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Superiorization with a Projected Subgradient Algorithm on the Solution Sets of Common Fixed Point Problems

Alexander J. Zaslavski

Department of Mathematics, The Technion—Israel Institute of Technology, 32000 Haifa, Israel; ajzasl@technion.ac.il

Abstract: In this work, we investigate a minimization problem with a convex objective function on a domain, which is the solution set of a common fixed point problem with a finite family of nonexpansive mappings. Our algorithm is a combination of a projected subgradient algorithm and string-averaging projection method with variable strings and variable weights. This algorithm generates a sequence of iterates which are approximate solutions of the corresponding fixed point problem. Additionally, either this sequence also has a minimizing subsequence for our optimization problem or the sequence is strictly Fejer monotone regarding the approximate solution set of the common fixed point problem.

Keywords: constrained minimization; common fixed point problem; dynamic string-averaging projections; subgradients

MSC: 90C25; 90C30; 65K10

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1. Introduction

The starting point of the fixed point theory of nonlinear operators is Banach's famous work [1], where the existence of a unique fixed point of a strict contraction was established. Since that, many important results were established in this area [2–24], which include the investigation of the asymptotic behavior of iterates of a nonlinear mappings. They also include the studies of feasibility, common fixed points, iterative methods, and variational inequalities and their applications in engineering, medical, and natural sciences [2,23–38].

Assume that (X, ρ) is a metric space. For every point $\eta \in X$ and every nonempty set $C \subset X$, define

$$\rho(\eta, C) := \inf\{\rho(\eta, \xi) : \ \xi \in C\}.$$

For every point $\eta \in X$ and every number $\Delta > 0$, define

$$B(\eta, \Delta) := \{\xi \in X : \ \rho(\eta, \xi) \leq \Delta\}.$$

For every operator $A : X \to X$, set $A^0(v) = v$ for all $v \in X$, $A^1 = A$ and $A^{i+1} = A \circ A^i$ for every nonnegative integer i.

A mapping $T : X \to X$ is called a strict contraction if there exists $\lambda \in (0, 1)$, such that

$$\rho(T(u), T(v)) \leq \lambda \rho(u, v)$$

for each $u, v \in X$.

According to the Banach's celebrated theorem [1], a strict contraction T has a fixed point $x_T \in X$ for which

$$T(x_T) = x_T$$

and which attracts every sequence of iterates of T. Moreover, it is known that this convergence of iterates of T is uniform on all bounded subsets of X.

In [18], A. M. Ostrowski investigated the influence of computational errors on the behavior of iterates of the strict contraction T. He proved that every sequence $\{u_i\}_{i=0}^{\infty} \subset X$ for which

$$\sum_{i=0}^{\infty} \rho(u_{i+1}, T(u_i)) < \infty$$

converges, and its limit is the fixed point of T.

A different approach was applied in [5] in order to generalize the result of [18] for a map $T : X \to X$, which is merely nonexpansive. We assumed that

$$\rho(T(v), T(u)) \leq \rho(v, u)$$

for all pairs of points $v, u \in X$, and showed that if all sequences of exact iterates of T converge, then all sequences of its inexact iterates with summable errors converge too.

This result has many applications and is an essential ingredient in superiorization and perturbation resilience of algorithms [25–28]. The superiorization technique was applied in [31,37], where an optimization problem with a convex objective function and with a feasible region was investigated, which is the intersection of a finite family of closed convex constraint sets. In this work, we investigate a minimization problem with a convex objective function on a domain, which is the solution set of a common fixed point problem with a finite family of nonexpansive mappings. Our algorithm is a combination of a projected subgradient algorithm and string-averaging projection method, with variable strings and variable weights. This algorithm generates a sequence of iterates which are approximate solutions of the corresponding fixed point problem. Additionally, either this sequence also has a minimizing subsequence for our optimization problem, or the sequence is strictly Fejer monotone regarding the approximate solution set of the common fixed point problem.

2. Common Fixed Point Problems in a Metric Space

Recall that (X, ρ) is a metric space. We prove the following result.

Theorem 1. *Assume that $\mathcal{A}$ is a nonempty set, for each $\alpha \in \mathcal{A}$, a map $T_\alpha : X \to X$ satisfies*

$$\rho(T_\alpha(x), T_\alpha(y)) \leq \rho(x, y) \tag{1}$$

for each $x, y \in X$ and that for each integer $i \geq 1$, a map $S_i : X \to X$ satisfies

$$\rho(S_i(x), S_i(y)) \leq \rho(x, y) \tag{2}$$

for each $x, y \in X$.

In addition, assume that for each integer $k \geq 1$, each $\alpha \in \mathcal{A}$, and each $x \in X$,

$$\rho(\prod_{j=k}^{k+n} S_j(x), T_\alpha(\prod_{j=k}^{k+n} S_j(x))) \to 0 \text{ as } n \to \infty, \tag{3}$$

$\{\Delta_i\}_{i=0}^{\infty} \subset (0, \infty)$ satisfies

$$\sum_{i=0}^{\infty} \Delta_i < \infty, \tag{4}$$

$\{x_i\}_{i=0}^{\infty} \subset X$, and that for each integer $i \geq 0$,

$$\rho(x_{i+1}, S_{i+1}(x_i)) \leq \Delta_i. \tag{5}$$

Then for each $\alpha \in \mathcal{A}$,

$$\lim_{i \to \infty} \rho(x_i, T_\alpha(x_i)) = 0.$$

Proof. Let $\epsilon \in (0,1)$. In view of (4), there exists a natural number k such that

$$\sum_{i=k}^{\infty} \Delta_i < \epsilon/4. \tag{6}$$

Set

$$y_k = x_k \tag{7}$$

and for each integer $i \geq k$ set

$$y_{i+1} = S_{i+1}(y_i). \tag{8}$$

By our assumptions and (3), (7) and (8), for each $\alpha \in \mathcal{A}$,

$$\lim_{i\to\infty} \rho(y_i, T_\alpha(y_i)) = 0. \tag{9}$$

We show that for each integer $p \geq k+1$,

$$\rho(x_p, y_p) \leq \sum_{i=k}^{p-1} \Delta_i. \tag{10}$$

It follows from (5) and (8) that

$$\rho(x_{k+1}, y_{k+1}) = \rho(x_{k+1}, S_{k+1}(y_k)) = \rho(x_{k+1}, S_{k+1}(x_k)) \leq \Delta_k$$

and (10) holds for $p = k+1$.

Assume that $p \geq k+1$ is an integer and (10) holds. By (2), (5), (8) and (10),

$$\rho(x_{p+1}, y_{p+1}) = \rho(x_{p+1}, S_{p+1}(y_p))$$

$$\leq \rho(x_{p+1}, S_{p+1}(x_p)) + \rho(S_{p+1}(x_p), S_{p+1}(y_p))$$

$$\leq \Delta_p + \rho(x_p, y_p) = \Delta_p + \sum_{i=k}^{p-1} \Delta_i = \sum_{i=k}^{p} \Delta_i.$$

Thus, we showed by induction that (10) holds for each integer $p \geq k+1$.

Let $\alpha \in \mathcal{A}$. In view of (3), (7) and (8), there exists an integer $k_1 > k$ such that

$$\rho(T_\alpha(y_i), y_i) < \epsilon/4 \text{ for each integer } i \geq k_1.$$

By (1), (6), (10) and the relation above, for each integer $i \geq k_1$,

$$\rho(T_\alpha(x_i), x_i) \leq \rho(T_\alpha(x_i), T_\alpha(y_i)) + \rho(T_\alpha(y_i), y_i) + \rho(y_i, x_i)$$

$$\leq 2\rho(y_i, x_i) + \rho(T_\alpha(y_i), y_i) \leq 2\sum_{j=k}^{\infty} \Delta_j + \epsilon/4 \leq 3\epsilon/4.$$

This completes the proof of Theorem 1. □

Theorem 1 is an extension of the result of [5], which was obtained for orbits of a nonexpansive mapping.

3. The Dynamic String-Averaging Projection Method

Let $(X, \|\cdot\|)$ be a normed space and $\rho(x,y) = \|x - y\|$, $x, y \in X$.

Suppose that m is a natural number, $P_j : X \to X$, $j = 1, \ldots, m$, for every $j \in \{1, \ldots, m\}$,

$$\|P_j(u) - P_j(v)\| \leq \|u - v\|,\ u, v \in X \tag{11}$$

and

$$\mathrm{Fix}(P_j) := \{\xi \in X : \ P_j(\xi) = \xi\} \neq \emptyset.$$

Set

$$F = \cap_{j=1}^m \mathrm{Fix}(P_j).$$

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$, put

$$F_\epsilon(P_i) = \{x \in X : \ \|x - P_i(x)\| \leq \epsilon\},$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ \rho(y, F_\epsilon(P_i)) \leq \epsilon\},$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i), \ \tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i).$$

Suppose that

$$F \neq \emptyset.$$

Let us now describe our dynamic string-averaging algorithm.

In the sequel, a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$ is called an index vector.

For every index vector $t = (t_1, \dots, t_q)$, define

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}.$$

Clearly, for every index vector t,

$$P[t](x) = x, \ x \in F, \tag{12}$$

$$\|P[t](x) - P[t](y)\| \leq \|x - y\| \tag{13}$$

for every pair $x, y \in X$.

Let $\mathcal{M}$ be the set of all (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{14}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x), \ x \in X. \tag{15}$$

It is not difficult to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{16}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| \leq \|x - y\| \tag{17}$$

for all $x, y \in X$.

We use the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given $x_k \in X$ choose

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}]$$

and an integer

$$\bar{q} \geq m.$$

Let $\mathcal{M}_*$ be the collection of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \ \text{ for all } t \in \Omega,$$

$$w(t) \geq \Delta \text{ for all } t \in \Omega.$$

Fix a natural number $\bar{N}$.

In order to find a point $x \in F$ we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k), \ k = 0, 1, \dots.$$

We assume that the following assumption holds.

(A1) For each $M > 0$ and each $\gamma > 0$, there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, M) \cap \mathrm{Fix}(P_i)$$

, and each $x \in B(0, M)$ satisfying $\|x - P_i(x)\| \geq \gamma$,

$$\|z - P_i(x)\| \leq \|z - x\| - \delta.$$

It should be mentioned that many mappings possess this property. For details see [22,23]. In particular, (A1) holds when our mappings are projection operators on closed convex sets in Hilbert spaces. In some classes of mappings most operators (in the sense of Baire category) have this property.

The following result was obtained in Chapter 4 of [23].

Theorem 2. *Let $M > 0$ satisfy*

$$B(0, M) \cap F \neq \emptyset$$

and let $\epsilon \in (0, 1)$. Then, there exists a constant $Q > 0$ such that for each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}),$$

each $x_0 \in B(0, M)$ and each sequences $\{x_i\}_{i=1}^{\infty} \subset X$ satisfying for each integer $i \geq 0$,

$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i)$$

the inequality

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \leq Q$$

holds.

In the sequel, we use the following lemma.

Lemma 1. *Let $\epsilon > 0$ and $i \in \{1, \dots, m\}$. Then*

$$\tilde{F}_\epsilon(P_i) \subset F_{3\epsilon}(P_i).$$

Proof. Let
$$x \in \tilde{F}_\epsilon(P_i)$$
and $\delta > 0$. Clearly, there exists
$$z \in B(x, \delta + \epsilon)$$
such that
$$\|z - P_i(z)\| \leq \epsilon.$$

By the relation above and (9),
$$\|x - P_i(x)\| \leq \|x - z\| + \|z - P_i(z)\| + \|P_i(z) - P_i(x)\|$$
$$\leq 2\|z - x\| + \|z - P_i(z)\| \leq 2(\epsilon + \delta) + \epsilon.$$

Since δ is any positive number, we conclude that
$$x \in F_{3\epsilon}(P_i).$$

Lemma 1 is proved. □

Corollary 1. *For each $\epsilon > 0$,*
$$\tilde{F}_\epsilon \subset F_{3\epsilon}.$$

Theorem 2 and Lemma 1 imply the following result.

Theorem 3. *Let $M > 0$ satisfy*
$$B(0, M) \cap F \neq \emptyset$$
and let $\epsilon \in (0, 1)$. Then, there exists a constant $Q > 0$ such that for each
$$\{(\Omega_i, w_i)\}_{i=1}^\infty \subset \mathcal{M}_*$$
which satisfies for each natural number j,
$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}),$$
each $x_0 \in B(0, M)$ and each sequences $\{x_i\}_{i=1}^\infty \subset X$, satisfying for each integer $i \geq 0$,
$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i)$$
the inequality
$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin F_\epsilon\}) \leq Q$$
holds.

Theorems 1 and 3 imply the following result which is an extension of Theorem 2 for the case of inexact iterates with summable computational errors.

Theorem 4. *Let*
$$\{(\Omega_i, w_i)\}_{i=1}^\infty \subset \mathcal{M}_*$$
satisfy for each natural number j,
$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}),$$
$\{\Delta_i\}_{i=0}^\infty \subset (0, \infty)$ satisfy
$$\sum_{i=0}^\infty \Delta_i < \infty$$

and $\{x_i\}_{i=0}^{\infty} \subset X$ *satisfy for each integer* $i \geq 0$,

$$\|x_{i+1} - P_{\Omega_{i+1}, w_{i+1}}(x_i)\| \leq \Delta_i.$$

Then, for each $\epsilon > 0$ *there exists an integer* $Q \geq 1$ *such that*

$$x_i \in F_\epsilon$$

for each integer $i \geq Q$.

Example 1. *The results of this section can be applied to the following common fixed point problem. Assume that* C_i, $i = 1, \dots, m$ *are nonempty, convex, closed sets in* X,

$$C = \cap_{i=1}^{m} C_i \neq \emptyset$$

and that for each $i \in \{1, \dots, m\}$, $P_{C_i} : X \to C_i$ *is a projection operator on* C_i*: for each* $x \in X$,

$$\|P_{C_i}(x) - x\| \leq \|y - x\|,\ y \in C_i.$$

Assume that for each $i \in \{1, \dots, m\}$, $\lambda_i \in (0, 1]$ *and*

$$P_i(x) = \lambda_i P_{C_i}(x) + (1 - \lambda_i)x,\ x \in X.$$

It is not difficult to see that all the assumptions made in the section hold and our results hold too. Note that if $\lambda_i = 1$, $i = 1, \dots, m$ *we have a feasibility problem. But, in the general case, we have a common fixed point problem with the solution set* C.

4. Superiorization

Assume that $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space equipped with an inner product that induces a norm

$$\|x\| = \langle x, x \rangle^{1/2},\ x \in X.$$

We continue to use all the notation, definitions, and assumptions introduced in Section 3. In particular, we assume that assumption (A1) holds.

Assume that $M > 0$ and that there exists $s_0 \in \{1, \dots, m\}$ such that

$$P_{s_0}(X) \subset B(0, M). \tag{18}$$

By (18),

$$F \subset B(0, M). \tag{19}$$

Denote by $\mathcal{M}_b$ the set of all $(\Omega, w) \in \mathcal{M}_*$ such that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega$ there exists $j \in \{1, \dots, p(t)\}$ such that

$$P_{t_j}(X) \subset B(0, M). \tag{20}$$

Assume that $L \geq 1$ and that $f : X \to R^1$ is a real-valued convex function such that

$$|f(u) - f(v)| \leq L\|u - v\| \text{ for all } u, v \in B(0, 3M + 1). \tag{21}$$

For each $u \in X$,

$$\partial f(u) = \{\eta \in X :\ \langle \eta, v - u \rangle \leq f(v) - f(u) \text{ for all } v \in X\} \tag{22}$$

is the subdifferential of the function f at the point u. We consider the minimization problem

$$f(x) \to \min,\ x \in F$$

and set

$$\inf(f,F) = \inf\}f(z) :\ z \in F\}. \tag{23}$$

Let us now describe our algorithm.

Suppose that

$$\{(\Omega_j, w_j)\}_{j=1}^{\infty} \subset \mathcal{M}_b,$$

$$a_j \in (0,1] \text{ for all nonnegative integers } j,$$

$$\sum_{j=0}^{\infty} a_j < \infty$$

and that for each natural number j,

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}),$$

let $x_0 \in X$ and let for each natural number j,

$$l_{j-1} \in \partial f(x_{j-1}) \tag{24}$$

and

$$x_j = P_{\Omega_j, w_j}(x_{j-1} - a_{j-1}l_{j-1}). \tag{25}$$

In this paper, we prove the following result.

Theorem 5. *For each integer $q \in \{1,\dots,m\}$,*

$$\lim_{i\to\infty} \|x_i - P_q(x_i)\| = 0$$

and at least one of the following cases holds:

(a) $\liminf_{i\to\infty} f(x_i) \le \inf(f,F)$;

(b) there exist a natural number k_0 and $\delta_0 > 0$ for each $x \in F$, satisfying

$$f(x) \le \inf(f,F) + \delta_0$$

and each integer $k \ge k_0$,

$$\|x_k - x\|^2 \le \|x_{k-1} - x\|^2 - \delta_0 a_{k-1}.$$

5. An Auxiliary Result

Lemma 2. *Let*

$$z \in F,\ a \in (0,1],\ d \ge 0,\ x \in B(0,3M), \tag{26}$$

satisfy

$$f(x) \ge f(z) + d, \tag{27}$$

$$u \in \partial f(x),\ (\Omega, w) \in \mathcal{M}_* \tag{28}$$

and let

$$y = P_{\Omega,w}(x - au). \tag{29}$$

Then

$$\|y - z\|^2 \le \|x - z\|^2 + a^2L^2 - 2ad.$$

Proof. In view of (21), (26) and (28),

$$\|u\| \le L. \tag{30}$$

By (22) and (28),

$$\langle u, z - x\rangle \le f(z) - f(x). \tag{31}$$

It follows from (9), (26), (27), and (29)–(31) that

$$\|y - z\|^2 = \|P_{\Omega,w}(x - au) - z\|^2$$
$$\leq \|x - au - z\|^2 \leq \|x - z\|^2 + a^2\|u\|^2 - 2a\langle u, x - z\rangle$$
$$\leq \|x - z\|^2 + a^2L^2 + 2a(f(z) - f(x)) \leq \|x - z\|^2 + a^2L^2 - 2ad.$$

Lemma 2 is proved. □

6. Proof of Theorem 5

Fix

$$z \in F. \tag{32}$$

In view of (25), for each natural number s,

$$x_s = P_{\Omega_s,w_s}(x_{s-1} - a_{s-1}l_{s-1}). \tag{33}$$

Let s be a natural number and

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_s. \tag{34}$$

By (20) and (34), there exists

$$j \in \{1, \ldots, p(t)\}$$

such that

$$P_{t_j}(X) \subset B(0, M). \tag{35}$$

In view of (35),

$$\prod_{i=1}^{j} P_{t_i}(x_{s-1} - a_{s-1}l_{s-1}) \in B(0, M). \tag{36}$$

It follows from (19), (32), (35), and (36) that

$$\|z - \prod_{i=1}^{j} P_{t_i}(x_{s-1} - a_{s-1}l_{s-1})\| \leq 2M. \tag{37}$$

Equations (9), (10), (37), and (38) imply that

$$\|P[t](x_{s-1} - a_{s-1}l_{s-1}) - z\| \leq \|\prod_{i=1}^{j} P_{t_i}(x_{s-1} - a_{s-1}l_{s-1}) - z\| \leq 2M.$$

Thus

$$\|P[t](x_{s-1} - a_{s-1}l_{s-1}) - z\| \leq 2M \text{ for all } t \in \Omega_s. \tag{38}$$

It follows from (14), (15), (38), and the convexity of the norm that

$$\|P_{\Omega_s,w_s}(x_{s-1} - a_{s-1}l_{s-1}) - z\|$$
$$\leq \sum_{t\in\Omega_s} w_s(t)\|P[t](x_{s-1} - a_{s-1}l_{s-1}) - z\| \leq 2M. \tag{39}$$

In view of (18), (32), (33), and (39),

$$\|x_s\| \leq 3M \tag{40}$$

for each integer $s \geq 1$. By (21), (24) and (40),

$$\|l_s\| \leq L \text{ for all integers } s \geq 1. \tag{41}$$

It follows from (9), (25) and (41) that for each integer $k \geq 2$,

$$\|x_k - P_{\Omega_k, w_k}(x_{k-1})\| = \|P_{\Omega_k, w_k}(x_{k-1} - a_{k-1} l_{k-1}) - P_{\Omega_k, w_k}(x_{k-1})\|$$
$$\leq a_{k-1}\|l_{k-1}\| \leq L a_{k-1}$$

and

$$\sum_{k=0}^{\infty} \|x_k - P_{\Omega_k, w_k}(x_{k-1})\| < \infty. \quad (42)$$

Theorem 4 and (42) imply that for each $q \in \{1, \ldots, m\}$,

$$\lim_{i \to \infty} \|P_q(x_i) - x_i\| = 0.$$

Assume that the case (a) does not hold. This implies that there exists $\delta_0 > 0$ such that

$$\liminf_{k \to \infty} f(x_k) > \inf(f, F) + 2\delta_0. \quad (43)$$

Since

$$\sum_{i=0}^{\infty} a_i < \infty$$

it follows from (43) that there exists a natural number k_0 such that for all integers $k \geq k_0$,

$$\alpha_{k-1} < L^{-2}\delta_0 \quad (44)$$

and

$$f(x_{k-1}) > \inf(f, F) + 2\delta_0. \quad (45)$$

Let

$$z \in F \quad (46)$$

and

$$f(z) \leq \inf(f, F) + \delta_0. \quad (47)$$

Let $k \geq k_0$ be an integer. By (24), (40), (44)–(47) and Lemma 2 applied with

$$a = a_{k-1},\ d = \delta_0,\ x = x_{k-1},\ u = l_{k-1},\ y = x_k,\ (\Omega, w) = (\Omega_k, w_k)$$

we have

$$\|x_k - z\|^2 \leq \|x_{k-1} - z\|^2 + a_{k-1}^2 L^2 - 2a_{k-1}\delta_0$$
$$\leq \|x_k - z\|^2 - a_{k-1}\delta_0.$$

Theorem 5 is proved.

7. Conclusions

In our work, we analyze a constrained minimization problem with a convex objective function on a region, which is the solution set of a common fixed point problem with a finite family of nonexpansive mappings. The goal was to generalize the result of [31] obtained for a convex minimization problem on the solution set of a convex feasibility problem. Note that a convex feasibility problem is a particular case of a common fixed point problem. We use a projected subgradient method combined with a dynamic string-averaging projection method, with variable strings and variable weights. This algorithm generates a sequence of iterates which are approximate solutions of the corresponding fixed point problem. Additionally, also either this sequence has a minimizing subsequence for our constrained minimization problem or the sequence is strictly Fejer monotone with respect to the approximate solution set of the common fixed point problem.

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