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Fast Method for Estimating the Parameters of Partial Differential Equations from Inaccurate Observations

Gurami Tsitsiashvili ^{1,*}, Alexey Gudimenko ¹ and Marina Osipova ^{1,2}

¹ Institute for Applied Mathematics, Far Eastern Branch of Russian Academy of Sciences, IAM FEB RAS, Radio Str. 7, 690041 Vladivostok, Russia; gudimenko@iam.dvo.ru (A.G.); mao1975@list.ru (M.O.)

² Institute for Applied Mathematics, Far Eastern Federal University, 690922 Vladivostok, Russia

* Correspondence: guram@iam.dvo.ru; Tel.: +7-914-693-2749

Abstract: In this paper, the problems of estimating the parameters of partial differential equations from numerous observations in the vicinity of some reference points are considered. The paper is devoted to estimating the diffusion coefficient in the diffusion equation and the parameters of one-soliton solutions of nonlinear partial differential equations. When estimating the diffusion coefficient, it was necessary to construct an estimate of the second derivative based on inaccurate observations of the solution of the diffusion equation. This procedure required consideration of two reference points when determining the first and second partial derivatives of the solution of the diffusion equation. To analyse one-soliton solutions of partial differential equations, a series of techniques have been developed that allow one to estimate the parameters of the solution itself, but not its equation. These techniques are used to estimate the parameters of one-soliton solutions of the equations kdv, mkdv, Sine–Gordon, Burgers and nonlinear Schrodinger. All the considered estimates were tested during computational experiments.

Keywords: reference points; experiment planning; one-soliton solution

MSC: 60J28



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1. Introduction

Estimating the parameters of differential equations based on inaccurate observations of their solution is the attention focus of many researchers due to the great fundamental and applied significance of this problem, especially in connection with problems of engineering mathematics. The problem is considered not only in statistical terms but also in a deterministic formulation as an inverse problem for equations of mathematical physics (see, for example, [1–3]). The task was solved under the assumption of a large number of observations [4] over a sufficiently long period of time and in a deterministic formulation [3]. When solving deterministic inverse problems for differential equations, the question of the influence of inaccurate observations on the estimation of parameters should be noted to be not raised.

An alternative approach based on a probabilistic model of observation errors was proposed in [4–8]. This approach includes both a functional central limit theorem [4,5] and a two-step optimisation procedure [6–8]. The two-step optimisation procedure has been significantly developed and continued. The approach focuses on the use of smoothing methods to develop and estimate the differential equations, following recent developments in functional data analysis and based on the methods described in [9,10].

The development of a two-stage optimisation procedure is based on parameter estimation using nonparametric estimators [11], taking into account time-varying parameters [12] and their Bayesian estimation [13], three- and four-stage modifications of a two-stage optimisation procedure [14–16], consideration of errors in the autoregressive model [17] and applications to chemical kinetics models [18].

This variety of approaches is largely determined by the task specifics of estimating the parameters of differential equations. Despite the differences in the approaches used to solve this problem, the general procedure in the proposed solutions is to minimize the deviation of the solution of the original differential equation/system from its solution at estimated parameter values.

In the traditional two-stage method, the distance between the approximation of the differential equation solution by observations and its exact value on the segment (or in another area) is minimized. In our modification of the two-stage method, it is proposed at the first stage to minimize the distances between estimates of the solution values and its derivatives at several reference points. And already at the second stage, the unknown parameters are estimated using these approximations, solving a system of nonlinear equations constructed using the method of moments (see, for example, [19]).

The method of moments for estimating the parameters of differential equations is conveniently illustrated by the example of the equation $\dot{x}(t) = ax(t)$. Let us choose a reference point t_0 and construct consistent estimates $\hat{x}(t_0)$, $\hat{\dot{x}}(t_0)$ of the values $\dot{x}(t_0)$, $x(t_0)$. Then, the evaluation of the parameter a becomes $\hat{a} = \hat{\dot{x}}(t_0)/\hat{x}(t_0)$. Let us call it the method of moments. In a more complex differential equation for estimating an unknown parameter, it is proposed to replace the values of the function and its derivative at the reference point with estimates using the least squares method. This paper presents the conditions under which the estimates of the function and its derivative at the reference point converge to exact values with an increase in the number of observations in the vicinity of the reference point/points. Then, the accuracy of the estimation of the parameter of the differential equation and its consistency are determined by the accuracy of the solution of the algebraic equation, which includes estimates of the function and derivative at the reference point/points (see [20] (Theorem 4)).

The results obtained in this way, in a certain sense, correlate with improved statistical pattern analysis based on Kalman filtering [21–23], but with fast estimation of the differential equation parameters. Estimating the value of a function and its derivative at a reference point requires a number of iterations proportional to the number of observations near this point. This can be done during observations and does not require additional time.

All the methods given in the articles referred to are based on stochastic optimisation. With a fixed step multiplier, this method does not provide convergence to the desired solution [24] (Theorem 2): “the gradient method converges on average not to the minimum point, but to some area around it”. To ensure the required convergence, it is necessary to consistently reduce the step multipliers, which makes the calculations very slow (inversely proportional to the number of iterations). And in our method, it is only required to find the root of a monotone function, for example, by dichotomous division, which is performed very quickly (as a decreasing geometric progression from the number of iterations). During computational experiments, parameters were estimated using the proposed method almost instantly (in a few seconds).

Application of analytical estimates of calculation errors based on probabilistic metrics allows us to control the accuracy of parameter estimation and to choose the number of measurements in the vicinity of control points, distances between neighbouring measurements and measurement accuracy. The use of probabilistic metrics gives an advantage when comparing the listed results and is important when designing a measuring system. The algorithm proposed allows us to control the accuracy of estimates of the function and its derivative according to the sufficient conditions of convergence in probability to their accuracy values and to choose the number of observations in the vicinity of the reference points, the distance between neighbouring observations around the reference point and the accuracy of observations. The result of such control is presented in numerical experiments, which give an affordable accuracy (several percent), especially in a nonlinear Schrodinger equation. This is important when designing a measuring system and determines the novelty of the proposed method. Such attempts to analyse observational systems are caused, in particular, by the results in the field of quantum physics (Nobel Prize on physics 2022).

The reference point number is usually equal to the number of unknown parameters [20]. In this paper, this approach allows us to construct fast running algorithms to estimate unknown parameters and apply them to estimating the parameters of partial differential equations and their solutions.

To estimate the solution based on inaccurate observations and its derivative at the reference point, the authors needed to level out both random observation errors and the deviation of the solution in a small vicinity of the reference point from the linear function. Random observation errors are leveled out by a large number of observation points in the immediate vicinity of the reference point. For this purpose, it is convenient to use linear regression analysis, which is based on the least squares method but applied to a nonlinear function. The deviation of the solution in a small neighbourhood of the reference point from the linear one is offset by the small distance between the points adjacent to the reference point. To determine the estimate error, the method of probabilistic metrics is used [25]. The distance between points adjacent to the reference point is related by a power law to the number of these points. And the power-law parameter is chosen in such a way as to ensure convergence in probability of estimates of the solution and its derivative at the reference point to exact values.

To apply this technique to estimating the parameters of partial differential equations and their solutions, it is necessary to select reference points. Using the method of moments makes it possible to significantly simplify this procedure, based on an analogy with the method of planning an experiment [26,27]. In this paper, the described modification of a two-step procedure for parameter inference in differential equations is used to estimate the diffusion coefficient in the diffusion equation and to estimate the parameters of one-soliton solutions of nonlinear partial differential equations.

In the first section, when determining the diffusion coefficient in the diffusion equation, the problem of estimating the function second derivative arises (see, for example, [28]). This problem solution needs a more complex algorithm, which requires two reference points with the distance between them being related by a power function to the number of points in the vicinity of the reference point. The power function exponent is determined by the requirement that the probability of estimating the second derivative converge to its value. Using this approach, an estimate of the diffusion coefficient is constructed and a computational experiment is carried out to confirm the sufficient accuracy of this estimate.

In the second section we consider kdv, mkdv, Sine–Gordon, Burgers, and nonlinear Schrodinger equation. One-soliton solutions of these equations are known [29–35] and were obtained by the Darboux transformation method [29]. However, the number of parameters in these equations is greater than the number of coefficients in them. This is because these parameters characterize certain spectral properties. The problem of reference point choosing stems from the fact that each reference point still needs to be compared with the partial derivative of the solution. Consequently, several partial derivatives can correspond to one reference point. For the equations KdV, mKdV, Sine–Gordon, and Burgers, this problem is solved quite simply, since you can take one reference point at which two partial derivatives are calculated. But for the nonlinear Schrodinger equation, the one-soliton solution of which contains four parameters [36], the task becomes more complicated, since it is necessary to select two reference points.

We find these reference points and obtain the necessary estimates for the parameters of one-soliton solutions in all of the listed nonlinear partial differential equations. The solutions obtained during the computational experiment turn out to be quite accurate.

2. Estimates of Diffusion Coefficient

2.1. Evaluation of the First Derivative

Suppose that inaccurate observations were obtained at points $\pm kh$, $h > 0$, $k = 0, 1, \dots, n$, for the state of some physical process described by the function $f(z) : f(\pm kh) + \varepsilon(\pm kh)$, $k = 0, 1, \dots, n$. Here, $\varepsilon(\pm kh)$, $k = 0, 1, \dots, n$, are independent identically distributed random variables with zero mean and finite variance δ^2 . At first we

estimate $f(0)$, $f_z(0)$ provided that for some Δ we have in interval $[-\Delta, \Delta]$ the inequality $|f_{zz}| < C_1 < \infty$.

Previously [37] (Theorem 3), estimates of the first derivative of the function $f(z)$ and its value at $z = 0$ were constructed with $y_k = f(kh) + \varepsilon_k$:

$$\hat{f}(0) = \frac{1}{2n+1} \sum_{k=-n}^n y_k, \hat{f}_z(0) = \frac{1}{\sum_{k=-n}^n (kh)^2} \sum_{k=-n}^n y_k kh. \tag{1}$$

Theorem 1. *If $\delta^2 < \infty$ and $h = n^{-\alpha}$, then, for $\alpha > 1$, the estimate of $\hat{f}(0)$ is an asymptotically unbiased and consistent estimate of the parameter $f(0)$. The estimate $\hat{f}_z(0)$ is an asymptotically unbiased estimate of the parameter $f_z(0)$. At $1 < \alpha < 3/2$, the estimate $\hat{f}_z(0)$ is a consistent estimate of $f_z(0)$.*

Proof of Theorem 1. Denote $\tilde{y}_k = f(0) + f_z(0)kh + \varepsilon_k$ and put

$$\tilde{f}(0) = \frac{\sum_{k=-n}^n \tilde{y}_k}{2n+1}, \tilde{f}_z(0) = \frac{\sum_{k=-n}^n \tilde{y}_k kh}{\sum_{k=-n}^n (kh)^2}.$$

Estimates of $\tilde{f}(0)$, $\tilde{f}_z(0)$ are obtained by the least squares method for $f(0)$, $f_z(0)$ of linear regression [19] and satisfy the following relations

$$E\tilde{f}(0) = f(0), E\tilde{f}_z(0) = f_z(0), Var\tilde{f}(0) = \frac{\delta^2}{2n+1}, Var\tilde{f}_z(0) = \frac{\delta^2}{\sum_{k=-n}^n (kh)^2}. \tag{2}$$

Here, symbols $E \dots$, $Var \dots$ denote the mathematical expectation of random variable... and its variance. In turn, the following equalities are almost surely fulfilled

$$\hat{f}(0) - \tilde{f}(0) = \frac{\sum_{k=-n}^n (y_k - \tilde{y}_k)}{2n+1}, \hat{f}_z(0) - \tilde{f}_z(0) = \frac{\sum_{k=-n}^n (y_k - \tilde{y}_k)kh}{\sum_{k=-n}^n (kh)^2}. \tag{3}$$

Moreover, the differences $y_k - \tilde{y}_k = f(kh) - f(0) - f_z(0)kh$, $k = 0, \pm 1, \dots, \pm n$ are deterministic quantities. Then, from the Taylor formula with a residual term in the Lagrange form, inequalities follow

$$|f(kh) - f(0) - f_z(0)kh| \leq C_1(kh)^2, k = 0, \pm 1, \dots, \pm n. \tag{4}$$

From Formulas (3) and (4) for $n \rightarrow \infty$, the relations follow

$$|\hat{f}(0) - \tilde{f}(0)| \leq \frac{\sum_{k=-n}^n |f(kh) - f(0) - f_z(0)kh|}{2n+1} \leq \frac{2C_1 h^2 \sum_{k=1}^n k^2}{2n+1} \sim \frac{C_1 h^2 n^2}{3} \rightarrow 0, \tag{5}$$

$$|\hat{f}_z(0) - \tilde{f}_z(0)| \leq \frac{\sum_{k=-n}^n |(f(kh) - f(0) - f_z(0)kh)kh|}{\sum_{k=-n}^n (kh)^2} \leq \frac{C_1 h^3 \sum_{k=1}^n k^3}{\sum_{k=1}^n h^2 k^2} \sim \frac{C_1 hn}{4} \rightarrow 0. \tag{6}$$

The Formulas (2), (5) and (6) lead to the relations

$$|E\hat{f}(0) - f(0)| = |E\hat{f}(0) - E\tilde{f}(0)| \leq \frac{C_1 h^2 n^2}{2}, Var\hat{f}(0) = Var\tilde{f}(0), \tag{7}$$

$$|E\hat{f}_z(0) - f_z(0)| = |E\hat{f}_z(0) - E\tilde{f}_z(0)| \leq \frac{3C_1 hn}{4}, Var\hat{f}_z(0) = Var\tilde{f}_z(0). \tag{8}$$

Here, $a_n \preceq b_n$ means that $\limsup_{n \rightarrow \infty} a_n/b_n \leq 1$. Then, from the condition $h = n^{-\alpha}$, $\alpha > 1$, and the relations (7) and (8) we have

$$|E\widehat{f}(0) - f(0)| \rightarrow 0, |E\widehat{f}_z(0) - f_z(0)| \rightarrow 0, n \rightarrow \infty, \tag{9}$$

that $\widehat{f}(0)$, $\widehat{f}_z(0)$ are asymptotic unbiased estimates of $f(0)$, $f_z(0)$.

From the Bieneme–Chebyshev inequality, the relations (5) and (7) and the conditions $h = n^{-\alpha}$, $\alpha > 1$, we obtain for any $\varepsilon > 0$, $n \rightarrow \infty$

$$\begin{aligned} P(|\widehat{f}(0) - f(0)| > \varepsilon) &\leq P(|\widehat{f}(0) - \tilde{f}(0)| + |\tilde{f}(0) - f(0)| > \varepsilon) = \\ &= P(|\tilde{f}(0) - f(0)| \geq \varepsilon - |\widehat{f}(0) - \tilde{f}(0)|) \leq \frac{\delta^2}{(2n + 1)(\varepsilon - |\widehat{f}(0) - \tilde{f}(0)|)^2} \rightarrow 0. \end{aligned}$$

Thus, for $h = n^{-\alpha}$, $\alpha > 1$, estimate \widehat{f}_0 is a consistent estimate of x_0 .

At the same time, from the relations (6), (8) and (9) for $h = n^{-\alpha}$, $1 < \alpha < 3/2$, we obtain for any $\varepsilon > 0$, $n \rightarrow \infty$

$$\begin{aligned} P(|\widehat{f}_z(0) - f_z(0)| > \varepsilon) &\leq P(|\widehat{f}_z(0) - \tilde{f}_z(0)| + |\tilde{f}_z(0) - f_z(0)| > \varepsilon) = \\ &= P(|\tilde{f}_z(0) - f_z(0)| > \varepsilon - |\widehat{f}_z(0) - \tilde{f}_z(0)|) \leq \frac{3\delta^2}{h^2 n^3 (\varepsilon - |\widehat{f}_z(0) - \tilde{f}_z(0)|)^2} \rightarrow 0. \end{aligned}$$

Therefore, if the condition $h = n^{-\alpha}$, $1 < \alpha < 3/2$ is true, the estimate $\widehat{f}_z(0)$ is a consistent estimate of $f_z(0)$. □

Theorem 1 contains sufficient conditions in which estimates of function and its derivative in reference point tend in probability to their accuracy meanings. And we represent in this subsection necessary relations between number of observations around some reference point and distance between neighbour observations using probability metrics and accuracy of observations, when estimates tend in probability to their accuracy meanings.

2.2. Evaluation of the Second Derivative

In this subsection, an algorithm for estimating the second derivative is constructed. To do this, in addition to the point 0, in the neighbourhood of which $2n + 1$ observations have been made to estimate the first derivative, it is necessary to consider the point h_1 , $h_1 = n^{-\beta}$ and make $2n + 1$ observations in its neighbourhood. In order to conduct such an analysis and obtain a fairly good estimate of the second derivative, it is necessary to choose the right ratio between the parameters α , β . Inequalities are constructed between the parameters α , β , to ensure the proper quality of the evaluation of the second derivative.

To estimate the second derivative, we will use the following formula

$$\widehat{f}_{zz}(0) = \frac{\widehat{f}_z(h_1) - \widehat{f}_z(0)}{h_1}. \tag{10}$$

If the condition is met $\sup_{z_0 - hn \leq z \leq z_0 + hn + h_1} |f_{zz}(z)| = C_2 < \infty$. Then, from the Taylor formula with a residual term in the Lagrange form, we obtain

$$\left| \frac{f_z(h_1) - f_z(0)}{h_1} - f_{zz}(0) \right| \leq C_2 h_1. \tag{11}$$

Thus, we obtain

$$\left| \frac{E\widehat{f}_z(0) - f_z(z_0)}{h_1} \right| \leq \frac{C_2 \sum_{k=-n}^n |kh|^3}{h_1 \sum_{k=-n}^n (kh)^2} \leq \frac{C_2 hn}{h_1} \preceq \frac{hn}{h_1}. \tag{12}$$

Therefore, from relations (11) and (12), we have

$$\begin{aligned}
 |E\hat{f}_{zz}(0) - f_{zz}(0)| &= \left| \frac{E\hat{f}_z(h_1) - E\hat{f}_z(0)}{h_1} - f_{zz}(0) \right| \leq \left| \frac{E\hat{f}_z(0) - f_z(0)}{h_1} \right| + \\
 &+ \left| \frac{f_z(h_1) - f_z(0)}{h_1} - f_{zz}(0) \right| + \left| \frac{E\hat{f}_z(h_1) - f_z(h_1)}{h_1} \right| \leq C_2 h_1 + \frac{2C_2 h n}{h_1} \leq h_1 + \frac{h n}{h_1}.
 \end{aligned}
 \tag{13}$$

It is not difficult to prove that

$$\text{Var } \hat{f}_{zz}(0) = \text{Var } \frac{\hat{f}_z(h_1)}{h_1} + \text{Var } \frac{\hat{f}_z(0)}{h_1} \leq \frac{2\sigma^2}{h^2 h_1^2 n^3}.
 \tag{14}$$

Assuming

$$h = n^{-\alpha}, \quad 1 < \alpha < 3/2; \quad h_1 = n^{-\beta}, \quad 0 < \beta < \min\{\alpha - 1, 3/2 - \alpha\},
 \tag{15}$$

from relations (13) and (14) and Chebyshev’s inequality for any $\varepsilon > 0$ and for $n \rightarrow \infty$ we have

$$\begin{aligned}
 P(|\hat{f}_{zz}(0) - f_{zz}(0)| > \varepsilon) &\leq P(|\hat{f}_{zz}(0) - E\hat{f}_{zz}(0)| + |E\hat{f}_{zz}(0) - f_{zz}(0)| > \varepsilon) = \\
 &= P(|\hat{f}_{zz}(0) - E\hat{f}_{zz}(0)| > \varepsilon - |E\hat{f}_{zz}(0) - f_{zz}(0)|) \leq \frac{\text{Var } \hat{f}_{zz}(0)}{(\varepsilon - |E\hat{f}_{zz}(0) - f_{zz}(0)|)^2} \rightarrow 0.
 \end{aligned}$$

This means convergence in probability (consistency of the constructed estimate)

$$\hat{f}_{zz}(0) \xrightarrow{P} f_{zz}(0), \quad n \rightarrow \infty.
 \tag{16}$$

Such complex conditions (15) of convergence in probability seem to be related to the possible presence of a phase transition and are consistent with the known results on the turbophoresis of inertial particles [38].

2.3. Numerical Experiment for Diffusion Equation

Consider the diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}.
 \tag{17}$$

As $\hat{u}_t(t_0, x_0)$, $\hat{u}_{xx}(t_0, x_0)$ are consistent estimates of $u_t(t_0, x_0)$, $u_{xx}(t_0, x_0)$ relatively, it follows from Equation (17) that

$$\hat{D} = \frac{\hat{u}_t(t_0, x_0)}{\hat{u}_{xx}(t_0, x_0)}$$

is a consistent estimate of the diffusion coefficient D .

The computational experiment was carried out for Equation (17) with the given boundary condition $u_x(t, 0) = u_x(t, 1) = 0, \quad t \geq 0$, and the initial condition $u(0, x) = \cos 2\pi x, \quad 0 \leq x \leq 1$, in the case when $D = 0.01$. Via the Fourier method [35], it is not difficult to find a solution of this equation: $u(t, x) = \exp[-Dt(2\pi)^2] \cos 2\pi x$.

Suppose that inaccurate observations are obtained at the points $(t_0 \pm kh, x_0), (t_0, x_0 \pm kh), (t_0, x_0 + h_1 \pm kh), k = 0, 1, \dots, n$. Denote $\hat{u}_t(t_0, x_0), \hat{u}_{xx}(t_0, x_0)$ consistent estimates of partial derivatives $u_t(t_0, x_0), u_{xx}(t_0, x_0)$, respectively. We believe that inaccurate observations are obtained at points $(t_0 \pm kh, x_0), (t_0, x_0 \pm kh), (t_0, x_0 + h_1 \pm kh), k = 0, 1, \dots, n, t_0 = 0.05, x_0 = 0.1, h_1 = n^{-40/41}, h = n^{-5/4}, n = 450,000 : u(t_0 \pm kh, x_0) + \varepsilon_1(\pm kh), u(t_0, x_0 \pm kh) + \varepsilon_2(\pm kh), u(t_0, x_0 + h_1 \pm kh) + \varepsilon_3(\pm kh)$, where $\varepsilon_i(\pm kh), \quad i = 1, 2, 3, k = 0, 1, \dots, n$, are

i.i.d.r.v.'s distributed uniformly on the segment $[1/8, 1/8]$. As a result, we obtain an estimate $\widehat{D} = 0.0113026$.

3. Estimates of One-Soliton Solution Parameters

3.1. Preliminaries

The solutions of equations, which we will deal with below, belong to the class of one-soliton solutions. Here, using the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{18}$$

as an example, we briefly consider a simple method for obtaining such solutions. Our goal is to show that the estimating parameters of one-soliton solutions (modulo the choice of initial conditions) are the spectral parameters of appropriate linear problems. Details, as well as other methods for solving soliton equations, can be found in [29–34]. Note that the KdV equation is the first equation in which single-soliton solutions were found. It arises naturally in plasma physics, solid state physics, biology and many other areas.

The method under consideration is based on two fundamental concepts: the Lax representation (more generally, the zero curvature representation [30]) and the Darboux transformation [29]. Let us discuss these concepts in application to Equation (18).

It is well known that an arbitrary soliton equation is represented as a condition for the joint solvability of two, generally speaking, matrix linear equations. For (18), we have a system of scalar equations

$$L\psi = \lambda\psi, \quad \psi_t = A\psi, \tag{19}$$

where $L = -\partial_x^2 + u$ and $A = -4\partial_x^3 + 6u\partial_x + 3u_x$. The solvability condition is obtained by differentiating the first equation in (19) with respect to t , then eliminating ψ_t with the help of the second equation and replacing $\lambda\psi$ by $L\psi$. This condition is written as

$$\partial_t L = AL - LA \tag{20}$$

and is called the Lax representation.

The first equation of the system (19) is a one-dimensional stationary Schrodinger equation. Its one-fold Darboux transformation is determined by the relation

$$L'T = TL, \tag{21}$$

where $L' = -\partial_x^2 + u'$ is a transformed Schrodinger operator and T is a first-order differential operator over x . It follows from (20) that $u' = u - 2\partial_x^2 \ln \varphi$ and $T = \partial_x - \varphi_x/\varphi$, where $\varphi = \varphi(t, x)$ is the eigenvector of the operator L , corresponding to the eigenvalue λ . For example, if $u = 0$, we can take

$$\varphi = \cosh[k(x - x_0) - \alpha], \quad \lambda = -k^2, \tag{22}$$

where $\alpha = \alpha(t)$ is an arbitrary real function.

Let us now show how, using the Lax representations and Darboux transformations, we can construct a one-soliton solution of Equation (18). We require that, along with (21), the following relation is fulfilled:

$$A'T = \partial_t T + TA, \tag{23}$$

where A' is an operator of the same type as A , but with u replaced by u' . It turns out that this requirement completely defines the operator A' and that the Lax representation (20) is valid for the corresponding dashed operators. The latter, in turn, means that $u' = u - 2\partial_x^2 \ln \varphi$, a solution of Equation (18). Choosing a function φ (22) from (23), we find $\alpha = 4k^3 t$ and

as a result we obtain the desired one-soliton solution (in which we replace for further convenience u' by u)

$$u = -2k^2 \cosh^{-2}[k(x - x_0) - 4k^3t]. \tag{24}$$

Multi-soliton solutions are obtained in a similar way by iterating the Darboux transformation.

Thus, using the KdV equation as an example, we have shown that the parameters of solutions of soliton equations estimated in this paper (excluding the initial conditions) are the spectral parameters of the corresponding linear problems. In the considered case, $-k^2$, the eigenvalue of the Schrodinger operator L , x_0 , the initial value of the coordinate x are employed. In a similar way, one-soliton solutions are sought in all the equations under consideration in this subsection.

3.2. Construction and Estimates of Parameters in One-Soliton Solutions

In this subsection, all random variables characterizing observation errors have a uniform distribution on the segment $[-1/8, 1/8]$.

KdV equation. Consequently for a KdV one-soliton solution we have

$$u_t = -16k^5w^{-3} \sinh(z), \quad z = k(x - 4k^2t - x_0), \quad w = \cosh(z),$$

$$u_x = 4k^3w^{-3} \sinh(z) \Rightarrow k = \sqrt{-\frac{u_t}{4u_x}}.$$

From (24), we obtain

$$w^2 = \frac{u_t}{2uu_x}.$$

Choosing reference point $(t, x) = (0, 0)$, we obtain

$$k = \sqrt{-\frac{u_t(0,0)}{4u_x(0,0)}}, \quad w = \sqrt{\frac{u_t(0,0)}{2u(0,0)u_x(0,0)}}, \quad x_0 = -\frac{\text{arch}(w)}{k}.$$

Then, using the method of moments we obtain

$$\hat{k} = \sqrt{-\frac{\hat{u}_t}{4\hat{u}_x}}, \quad \hat{w} = \sqrt{\frac{\hat{u}_t(0,0)}{2\hat{u}(0,0)\hat{u}_x(0,0)}}, \quad \hat{x}_0 = -\frac{\text{arch}(\hat{w})}{\hat{k}}.$$

The computational experiment was carried out in the case of $x_0 = k = 1$ with the number $n = 300,000$. The following results were obtained

$$\hat{k} = 0.997589, \quad \hat{x}_0 = 0.989265.$$

If in the KdV equation with $k = 1$ we increase n from 300,000 to 3,000,000, we obtain a change of estimate \hat{k} from 0.997589 to 1.00075.

mKdV equation. Consider now mKdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{25}$$

This equation is used in descriptions of isotropic media dimensionally quantized films, acoustic waves in plasma and internal waves in a symmetric stratified liquid. The one-soliton solution of Equation (25) is

$$u(t, x) = sA \sinh(A(x - A^2t - x_0)), \quad s = \pm 1,$$

then

$$u_x = sA^2 \cosh(A(x - A^2t - x_0)), \quad u_t = -sA^4 \cosh(A(x - A^2t - x_0)) \Rightarrow A = \sqrt{-\frac{u_t}{u_x}}.$$

Choosing reference point $(t, x) = (0, 0)$, we obtain from (25)

$$\frac{u}{sA} = \sinh(-Ax_0) = -\sinh(Ax_0)$$

and so

$$A = \sqrt{-\frac{u_t(0,0)}{u_x(0,0)}}, \quad x_0 = -\frac{1}{A} \operatorname{arsh} \left(\frac{u(0,0)}{s} \sqrt{-\frac{u_x(0,0)}{u_t(0,0)}} \right).$$

Using the method of moments, we obtain

$$\hat{A} = \sqrt{-\frac{\hat{u}_t(0,0)}{\hat{u}_x(0,0)}}, \quad \hat{x}_0 = -\frac{1}{\hat{A}} \operatorname{arsh} \left(\frac{\hat{u}(0,0)}{s} \sqrt{-\frac{\hat{u}_x(0,0)}{\hat{u}_t(0,0)}} \right).$$

The computational experiment was carried out in the case $s = 1$ and for $x_0 = A = 1$ at the point $(x, t) = (0, 0)$ with the number $n = 300,000$. The following results were obtained

$$\hat{A} = 1.00114, \quad \hat{x}_0 = 1.00855.$$

But it is possible to improve results of numerical experiments if we assume that accuracy of observations is higher (random variables characterizing observations have a uniform distribution on the segment $[-d, d]$ with $d < 1/8$). If in the mKdV equation with $A = 1$ we change d from $1/8$ to $1/80$, we obtain a change of estimate \hat{A} from 1.00114 to 1.00016.

These experiments show that not only number n but accuracy of observations influence accuracy of estimates.

Sine–Gordon equation. Consider now the Sine–Gordon equation

$$\varphi_{xx} - \varphi_{tt} = m^2 \sin \varphi.$$

This equation is used in descriptions of Bloch wall motion in ferromagnetic crystals, Jackson constants in superconductivity and nonlinear optics. Its one-soliton solution is

$$\varphi(t, x) = 4 \arctan(\exp(m\gamma(x - vt) + \delta)), \quad \gamma^2 = \frac{1}{1 - v^2}. \tag{26}$$

Denote $w = \exp(m\gamma(x - vt) + \delta)$; then

$$\varphi_t = -4m\gamma v \frac{w}{1 + w^2}, \quad \varphi_x = 4m\gamma \frac{w}{1 + w^2} \Rightarrow v = -\frac{\varphi_t}{\varphi_x}, \quad \gamma = \sqrt{\frac{\varphi_x^2}{\varphi_x^2 - \varphi_t^2}}.$$

Choosing reference point $(t, x) = (0, 0)$, we obtain

$$\varphi(0, 0) = 4 \arctan e^\delta \Rightarrow \delta = \ln \tan \frac{\varphi(0, 0)}{4}, \quad \gamma = \sqrt{\frac{\varphi_x^2(0, 0)}{\varphi_x^2(0, 0) - \varphi_t^2(0, 0)}}.$$

Choosing another reference point $(t, x) = (0, 1)$, we obtain

$$m\gamma + \delta = \ln \tan \frac{\varphi(0, 1)}{4} \Rightarrow m = \frac{1}{\gamma} \left(\ln \tan \frac{\varphi(0, 1)}{4} - \delta \right).$$

Then, we obtain

$$\hat{\delta} = \ln \tan \frac{\hat{\varphi}(0, 0)}{4}, \quad \hat{\gamma} = \sqrt{\frac{\hat{\varphi}_x^2(0, 0)}{\hat{\varphi}_x^2(0, 0) - \hat{\varphi}_t^2(0, 0)}}, \quad \hat{m} = \frac{1}{\hat{\gamma}} \left(\ln \tan \frac{\hat{\varphi}(0, 1)}{4} - \hat{\delta} \right).$$

The computational experiment was carried out in the case of $\gamma = 1.1575$, $\delta = 1$, $m = 2$ with the number $n = 300,000$. The following results were obtained

$$\hat{\gamma} = 1.15352, \delta = 0.96641, \hat{m} = 2.03116.$$

Burgers equation. Consider Burgers equation

$$u_t + uu_x = u_{xx}, \tag{27}$$

Equation (27) is used in hydrodynamics, dislocation theory and visco-elasticity theory. Its one-soliton solution may be represented in the form

$$u(t, x) = a\delta(1 - \tanh(z)), \quad z = \frac{1}{2}(ax - \delta a^2 t),$$

then

$$u_x = -\frac{a^2\delta}{2 \cosh^2(z)}, \quad u_t = \frac{a^3\delta^2}{2 \cosh^2(z)} \Rightarrow \delta a = -\frac{u_t}{u_x}.$$

Choosing reference point $(t, x) = (0, 0)$, we obtain parameters a, δ :

$$u_t(0, 0) = \frac{a^3\delta^2}{2}, \quad u_x(0, 0) = -\frac{\delta a^2}{2} \Rightarrow a = \frac{2u_x^2(0, 0)}{u_t(0, 0)}, \quad \delta = -\frac{u_t^2(0, 0)}{2u_x^3(0, 0)}.$$

Then, using the moments method, we obtain

$$\hat{a} = \frac{2\hat{u}_x^2(0, 0)}{\hat{u}_t(0, 0)}, \quad \hat{\delta} = -\frac{\hat{u}_t^2(0, 0)}{2\hat{u}_x^3(0, 0)}.$$

The computational experiment was carried out in the case of $a = 1$, $\delta = 1$ with the number $n = 300,000$. The following results were obtained

$$\hat{a} = 1.00355, \quad \hat{\delta} = 0.992503.$$

Nonlinear Schrodinger equation. The nonlinear Schrodinger equation has the following form

$$iu_t + u_{xx} + v|u|^2u = 0. \tag{28}$$

and is used in nonlinear optics and plasma physics. The one-soliton solution of this equation has the form [39]

$$u(x, t) = \exp\{irx - ist\}v(x - Ut) \tag{29}$$

where r, s, U are constants connected by the relations:

$$r = \frac{U}{2}, \quad s = \frac{U^2}{4} - \alpha \tag{30}$$

and the function $v(q)$ satisfies an ordinary differential equation of the form $\ddot{v} - \alpha v + v v^3 = 0$ with $\alpha = r^2 - s$.

The one-soliton solution of this equation has the form

$$|u(x - Ut)| = v(x - Ut) = \sqrt{\frac{2\alpha}{v}} \cosh^{-1}[\sqrt{\alpha}(x - Ut)]. \tag{31}$$

Thus, the parameter α determines the amplitude of the waves and the parameter U determines their speed. Next, we assume that the parameters of Equation (29) are equal to:

$$U = 1, \alpha = 1, v = 4, \rho = \frac{\alpha}{v} = \frac{1}{4},$$

the parameters of r, s are determined by the equalities (30) $r = 0.5, s = -0.75$. From Formula (31), it follows that

$$U = -\frac{v_t(x - Ut)}{v_x(x - Ut)}.$$

If $(x, t) = (1, -1)$, then we obtain $U = -\frac{v_t(1, -1)}{v_x(1, -1)}$, where the estimate is as follows

$$\hat{U} = -\frac{\hat{v}_t(1, -1)}{\hat{v}_x(1, -1)}. \tag{32}$$

So if $(x, t) = (0, 0)$, then $2\rho = v^2(0, 0)$ and consequently

$$\hat{\rho} = \frac{\hat{v}^2(0, 0)}{2}. \tag{33}$$

In turn, for $(x, t) = (1, -1)$, we obtain from Formula (31) the equation

$$\frac{v_t(1, -1)}{v^2(1, -1)} = \sqrt{\frac{v}{2}} U \sinh(\sqrt{\rho v}(1 + U)).$$

It follows that in order to estimate the parameter v , it is necessary to solve the equation

$$\frac{\hat{v}_t(1, -1)}{\hat{v}^2(1, -1)} = \sqrt{\frac{v}{2}} \hat{U} \sinh\left(\sqrt{\hat{\rho} \cdot v}(1 + \hat{U})\right). \tag{34}$$

Inserting estimates $\hat{U}, \hat{\rho}$ from (32) and (33) into Equation (34) and solving this equation by v , we find the estimate \hat{v} . Estimates of listed parameters for the number $n = 300,000$ are $\hat{U} = 1.03146, \hat{v} = 3.84376, \hat{\alpha} = 0.960547, \hat{r} = 0.51573, \hat{s} = -0.69457$.

It should be noted that most of the one-soliton solutions have the form of travelling waves. Therefore, by choosing the almost arbitrary reference point (t_1, x_1) , it is not difficult to estimate the velocity of a travelling wave through partial derivatives of the solution by the coordinates t, x at the point (t_1, x_1) . Moreover, for the nonlinear Schrodinger equation, the point $(t_2, x_2) = (0, 0)$ can serve as another reference point. Using this reference point allows us to estimate all the parameters of this equation. These rather simple considerations can be used to estimate the parameters of one-soliton solutions and other nonlinear partial differential equations.

When we calculate the ratio a/b with the inaccuracy estimates \hat{a}, \hat{b} (we used this calculation in all the examples from Section 3.2), the accuracy of the calculations is quite high. But when we calculate the roots of hyperbolic functions with small derivatives, the result may be worse in some (not all) numerical experiments. This is due to the inaccuracy estimates at the first step of the proposed algorithm and the small derivatives of the functions whose roots we calculate. Nevertheless, our proposed algorithm allows us to control the accuracy of estimates of the function and its derivative under the conditions of convergence in probability to their exact values and to choose the number of observations in the vicinity of the reference points, the distance between neighbouring observations around the reference point and the accuracy of observations. The result of such control is presented in numerical experiments, which give an affordable accuracy (several percent), especially in nonlinear Schrodinger equation. This is important when designing a measuring system. Similar problems have arisen, for example, when measuring hidden parameters in quantum physics (Nobel Prize on Physics, 2022).

4. Discussion

The technique of the moment method developed in this work is also applicable to estimating the parameters of functions that are not necessarily related to differential equations. This formulation of the question allows us to consider some problems of

reliability theory. And finally, in some cases there is no need to estimate the parameters of differential equations, for example, in problems about solitons. It is only necessary to estimate the solution parameters of these equations. Moreover, in the problem of solitons one can consider not only one-soliton, but also multi-soliton solutions. And finally, the method proposed in this paper allows us to estimate the coefficients of nonlinear partial differential equations. This issue is especially important when solving inverse problems of mathematical physics and requires additional and more detailed consideration. In particular, for differential equation systems in a small neighbourhood of a certain point, the theorems of solution existence and uniqueness can be used (see, for example, [40,41]).

5. Conclusions

This paper considers the problem of estimating the differential equation parameters from numerous observations near certain reference points. Conditions are obtained under which the constructed estimates are consistent. The review includes both linear and nonlinear differential equations. The greatest computational difficulties arose when estimating the diffusion coefficient in Section 2. According to our data, it took several minutes to solve the problems in question in the Mathematics package. The described algorithms were quite successfully used to work with FEFU students (educational program: computer design) in the subject of systems analysis. This work mainly considers systems of differential equations with analytical solutions (with the exception of Section 3.1). But there are many differential equations and systems that cannot be solved analytically. In the future, when developing this topic, it is planned to consider such equations and systems, the solutions of which can only be obtained using computer calculations. The purpose of this article is to initiate the development of the proposed method in various directions. In particular, using high computing speed, you can choose a model of the system under consideration from several options. As considered in this work, the problems of estimating the differential equation parameters are close to the problems of Kalman filtering. This circumstance allows us to pose the question of the advisability of conducting a large number of observations in the vicinity of the reference point.

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