 7. Isolines of the velocity potential for a hemispherical drop for $k = 1$, $\beta = 0.001$ (**left**) and $k = 3$, $\beta = 0.009$ (**right**). Step of isolines on the left 0.05, on the right 0.2.

The dependence of the drop oscillation amplitude on the parameter k is determined. Similar to the solution found without taking into account surface tension, when the substrate vibration frequency coincides with the acoustic oscillation frequencies of the drop, resonance is observed. In addition, new resonances appear, corresponding to shape oscillation. In Figure 8 in addition to two resonances of the acoustic mode (at $k = 3.14$; 5.76), five shape resonances are visible. In the given example, the fourth resonance, corresponding to shape oscillation, and the first acoustic resonance occur at fairly close frequencies.

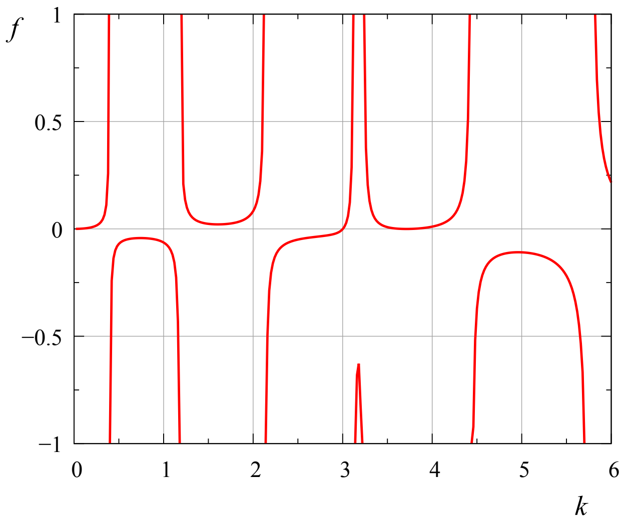

Figure 8. The dependence of the amplitude of oscillations of the top point of the drop (see (19)) on frequency at $\beta = 0.02$, $\varepsilon = 0.02$.

As was mentioned above (see discussion of (17)), β can be considered small. Let us analyze solution (78) in the limit of small β. For finite values of n, the term containing β can be neglected (see Section 4.4).

For sufficiently large values of the meridional number n $\left(\beta n^3 \sim 1\right)$, surface tension can play a decisive role, and compressibility effects can be neglected. Then, by expanding the spherical Bessel functions and neglecting the particular solution of the inhomogeneous equation w_n, we obtain

$$c_n = \frac{C_n k^{2n-1}}{(4n+1)!!} = \frac{k^2}{\beta\Omega_n^2 - k^2} \approx -\frac{N^3}{(N+\mu-n)(N^2+n^2+Nn)}. \quad (98)$$

Here, the notation $k^2/\beta = 8(N+\mu)^3$, $2N$ is the number of the resonant Rayleigh mode for the given substrate vibration frequency, μ ($|\mu| \leq 1/2$) characterizes the difference between frequency and resonance. The parameter μ can be neglected compared to N; large n $\Omega_n^2 \approx 8n^3$ is also considered approximately.

However, series (96) will also be cut off by viscosity effects, which are the most pronounced for higher harmonics. Let us estimate what should be the ratio of small parameters β and $\delta^2 = \nu/cR$ so that resonant excitation of shape oscillations become suppressed by viscosity. It is quite obvious that for a small-scale flow, the main contribution will be made by volume attenuation, and not by the presence of a boundary layer near a solid or free surface. The damping of the N-th harmonic over a period will be substantional if

$$\omega \ll \frac{\nu}{R^2} N^2. \quad (99)$$

By substituting the value of N into (99), we obtain

$$\beta^2 \gg k\delta^6, \quad (100)$$

or, in dimensional form,

$$\omega \ll \frac{\sigma^2}{\rho^2 \nu^3}. \quad (101)$$

In this case, the effect of surface tension can be neglected.

5. Average Shape of the Drop

In this case of rather weak vibrations, then vibrational parameter Q is small, the drop shape can be given by the following equation:

$$F_0 = r - 1 - Q\zeta(\vartheta), \quad (102)$$

where $\zeta(\vartheta)$ is the deviation of the averaged surface. Then, the divergence of the normal vector to this surface is written as:

$$\mathrm{div}\vec{n} = 2 - Q(\Delta_\vartheta \zeta + 2\zeta). \quad (103)$$

Here, $\Delta_\vartheta = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right)$ is the spherical part of the Laplace operator. As a result, the boundary condition (38) for the quasi-equilibrium state will be rewritten as follows:

$$r = 1: \ \frac{1}{4}\left[(\nabla_n \varphi)^2 - (\nabla_\tau \varphi)^2 + k^2 q^2 - 2\gamma_n \nabla_n \varphi\right] = \Delta_\vartheta \zeta + 2\zeta. \quad (104)$$

The pulsation velocity potential on the free surface of the drop is known (95); therefore, the tangential component of the potential gradient can be written as:

$$\nabla_\tau \varphi = \vec{\tau} \cdot \nabla z = \vec{\tau} \cdot \vec{\gamma} = \gamma_\tau, \quad (105)$$

where $\vec{\tau}$ is a unit vector directed tangentially to the droplet surface. It is obvious that

$$\gamma_\tau^2 = 1 - \gamma_n^2. \quad (106)$$

Using Formulae (105) and (106), we rewrite the first three terms of Equation (104) as follows:

$$(\nabla_n \varphi)^2 - (\nabla_\tau \varphi)^2 - 2\gamma_n \nabla_n \varphi = (\nabla_n \varphi - \gamma_n)^2 - 1. \quad (107)$$

For a hemispherical drop, the normal to the surface component of the vector $\vec{\gamma}$ is equal to $\cos\vartheta$. The pulsating part of the pressure is equal to:

$$q = \cos\vartheta - \varphi. \quad (108)$$

Thus, Equation (104) will be rewritten as follows:

$$r = 1: \ \frac{1}{4}(\nabla_n \varphi - \cos\vartheta)^2 + \frac{1}{4}k^2(\varphi - \cos\vartheta)^2 = \Delta_\vartheta \zeta + 2\zeta. \quad (109)$$

Since the surface of the drop is close to spherical, deviations in the average surface can be written as:

$$\zeta(\vartheta) = \sum_{i=0}^{\infty} \zeta_i P_{2i}(\cos\vartheta). \quad (110)$$

It is known that the Legendre polynomials are eigenfunctions for Δ_ϑ, i.e.,

$$\Delta_\vartheta P_{2i}(\Theta) = \left[\left(1 - \Theta^2\right) P_{2i}{}'(\Theta)\right]' = -2i(2i+1)P_{2i}(\Theta), \quad (111)$$

where $\Theta = \cos\vartheta$, the prime denotes the derivative with respect to Θ. Thus, by substituting expansion (110) into (109), we transform the left side of this equation as follows:

$$\Delta_\vartheta \zeta + 2\zeta = -\sum_{i=0}^{\infty} (2i-1)(2i+2)\zeta_i P_{2i}(\Theta). \quad (112)$$

We write the right side of Equation (109) in the form:

$$\frac{1}{4}(\nabla_n \varphi - \cos\vartheta)^2 + \frac{1}{4}k^2(\varphi - \cos\vartheta)^2 = \frac{1}{4}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} D_{il} P_{2i}(\Theta) P_{2l}(\Theta), \quad (113)$$

where

$$D_{il} = \left[(\varphi_i{}' - \alpha_i)(\varphi_l{}' - \alpha_l) + k^2(\varphi_i - \alpha_i)(\varphi_l - \alpha_l)\right]\Big|_{r=1}. \quad (114)$$

By substituting expressions (112), (113) into Equation (109), we multiply both parts of the resulting relation by $P_{2m}(\Theta)$ and integrate over Θ:

$$\begin{aligned} &\frac{1}{4}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} D_{il} \int_0^1 P_{2i}(\Theta) P_{2l}(\Theta) P_{2m}(\Theta) d\Theta \\ &\qquad = -\sum_{i=0}^{\infty} (2i-1)(2i+2)\zeta_i \int_0^1 P_{2i}(\Theta) P_{2m}(\Theta) d\Theta. \end{aligned} \quad (115)$$

In accordance with the condition of orthogonality of Legendre polynomials:

$$\int_0^1 P_{2i}(\Theta) P_{2l}(\Theta) d\Theta = \frac{1}{4l+1}\delta_{il}, \quad (116)$$

where δ_{il} is the Kronecker symbol ($\delta_{il} = 1$ if $i = l$; $\delta_{il} = 0$ if $i \neq l$). As a result, from (115), it follows that

$$\zeta_i = \frac{(4i+1)}{4(2i-1)(2i+2)} \sum_{l,m} D_{lm} \int_0^1 P_{2i}(\Theta) P_{2l}(\Theta) P_{2m}(\Theta) d\Theta. \quad (117)$$

It is known that the integral of the product of three Legendre polynomials in (117) is equal to the square of the Wigner 3j-symbol [28,29]:

$$\int_0^1 P_{2i}(\Theta)P_{2l}(\Theta)P_{2m}(\Theta)d\Theta = \begin{pmatrix} 2i & 2l & 2m \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{118}$$

This 3j-symbol can be calculated using the following formula [28,29]:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^p \left[\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(2p+1)!} \right]^{\frac{1}{2}} \times \frac{p!}{(p-j_1)!(p-j_2)!(p-j_3)!}, \tag{119}$$

where $2p = j_1 + j_2 + j_3$ is an even number (if $2p$ is odd, then the 3j-symbol is equal to zero). Thus, by calculating the coefficients ζ_i using Formula (117) and summing up series (110), we find the deviation of the quasi-equilibrium surface for a nearly hemispherical drop.

Our calculations demonstrated that under vibrations, drop height decreases, and the base area increases (see Figure 9). This result was predictable, since the tendency to maximize the cross-sectional area perpendicular to the vibration axis is typical for many fluid systems [4,13,27]. However, the dependence of the amplitude of the top point of the drop on frequency, shown in Figure 10, is not monotonic near resonance of drop volume oscillations. As a result, near the resonance, the opposite behavior takes place; the drop height grows and the liquid–solid contact area decreases (see Figure 11).

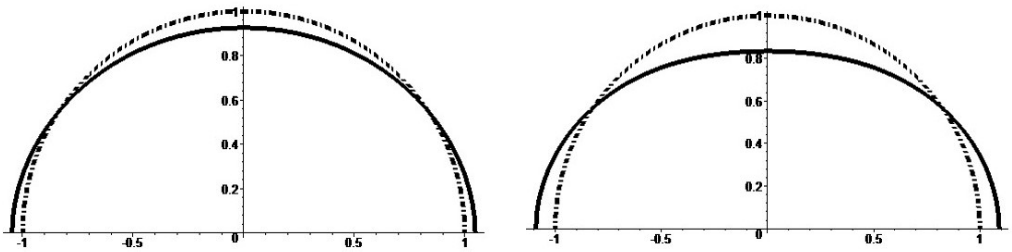

Figure 9. Averaged shape of the drop near resonance for $Q = 1$, $k = 0.01$ (**left**) and $k = 1$ (**right**). The dashed line shows the semispherical drop shape without vibrations.

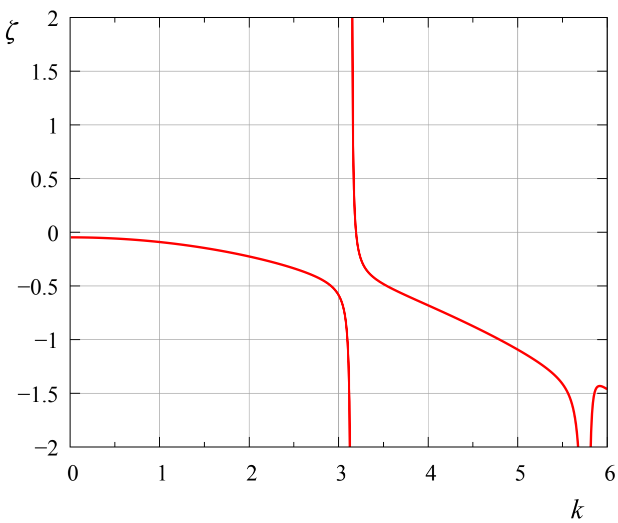

Figure 10. The dependence of the amplitude of the top point of the drop (see (102)) on dimensionless frequency.

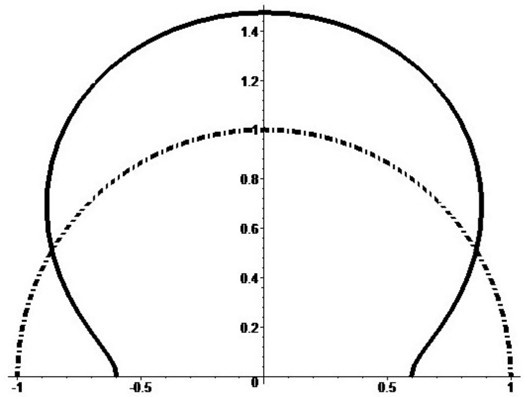

Figure 11. Averaged shape of the drop for $Q = 1$, $k = 3.17$. The dashed line shows the semispherical drop shape without vibrations.

In the case of finite values of vibrational parameter Q, the shape of the drop can be calculated using the variational principle (69).

6. Conclusions

The influence of high-frequency vibrations on the shape of a drop placed on a vibrating solid substrate is studied in this work. The solution to the problem is analytically determined as a series in Legendre polynomials. Natural frequencies of sound oscillations of a hemispherical axisymmetric drop are obtained. Resonances of the acoustic mode of drop oscillations are found.

We obtained the solution to the pulsation problem by taking into account the surface tension of the liquid. It is shown that in the presence of surface forces, surface deviations are limited. In this case, high oscillation modes of the drop shape are excited in a resonant manner, and small-scale flows appear near the drop surface.

The problem of forced oscillations of a drop in the limit of a weakly compressible liquid is considered. It is shown that the increase in the amplitude of oscillations with an increase in the frequency of vibrations occurs according to quadratic law.

The influence of vibrations on the quasi-equilibrium shape of a drop is studied at a small value of the vibrational parameter. Our calculations showed that in response to vibrations, the drop height decreases, and the area of its base increases for all vibrational frequencies, except the resonant ones, for which the opposite behavior is possible. The deformation of the surface changes in proportion to the vibration parameter.

Author Contributions: Conceptualization, A.I. and D.L.; Methodology, T.L. and D.L.; Software, A.I. and G.K.; Validation, A.I. and T.L.; Formal analysis, A.I. and G.K.; Investigation, A.I. and T.L.; Data curation, A.I. and G.K.; Writing—original draft, A.I. and T.L.; Writing—review & editing, A.I., T.L. and G.K.; Visualization, A.I. All authors have read and agreed to the published version of the manuscript.

Funding: The authors gratefully acknowledge financial support provided by the Ministry of Science and High Education of Russia (theme no.121031700169-1).

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jazayeri, S.; Pourahmad, A.; Abdollahi, S.A.; Hassanvand, A.; Alobaid, F.; Aghel, B. Experimental Investigation and CFD Simulation of Cryogenic Condenser. *Processes* **2023**, *11*, 1845. [CrossRef]
2. Ono, J.; Unno, N.; Yuki, K.; Taniguchi, J.; Satake, S.i. Visualization and Sound Measurements of Vibration Plate in a Boiling Bubble Resonator. *Fluids* **2021**, *6*, 443. [CrossRef]
3. Pavlenko, I.; Sklabinskyi, V.; Piteľ, J.; Židek, K.; Kuric, I.; Ivanov, V.; Skydanenko, M.; Liaposhchenko, O. Effect of Superimposed Vibrations on Droplet Oscillation Modes in Prilling Process. *Processes* **2020**, *8*, 566. [CrossRef]

4. Lyubimov, D.V.; Cherepanov, A.A.; Lyubimova, T.P.; Roux, B. Interface orienting by vibration. *C. R. l'Acad. Sci.-Ser. IIB-Mech.-Phys.-Chem.-Astron.* **1997**, *325*, 391–396. [CrossRef]
5. Kozlov, N. Steady flow in an annulus with a varying number of deflectors at rotational vibration. *Fluid Dyn. Res.* **2018**, *50*, 051402. [CrossRef]
6. Kozlov, N.; Vjatkin, A.; Schipitsyn, V.; Stambouli, M. Steady flows excited by local oscillations of flexible boundary of a container with fluid. *Microgravity Sci. Technol.* **2019**, *31*, 821–831. [CrossRef]
7. Wolf, G. The dynamic stabilization of Rayleigh-Taylor instability and corresponding dynamic equilibrium. *Z. Phys.* **1969**, *B227*, 291–300. [CrossRef]
8. Brunet, P.; Eggers, J.; Deegan, R.D. Vibration-Induced Climbing of Drops. *Phys. Rev. Lett.* **2007**, *99*, 144501. [CrossRef]
9. Noblin, X.; Buguin, A.; Brochard-Wyart, F. Vibrations of sessile drops. *Eur. Phys. J. Spec. Top.* **2009**, *166*, 7–10. [CrossRef]
10. Sartori, P.; Quagliati, D.; Varagnolo, S.; Pierno, M.; Mistura, G.; Magaletti, F.; Casciola, C. Drop motion induced by vertical vibrations. *New J. Phys.* **2015**, *17*, 113017. [CrossRef]
11. Xu, J.; Ma, J.; Ni, J.; Fan, S.; Zhang, L.; Wang, R. Morphological Evolution and Interfacial Effects Analysis of Drop Motion in Transverse Vibration of Inclined Plate. *Coatings* **2020**, *10*, 845. [CrossRef]
12. Holmes, H.R.; Böhringer, K.F. Vibration Induced Transport of Enclosed Droplets. *Micromachines* **2019**, *10*, 69. [CrossRef] [PubMed]
13. Lyubimova, T.P.; Fomicheva, A.A.; Ivantsov, A.O. Dynamics of a bubble in oscillating viscous liquid. *Philos. Trans. R. Soc. A Math. Phys. Eng. Sci.* **2023**, *381*, 20220085. [CrossRef] [PubMed]
14. Vukasinovic, B.; Smith, M.K.; Glezer, A. Dynamics of a sessile drop in forced vibration. *J. Fluid Mech.* **2007**, *587*, 395–423. [CrossRef]
15. Löwe, J.M.; Hinrichsen, V.; Roisman, I.; Tropea, C. Behavior of charged and uncharged drops in high alternating tangential electric fields. *Phys. Rev. E* **2020**, *101*, 023102. [CrossRef]
16. Kashina, M.; Alabuzhev, A. The Forced Oscillations of an Oblate Drop Sandwiched Between Different Inhomogeneous Surfaces under AC Vibrational Force. *Micrograviy Sci. Technol.* **2021**, *33*, 35. [CrossRef]
17. Tankovsky, N.; Zografov, N.; Russev, S. Electrically driven resonant oscillations of pendant hemispherical liquid droplet and possibility to evaluate the surface tension in real time. *Z. Phys. Chem.* **2013**, *227*, 1759–1766. [CrossRef]
18. Pyankova, M.; Alabuzhev, A. Influence of the properties of the plate surface on the oscillations of the cramped drop. *Phys. Fluids* **2022**, *34*, 092015. [CrossRef]
19. Alabuzhev, A. Forced axisymmetric oscillations of a drop, which is clamped between different surfaces. *Microgravity Sci. Technol.* **2020**, *32*, 545–553. [CrossRef]
20. Alabuzhev, A.; Pyankova, M. Dynamics of a clamped drop under translational vibrations. *Comput. Contin. Mech.* **2023**, *16*, 78–88. [CrossRef]
21. Boughzala, M.; Stéphan, O.; Bossy, E.; Dollet, B.; Marmottant, P. Polyhedral Bubble Vibrations. *Phys. Rev. Lett.* **2021**, *126*, 054502. [CrossRef] [PubMed]
22. Vos, H.J.; Dollet, B.; Bosch, J.G.; Versluis, M.; de Jong, N. Nonspherical vibrations of microbubbles in contact with a wall: A pilot study at low mechanical index. *Ultrasound Med. Biol.* **2008**, *34*, 685–688. [CrossRef]
23. Geng, D.L.; Xie, W.J.; Yan, N.; Wei, B. Vertical vibration and shape oscillation of acoustically levitated water drops. *Appl. Phys. Lett.* **2014**, *105*, 104101. [CrossRef]
24. Lyubimov, D.V.; Lyubimova, T.P.; Shklyaev, S.V. Behavior of a drop on an oscillating solid plate. *Phys. Fluids* **2006**, *18*, 012101. [CrossRef]
25. Shklyaev, S.; Straube, A.V. Linear oscillations of a compressible hemispherical bubble on a solid substrate. *Phys. Fluids* **2007**, *20*, 052102. [CrossRef]
26. Neshasteh, H.; Ravaro, M.; Favero, I. Fluid–structure model for disks vibrating at ultra-high frequency in a compressible viscous fluid. *Phys. Fluids* **2023**, *35*, 052008. [CrossRef]
27. Lyubimov, D.; Lyubimova, T.; Cherepanov, A. *Dynamics of Interfaces in Vibration Fields*; FizMatLit: Moscow, Russia, 2003. (In Russian)
28. Landau, L.D.; Lifshits, E.M. *Quantum Mechanics: Non-Relativistic Theory*; Course of Theoretical Physics; Butterworth-Heinemann: Oxford, UK, 1991.
29. Wigner, E.P. On the matrices which reduce the Kronecker products of representations of S. R. Groups. In *The Collected Works of Eugene Paul Wigner: Part A: The Scientific Papers*; Wightman, A.S., Ed.; Springer: Berlin/Heidelberg, Germany, 1993; pp. 608–654. [CrossRef]