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Stochastic Quasi-Geostrophic Equation with Jump Noise in L_p Spaces

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Abstract: In this paper, we consider a 2D stochastic quasi-geostrophic equation driven by jump noise in a smooth bounded domain. We prove the local existence and uniqueness of mild $L^p(D)$-solutions for the dissipative quasi-geostrophic equation with a full range of subcritical powers $\alpha \in (\frac{1}{2}, 1]$ by using the semigroup theory and fixed point theorem. Our approach, based on the Yosida approximation argument and Itô formula for the Banach space valued processes, allows for establishing some uniform bounds for the mild solutions and we prove the global existence of mild solutions in $L^\infty(0,T;L^p(D))$ space for all $p > \frac{2}{2\alpha-1}$, which is consistent with the deterministic case.

Keywords: Poisson random measure; stochastic quasi-geostrophic equation; global mild solution; semigroup

MSC: 35Q55; 60H15; 60G51

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1. Introduction

The two-dimensional quasi-geostrophic (QG) equation

$$\partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0$$

is considered as an essential model in geophysical fluid dynamics and has been widely used in oceanography and meteorology for modeling and forecasting the mid-latitude oceanic and atmospheric circulation. Physically, the scalar θ represents the potential temperature and u is the fluid velocity. When $\alpha = 1/2$, the behavior of the solution of the QG equation shares similar features with the potentially singular solutions of the three-dimensional fluid motion equations, so it sometimes serves as a lower dimensional model of the 3D Navier–Stokes equations. The cases of $\alpha < 1/2$, $\alpha = 1/2$, $\alpha > 1/2$ are called supercritical, critical and subcritical cases, respectively. Since the pioneering work by Constantin, Majda and Tabak [1], and also the work of Resniak [2], the QG equation has been studied from a wide variety of perspectives (see, e.g., ref. [3] for the critical dissipative QG equation on $\mathbb{R}^n$, ref. [4] for the 2D QG equation with critical and sub-critial cases, ref. [5] for the 2D dissipative QG equation in the Sobolev space, and ref. [6] for the global well-posedness for the 2D dissipative QG equation, to name just a few).

1.1. Motivation

Our motivation behind studying stochastic quasi-geostrophic equations with jump noise mostly stems from four recents works [7–10].

For the deterministic case, Wu [9] studied the dissipative QG equation with $\alpha > 1/2$ on the whole space $\mathbb{R}^2$. In the case of the whole space, the kernel of the convolution operator of the fractional Laplacian has an explicit form and possesses similar properties to the heat kernel. By using the method of integral equations and a contraction mapping argument, Wu proved the local existence and uniqueness of solutions of the dissipative sub-critical QG equation in $L^q([0,T];L^p(\mathbb{R}^2))$, $p > \frac{2}{2\alpha-1}$, $\frac{1}{p} + \frac{\alpha}{q} = \alpha - \frac{1}{2}$.

The random fluctuations or noise are ubiquitous to physical systems in nature, which might be either due to spatial and temporal fluctuations of certain parameters, thermal fluctuations, fluids flow through a random media, or random initial conditions, etc. By taking into account the stochastic behavior of the system, Röckner, Zhu and Zhu [8] conducted a study of the 2D stochastic quasi-geostrophic equation with multiplicative Gaussian noise on $\mathbb{T}^2$. They established the existence of a martingale solution for all $\alpha \in (0,1)$, and the existence and uniqueness of (probabilistically) strong solutions in the subcritical cases. Brzézniak and Motyl [7] proved the existence of a martingale solution for a stochastic quasi-geostrophic equation on $\mathbb{R}^d$, $d \in \mathbb{N}$ with multiplicative Gaussian noise. In the two-dimensional subcritical case, they also established the pathwise uniqueness of the solutions.

In some circumstances, Gaussian noise may not fully capture the possibility of having sudden and large moves, which occur commonly in real-world models. Jump-type perturbations come to the stage to reproduce the performance of those natural phenomena. Compared to the case of Gaussian noise, there has been considerably fewer results regarding the study of stochastic partial differential equations with jump noise. This motivates us, including discontinuous random perturbation effects in the QG equation models. To the best of our knowledge, there is only one article so far which employs the well-posedness of the mild solutions for 2D stochastic QG equations with jump noise. In reference [10], the first-named author, Brzézniak and Liu considered the 2D stochastic quasi-geostrophic equation with $\alpha = 1$ driven by the compensated Poisson measures on $\mathbb{R}^2$ and established the existence and uniqueness of mild solutions in $L^4([0,T];L^4(\mathbb{R}^2))$.

The current paper is motivated by similar questions and we will study the stochastic quasi-geostrophic equation driven by jump noise in a bounded domain and generalize the results in references [10] to all subcritical ranges $\alpha \in (\frac{1}{2}, 1]$. Unlike the case of the full space $\mathbb{R}^d$ in references [7,9,10], we consider the case of a bounded domain and QG equations are less well studied on bounded domains. It is worth mentioning that we obtain the global well-posedness of the solution of the stochastic QG equation in $L^\infty([0,T];L^p(D))$ for all $p > \frac{2}{2\alpha-1}$, which is consistent with the deterministic case presented in reference [9].

1.2. Formulation of the Problem

Let $D \subset \mathbb{R}^2$ be a bounded open domain with a sufficiently smooth boundary. In this paper, we consider the following two-dimensional stochastic quasi-geostrophic equation driven by Lévy-type noise on $D \subset \mathbb{R}^2$

$$\begin{aligned} &\mathrm{d}\theta(t) + [(v(t)\cdot\nabla)\,\theta(t) + \kappa(-\Delta)^\alpha\,\theta(t)]\,\mathrm{d}t = \int_Z \xi(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \quad t \geq 0, \\ &\theta(t,\cdot) = 0 \text{ on } \partial D, \\ &\theta(0) = \theta_0, \end{aligned} \tag{1}$$

where $\kappa \geq 0$, $\alpha \in [0,1]$, and $\theta : \mathbb{R}^+ \times D \to \mathbb{R}$ represents the potential temperature and the velocity v is determined from θ using a stream function ψ via the auxiliary relations

$$v = (v_1, v_2) = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right) \quad \text{and} \quad (-\Delta)^{1/2}\psi = -\theta. \tag{2}$$

Let $\Lambda := (-\Delta)^{\frac{1}{2}}$ and $\nabla^\perp = (-\partial_2, \partial_1)$ be the gradient rotated by $\frac{\pi}{2}$. Using the notations Λ and $\nabla^\perp$, the relations in (2) can be combined into

$$v = \nabla^\perp\Lambda^{-1}\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta),$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the usual Riesz transforms. Here, we assume that $(Z, \mathcal{Z})$ is a measurable space, ν is a non-negative σ-finite measure on it and $\tilde{N}$ is a time homogeneous compensated Poisson random measure defined on a given complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with an intensity measure ν on Z, where $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual condition. The noise coefficient $\xi : [0,T] \times Z \to L^p(D)$ is a deterministic measurable function such that $\xi \in L^2([0,T] \times Z; L^p(D))$. The nonnegative number κ distinguishes between the inviscid quasi-geostrophic equation ($\kappa = 0$) and the dissipative quasi-geostrophic equation ($\kappa > 0$). We will assume that $\kappa = 1$.

Compared to references [7,9,10], we consider the stochastic QG equation in a bounded domain. There are significant differences between the well-posedness problems in the whole space and in bounded domains. In the case of the whole space, quasi-geostrophic equations possess certain symmetries, such as translation invariance, which can be exploited to establish the well-posedness results (see [11,12]) for further discussion. Additionally, the Fourier transform plays a crucial rule in the study of the quasi-geostrophic equation in the whole space, as it can be used to derive an explicit kernel for the fractional Laplacian (see, e.g., reference [9]). In contrast, the well-posedness of a quasi-geostrophic equation in bounded domains is more delicate due to the presence of boundaries and the lack of translation invariance and powerful tools of Fourier analysis. To establish well-posedness in bounded domains, specialized techniques and tools are required. In particular, we need to establish appropriate estimates for the nonlinearity of the equation.

Furthermore, due to the presence of a discontinuous driving term in the equation, in contrast to references [7,8], it is necessary to employ alternative analytical techniques to handle the jump noise term. Rather than utilizing the Galerkin approximation, as in references [7,8], we employ a regularization procedure based on Yosida approximations. The Itô formula for Banach space valued processes driven by a compensated Poisson random measure is adopted in the proof. The following is our main result regarding the global existence and uniqueness:

Theorem 1. *Let $T > 0$, $\alpha \in (\frac{1}{2}, 1]$, and $\frac{2}{2\alpha-1} < p < \infty$. Assume that $\theta_0 \in L^p(D)$ and $\xi \in L^2([0,T] \times Z; L^p(D)) \cap L^p([0,T] \times Z; L^p(D))$. Then, there exists a unique global mild solution $\theta \in L^\infty(0,T; L^p(D))$ of (1).*

1.3. Plan of the Paper

The paper is organized as follows. In Section 2, we introduce some basic notations and collect various facts about the fractional Laplacian with Dirichlet boundary condition and the semigroup on $L^p(D)$. In Section 3, the problem (1) is reformulated as an abstract stochastic equation. Based on the fixed point argument and semigroup theory, the local existence and uniqueness of mild solution in $L^\infty(0,T; L^p(D))$ is established (see Theorem 2). In Section 4, we give the proof of our global well-posedness result (Theorem 1). The conclusion is given in Section 5.

2. Preliminaries

In this section, we introduce some preliminary result that will be useful for the proof of our main results.

Let $D \subset \mathbb{R}^2$ be an arbitrary open set and E be a finite dimensional space. We denote by $L^p(D; E)$, $p \in (1, \infty)$ the space of E-valued functions on D, such that

$$\|u\|_{L^p} := \left(\int_D \|u(x)\|_E^p dx \right)^{1/p}$$

is finite. If $E = \mathbb{R}$, then we will simply write $L^p(D)$ instead of $L^p(D; \mathbb{R})$.

For $1 \leqslant p \leqslant \infty$ and $-\infty < s < \infty$ the Sobolev space $H^{s,p}(\mathbb{R}^2)$ is defined by

$$H^{s,p}(\mathbb{R}^2) := \left\{ (I - \Delta)^{-\alpha/2} g : g \in L^p(\mathbb{R}^2) \right\},$$

with norm
$$\|u\|_{H^{s,p}(\mathbb{R}^2)} = \left\|(I-\Delta)^{\alpha/2}u\right\|_{L^p(\mathbb{R}^2)}.$$

For $s \geqslant 0$, define the Sobolev space $H^{s,p}(D)$ as the space of restrictions of functions in $H^{s,p}(\mathbb{R}^2)$ to D. That is, a function $u \in L_p(D)$ is in $H^{s,p}(D)$ if and only if there exists a function $U \in H^{s,p}(\mathbb{R}^2)$ such that $U_{|D} = u$ almost everywhere in D. For $u \in H^{s,p}(D)$, its $H^{s,p}$-norm is defined by

$$\|u\|_{H^{s,p}(D)} = \inf_{U\in H^{s,p}(\mathbb{R}^2), U_{|D}=u} \|U\|_{H^{s,p}(\mathbb{R}^2)}.$$

With this norm, $H^{s,p}(D)$ is a reflexive Banach space. Let $1 < p < \infty$ and $0 \le s_0 < s_1 < \infty$. Then, for $0 \le \theta \le 1$ and $s = (1-\theta)s_0 + \theta s_1$,

$$[H^{s_0,p}(D), H^{s_1,p}(D)]_\theta = H^{s,p}(D) \quad \text{with norm equivalence,}$$

where $[H^{s_0,p}(D), H^{s_1,p}(D)]_\theta$ denotes the complex interpolation space. Define

$$H_0^{s,p}(D) := \text{the completion of } C_0^\infty(D) \text{ in } H^{s,p}(D).$$

For $0 \le s \le \frac{1}{p}$ with $1 < p < \infty$, the space $C_0^\infty(D)$ is dense in $H^{s,p}(D)$ and hence

$$H^{s,p}(D) = H_0^{s,p}(D), \quad \text{for} \quad 0 \le s \le \frac{1}{p}.$$

The domain of the minus Laplace operator with inhomogeneous Dirichlet boundary condition on $L^p(D)$ is

$$\begin{aligned} \mathscr{D}(A_p) &= H_0^{1,p}(D) \cap H^{2,p}(D), \\ A_p u &= -\Delta u, \quad u \in \mathscr{D}(A_p). \end{aligned} \tag{3}$$

Lemma 1 ([13], Theorem 2.4.4)**.** *The operator A_p is densely defined and non-negative and $-A_p$ generates a contractive and analytic C_0-semigroup.*

For all $s > 0$, we define A_p^s, the fractional power of the operator A_p, as the inverse of

$$A_p^{-s} := \frac{1}{\Gamma(s)} \int_0^\infty z^{s-1} e^{-zA_p} dz.$$

The domain of A_p^s is defined by the complex interpolation

$$\mathscr{D}(A_p^s) = \left[L^p(D), \mathscr{D}(A_p)\right]_s.$$

It is a Banach space equipped with the graph norm $\|\cdot\|_{\mathscr{D}(A_p^s)} := \|A_p^s \cdot\|_{L^p}$. According to references [14,15], we have

$$\mathscr{D}(A_p^s) = \begin{cases} H^{2s,p}(D) & \text{if } 0 \le s < \frac{1}{2p} \\ \tilde{H}^{2s,p}(D) = \left\{u \in H^{2s,p}(D); \gamma u = 0 \text{ on } \partial\Omega\right\} & \text{if } \frac{1}{2p} < s \le 1, \ s \ne \frac{p+1}{2p}. \end{cases}$$

Since $\tilde{H}^{2s,p}(D) = H_0^{2s,p}(D)$ when $\frac{1}{2p} < s < \frac{1}{2} + \frac{1}{2p}$ (see, e.g., reference ([16], Proposition 3.3) or ([17], Theorem 4.7.1), we have

$$\mathscr{D}\left(A_p^s\right) = \begin{cases} H^{2s,p}(D) & \text{for} \quad 0 \le s < \frac{1}{2p}, \\ H_0^{2s,p}(D) & \text{if} \quad \frac{1}{2p} < s < \frac{1}{2} + \frac{1}{2p}, \\ H^{2s,p}(D) \cap H_0^{1,p}(D) & \text{if} \quad \frac{1}{2} + \frac{1}{2p} < s \le 1. \end{cases}$$

Lemma 2. *For $1 < p < \infty$, the embedding*

$$\mathscr{D}(A_p^s) \hookrightarrow H^{2s,p}(D) \tag{4}$$

is continuous for $s \geq 0$.

Proof. The case $0 \leq s \leq 2$ immediately follows from ([15], Theorem 16.15). For the case $s > 2$, using the general interpolation argument, it is sufficient to prove (4) for $s = 2, 3, 4, \cdots$. So, next assume by induction that (4) holds for $s \leq k$ with $k \geq 2$. Let $\theta \in \mathscr{D}(A_p^{k+1})$. Then, $v = A_p\theta \in \mathscr{D}(A_p^k)$. Hence, using the induction assumption, $v \in H^{2k,p}(D)$ and

$$\|A_p\theta\|_{H^{2k,p}} \leq C\|A_p^k(A_p\theta)\|_{L^p}.$$

According to ([16], Theorem 0.3), we have

$$\|\theta\|_{H^{2(k+1),p}} \leq C\|A_p\theta\|_{H^{2k,p}} \leq C\|A_p^k(A_p\theta)\|_{L^p} = \|A_p^{k+1}\theta\|_{L^p},$$

which shows that (4) holds for $s = k+1$. □

Lemma 3 ([15,18])**.** *The operator $-A_p^\alpha$, $\alpha \in (0,1]$ generates a contractive and analytic semigroup $S_\alpha(t) := e^{-tA_p^\alpha}$ satisfying for $\beta \geq 0$,*

$$\left\| A_p^\beta S_\alpha(t) \right\|_{\mathcal{L}(L^p)} \leq C_{\alpha,\beta,p}\, t^{-\frac{\beta}{\alpha}},$$

with some constant $C_{\alpha,\beta,p}$ depending on α, β and p.

Lemma 4. *For every $s \geq \frac{1}{p}$, the operator A_p^{-s} extends uniquely to a bounded operator from $L^{\frac{p}{2}}(D)$ to $L^p(D)$.*

Proof. Take $r = \frac{p}{2}$. Let $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. By applying Lemma 2 and Sobolev embedding ([17], Theorem 4.6.1), we have

$$\mathscr{D}(A_{p'}^s) \hookrightarrow H^{2s,p'}(D) \hookrightarrow L^{r'}(D),$$

continuously for every $s \geq \frac{1}{p}$, which implies that there exists a unique bounded extension of $A_{p'}^s$ from $L^{r'}(D)$ to $L^{p'}(D)$. It follows that the operator $A_{p'}^{-s}$ is bounded from $L^{p'}(D)$ to $L^{r'}(D)$. By duality, this implies that A_p^{-s} is a bounded operator form $L^r(D)$ to $L^p(D)$. □

For $j = 1, 2$, define the Riesz transform as follows:

$$\mathcal{R}_j\theta := \frac{\partial}{\partial x_j} A_p^{-\frac{1}{2}}\theta, \qquad \theta \in L^p(D), \;\; j = 1,2. \tag{5}$$

Denote $\mathscr{R}\theta := (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$. Because of the form of $v = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$, the following equality holds:

$$v \cdot \nabla\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) \cdot \left(\frac{\partial\theta}{\partial x_1}, \frac{\partial\theta}{\partial x_2} \right) = \nabla \cdot (v\theta). \tag{6}$$

Recall from ([13], p. 300) that for $\psi \in \mathscr{D}(A_p^\beta), \beta > 0, j = 1, 2$,

$$A_p^\beta \mathcal{R}_j\psi = \mathcal{R}_j A_p^\beta \psi. \tag{7}$$

Lemma 5 ([4], Proposition A.1)**.** *For each $j = 1,2$ and $1 < p < \infty$, $\mathcal{R}_j$ is a bounded linear operator from $L^p(D)$ to $L^p(D)$, and the operator $A_p^{-\frac{1}{2}} \frac{\partial}{\partial x_j}$ extends uniquely to a bounded linear operator from $L^p(D)$ to $L^p(D)$.*

3. Local Existence and Uniqueness

With the above notations, Equation (1) can be written as

$$\begin{aligned} &\mathrm{d}\theta(t) = -A_p^{\alpha}\theta(t)\mathrm{d}t - (\mathscr{R}\theta(t)\cdot\nabla)\theta(t))\,\mathrm{d}t + \int_Z \xi(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \quad t \geq 0, \\ &\theta(0) = \theta_0. \end{aligned} \tag{8}$$

We propose the following definition of a mild solution to problem (8).

Definition 1. *Assume that $T > 0$, $\alpha \in (\frac{1}{2}, 1]$ and $\frac{2}{2\alpha-1} < p < \infty$. We call an $\mathbb{F}$-progressively measurable process $\theta : [0,T] \times \Omega \to L^p(D)$ a mild solution in $L^\infty(0,T;L^p(D))$ to (8) if the following conditions hold:*

1. *θ has càdlàg paths $\mathbb{P}$-a.s.;*
2. *$\theta \in L^\infty(0,T;L^p(D))$, $\mathbb{P}$-a.s.;*
3. *for all $t \in [0,T]$, the following equality holds $\mathbb{P}$-a.s.*

$$\theta(t) = S_\alpha(t)\theta_0 - \int_0^t S_\alpha(t-s)((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))\,\mathrm{d}s + \int_0^t\int_Z S_\alpha(t-s)\xi(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z). \tag{9}$$

We first consider the following stochastic linear equation driven by a compensated Poisson random measure:

$$\begin{aligned} &\mathrm{d}Z = -A_p^{\alpha} Z\mathrm{d}t + \int_Z \xi(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \\ &Z(0) = 0. \end{aligned} \tag{10}$$

The mild solution of (10) is defined to be

$$Z(t) = \int_0^t\int_Z S_\alpha(t-s)\xi(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z), \qquad t \in [0,T]. \tag{11}$$

Proposition 1. *Assume that $\xi \in L^2([0,T]\times Z; L^p(D))$, $\frac{2}{2\alpha-1} < p < \infty$. Then, the process Z given by (11) has a càdlàg version satisfying*

$$\mathbb{E}\sup_{0\leq s\leq T}\|Z(t)\|_{L^p} \leq C\Big(\int_0^T\int_Z \|\xi(s,z)\|_{L^p}^2\nu(\mathrm{d}z)\mathrm{d}s\Big)^{\frac{1}{2}} < \infty.$$

Proof. Since $L^p(D)$, $p > \frac{2}{2\alpha-1} \geq 2$, is a martingale type 2 Banach space and $S_\alpha(t)$ is a contractive and analytic semigroup, by applying ([10], Theorem 3.1) we find that Z has a càdlàg modification and

$$\mathbb{E}\sup_{0\leq s\leq T}\|Z(t)\|_{L^p} \leq C\Big(\int_0^T\int_Z \|\xi(s,z)\|_{L^p}^2\nu(\mathrm{d}z)\mathrm{d}s\Big)^{\frac{1}{2}} < \infty.$$

□

Proposition 2. *For any u, u_1, $u_2 \in L^p(D)$, we have*

$$\Big\|A^{-\frac{1}{2}-\frac{1}{p}}(\mathscr{R}u\cdot\nabla)u\Big\|_{L^p} \leq M_p\|u(s)\|_{L^p}^2, \tag{12}$$

$$\left\| A^{-\frac{1}{2}-\frac{1}{p}}(\mathscr{R}u_1 \cdot \nabla)u_1 - A^{-\frac{1}{2}-\frac{1}{p}}(\mathscr{R}u_2 \cdot \nabla)u_2 \right\|_{L^p} \leq M_p \|u_1 - u_2\|_{L^p} \big(\|u_1\|_{L^p} + \|u_2\|_{L^p} \big). \tag{13}$$

with some constant M_p depending only on p.

Proof. Observe that

$$((\mathscr{R}(u)) \cdot \nabla)u = \nabla \cdot ((\mathscr{R}u)u).$$

Recall from Lemma 3 that $A_p^{-\frac{1}{p}}$ is a bounded operator from $L^{\frac{p}{2}}(D)$ to $L^p(D)$. Using Lemma 5, we obtain

$$\begin{aligned}
&\left\| A_p^{-\frac{1}{2}-\frac{1}{p}}(\mathscr{R}u(s) \cdot \nabla)u(s) \right\|_{L^p} \\
&= \left\| A_p^{-\frac{1}{p}} A_p^{-\frac{1}{2}} \Big(-\frac{\partial}{\partial x_1}\mathcal{R}_2(u)u + \frac{\partial}{\partial x_2}\mathcal{R}_1(u)u \Big) \right\|_{L^p} \\
&\leq C_p \Big(\left\|\mathcal{R}_2(u)u\right\|_{L^{p/2}} + \left\|\mathcal{R}_1(u)u\right\|_{L^{p/2}} \Big) \\
&\leq C_p K \|u\|^2_{L^p},
\end{aligned}$$

where we also used the Cauchy–Schwartz inequality and the boundedness of $\mathcal{R}_j$ with $\|\mathcal{R}_j\|_{\mathcal{L}(L^p(D))} \leq \frac{K}{2}$ in the last inequality. Similar arguments show that

$$\begin{aligned}
&\left\| A_p^{-\frac{1}{2}-\frac{1}{p}}((\mathscr{R}u_1(s) \cdot \nabla)u_1(s) - (\mathscr{R}u_2(s) \cdot \nabla)u_2(s)) \right\|_{L^p} \\
\leq& \left\| A_p^{-\frac{1}{p}} A_p^{-\frac{1}{2}} \Big(-\frac{\partial}{\partial x_1}\mathcal{R}_2(u_1)(u_1-u_2) + \frac{\partial}{\partial x_2}\mathcal{R}_1(u_1)(u_1-u_2) \Big) \right\|_{L^p} \\
&+ \left\| A_p^{-\frac{1}{p}} A_p^{-\frac{1}{2}} \Big(-\frac{\partial}{\partial x_1}\mathcal{R}_2(u_1-u_2)u_2 + \frac{\partial}{\partial x_2}\mathcal{R}_1(u_1-u_2)u_2 \Big) \right\|_{L^p} \\
\leq& C_p \Big[\|\mathcal{R}_2(u_1)(u_1-u_2)\|_{L^{p/2}} + \|\mathcal{R}_1(u_1)(u_1-u_2)\|_{L^{p/2}} \\
&\qquad + \|\mathcal{R}_2(u_1-u_2)u_2\|_{L^{p/2}} + \|\mathcal{R}_1(u_1-u_2)u_2\|_{L^{p/2}} \Big] \\
\leq& C_p K (\|u_1\|_{L^p} + \|u_2\|_{L^p}) \|u_1(s) - u_2(s)\|_{L^p}.
\end{aligned}$$

□

Theorem 2. *Let $\alpha \in (\frac{1}{2}, 1]$, $T > 0$, and $\frac{2}{2\alpha-1} < p < \infty$. Assume that $\theta_0 \in L^p(D)$ and $\xi \in L^2([0,T] \times Z; L^p(D))$. Then, there exists a stopping time τ^* taking values $\mathbb{P}$-a.s. in $(0,T]$, such that Equation (8) has a unique mild solution $\theta \in L^\infty(0,\tau^*; L^p(D))$, $\mathbb{P}$-a.s.*

Proof. If a process $\theta = (\theta(t))_{t\in[0,T]}$, is a mild solution to Problem (8), then a process $Y = ((Y(t))_{t\in[0,T]}$ defined by

$$Y(t) = \theta(t) - Z(t), \quad t \in [0,T],$$

satisfies, on a heuristic level, the following nonlinear equation:

$$\begin{aligned}
&\mathrm{d}Y(t) = -A_p^\alpha Y(t)\mathrm{d}t - (\mathscr{R}(Y(t)+Z(t)) \cdot \nabla)(Y(t)+Z(t))\,\mathrm{d}t, \quad t \in (0,T] \\
&Y(0) = \theta_0.
\end{aligned} \tag{14}$$

Since Proposition 1 guarantees the existence of a mild solution Z in $L^\infty(0,T; L^p(D))$ of (10), solving for θ of (8) in $L^\infty(0,T; L^p(D))$ is equivalent to solving for Y of (14) in

$L^\infty(0,T;L^p(D))$. We will establish the existence and uniqueness of a local mild solution of (14) in $L^\infty(0,T;L^p(D))$ by using the fixed point arguments.

Let us choose and fix $T>0$ and $\theta_0 \in L^p(D)$. Define the operator Γ by

$$\Gamma(Y) := S_\alpha(t)\theta_0 - \int_0^t S_\alpha(t-s)((\mathscr{R}(Y(s)+Z(s)))\cdot\nabla)(Y(s)+Z(s))\mathrm{d}s, \quad t\in[0,T].$$

Let τ be a stopping time where its value will be determined later on. Define a random constant

$$N_0 = \|\theta_0\|_{L^p} + \sup_{t\in[0,T]} \|Z(t)\|_{L^p},$$

and set $R = 2N_0$. Let B_R^τ be the closed ball with radius R centered at the origin in $L^\infty(0,\tau;L^p(D))$.

To establish the existence of mild solutions to (14), it is equivalent to finding a fixed point for the operator Γ in $L^\infty(0,\tau;L^p(D))$. In other words, we are looking for a random time τ such that $\Gamma(B_R^\tau)\subset B_R^\tau$ and Γ is a contraction map on B_R^τ.

Let $Y\in B_R^\tau$. Since S_α is a contractive and analytic semigroup on $L^p(D)$, we infer

$$\|S_\alpha(t)\theta_0\|_{L^p} \le \|\theta_0\|_{L^p}.$$

To estimate the nonlinear term, observe first that

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)((\mathscr{R}(Y(s)+Z(s)))\cdot\nabla)(Y(s)+Z(s))\mathrm{d}s\right\|_{L^p}\\
&\le \int_0^t \left\|S_\alpha(t-s)((\mathscr{R}(Y(s)+Z(s)))\cdot\nabla)(Y(s)+Z(s))\right\|_{L^p}\mathrm{d}s\\
&= \int_0^t \left\|A_p^{\frac12+\frac1p}S_\alpha(t-s)A_p^{-\frac12-\frac1p}((\mathscr{R}(Y(s)+Z(s)))\cdot\nabla)(Y(s)+Z(s))\right\|_{L^p}\mathrm{d}s.
\end{aligned}$$

Using Lemma 3 and Proposition 2, we obtain

$$\begin{aligned}
&\left\|A_p^{\frac12+\frac1p}S_\alpha(t-s)A_p^{-\frac12-\frac1p}((\mathscr{R}(Y(s)+Z(s)))\cdot\nabla)(Y(s)+Z(s))\right\|_{L^p}\\
&\le C_{\alpha,p}(t-s)^{-\frac1\alpha\left(\frac12+\frac1p\right)}\left\|A_p^{-\frac12-\frac1p}((\mathscr{R}(Y(s)+Z(s)))\cdot\nabla)(Y(s)+Z(s))\right\|_{L^p}\\
&\le 2M_pC_{\alpha,p}(t-s)^{-\frac1\alpha\left(\frac12+\frac1p\right)}\left(\|Y(s)\|_{L^p}^2+\|Z(s)\|_{L^p}^2\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)((\mathscr{R}(Y(s)+Z(s)))\cdot\nabla)(Y(s)+Z(s))\mathrm{d}s\right\|_{L^p}\\
&\le 2M_pC_{\alpha,p}\int_0^t (t-s)^{-\frac1\alpha\left(\frac12+\frac1p\right)}\left(\|Y(s)\|_{L^p}^2+\|Z(s)\|_{L^p}^2\right)\mathrm{d}s.
\end{aligned}$$

Here, all the constants are independent of T. For $\alpha>\frac12$ and $p>\frac{2}{2\alpha-1}$,

$$0<\frac1\alpha\left(\frac12+\frac1p\right)<1,$$

we obtain

$$\sup_{0\le t\le\tau}\|\Gamma(Y)(t)\|_{L^p} \le \|\theta_0\|_{L^p(D)} + \frac{\tau^{1-\frac1\alpha(\frac12+\frac1p)}}{1-\frac1\alpha(\frac12+\frac1p)}2M_pC_{\alpha,p}\left(\sup_{t\in[0,\tau]}\|Y(s)\|_{L^p}^2+\sup_{t\in[0,\tau]}\|Z(s)\|_{L^p}^2\right)$$

$$\le N_0 + \frac{\tau^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})}}{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})} 2M_pC_{\alpha,p}\Big((2N_0)^2 + N_0^2\Big)$$
$$\le 2N_0\Big(\frac{1}{2} + \frac{\tau^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})}}{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})} 6M_pC_{\alpha,p}N_0\Big).$$

The stopping time τ is defined as follows:

$$\tau := \inf\left\{ t \in [0,T] : \frac{t^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})}}{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})} 6M_pC_{\alpha,p}\Big(\|\theta_0\|_{L^p} + \sup_{t\in[0,T]}\|Z(t)\|_{L^p}\Big) > \frac{1}{2}\right\} \wedge T. \quad (15)$$

It follows that

$$\sup_{0\le t\le\tau}\|\Gamma(Y)\|_{L^p} \le 2N_0\Big(\frac{1}{2} + \frac{\tau^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})}}{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})} 6M_pC_{\alpha,p}N_0\Big) \le 2N_0.$$

Hence, $\Gamma(Y) \in B_R^\tau$.

Next, we shall show that Γ is a contraction on B_R^τ. Take $Y_1, Y_2 \in B_R^\tau$. Let $\theta_i = Y_i + Z$, $i = 1,2$. Applying Lemma 3 and Proposition 2 again, we obtain

$$\|\Gamma(Y_1)(t) - \Gamma(Y_2)(t)\|_{L^p}$$
$$= \left\|\int_0^t S_\alpha(t-s)\big((\mathscr{R}(\theta_1(s))\cdot\nabla)\theta_1(s) - (\mathscr{R}(\theta_2(s))\cdot\nabla)\theta_2(s)\big)\mathrm{d}s\right\|_{L^p}$$
$$\le \int_0^t C_{\alpha,p}(t-s)^{-\frac{1}{\alpha}\left(\frac{1}{2}-\frac{1}{p}\right)}\left\|A_p^{-\frac{1}{2}-\frac{1}{p}}\big((\mathscr{R}(\theta_1(s))\cdot\nabla)\theta_1(s) - (\mathscr{R}(\theta_2(s))\cdot\nabla)\theta_2(s)\big)\right\|_{L^p}\mathrm{d}s$$
$$\le M_pC_{\alpha,p}\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{1}{2}-\frac{1}{p}\right)}(\|Y_1\|_{L^p} + \|Y_2\|_{L^p} + 2\|Z\|_{L^p})\|Y_1 - Y_2\|_{L^p}\mathrm{d}s.$$

It follows that

$$\sup_{t\in[0,\tau]}\|\Gamma(Y_1)(t) - \Gamma(Y_2)(t)\|_{L^p} \le \frac{\tau^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})}}{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})} 6M_pC_{\alpha,p}N_0 \sup_{t\in[0,\tau]}\|Y_1 - Y_2\|_{L^p}.$$

According to the definition of τ, we find $\frac{\tau^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})}}{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{p})} 6M_pC_{\alpha,p}N_0 < 1$. Using the Banach fixed point theorem, there exits a unique local mild solution $Y \in L^\infty(0,\tau; L^p(D))$ to the Equation (14). Since $\theta = Y + Z$ and $Z \in L^\infty(0,T;L^p(D))$, $\mathbb{P}$-a.s., we obtain a unique local mild solution $\theta \in L^\infty(0,\tau;L^p(D))$ for Equation (8).

To extend the solution to a larger interval, we can repeat the argument above and consider the equation with initial data $Y(\tau) \in L^p(D)$. Define $N^\tau(t,A) := N(\tau + t, A) - N(\tau, A)$. Then, it is a Poisson random measure with respect to $(\mathcal{F}_{t+\tau})_{t\ge0}$ and independent of $\mathcal{F}_\tau$. Let

$$Z^\tau(t) = \int_0^t \int_Z S_\alpha(t-s)\xi(s,z)\tilde{N}^\tau(\mathrm{d}s,\mathrm{d}z), \qquad t \in [0,T]. \quad (16)$$

It is easy to prove that

$$Z^\tau(t) = \int_0^t \int_Z S_\alpha(t-s)\xi(s,z)\tilde{N}^\tau(\mathrm{d}s,\mathrm{d}z) = \int_\tau^{\tau+t}\int_Z S_\alpha(\tau+t-s)\xi(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z). \quad (17)$$

Consider

$$\mathrm{d}Y^{(1)}(t) = -A_p^\alpha Y^{(1)}(t)\mathrm{d}t - (\mathscr{R}(Y^{(1)}(t) + Z^\tau(t))\cdot\nabla)(Y^{(1)}(t) + Z^\tau(t))\,\mathrm{d}t, \ t \in (0,T] \quad (18)$$
$$Y^{(1)}(0) = Y_\tau. \quad (19)$$

Similar arguments to before show that there exists a stopping time $\tau_1 > \tau_0$ and a function $Y^{(1)} \in L^\infty(0, \tau_1; L^p(D))$, such that $Y^{(1)}$ is a unique local mild solution to the Equation (19) on $[0, \tau_1]$. Set

$$\theta(t) = \begin{cases} Y(t) + Z(t), & \text{for} \quad t \in [0, \tau] \\ Y^{(1)}(t-\tau) + Z^{\tau}(t-\tau), & \text{for} \quad t \in [\tau, \tau_1]. \end{cases}$$

In this way, we extend our solution to the time interval $[0, \tau_1]$. Repeating this procedure a finite number of times leads to a mild solution θ in $L^q(0, \tau^*; L^p(D))$ to the Equation (8) on $[0, \tau^*]$, where τ^* is the supremum time over $[0, T]$ upon which the solution exists. □

4. Global Solution

Our next task in this section is to extend the local time solution established in Theorem 2 to a global solution.

Proposition 3 ([19], Corollary 2.3)**.** *For $\alpha \in [0,1]$, $p \in [2, +\infty)$, $\phi \in \mathscr{D}(A_p^\alpha)$ and $|\phi|^{p-1} \in \mathscr{D}(A^{\alpha/2})$, the following estimate holds:*

$$\int_D |\phi|^{p-2}\phi A_p^\alpha \phi \, dx \geq \frac{4(p-1)}{p^2} \int_D \left[A_p^{\frac{\alpha}{2}} \left(|\theta|^{\frac{p}{2}} \right) \right]^2 \mathrm{d}x \geq 0.$$

In the sequel, we drop the subscript p attached to A for simplicity of notations.

Proposition 4. *For $\alpha \in [0,1]$, define $A_\lambda^\alpha := \lambda A^\alpha (\lambda + A^\alpha)^{-1}$, $\lambda > 0$. Then, for $p \in [2, +\infty)$, $\phi \in \mathscr{D}(A^\alpha)$ and $|\phi|^{p-1} \in \mathscr{D}(A^{\alpha/2})$,*

$$\int_D |\phi|^{p-2}\phi A_\lambda^\alpha \phi \, \mathrm{d}x \geq 0.$$

Proof. According to ([19], p. 475) and ([20], p. 320), we have

$$\begin{aligned} &\int_D |\phi|^{p-2}\phi A_\lambda^\alpha \phi \, dx \\ &= \int_D |\phi|^{p-2}\phi (\lambda A^\alpha (\lambda + A^\alpha)^{-1}) \phi \, dx \\ &= \lambda \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \frac{\lambda s^{\alpha-1}}{\lambda^2 + 2\lambda s^\alpha \cos \pi\alpha + s^{2\alpha}} \int_D |\phi|^{p-2}\phi (I - s(sI + A)^{-1}) \phi \, \mathrm{d}x \, \mathrm{d}s \\ &\geq 0. \end{aligned}$$

□

Proof of Theorem 1. Let τ^* be the supremum time over $[0, T]$ upon which the mild solution θ exists. We will prove that $\mathbb{P}(\tau^* = T) = 1$. Define the following stopping times:

$$\tau_n = \inf\{t \in [0, T] : \|\theta(t)\|_{L^p(D)} > n\} \wedge T, \quad n \in \mathbb{N}.$$

We first approximate (8) using equations with strong solutions. For this purpose, we will use a regularization procedure via Yosida approximations $A_\lambda^\alpha = \lambda A^\alpha (\lambda + A^\alpha)^{-1}$, and $J_\lambda := \lambda(\lambda + A^\alpha)^{-1}$, where $\lambda > 0$ is sufficiently large. Then, A_λ^α is the infinitesimal generator of a uniformly continuous semigroup of contractions

$$S_\alpha^\lambda(t) = e^{-tA_\lambda^\alpha}, \quad t \in [0, T],$$

and

$$\lim_{\lambda \to \infty} A_\lambda^\alpha x = A^\alpha x, \quad \text{for} \quad u \in \mathcal{D}(A^\alpha) \tag{20}$$

$$\lim_{\lambda\to\infty} S_\alpha^\lambda(t)u = S_\alpha(t)u, \quad \text{for} \quad u \in L^p(D), \quad \text{uniformly in } t \text{ on } [0,T]. \tag{21}$$

The Yosida approximation operator J_λ has the following fundamental properties on the space $L^p(D)$, $1 < p < \infty$:

$$\begin{aligned} &J_\lambda u \in \mathcal{D}(A^\alpha), \\ &\|J_\lambda\|_{\mathcal{L}(L^p)} \le 1, \quad \text{for all} \quad \lambda > 0, \\ &J_\lambda u \to u \text{ in } L^p(D) \text{ for each } u \in L^p(D), \text{ as } \lambda \to \infty. \end{aligned} \tag{22}$$

Consider the following equation:

$$\begin{aligned} &\Theta(t) = -A^\alpha\Theta(t)\mathrm{d}t - 1_{[0,\tau_n)}(t)((\mathscr{R}\theta(t)\cdot\nabla)\theta(t))\,\mathrm{d}t + \int_Z 1_{[0,\tau_n]}(t)\xi(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z) \\ &\Theta(0) = \theta_0, \end{aligned}$$

which has a unique mild solution and we have

$$\begin{aligned} \Theta(t) &= S_\alpha(t)\theta_0 - \int_0^t S_\alpha(t-s)1_{[0,\tau_n)}(s)((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))\mathrm{d}s \\ &\quad + \int_0^t\int_Z S_\alpha(t-s)1_{[0,\tau_n]}(t)\xi(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z) \\ &= \theta(t), \qquad \text{for } 0 \le t \le \tau_n. \end{aligned}$$

According to Theorem 2, the regularized equation

$$\begin{aligned} &\theta_\lambda(t) = -A_\lambda^\alpha\theta_\lambda(t)\mathrm{d}t - 1_{[0,\tau_n)}(t)((\mathscr{R}\theta(t)\cdot\nabla)\theta_\lambda(t))\,\mathrm{d}t + \int_Z 1_{[0,\tau_n]}(t)J_\lambda\xi(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z) \\ &\theta_\lambda(0) = \theta_0, \end{aligned}$$

has a unique mild solution θ_λ. In addition, since A_λ^α is bounded, θ_λ is also a strong solution.

Observe that, for $0 \le t \le T$,

$$\begin{aligned} &\Theta(t) - \theta_\lambda(t) \\ &= \Big[S_\alpha(t) - S_\alpha^\lambda(t)\Big]\theta_0 \\ &\quad - \int_0^t \Big[S_\alpha(t-s)1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta(s)) - S_\alpha^\lambda(t-s)1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta_\lambda(s))\Big]\,\mathrm{d}s \\ &\quad + \int_0^t\int_Z \Big[S_\alpha(t-s)\xi(s,z) - S_\alpha^\lambda(t-s)J_\lambda\xi(s,z)\Big]1_{[0,\tau_n]}\tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &= \Big[S_\alpha(t) - S_\alpha^\lambda(t)\Big]\theta_0 \\ &\quad - \int_0^t \Big[S_\alpha(t-s) - S_\alpha^\lambda(t-s)\Big]1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))\,\mathrm{d}s \\ &\quad - \int_0^t S_\alpha^\lambda(t-s)1_{[0,\tau_n)}\Big[(\mathscr{R}\theta(s)\cdot\nabla)\theta(s) - (\mathscr{R}\theta(s)\cdot\nabla)\theta_\lambda(s))\Big]\,\mathrm{d}s \\ &\quad + \int_0^t\int_Z \Big[S_\alpha(t-s)\xi(s,z) - S_\alpha^\lambda(t-s)J_\lambda\xi(s,z)\Big]1_{[0,\tau_n]}\tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &:= \Big[S_\alpha(t) - S_\alpha^\lambda(t)\Big]\theta_0 + N_1^\lambda(t) + N_2^\lambda(t) + M^\lambda(t). \end{aligned}$$

Since

$$\|A_\lambda^\gamma S_\alpha^\lambda(t)\|_{\mathcal{L}(L^p)} \le C_\gamma t^{-\frac{\gamma}{\alpha}}$$

where C_γ is independent of λ, and similar to arguments as in the proofs of Theorem 2 and Proposition 2, we infer

$$\begin{aligned}\|N_2^\lambda(t)\|_{L^p} &= \Big\| \int_0^t S_\alpha^\lambda(t-s)1_{[0,\tau_n)}\Big[(\mathscr{R}\theta(s)\cdot\nabla)\theta(s) - (\mathscr{R}\theta(s)\cdot\nabla)\theta_\lambda(s))\Big]\mathrm{d}s\Big\|_{L^p} \\ &\le M_p C_{\alpha,p}\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{p}\right)}1_{[0,\tau_n)}\|\theta(s)\|_{L^p}\|\theta(s)-\theta_\lambda(s)\|_{L^p}\,\mathrm{d}s.\end{aligned}$$

Consequently, for $0\le t\le\tau_n$,

$$\begin{aligned}&\|\theta(t)-\theta_\lambda(t)\|_{L^p} \\ &= \|\Theta(t)-\theta_\lambda(t)\|_{L^p} \\ &\le \Big\|\big[S_\alpha(t)-S_\alpha^\lambda(t)\big]\theta_0\Big\|_{L^p} + \|N_1^\lambda(t)\|_{L^p} \\ &\quad + M_p\,C_{\alpha,p}\int_0^t (t-s)^{-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{p}\right)}1_{[0,\tau_n)}\|\theta(s)\|_{L^p}\|\theta(s)-\theta_\lambda(s)\|_{L^p}\,\mathrm{d}s + \|M^\lambda(t)\|_{L^p}.\end{aligned}$$

It follows from the Gronwall inequality that, for $0\le t\le\tau_n$,

$$\begin{aligned}&\|\theta(t)-\theta_\lambda(t)\|_{L^p} \\ &\le \left(\Big\|\big[S_\alpha(t)-S_\alpha^\lambda(t)\big]\theta_0\Big\|_{L^p} + \|N_1^\lambda(t)\|_{L^p} + \|M^\lambda(t)\|_{L^p}\right)e^{M_pC_{\alpha,p}\left(\int_0^t(t-s)^{-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{p}\right)}1_{[0,\tau_n)}\|\theta\|_{L^p}\,\mathrm{d}s\right)} \\ &\le \left(\Big\|\big[S_\alpha(t)-S_\alpha^\lambda(t)\big]\theta_0\Big\|_{L^p} + \|N_1^\lambda(t)\|_{L^p} + \|M^\lambda(t)\|_{L^p}\right)e^{M_pC_{\alpha,p}nqT^{\frac{1}{q}}},\end{aligned} \tag{23}$$

where $\frac{1}{q} = 1-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{p}\right)$. Since $\Big\|\big[S_\alpha(t)-S_\alpha^\lambda(t)\big]\theta_0\Big\|_{L^p}\to 0$, as $\lambda\to\infty$ uniformly on $[0,T]$, we have

$$\lim_{\lambda\to\infty}\sup_{t\in[0,T]}\Big\|\big[S_\alpha(t)-S_\alpha^\lambda(t)\big]\theta_0\Big\|_{L^p} = 0.$$

Meanwhile, due to

$$\begin{aligned}&\Big\|\Big[S_\alpha(t-s)-S_\alpha^\lambda(t-s)\Big]1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))\Big\|_{L^p} \\ &\le M_pC_{\alpha,p}(t-s)^{-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{p}\right)}1_{[0,\tau_n)}(s)\|\theta(s)\|_{L^p}^2,\end{aligned}$$

we obtain

$$\begin{aligned}\sup_{t\in[0,T]}\|N_1^\lambda(t)\|_{L^p(D)} &\le \sup_{t\in[0,T]}\Big\|\int_0^t\Big[S_\alpha(t-s)-S_\alpha^\lambda(t-s)\Big]1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))\,\mathrm{d}s\Big\|_{L^p(D)} \\ &\le M_p\,C_{\alpha,p}\,q\,T^{\frac{1}{q}}\sup_{s\in[0,\tau_n)}\|\theta(s)\|_{L^p(D)}^2 \le M_p\,C_{\alpha,p}\,q\,T^{\frac{1}{q}}n^2.\end{aligned}$$

Since $\Big\|\Big[S_\alpha(t-s)-S_\alpha^\lambda(t-s)\Big]1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))\Big\|_{L^p}\to 0$, as $\lambda\to\infty$, uniformly on finite intervals, for every s, we can apply the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}&\sup_{t\in[0,T]}\|N_1^\lambda(t)\|_{L^p(D)} \\ &= \sup_{t\in[0,T]}\Big\|\int_0^t\Big[S_\alpha(t-s)-S_\alpha^\lambda(t-s)\Big]1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))\,\mathrm{d}s\Big\|_{L^p(D)} \\ &\le \int_0^T\sup_{t\in[0,T]}\big\|\big[S_\alpha(t-s)-S_\alpha^\lambda(t-s)\big]1_{[0,\tau_n)}((\mathscr{R}\theta(s)\cdot\nabla)\theta(s))1_{[0,t]}(s)\big\|_{L^p(D)}\,\mathrm{d}s\end{aligned}$$

$$\to 0, \quad \text{as} \quad \lambda \to \infty.$$

Note that, for $v \in \mathcal{D}(A^\alpha)$,

$$\begin{aligned} S_\alpha(t)v - S_\alpha^\lambda(t)v &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} S_\alpha(t\theta) S_\alpha^\lambda(t(1-\theta))v \, \mathrm{d}\theta \\ &= \int_0^1 t S_\alpha(t\theta) S_\alpha^\lambda(t(1-\theta))(A^\alpha v - A_\lambda^\alpha v) \, \mathrm{d}\theta, \end{aligned}$$

and hence

$$\|S_\alpha(t)v - S_\alpha^\lambda(t)v\|_{L^p} \leq \|t(A^\alpha v - A_\lambda^\alpha v)\|_{L^p}. \tag{24}$$

The C_0-semigroups S_α and S_α^λ of contractions can be extended to C_0-groups of contractions, still denoted by S_α and S_α^λ, respectively. Since $J_\lambda \xi \in \mathcal{D}(A^\alpha)$, using the maximal inequality ([10], Theorem 3.1) and (24), we see that

$$\begin{aligned} &\mathbb{E} \sup_{0 \leq t \leq T} \left\| M^\lambda(t) \right\|_{L^p} \\ &\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t \int_Z S_\alpha(t-s) \Big[\xi(s,z) - J_\lambda \xi(s,z) \Big] 1_{[0,\tau_n]} \tilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|_{L^p} \\ &\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t \int_Z \Big[S_\alpha(t-s) - S_\alpha^\lambda(t-s) \Big] J_\lambda \xi(s,z) 1_{[0,\tau_n]} \tilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|_{L^p} \\ &\leq C_p \left(\int_0^T \int_Z \|\xi(s,z) - J_\lambda \xi(s,z)\|_{L^p}^2 \nu(\mathrm{d}z) \mathrm{d}s \right)^{\frac{1}{2}} \\ &\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t \int_Z (S_\alpha(t) - S_\alpha^\lambda(t)) S_\alpha(-s) J_\lambda \xi(s,z) 1_{[0,\tau_n]} \tilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|_{L^p} \\ &\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left\| S_\alpha^\lambda(t) \int_0^t \int_Z \Big[S_\alpha(-s) - S_\alpha^\lambda(-s) \Big] J_\lambda \xi(s,z) 1_{[0,\tau_n]} \tilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|_{L^p} \\ &\leq C_p \left(\int_0^T \int_Z \|\xi(s,z) - J_\lambda \xi(s,z)\|_{L^p}^2 \nu(\mathrm{d}z) \mathrm{d}s \right)^{\frac{1}{2}} \\ &\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^1 t S_\alpha(t\theta) S_\alpha^\lambda(t(1-\theta)) \int_0^t \int_Z (A^\alpha - A_\lambda^\alpha) S_\alpha(-s) J_\lambda \xi(s,z) 1_{[0,\tau_n]} \tilde{N}(\mathrm{d}s, \mathrm{d}z) \, d\theta \right\|_{L^p} \\ &\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t \int_Z \Big[S_\alpha(-s) - S_\alpha^\lambda(-s) \Big] J_\lambda \xi(s,z) 1_{[0,\tau_n]} \tilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|_{L^p(D)} \\ &\leq C_p \left(\int_0^T \int_Z \|\xi(s,z) - J_\lambda \xi(s,z)\|_{L^p}^2 \nu(\mathrm{d}z) \mathrm{d}s \right)^{\frac{1}{2}} \\ &\quad + T \left(\int_0^T \int_Z \left\| (A_\lambda^\alpha - A^\alpha) J_\lambda \xi(s,z) 1_{[0,\tau_n]} \right\|_{L^p}^2 \nu(\mathrm{d}z) \mathrm{d}s \right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^T \int_Z \left\| s (A_\lambda^\alpha - A^\alpha) J_\lambda \xi(s,z) 1_{[0,\tau_n]} \right\|_{L^p}^2 \nu(\mathrm{d}z) \mathrm{d}s \right)^{\frac{1}{2}}. \end{aligned}$$

The three terms on the right hand side go to zero as $\lambda \to \infty$, according to (20), (22) and the Dominated Convergence Theorem.

From (23), we infer that

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0,\tau_n]} \|\theta(t) - \theta_\lambda(t)\|_{L^p(D)} \\ &\leq \left(\sup_{t \in [0,T]} \left\| \big[S_\alpha(t) - S_\alpha^\lambda(t) \big] \theta_0 \right\|_{L^p} + \mathbb{E} \sup_{t \in [0,T]} \|N_1^\lambda(t)\|_{L^p} + \mathbb{E} \sup_{t \in [0,T]} \|M^\lambda(t)\|_{L^p} \right) e^{M_p C_{\alpha,p} n q T^{\frac{1}{q}}}. \end{aligned}$$

Therefore,

$$\lim_{\lambda\to\infty}\mathbb{E}\sup_{t\in[0,\tau_n]}\|\theta_\lambda(t)-\theta(t)\|_{L^p(D)}=0. \tag{25}$$

Next, put

$$\Psi(u):=\|u\|_{L^p}^p,\quad u\in L^p(D).$$

Then, for all $u\in L^p(D)$ and $h,h_1,h_2\in L^p(D)$, we have

$$\Psi'(u)h=p\int_D|u(x)|^{p-2}u(x)h(x)\mathrm{d}x$$

and

$$\Psi''(u)(h_1,h_2)=p(p-1)\int_D|u(x)|^{p-2}h_1(x)h_2(x)\mathrm{d}x.$$

Observe that

$$\begin{aligned}
&p\int_D|\theta_\lambda|^{p-2}\theta_\lambda(\mathscr{R}\theta\cdot\nabla)\theta_\lambda dx\\
&=\int_D p|\theta_\lambda|^{p-2}\theta_\lambda\left(\frac{\partial\theta_\lambda}{\partial x_1}\frac{\partial}{\partial x_2}\left[A^{-\frac12}\theta\right]-\frac{\partial\theta_\lambda}{\partial x_2}\frac{\partial}{\partial x_1}\left[A^{-\frac12}\theta\right]\right)dx\\
&=\int_D\left(\frac{\partial(|\theta_\lambda|^p)}{\partial x_1}\frac{\partial}{\partial x_2}\left[A^{-\frac12}\theta\right]-\frac{\partial(|\theta_\lambda|^p)}{\partial x_2}\frac{\partial}{\partial x_1}\left[A^{-\frac12}\theta\right]\right)dx=0,
\end{aligned}\tag{26}$$

where we used integration by parts. Moreover, for all $u,\ h\in L^p(D)$ we have

$$|\Psi'(u)h|=p\left|\int_D|u(x)|^{p-2}u(x)h(x)\mathrm{d}x\right|\le p\|u\|_{L^p}^{p-1}\|h\|_{L^p}. \tag{27}$$

Applying the Itô formula (see, e.g., reference [10]) to the process θ_λ and the function $\|\cdot\|_{L^p}^p$, we obtain for all $t\in[0,T]$,

$$\begin{aligned}
&\|\theta_\lambda(t)\|_{L^p}^p-\|\theta_0\|_{L^p}^p\\
&=-\int_0^t\int_D p|\theta_\lambda(s,x)|^{p-2}\theta_\lambda(s,x)A_\lambda^\alpha\theta_\lambda(s,x)\mathrm{d}x\mathrm{d}s\\
&\quad-\int_0^t\int_D p|\theta_\lambda(s,x)|^{p-2}\theta_\lambda(s,x)(\mathscr{R}\theta(s,x)\cdot\nabla)\theta_\lambda(s,x)\mathrm{d}x\mathrm{d}s\\
&\quad+\int_0^t\int_Z\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\,\tilde N(\mathrm{d}s,\mathrm{d}z)\\
&\quad+\int_0^t\int_Z\|\theta_\lambda(s-)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s-)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\,N(\mathrm{d}s,\mathrm{d}z)\\
&\le\int_0^t\int_Z\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\,\tilde N(\mathrm{d}s,\mathrm{d}z)\\
&\quad+\int_0^t\int_Z\|\theta_\lambda(s-)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s-)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\,N(\mathrm{d}s,\mathrm{d}z)\\
&:=I_1(t)+I_2(t)
\end{aligned}$$

where we also used Proposition 3 and (26).

For the first term I_1, by using the Doob inequality (27) and the Young inequality, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|I_1(t)|\\
&\le 3\mathbb{E}\Big(\int_0^T\int_Z|\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)|^2\nu(\mathrm{d}z)\mathrm{d}s\Big)^{\frac12}
\end{aligned}$$

$$\le 3\mathbb{E}\Big(\int_0^T\int_Z p^2\|\theta_\lambda(s)\|_{L^p}^{2p-2}\|J_\lambda\xi(s,z)\|_{L^p}^2\nu(\mathrm{d}z)\mathrm{d}s\Big)^{\frac{1}{2}}$$
$$\le 3p\mathbb{E}\left[\Big(\sup_{t\in[0,T]}\|\theta_\lambda(t)\|_{L^p}^p\Big)^{\frac{1}{2}}\Big(\int_0^{t\wedge\sigma_k}\int_Z\|\theta_\lambda(s)\|_{L^p}^{p-2}\|J_\lambda\xi(s,z)\|_{L^p}^2\,\nu(\mathrm{d}z)\mathrm{d}s\Big)^{\frac{1}{2}}\right]$$
$$\le \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|\theta_\lambda(t)\|_{L^p}^p\Big)+\frac{9p^2}{2}\mathbb{E}\Big(\int_0^T\int_Z\|\theta_\lambda(s)\|_{L^p}^{p-2}\|J_\lambda\xi(s,z)\|_{L^p}^2\,\nu(\mathrm{d}z)\mathrm{d}s\Big)$$
$$\le \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|\theta_\lambda(t)\|_{L^p}^p\Big)+\frac{9p^2}{2}\mathbb{E}\Big(\int_0^T\Big(\Big(1-\frac{2}{p}\Big)\|\theta_\lambda(s)\|_{L^p}^p+\frac{2}{p}\Big)\int_Z\|J_\lambda\xi(s,z)\|_{L^p}^2\,\nu(\mathrm{d}z)\mathrm{d}s\Big)$$
$$\le \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|\theta_\lambda(t)\|_{L^p}^p\Big)+\frac{9p(p-2)}{2}\int_0^T\mathbb{E}\Big(\sup_{r\in[0,s]}\|\theta_\lambda(r)\|_{L^p}^p\Big)\int_Z\|\xi(s,z)\|_{L^p}^2\,\nu(\mathrm{d}z)\mathrm{d}s+9pK_\xi^1,$$

where $K_\xi^1 := \int_0^T\int_Z\|\xi(s,z)\|_{L^p}^2\nu(\mathrm{d}z)\mathrm{d}s<\infty$, since $\xi\in L^p([0,T]\times Z;L^p(D))\cap L^2([0,T]\times Z;L^p(D))$. Note that

$$\mathbb{E}\sup_{t\in[0,T]}|I_2(t)|$$
$$=\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\int_Z\|\theta_\lambda(s)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\ N(\mathrm{d}s,\mathrm{d}z)\right|$$
$$\le\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_Z\left|\|\theta_\lambda(s)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\right|N(\mathrm{d}s,\mathrm{d}z)$$
$$\le\mathbb{E}\int_0^T\int_Z\left|\|\theta_\lambda(s)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\right|N(\mathrm{d}s,\mathrm{d}z)$$
$$=\mathbb{E}\int_0^T\int_Z\left|\|\theta_\lambda(s)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\right|\nu(\mathrm{d}z)\mathrm{d}s.$$

According to the mean value theorem, there exists $\gamma\in(0,1)$ such that

$$\left|\|\theta_\lambda(s)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)\right|$$
$$\le p(p-1)\|\theta_\lambda(s)+\gamma J_\lambda\xi(s,z)\|_{L^p}^{p-2}\|J_\lambda\xi(s,z)\|_{L^p}^2$$
$$\le p(p-1)2^{p-2}(\|\theta_\lambda(s)\|_{L^p}^{p-2}+\|J_\lambda\xi(s,z)\|_{L^p}^{p-2})\|J_\lambda\xi(s,z)\|_{L^p}^2.$$

It then follows that

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\int_Z\|\theta_\lambda(s)+J_\lambda\xi(s,z)\|_{L^p}^p-\|\theta_\lambda(s)\|_{L^p}^p-\Psi'(\theta_\lambda(s-))J_\lambda\xi(s,z)N(\mathrm{d}s,\mathrm{d}z)\right|$$
$$\le p(p-1)2^{p-2}\mathbb{E}\int_0^T\int_Z\|\theta_\lambda(s)\|_{L^p}^{p-2}\|J_\lambda\xi(s,z)\|_{L^p}^2\nu(\mathrm{d}z)\mathrm{d}s$$
$$+p(p-1)2^{p-2}\int_0^T\int_Z\|J_\lambda\xi(s,z)\|_{L^p}^p\nu(\mathrm{d}z)\mathrm{d}s$$
$$\le p(p-1)2^{p-2}\mathbb{E}\int_0^T\int_Z\Big(\Big(1-\frac{2}{p}\Big)\|\theta_\lambda(s)\|_{L^p}^p+\frac{2}{p}\Big)\|J_\lambda\xi(s,z)\|_{L^p}^2\nu(\mathrm{d}z)\mathrm{d}s$$
$$+p(p-1)2^{p-2}\int_0^T\int_Z\|J_\lambda\xi(s,z)\|_{L^p}^p\nu(\mathrm{d}z)\mathrm{d}s$$
$$\le(p-1)(p-2)2^{p-2}\int_0^t\mathbb{E}\Big(\sup_{r\in[0,s]}\|\theta_\lambda(r)\|_{L^p}^p\Big)\int_Z\|\xi(s,z)\|_{L^p}^2\nu(\mathrm{d}z)\mathrm{d}s+p(p-1)2^{p-2}K_\xi^2,$$

where $K_\xi^2 := \int_0^T \int_Z \|\xi(s,z)\|_{L^p}^2 \nu(\mathrm{d}z)\mathrm{d}s + \int_0^T \int_Z \|\xi(s,z)\|_{L^p}^p \nu(\mathrm{d}z)\mathrm{d}s < \infty$, since $\xi \in L^p([0,T] \times Z; L^p(D)) \cap L^2([0,T] \times Z; L^p(D))$. Combining the above estimates gives

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]} \|\theta_\lambda(r)\|_{L^p}^p\Big) \\
\le& \|\theta(0)\|_{L^p}^p + \frac{9p(p-2)}{2}\int_0^T \mathbb{E}\Big(\sup_{r\in[0,s]} \|\theta_\lambda(r)\|_{L^p}^p\Big) \int_Z \|\xi(s,z)\|_{L^p}^2 \nu(\mathrm{d}z)\mathrm{d}s + 9pK_\xi^1 \\
&+ (p-1)(p-2)2^{p-2}\int_0^T \mathbb{E}\Big(\sup_{r\in[0,s]} \|\theta_\lambda(r)\|_{L^p}^p\Big) \int_Z \|\xi(s,z)\|_{L^p}^2 \nu(\mathrm{d}z)\mathrm{d}s \\
&+ p(p-1)2^{p-2}K_\xi^2 \\
\le& \|\theta(0)\|_{L^p}^p + C_p^1 \int_0^T \mathbb{E}\Big(\sup_{r\in[0,s]} \|\theta_\lambda(r)\|_{L^p}^p\Big) \int_Z \|\xi(s,z)\|_{L^p}^2 \nu(\mathrm{d}z)\mathrm{d}s + C_p^2 K_\xi^2,
\end{aligned}$$

where $C_p^1 = (p-2)\Big(\frac{9p}{2} + (p-1)2^{p-2}\Big)$ and $C_p^2 = 9p + p(p-1)2^{p-2}$. Using the Gronwall inequality, we infer

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} |\theta_\lambda(t)|_{L^p}^p &\le \Big[2\|\theta(0)\|_{L^p}^p + 2C_p^2K_\xi^2\Big] \exp\Big(2C_p^1 \int_0^T \int_Z \|\xi(s,z)\|_{L^p}^2 \nu(dz)ds\Big) \\
&\le \Big[2\|\theta(0)\|_{L^p}^p + 2C_p^2K_\xi^2\Big] \exp\Big(2C_p^1 K_\xi^1\Big) < \infty. \qquad (28)
\end{aligned}$$

It follows from (25) and (28) that

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau_n]} \|\theta(t)\|_{L^p(D)}^p &\le 2^p \lim_{\lambda\to\infty} \mathbb{E}\sup_{t\in[0,T]} \|\theta_\lambda(t)\|_{L^p}^p + 2^p \lim_{\lambda\to\infty} \mathbb{E}\sup_{t\in[0,\tau_n]} \|\theta(t) - \theta_\lambda(t)\|_{L^p(D)}^p \\
&\le 2^p\Big[2\|\theta(0)\|_{L^p}^p + 2C_p^2K_\xi^2\Big] \exp\Big(2C_p^1K_\xi^1\Big) := \rho < \infty,
\end{aligned}$$

where ρ depends only on θ_0, p, K_ξ^1 and K_ξ^2. Then, we infer

$$\mathbb{P}\{\tau_n = T\} \ge \mathbb{P}(\sup_{t\in[0,\tau_n]} \|\theta(t)\|_{L^p} \le n) \ge 1 - \frac{\rho}{n^p}.$$

Therefore, using the continuity of the measure, we obtain

$$\mathbb{P}(\tau^* = T) \ge \mathbb{P}\Big(\bigcup_{n\in\mathbb{N}} \{\tau_n = T\}\Big) = \lim_{n\to\infty} \mathbb{P}(\tau_n = T) = 1.$$

□

5. Conclusions

We prove the global existence and uniqueness of $L^\infty(0,T; L^p(D))$-mild solutions, for all $p > \frac{2}{2\alpha-1}$, of 2D stochastic QG equations driven by jump-type noise with subcritical powers $\alpha \in (\frac{1}{2}, 1]$. Unlike references [7,9,10], we consider the case of a bounded domain. Due to the presence of boundaries, well-posedness issues become more delicate. To prove the local well-posedness, we break up the system into a linear stochastic system and a nonlinear partial differential equation. By using the maximal inequality for stochastic convolution, we prove that the mild solution to the linear stochastic system is in $L^\infty([0,T]; L^p(D))$, $\mathbb{P}$-a.s. A fixed point argument is applied to prove the local existence and uniqueness of the mild solutions. To prove the global well-posedness, we employ the regularization procedure based on Yosida approximations, which differs from references [7,8], where Galerkin approximation is used. By applying the Itô formula for Banach space valued processes and the Burkholder–Davis–Gundy inequality to the stochastic term, we obtain suitable uniform bounds for the Yosida equations. The global existence of mild solutions in $L^\infty(0,T;L^p(D))$ space is established for all $p > \frac{2}{2\alpha-1}$.

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