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# Study of a Transmission Problem with Friction Law and Increasing Continuous Terms in a Thin Layer

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**Abstract:** The aim of this paper is to establish the asymptotic analysis of nonlinear boundary value problems. The non-stationary motion is given by the elastic constructive law. The contact is described with a version of Tresca's law of friction. A variational formulation of the model, in the form of a coupled system for the displacements and the nonlinear source terms, is derived. The existence of a unique weak solution of the model is established. We also give the problem in transpose form, and we demonstrate different estimates of the displacement and of the source term independently of the small parameter. The main corresponding convergence results are stated in the different theorems of the last section.

**Keywords:** dynamic regime; elastic bodies; frictionless contact; mathematical operators; Newtonian fluids; partial differential equations; Tresca law; variational inequalities

**MSC:** 35R35; 76F10; 78M35

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## 1. Introduction

This present article is devoted to the study of the solution of a transmission problem in a non-stationary regime in a 3D thin layer with Tresca's friction law. More specifically and for the ease of the reader, we give notations that specify our domain: we suppose that the nonhomogeneous $\Omega^\varepsilon$ is composed of two homogeneous bodies $\Omega_1^\varepsilon$ and $\Omega_2^\varepsilon$ of $\mathbb{R}^3$. Throughout this work, the index $l$ indicates that a quantity is associated with the domain $\Omega_l^\varepsilon, l = 1,2$, where $(0 < \varepsilon < 1)$ is the thickness that becomes infinitely small, which will tend to zero. Suppose also that the boundary $\partial\Omega_l^\varepsilon = \overline{\omega} \cup \overline{\Gamma}_l^\varepsilon \cup \overline{\Gamma}_{L_l}^\varepsilon$ of the domain $\Omega_l^\varepsilon$ is partitioned into three disjoint measurable parts and belongs to $C^1$, where $\omega$ is a fixed region in the plane $x' = (x_1, x_2) \in \mathbb{R}$. The upper surface $\Gamma_1^\varepsilon$ is defined by $x_3 = \varepsilon h(x')$, and $\Gamma_2^\varepsilon$ is defined by $x_3 = -\varepsilon h(x')$. Additionally, $h$ is a bounded continuous function with $0 < h_\star \leq h(x') \leq h^*$ for all $(x', 0) \in \omega$, and $\Gamma_{L_l}^\varepsilon$, $l = 1,2$ is a lateral boundary. For any function $\mathbf{u}^\varepsilon$ defined on $\Omega^\varepsilon$, we designate by $\mathbf{u}_1^\varepsilon = (u_{1i}^\varepsilon)_{1\leq i\leq 3}$ (resp.,$\mathbf{u}_2^\varepsilon = (u_{2i}^\varepsilon)_{1\leq i\leq 3}$) its restriction on $\Omega_1^\varepsilon$ (resp., on $\Omega_2^\varepsilon$).

During the last decades, many authors have studied the problems of contact with the various laws of behavior as well as the various conditions of friction close to this study. In [1–3], the authors devoted their studies to the convergence of the solutions of the linearized elasticity system with different boundary conditions to generalized weak equations in the plane. In [4,5], the authors show the reduction of the 3D-1D dimension in anisotropic heterogeneous linearized elasticity. This work is devoted only to strong solutions, with the absence of a friction law. This type of study, governed by the different models of the mechanics of continuum in thin layers is essentially based on the theory of variational inequalities which represents, in a very natural generalization of the theory of boundary problems, and makes it possible to consider new models from many areas of applied mathematics. The variational analysis, existence, uniqueness, and regularity results in the study of a new class of variational inequalities were proved in [6] (see also, e.g., [7–9] and references therein). In the case of linear thin elasticity and in a non-stationary regime, Benseridi et al., in [10,11], gave the asymptotic analysis of the solutions whose influence (or not) of the heat on the model with friction did not increase the continuous terms. Several studies of the asymptotic convergence of Newtonian and non-Newtonian fluids are considered in [12–15], of which the authors have shown that the initial problems are converging towards limit problems represented by weak forms (Reynolds equations). A significant number of researchers have devoted their work to the study of transmission problems in different functional spaces with several types of boundary conditions. For example, Manaa et al., in [16], proved the reduction of the 3D-2D dimension of an interface problem with a dissipative term in a dynamic regime. We would like readers to note that, in this study, the authors are interested in a very particular body that follows Hooke's law (an isotropic case of elastic materials). The asymptotic study of a transmission problem governed by an elastic body in a stationary regime with Tresca has been studied in [17]. Another work analogous to this present study, but relating only to the study of the existence and uniqueness of the weak solution of a frictionless contact problem between an elastic body and a rigid foundation, is given by [18]. Other recent works on the contact problems are given in [19–23].

In this study, the objective is to make an extension of our previous works [16,17]. The novelty of our study can be summarized in the following two major points. First, we take into account a generalized stress tensor compared to what is given in [16]:

$$\sigma_l^\varepsilon = \mathcal{E}^l e(\mathbf{u}_l^\varepsilon) \text{ or } (\sigma_l^\varepsilon)_{ij}(u_l^\varepsilon) = \mathcal{E}_{ijpq}^l \, e_{pq}(u_l^\varepsilon),$$

where $\mathcal{E}^l$ is a bounded symmetric positive definite fourth-order tensor that describes the elastic properties of the material and $e(\mathbf{u}_l^\varepsilon)$ is the linearized strain tensor. Second, we study the asymptotic behavior of the considered problem with the Tresca friction and the presence of the nonlinear source terms in a non-stationary regime compared to what is given in [17]. This choice will create different difficulties in the next section of this study, especially in Theorems 5–7 and the uniqueness theorem. Because the study of the asymptotic analysis is more difficult since in general, the limit problem involves an equation that takes into account the anisotropy of the medium, and it is therefore important to identify the elastic components of $(\mathcal{E}_{ijpq}^l)$ that appear in the (2D) equation model.

The remainder of our paper is organized as follows: Section 2 will summarize the description of the problem and the basic equations. Moreover, we introduce some notations and preliminaries that will be used in other sections. Section 3 is reserved for the proof of the related weak formulation. We also give the problem in transpose form, and we establish some estimates of the displacement that do not depend on the parameter $\varepsilon$ in Section 4. The corresponding main convergence results are stated in different theorems in Section 5.

## 2. The Domain and Notations

We denote by $\mathbf{S}_3$ the space of the second-order symmetric tensor on $\mathbb{R}^3$, and $|.|$ is the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbf{S}_3$, respectively. In addition, $\forall\, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u}.\mathbf{v} = u_i.v_i, |\mathbf{u}| = (\mathbf{u}.\mathbf{u})^{\frac{1}{2}}$, and $\sigma.\top = \sigma_{ij}\top_{ij}, |\top| = (\top.\top)^{\frac{1}{2}}, \forall\, \sigma, \top \in \mathbf{S}_3$. Throughout this article, $i, j, p, q = 1, 2, 3$, repeated indices are implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $x$.

Following the notations presented in the introduction, we denote by $\Omega^\varepsilon$ the domain $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$, where

$$\Omega_1^\varepsilon = \Big\{ x = (x', x_3) \in \mathbb{R}^3, (x', 0) \in \omega,\ 0 < x_3 < \varepsilon h(x') \Big\},$$
$$\Omega_2^\varepsilon = \Big\{ x = (x', x_3) \in \mathbb{R}^3, (x', 0) \in \omega,\ -\varepsilon h(x') < x_3 < 0 \Big\}.$$

We assume that the boundary $\partial\Omega_l^\varepsilon = \overline{\omega} \cup \overline{\Gamma}_l^\varepsilon \cup \overline{\Gamma}_{L_l}^\varepsilon$ of the domain $\Omega_l^\varepsilon$ is partitioned into three disjoint measurable parts and belongs to $C^1$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, by:

$$\mathbf{v}_{l\nu}^\varepsilon = \mathbf{v}_l^\varepsilon.\nu_l, \quad \mathbf{v}_{l\tau}^\varepsilon = \mathbf{v}_l^\varepsilon - \mathbf{v}_{l\nu}^\varepsilon \nu_l, \text{ with } \quad \nu = \nu_1 = -\nu_2.$$

$$\sigma_{l\nu}^\varepsilon = (\sigma_l^\varepsilon \nu_l)\nu_l, \quad \sigma_{l\tau}^\varepsilon = \sigma_l^\varepsilon \nu_l - \sigma_{l\nu}^\varepsilon \nu_l.$$

For the displacement field, we use three Hilbert spaces

$$H^1(\Omega_l^\varepsilon)^3 = \left\{\chi \in \left(L^2(\Omega_l^\varepsilon)\right)^3 : \frac{\partial \chi_i}{\partial x_j} \in L^2(\Omega_l^\varepsilon)\right\},$$
$$W(\Omega_l^\varepsilon) = \left\{\chi \in \left(H^1(\Omega_l^\varepsilon)\right)^3 : \chi = 0 \text{ on } \Gamma_l^\varepsilon \cup \Gamma_{L_l}^\varepsilon, l = 1,2\right\},$$
$$W^\varepsilon = \{(\chi_1, \chi_2) \in W(\Omega_1^\varepsilon) \times W(\Omega_2^\varepsilon) : \chi_1.\nu_1 + \chi_2.\nu_2 = 0 \text{ on } \omega\},$$

where $H^1(\Omega_l^\varepsilon)^3$ is endowed with the inner products $(.,.)_{1,\Omega_l^\varepsilon}$ and the associated norms $\|.\|_{1,\Omega_l^\varepsilon}$. $W^\varepsilon$ is endowed with the canonical inner product $(.,.)_{W^\varepsilon}$ and the associated norm $\|.\|_{W^\varepsilon}$, which are defined by

$$\|(\chi_1, \chi_2)\|_{W^\varepsilon} = \left(\|\chi_1\|^2_{W(\Omega_1^\varepsilon)} + \|\chi_2\|^2_{W(\Omega_2^\varepsilon)}\right)^{\frac{1}{2}}.$$

For the stress, we use the real Hilbert space

$$Q = \left\{\mathsf{T} = (\mathsf{T}_{ij}) : \mathsf{T}_{ij} = \mathsf{T}_{ji} \in L^2(\Omega_l^\varepsilon), \forall i,j = 1,2,3\right\},$$

endowed with the inner product

$$(\sigma, \mathsf{T})_Q = \int_{\Omega_l^\varepsilon} \sigma_{ij}\mathsf{T}_{\mathbf{ij}} dx = \int_{\Omega_l^\varepsilon} \sigma.\mathsf{T} dx,$$

Likewise, for the displacement variable, we use the real Hilbert space

$$H = \left\{u = (u_i) \in L^2(\Omega_l^\varepsilon)^3 : e(\mathbf{u}) \in Q\right\}, \ l = 1,2$$

endowed with the inner product

$$(\mathbf{w}_1, \mathbf{w}_2)_H = (\mathbf{w}_1, \mathbf{w}_2)_{L^2(\Omega)^3} + (e(\mathbf{w}_1), e(\mathbf{w}_2))_Q$$

and the norm $\|.\|_H$, where the deformation operator $e(\mathbf{u}) = (e_{ij}(\mathbf{u}))$ and $e_{ij}(\mathbf{u}) = \left(u_{i,j} + u_{j,i}\right)/2$.

We denote by $Q_\infty$ the real Banach space (see [6]):

$$Q_\infty = \{\mathcal{E}^l = (\mathcal{E}^l_{ijpq}) : \mathcal{E}^l_{ijpq} = \mathcal{E}^l_{jipq} = \mathcal{E}^l_{pqij} \in L^\infty(\Omega_l^\varepsilon), \, i,j,p,q = 1,2,3, \, l = 1,2\}.$$

endowed with the norm

$$\|\mathcal{E}^l\|_{Q_\infty} = \max_{0 \le i,j,p,q \le 3} \|\mathcal{E}^l_{ijpq}\|_{L^\infty(\Omega_l^\varepsilon)}, \, l = 1,2 \tag{1}$$

and, moreover,

$$\|\mathcal{E}^l \mathsf{T}\|_Q \le 3\|\mathcal{E}^l\|_{Q_\infty}\|\mathsf{T}\|_Q, \qquad \forall \mathcal{E}^l \in Q_\infty, \, \mathsf{T} \in Q, \, l = 1,2. \tag{2}$$

Finally, for a real Banach space $(X, \|.\|_X)$, we use the usual notation for the spaces $L^p(0,T;X)$, where $1 \leq p \leq \infty$; we also denote by $C(0,T;X)$ and $C^1(0,T;X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in $X$.

## 3. The Problem Statement and Weak Variational Formulation

We consider two bodies made of an elastic material that occupy the domain $\Omega^\varepsilon = (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon)$ of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega_l^\varepsilon = \overline{\omega} \cup \overline{\Gamma}_l^\varepsilon \cup \overline{\Gamma}_{L_l}^\varepsilon$ and a unit outward normal $\nu$. For any displacement vectors $\mathbf{u}^\varepsilon$ defined on $\Omega^\varepsilon$, we designate by $\mathbf{u}_1^\varepsilon = (u_{1i}^\varepsilon)_{1\leq i\leq 3}$ (resp., $\mathbf{u}_2^\varepsilon = (u_{2i}^\varepsilon)_{1\leq i\leq 3}$) its restriction on $\Omega_1^\varepsilon$ (resp., on $\Omega_2^\varepsilon$). The notation $\sigma_l^\varepsilon = \left(\sigma_l^\varepsilon\right)_{ij}, 1 \leq i,j \leq 3$, $l = 1,2$, represents the stress tensor.

The stress–strain relation is expressed as

$$\sigma_l^\varepsilon = \mathcal{E}^l e(\mathbf{u}_l^\varepsilon) \text{ or } (\sigma_l^\varepsilon)_{ij}(u_l^\varepsilon) = \mathcal{E}_{ijpq}^l \, e_{pq}(u_l^\varepsilon), 1 \leq l \leq 2,, \text{and} \quad i,j,p,q \in \{1,2,3\},$$

where the elasticity operator $\mathcal{E}^l$ is assumed to satisfy the conditions:

$$\begin{cases} (\mathrm{H}_1)\ \mathcal{E}^l : \Omega_l^\varepsilon \times \mathbf{S}_3 \to \mathbf{S}_3. \\ (\mathrm{H}_2)\ \text{There exists } L_{\mathcal{E}^l} > 0 \text{ such that} \\ \quad |\mathcal{E}^l e_1 - \mathcal{E}^l e_2)| \leq L_{\mathcal{E}^l}\, |e_1 - e_2| \\ \quad\quad \forall\, e_1,\, e_2 \in \mathbf{S}_3, \text{ a.e. } x \in \Omega_l^\varepsilon. \\ (\mathrm{H}_3)\ \text{There exists } m_{\mathcal{E}^l} > 0 : \forall e \in \mathbf{S}_3 \\ \quad\quad \mathcal{E}^l e \cdot e \geq m_{\mathcal{E}^l} |e|^2 \text{ a.e. in } \Omega^\varepsilon, l = 1,2. \\ (\mathrm{H}_4)\ \text{The mapping } x \longmapsto \mathcal{E}^l(x,e) \text{ is measurable on } \Omega_l^\varepsilon,\ \forall\, e \in \mathbf{S}_3, \\ \\ (\mathrm{H}_5)\ \text{The mapping } x \longmapsto \mathcal{E}^l(x,0) \in Q. \end{cases}$$

Next, we adopt these assumptions:

- On $\Gamma_l^\varepsilon \times ]0,T[$, the upper surface is assumed to be fixed:

$$\mathbf{u}_l^\varepsilon = 0, \quad l = 1,2.$$

- On $\Gamma_{L_l}^\varepsilon \times ]0,T[$, the displacement is known and parallel to the $w$-plane:

$$\mathbf{u}_l^\varepsilon = 0, \quad l = 1,2.$$

- On $\omega \times ]0,T[$, we suppose that the normal velocity is bilateral, that is:

$$\dot{\mathbf{u}}_1^\varepsilon.\nu_1 + \dot{\mathbf{u}}_2^\varepsilon.\nu_2 = 0 \text{ on } \omega \times ]0,T[.$$

Therefore,

$$\nu_1 = -\nu_2 \text{ and } \sigma_1^\varepsilon.\nu_1 = -\sigma_2^\varepsilon.\nu_2 \text{ on } \omega \times ]0,T[.$$

Consequently,

$$\sigma_\nu^\varepsilon = \sigma_{1\nu}^\varepsilon = \sigma_{2\nu}^\varepsilon \text{ and } \sigma_\tau^\varepsilon = \sigma_{1\tau}^\varepsilon = -\sigma_{2\tau}^\varepsilon \text{ on } \omega \times ]0,T[.$$

Let us suppose that we have the condition of the Tresca friction law on the part $\omega \times [0,T]$ with $\kappa^\varepsilon$ being the friction coefficient:

$$\begin{cases} |\sigma_\tau^\varepsilon| < \kappa^\varepsilon \Rightarrow \left(\dot{\mathbf{u}}_1^\varepsilon\right)_\tau - \left(\dot{\mathbf{u}}_2^\varepsilon\right)_\tau = s, \\ |\sigma_\tau^\varepsilon| = \kappa^\varepsilon \Rightarrow \exists\lambda \geq 0, \left(\dot{\mathbf{u}}_1^\varepsilon\right)_\tau - \left(\dot{\mathbf{u}}_2^\varepsilon\right)_\tau = s - \lambda\sigma_\tau^\varepsilon, \end{cases} \quad \text{on } \omega \times ]0,T[.$$

Solving the problem posed is equivalent to finding $\mathbf{u} = (\mathbf{u}_1^\varepsilon, \mathbf{u}_2^\varepsilon)$ satisfying the constitutive law and the boundary conditions, using the following initial conditions:

$$\mathbf{u}_l^\varepsilon(x,0) = \mathbf{u}_l^0(x), \quad \dot{\mathbf{u}}_l^\varepsilon(x,0) = \mathbf{u}_l^1(x), \quad \forall x \in \Omega_l^\varepsilon,\ l = 1,2$$

Finally, the dissipative terms $g_{li} : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1,2,3,\ l = 1,2$ are a continuous increasing function and satisfy the following hypothesis:

1. $g_l, \quad l = 1,2$ is monotonous, i.e., $(g_l(u) - g_l(v), u - v) \geq 0,\ \forall u, v \in \mathbb{R}^3$;
2. $g_l(0_3) = 0_3$ for $\quad l = 1,2$;
3. For all $w_1, w_2 \in \mathbb{R}$, there exists a positive constant $c_g$ independent of $w_1$ and $w_2$, such that
$$|g_{li}(w_1) - g_{li}(w_2)| \leq c_g |w_1 - w_2|, i \in \{1,2,3\}, 1 \leq l \leq 2.$$

For the given body forces $\mathbf{f}_l^\varepsilon, l = 1,2$, the classical model for the process is as follows.

**Problem 1** ($\mathcal{P}^\varepsilon$). *Find a displacement field* $(\mathbf{u}_1^\varepsilon, \mathbf{u}_2^\varepsilon) = ((\mathbf{u}_{1i}^\varepsilon, \mathbf{u}_{2i}^\varepsilon)_{1\leq i\leq 3} : \Omega_1^\varepsilon \times \Omega_2^\varepsilon \times ]0, T[ \to \mathbb{R}^3$ *such that*

$$\ddot{\mathbf{u}}_1^\varepsilon(t) - \mathrm{Div}\Big(\mathcal{E}^1 e(\mathbf{u}_1^\varepsilon)\Big) + (\delta_1^\varepsilon g_1)\dot{\mathbf{u}}_1^\varepsilon(t) = \mathbf{f}_1^\varepsilon, \quad in\ \Omega_1^\varepsilon \times ]0, T[, \tag{3}$$

$$\ddot{\mathbf{u}}_2^\varepsilon(t) - \mathrm{Div}\Big(\mathcal{E}^2 e(\mathbf{u}_2^\varepsilon)\Big) + (\delta_2^\varepsilon g_2)\dot{\mathbf{u}}_2^\varepsilon(t) = \mathbf{f}_2^\varepsilon, \quad in\ \Omega_2^\varepsilon \times ]0, T[, \tag{4}$$

$$\sigma_1^\varepsilon(\mathbf{u}_1^\varepsilon) = \mathcal{E}^1 e(\mathbf{u}_1^\varepsilon), \quad in\ \Omega_1^\varepsilon \times ]0, T[, \tag{5}$$

$$\sigma_2^\varepsilon(\mathbf{u}_2^\varepsilon) = \mathcal{E}^2 e(\mathbf{u}_2^\varepsilon), \quad in\ \Omega_2^\varepsilon \times ]0, T[, \tag{6}$$

$$\mathbf{u}_1^\varepsilon = 0, \quad on\ (\Gamma_1^\varepsilon \cup \Gamma_{L_1}^\varepsilon) \times ]0, T[, \tag{7}$$

$$\mathbf{u}_2^\varepsilon = 0, \quad on\ (\Gamma_2^\varepsilon \cup \Gamma_{L_2}^\varepsilon) \times ]0, T[, \tag{8}$$

$$\dot{\mathbf{u}}_1^\varepsilon.\nu - \dot{\mathbf{u}}_2^\varepsilon.\nu = 0, \quad on\ \omega \times ]0, T[, \tag{9}$$

$$(\sigma_1^\varepsilon(\mathbf{u}_1^\varepsilon)).\nu - (\sigma_2^\varepsilon(\mathbf{u}_2^\varepsilon)).\nu = 0, \quad on\ \omega \times ]0, T[, \tag{10}$$

$$\left\{ \begin{array}{l} |\sigma_\tau^\varepsilon| < \kappa^\varepsilon \Rightarrow \left(\dot{\mathbf{u}}_1^\varepsilon\right)_\tau - (\dot{\mathbf{u}}_2^\varepsilon)_\tau = s, \\ |\sigma_\tau^\varepsilon| < \kappa^\varepsilon \Rightarrow \exists \lambda \geq 0, \left(\dot{\mathbf{u}}_1^\varepsilon\right)_\tau - (\dot{\mathbf{u}}_2^\varepsilon)_\tau = s - \lambda \sigma_\tau^\varepsilon, \end{array} \right. \quad on\ \ \omega \times ]0, T[, \tag{11}$$

$$\mathbf{u}_l^\varepsilon(x, 0) = \mathbf{u}_l^0, \quad \dot{\mathbf{u}}_l^\varepsilon(x, 0) = \mathbf{u}_l^1(x, 0),\ \delta_l^\varepsilon \in \mathbb{R}_+, \quad l = 1,2, \tag{12}$$

**Theorem 1.** *If* $(\mathbf{u}_1^\varepsilon, \mathbf{u}_2^\varepsilon)$ *solution of the problem* $\mathcal{P}^\varepsilon$*, then it is also a solution of the following variational problem:*

**Problem 2** ($\mathcal{P}_v^\varepsilon$). *Find* $(\mathbf{u}_1^\varepsilon, \mathbf{u}_2^\varepsilon)$ *where* $(\dot{\mathbf{u}}_1^\varepsilon(t), \dot{\mathbf{u}}_2^\varepsilon(t)) \in W^\varepsilon,\ \forall t \in [0, T]$ *such that*

$$\left\{ \begin{array}{l} \left(\ddot{\mathbf{u}}_1^\varepsilon(t), \mathbf{v}_1 - \dot{\mathbf{u}}_1^\varepsilon(t)\right) + \left(\ddot{\mathbf{u}}_2^\varepsilon(t), \mathbf{v}_2 - \dot{\mathbf{u}}_2^\varepsilon(t)\right) + \sum_{l=1}^{l=2} a(\mathbf{u}_l^\varepsilon, \mathbf{v}_l - \dot{\mathbf{u}}_l^\varepsilon(t)) \\ + \sum_{l=1}^{l=2} \left(\delta_l^\varepsilon g_l(\dot{\mathbf{u}}_l^\varepsilon(t)), (\mathbf{v}_l - \dot{\mathbf{u}}_l^\varepsilon(t))\right) + J^\varepsilon(\mathbf{v}_1, \mathbf{v}_2) - J^\varepsilon(\dot{\mathbf{u}}_1^\varepsilon(t), \dot{\mathbf{u}}_2^\varepsilon(t)) \geq \sum_{l=1}^{l=2} \left(\mathbf{f}_l^\varepsilon, (\mathbf{v}_l - \dot{\mathbf{u}}_l^\varepsilon(t))\right), \\ \mathbf{u}_l^\varepsilon(x, 0) = \mathbf{u}_l^0, \quad \dot{\mathbf{u}}_l^\varepsilon(x, 0) = \mathbf{u}_l^1(x, 0), \quad l = 1,2, \end{array} \right. \tag{13}$$

*where* $(\mathbf{v}_1, \mathbf{v}_2) \in W^\varepsilon$ *is the test function and*

$$\begin{aligned} a(\mathbf{u}_l^\varepsilon, \mathbf{v}_l) &= \int_{\Omega_l^\varepsilon} (\sigma_l^\varepsilon(\mathbf{u}_l^\varepsilon))_{ij} e_{ij}(\mathbf{v}_l) dx' dx_3 = \int_{\Omega_l^\varepsilon} \mathcal{E}_{ijpq}^l \, e_{pq}(\mathbf{u}_l^\varepsilon) e_{ij}(\mathbf{v}_l) dx' dx_3, \\ (\delta_l^\varepsilon g_l(\dot{\mathbf{u}}_l^\varepsilon(t)), \mathbf{v}) &= \int_{\Omega_l^\varepsilon}^{\varepsilon} \delta_l^\varepsilon g_l . \dot{\mathbf{u}}_l^\varepsilon(t) . \mathbf{v} \, dx' dx_3, \\ (\mathbf{f}_l^\varepsilon, \mathbf{v}) &= \int_{\Omega_l^\varepsilon} \mathbf{f}_l^\varepsilon \mathbf{v} \, dx' dx_3 \text{ and } J^\varepsilon(\mathbf{v}_1, \mathbf{v}_2) = \int_w \kappa^\varepsilon |\mathbf{v}_{1\tau} - \mathbf{v}_{2\tau} - s| dx'. \end{aligned}$$

**Remark 1.** *Using the previous properties and by Korn's inequality (as in [6]), one easily checks that the bilinear form* $a(.,.)$ *is coercive and continuous, i.e.,*

$$a(\mathbf{u}_l, \mathbf{u}_l) \geq \mu_l C_K \|\nabla \mathbf{u}_l\|^2_{L^2(\Omega_l^\varepsilon)} \quad \forall \mathbf{u}_l \in W(\Omega_l^\varepsilon), \ l = 1, 2$$

$$|a(\mathbf{u}_l, \mathbf{v}_l)| \leq M \|\nabla \mathbf{u}_l\|_{L^2(\Omega_l^\varepsilon)} \|\nabla \mathbf{v}_l\|_{L^2(\Omega_l^\varepsilon)} \ \forall \mathbf{u}_l, \mathbf{v}_l \in W(\Omega_l^\varepsilon),$$

*where* $M = \max_{1 \leq i,j,p,q \leq 3} \left\| \mathcal{E}_{ijpq}^l \right\|_{L^\infty(\Omega_l^\varepsilon)}$ *and* $C_K$ *denoting a positive constant depends on* $\Omega_l^\varepsilon$, $\Gamma_l^\varepsilon$, $\Gamma_{L_l}^\varepsilon$, $l = 1, 2$.

**Proof of Theorem 1.** Let $(\mathbf{u}_1^\varepsilon, \mathbf{u}_2^\varepsilon)$ be a solution to problem $\mathcal{P}^\varepsilon$. Multiply (3) by $(\mathbf{v}_1 - \dot{\mathbf{u}}_1^\varepsilon(t))$ and (4) by $(\mathbf{v}_2 - \dot{\mathbf{u}}_2^\varepsilon(t))$ where $(\mathbf{v}_1, \mathbf{v}_2) \in V^\varepsilon$. Using the integral by parts on $\Omega_1^\varepsilon$ and $\Omega_2^\varepsilon$, and then using Green's formula, the results of Remark 1, and (7)–(12), we obtain the variational problem, (13). □

The existence and unique results of the weak solution to problem (13) are obtained in the following Theorem.

**Theorem 2.** *If the following assumptions are realized*

$$\left.\begin{array}{c} (f_1^\varepsilon, f_2^\varepsilon), \ \left(\frac{\partial f_1^\varepsilon}{\partial t}, \frac{\partial f_2^\varepsilon}{\partial t}\right), \ \left(\frac{\partial^2 f_1^\varepsilon}{\partial t^2}, \frac{\partial^2 f_2^\varepsilon}{\partial t^2}\right) \in L^2(0, T; L^2(\Omega_1^\varepsilon)^3 \times L^2(\Omega_2^\varepsilon)^3); \\ \kappa^\varepsilon \in C_0^\infty(\omega), \quad \kappa^\varepsilon > 0 \ \text{ is independent of } t; \\ (\mathbf{u}_1^0, \mathbf{u}_2^0) \in H^2(\Omega_1^\varepsilon)^3 \times H^2(\Omega_2^\varepsilon)^3, \quad (\mathbf{u}_1^1, \mathbf{u}_2^1) \in H^1(\Omega_1^\varepsilon)^3 \times H^1(\Omega_2^\varepsilon)^3. \end{array}\right\} \tag{14}$$

*There exists a unique solution* $(\mathbf{u}_1^\varepsilon, \mathbf{u}_2^\varepsilon)$ *to problem* $P_v^\varepsilon$ *with*

$$\begin{aligned} (\mathbf{u}_1^\varepsilon, \mathbf{u}_2^\varepsilon), \ (\dot{\mathbf{u}}_1^\varepsilon(t), \dot{\mathbf{u}}_2^\varepsilon(t)) &\in L^\infty(0, T; H^1(\Omega_1^\varepsilon)^3 \times H^1(\Omega_2^\varepsilon)^3), \\ (\ddot{\mathbf{u}}_1^\varepsilon(t), \ddot{\mathbf{u}}_2^\varepsilon(t)), (g_1 \dot{\mathbf{u}}_1^\varepsilon(t), g_2 \dot{\mathbf{u}}_2^\varepsilon(t)) \dot{\mathbf{u}}_1^\varepsilon(t) &\in L^\infty(0, T; L^2(\Omega_1^\varepsilon)^3 \times L^2(\Omega_2^\varepsilon)^3). \end{aligned}$$

**Proof.** Since the function $J_\xi$ is not regularized, then we will regularize it by $J_\xi^\varepsilon$:

$$J_\xi^\varepsilon(\mathbf{v}_1, \mathbf{v}_2) = \int_\omega \kappa^\varepsilon(x) \psi_\xi(|\mathbf{v}_{1\tau} - \mathbf{v}_{2\tau} - s|^2) dx', \ \text{ with } \ \psi_\xi(\beta) = \frac{1}{1+\xi} |\beta|^{(1+\xi)}, \ \xi > 0.$$

Next, we formulate the associated approximate problem

$$\left\{ \begin{array}{l} \left(\ddot{\mathbf{u}}_{1\xi}^\varepsilon(t), \mathbf{v}_1\right) + \left(\ddot{\mathbf{u}}_{2\xi}^\varepsilon(t), \mathbf{v}_2\right) + a(\mathbf{u}_{1\xi}^\varepsilon, \mathbf{v}_1) + a(\mathbf{u}_{2\xi}^\varepsilon, \mathbf{v}_2) + \\ \sum_{l=1}^{l=2} \left(\delta_l^\varepsilon g_l \left(\dot{\mathbf{u}}_{1\xi}^\varepsilon(t)\right), \mathbf{v}_l\right) + \left((J_\xi^\varepsilon)' \left(\dot{\mathbf{u}}_{1\xi}^\varepsilon(t), \dot{\mathbf{u}}_{2\xi}^\varepsilon(t)\right), (\mathbf{v}_1, \mathbf{v}_2)\right) = \sum_{l=1}^{l=2} (\mathbf{f}_l^\varepsilon, \mathbf{v}_l), \\ \mathbf{u}_{l\xi}^\varepsilon(x, 0) = \mathbf{u}_l^0, \quad \dot{\mathbf{u}}_{l\xi}^\varepsilon(t)(x, 0) = \mathbf{u}_l^1(x, 0), \quad \forall \mathbf{v} \in W^\varepsilon, \ \forall t \in [0, T] \end{array} \right. \tag{15}$$

For the rest of the proof, we apply Galerkin's method as in ([24,25]), with hypothesis $(H_1) - (H_5)$. We begin to show that problem (15) admits a unique solution denoted by $\mathbf{u}_\xi^\varepsilon = (\mathbf{u}_{1\xi}^\varepsilon, \mathbf{u}_{2\xi}^\varepsilon)$.

In the last step, it is easy to verify that the limit of $\mathbf{u}^{\varepsilon}_{\zeta}$ to $\mathbf{u}^{\varepsilon}$ when $\zeta \to 0$ is a solution of (13). □

## 4. The Problem in a Fixed Domain

In this section, we use the dilatation in the variable $x_3$ given by $x_3 = \varepsilon z$; then, our problem will be defined on a domain $\Omega^{\varepsilon}$, which is independent of $\varepsilon$. So for $(x, x_3)$ in $\Omega^{\varepsilon}_l$, $l = 1, 2$, we have $(x, z)$ in $\Omega_l$, where

$$\begin{aligned} \Omega_1 &= \{x = (x', z) \in \mathbb{R}^3, \, (x', 0) \in \omega, \quad 0 < z < h(x')\}, \\ \Omega_2 &= \{x = (x', z) \in \mathbb{R}^3, \, (x', 0) \in \omega, \quad -h(x') < z < 0\}, \end{aligned}$$

with $\partial\Omega_l = \overline{\omega} \cup \overline{\Gamma}_l \cup \overline{\Gamma}_{L_l}$ being the boundary of $\Omega_l$, $1 \le l \le 2$.

To simplify the notation, everywhere in the sequel, $\alpha, \beta, \gamma, \theta = 1, 2$. According to this convention, when an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.
So, we define the following functions in $\Omega_l$

$$\begin{cases} \hat{u}^{\varepsilon}_{l\alpha}(x', z, t) = u^{\varepsilon}_{l\alpha}(x', x_3, t), & 1 \le l \le 2, \\ \hat{u}^{\varepsilon}_{l3}(x', z, t) = \varepsilon^{-1} u^{\varepsilon}_{l3}(x', x_3, t), \end{cases} \tag{16}$$

For the data of problems (3)–(12), it is assumed that they depend on $\varepsilon$ as follows:

$$\begin{cases} \hat{\mathbf{f}}_l(x', z, t) = \varepsilon^2 \mathbf{f}^{\varepsilon}_l(x', x_3), \\ \hat{\kappa} = \varepsilon \kappa^{\varepsilon}, \\ \hat{\delta}_l = \varepsilon^2 \delta^{\varepsilon}_l, & 1 \le l \le 2, \\ \hat{g}_l(x', z, t) = g^{\varepsilon}_l(x', x_3, t), \end{cases} \tag{17}$$

with $\hat{\mathbf{f}}_l, \hat{\kappa}, \hat{\delta}_l$, and $\hat{g}_l$ not depending on $\varepsilon$. We introduce the following spaces:

$$\begin{aligned} W(\Omega_l) &= \{\hat{\mathbf{v}} \in H^1(\Omega_l)^3 : \hat{\mathbf{v}} = 0 \quad \text{on} \quad \Gamma_l \cup \Gamma_{\mathrm{L}_l}, l = 1, 2\}, \\ W &= \{(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in W(\Omega_1) \times W(\Omega_2) : \hat{\mathbf{v}}_1.\nu - \hat{\mathbf{v}}_2.\nu = 0 \quad \text{on} \quad \omega\}, \\ \Pi(W) &= \{\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2) \in H^1(\Omega_1)^2 \times H^1(\Omega_2)^2 : \hat{\phi} = 0 \quad \text{on} \quad \Gamma_l \cup \Gamma_{L_l}, 1 \le l \le 2\}, \\ H_z(\Omega_l) &= \{\hat{\mathbf{v}}_l = (\hat{\mathbf{v}}_{l1}, \hat{\mathbf{v}}_{l2}) \in L^2(\Omega_l)^2 : \frac{\partial \hat{\mathbf{v}}_{l\alpha}}{\partial z} \in L^2(\Omega_l), \quad \hat{\mathbf{v}}_l = 0 \text{ on} \, \Gamma_l, \; 1 \le l \le 2\}, \\ H_z &= H_z(\Omega_1) \times H_z(\Omega_2), \\ H^1_{\Gamma_l \cup \Gamma_{L_l}(\Omega_l)} &= \{\hat{v}_l \in H^1(\Omega_l) : v_l = 0 \text{ on } \Gamma_l \cup \Gamma_{L_l}, 1 \le l \le 2\}. \end{aligned}$$

endowed with the norms, respectively:

$$\begin{aligned} \|\hat{\mathbf{v}}_l\|^2_{H_z(\Omega_l)} &= \sum_{i=1}^{2} \left( \|\hat{\mathbf{v}}_{li}\|^2_{L^2(\Omega_l)} + \|\frac{\partial \hat{\mathbf{v}}_{li}}{\partial z}\|^2_{L^2(\Omega_l)} \right), \\ \|(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2)\|^2_{H_z} &= \|\hat{\mathbf{v}}_1\|^2_{H_z(\Omega_1)} + \|\hat{\mathbf{v}}_2\|^2_{H_z(\Omega_2)}. \end{aligned}$$

Using the symmetry of $\mathcal{E}^l_{ijpq}$, the variational problem (13) is reformulated on the fixed domain as follows:

**Problem 3** ($\hat{\mathcal{P}}^\varepsilon$)**.** *Find* $(\hat{\mathbf{u}}_1^\varepsilon, \hat{\mathbf{u}}_2^\varepsilon) \in W$, *with* $\left(\frac{\partial \hat{\mathbf{u}}_1^\varepsilon}{\partial t}(t), \frac{\partial \hat{\mathbf{u}}_2^\varepsilon}{\partial t}(t)\right) \in W, \forall t \in [0,T]$, *provided that*

$$\left.\begin{array}{c}
\varepsilon^2 \sum_{l=1}^{2}\sum_{\alpha=1}^{2}\left(\frac{\partial^2 \hat{u}_{l\alpha}^\varepsilon}{\partial t^2}, \hat{\varphi}_{l\alpha} - \frac{\partial \hat{u}_{l\alpha}^\varepsilon}{\partial t}\right) + \varepsilon^4 \sum_{l=1}^{2}\left(\frac{\partial^2 \hat{u}_{l3}^\varepsilon}{\partial t^2}, \hat{v}_{l3} - \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t}\right) \\
+\sum_{l=1}^{2}\sum_{\alpha=1}^{2} \hat{\delta}_l \hat{g}_{l\alpha}\left(\frac{\partial \hat{u}_{l\alpha}^\varepsilon}{\partial t}, \hat{\varphi}_{l\alpha} - \frac{\partial \hat{u}_{l\alpha}^\varepsilon}{\partial t}\right) + \varepsilon^2 \sum_{l=1}^{2} \hat{\delta}_l g_{l3}\left(\frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t}, \hat{\varphi}_{l3} - \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t}\right) \\
+\sum_{l=1}^{2} \widehat{a}\left(\hat{\mathbf{u}}_l^\varepsilon, \hat{\varphi}_l - \frac{\partial \hat{\mathbf{u}}_l^\varepsilon}{\partial t}\right) + \hat{J}(\hat{\varphi}_1, \hat{\varphi}_2) - \hat{J}\left(\frac{\partial \hat{\mathbf{u}}_1^\varepsilon}{\partial t}, \frac{\partial \hat{\mathbf{u}}_2^\varepsilon}{\partial t}\right) \geq \\
\sum_{l=1}^{2}\sum_{\alpha=1}^{2}\left(\hat{f}_{l\alpha}, \hat{\varphi}_{1\alpha} - \frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial t}\right) + \varepsilon \sum_{l=1}^{2}\left(\hat{f}_{l3}, \hat{\varphi}_{l3} - \frac{\partial \hat{u}_{li}^\varepsilon}{\partial t}\right), \forall (\hat{\varphi}_1, \hat{\varphi}_2) \in W, \\
\hat{\mathbf{u}}_l^\varepsilon = \hat{\mathbf{u}}_l^0, \quad \frac{\partial \hat{\mathbf{u}}_l^\varepsilon}{\partial t}(x,0) = \hat{\mathbf{u}}_l^1(x,0), \quad l = 1,2,
\end{array}\right\} \tag{18}$$

*where*

$$\hat{J}(\hat{\varphi}_1, \hat{\varphi}_2) = \int_\omega \hat{\kappa} |\hat{\varphi}_{1\tau} - \hat{\varphi}_{2\tau} - s| dx',$$

$$\begin{aligned}
\widehat{a}(\hat{\mathbf{u}}_l^\varepsilon, \hat{\varphi}_l) \quad = \quad & \varepsilon^2 \int_{\Omega_l} \widehat{\mathcal{E}}_{\alpha\beta\gamma\theta}^l \hat{e}_{\gamma\theta}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_\alpha}{\partial x_\beta} dx' dz + 2\varepsilon \int_{\Omega_l} \widehat{\mathcal{E}}_{\alpha 3\gamma\theta}^l \hat{e}_{\gamma\theta}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_\alpha}{\partial z} dx' dz \\
& + 2\varepsilon^2 \int_{\Omega_l} \widehat{\mathcal{E}}_{\alpha\beta\gamma 3} \hat{e}_{\gamma 3}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_\alpha}{\partial x_\beta} dx' dz + 4\varepsilon \int_{\Omega_l} \widehat{\mathcal{E}}_{\alpha 3\gamma 3}^l \hat{e}_{\gamma 3}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_\alpha}{\partial z} dx' dz \\
& + \varepsilon^2 \int_{\Omega_l} \widehat{\mathcal{E}}_{\alpha\beta 33}^l \hat{e}_{33}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_\alpha}{\partial x_\beta} dx' dz + \varepsilon^2 \int_{\Omega_l} \widehat{\mathcal{E}}_{33\alpha\beta}^l \hat{e}_{\alpha\beta}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_3}{\partial z} dx' dz \\
& + 2\varepsilon \int_{\Omega_l} \widehat{\mathcal{E}}_{\alpha 333}^l \hat{e}_{33}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_\alpha}{\partial z} dx' dz + 2\varepsilon^2 \int_{\Omega_l} \widehat{\mathcal{E}}_{33\alpha 3}^l \hat{e}_{\alpha 3}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_3}{\partial z} dx' dz \\
& + \varepsilon^2 \int_{\Omega_l} \widehat{\mathcal{E}}_{3333}^l \hat{e}_{33}(\hat{\mathbf{u}}_l^\varepsilon) \frac{\partial \hat{\varphi}_3}{\partial z} dx' dz
\end{aligned}$$

*and* $\hat{e}\left(\hat{\mathbf{u}}_l^\varepsilon\right) = \left(\hat{e}_{ij}\left(\hat{\mathbf{u}}_l^\varepsilon\right)\right)_{ij}$ *are given by the relations*

$$\left\{\begin{array}{lll}
\hat{e}_{ij}\left(\hat{\mathbf{u}}_l^\varepsilon\right) = \frac{1}{2}\left(\frac{\partial \hat{u}_{li}^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_{lj}^\varepsilon}{\partial x_i}\right) & i,j,l = 1,2 & \\
\hat{e}_{i3}\left(\hat{\mathbf{u}}_l^\varepsilon\right) = \hat{e}_{3i}\left(\hat{\mathbf{u}}_l^\varepsilon\right) = \frac{1}{2}\left(\frac{1}{\varepsilon}\frac{\partial \hat{u}_{li}^\varepsilon}{\partial z} + \varepsilon \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial x_i}\right) & i,l = 1,2 & \left(\mathrm{Def}_{\hat{e}_{ij}\left(\hat{u}_l^\varepsilon\right)}\right) \\
\hat{e}_{33}\left(\hat{\mathbf{u}}_l^\varepsilon\right) = \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial z} & l = 1,2 &
\end{array}\right.$$

*In the next section, we establish some estimates for the solutions to the variational problem* (18).

**Theorem 3.** *If the hypotheses of Theorem 2 hold, then there exists a positive constant C that does not depend on ε, such that we have:*

$$\begin{aligned}
& \sum_{\alpha=1}^{2}\left(\left\|\left(\frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial z}, \frac{\partial \hat{u}_{2\alpha}^\varepsilon}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 + \left\|\varepsilon\left(\frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial t}, \frac{\partial \hat{u}_{2\alpha}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2\right) \\
& + \sum_{\alpha=1}^{2}\left\|\varepsilon^2\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial x_\alpha}, \frac{\partial \hat{u}_{23}^\varepsilon}{\partial x_\alpha}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 + \sum_{\alpha,\beta=1}^{2}\left\|\varepsilon\left(\frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial x_\beta}, \frac{\partial \hat{u}_{2\alpha}^\varepsilon}{\partial x_\beta}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 \\
& + \left\|\varepsilon\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial z}, \frac{\partial \hat{u}_{23}^\varepsilon}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 + \left\|\varepsilon^2\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial t}, \frac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 \leq C,
\end{aligned} \tag{19}$$

$$\sum_{l,\alpha=1}^{2} \int_0^t \int_{\Omega_l} g_l\left(\frac{\partial \hat{u}_{l\alpha}^\varepsilon}{\partial t}\right) \frac{\partial \hat{u}_{l\alpha}^\varepsilon}{\partial t} dx dr + \sum_{l=1}^{2} \varepsilon \int_0^t \int_{\Omega_l} g_l\left(\varepsilon \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t}\right) \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t} dx dr \leq C, \tag{20}$$

$$\sum_{\alpha=1}^{2}\left(\left\|\left(\frac{\partial^2 \hat{u}^\varepsilon_{1\alpha}}{\partial z\partial t},\frac{\partial^2 \hat{u}^\varepsilon_{2\alpha}}{\partial z}\right)\right\|^2_{0,\Omega_1\times\Omega_2}+\left\|\varepsilon\left(\frac{\partial^2 \hat{u}^\varepsilon_{1\alpha}}{\partial t^2},\frac{\partial^2 \hat{u}^\varepsilon_{2\alpha}}{\partial t^2}\right)\right\|^2_{0,\Omega_1\times\Omega_2}\right)$$
$$+\sum_{\alpha=1}^{2}\left\|\varepsilon^2\left(\frac{\partial^2 \hat{u}^\varepsilon_{13}}{\partial x_\alpha\partial t},\frac{\partial^2 \hat{u}^\varepsilon_{23}}{\partial x_\alpha\partial t}\right)\right\|^2_{0,\Omega_1\times\Omega_2}+\sum_{\alpha,\beta=1}^{2}\left\|\varepsilon\left(\frac{\partial^2 \hat{u}^\varepsilon_{1\alpha}}{\partial x_\beta\partial t},\frac{\partial^2 \hat{u}^\varepsilon_{2\alpha}}{\partial x_\beta\partial t}\right)\right\|^2_{0,\Omega_1\times\Omega_2} \tag{21}$$
$$+\left\|\varepsilon\left(\frac{\partial^2 \hat{u}^\varepsilon_{13}}{\partial z\partial t},\frac{\partial^2 \hat{u}^\varepsilon_{23}}{\partial z\partial t}\right)\right\|^2_{0,\Omega_1\times\Omega_2}+\left\|\varepsilon^2\left(\frac{\partial^2 \hat{u}^\varepsilon_{13}}{\partial t^2},\frac{\partial^2 \hat{u}^\varepsilon_{23}}{\partial t^2}\right)\right\|^2_{0,\Omega_1\times\Omega_2}\le C.$$

**Proof.** Suppose that the problem $\mathcal{P}^\varepsilon_v$ admits a solution denoted by $(\mathbf{u}^\varepsilon_1,\mathbf{u}^\varepsilon_2)$, then we have

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{1\le l\le 2}\|\dot{\mathbf{u}}^\varepsilon_l(t)\|^2_{L^2(\Omega^\varepsilon_l)}+\sum_{1\le l\le 2}a(\mathbf{u}^\varepsilon_l(t),\mathbf{u}^\varepsilon_l(t))\right)$$
$$+\delta^\varepsilon_l\sum_{1\le l\le 2}\int_{\Omega^\varepsilon_l}g_l|\dot{\mathbf{u}}^\varepsilon_l(t)|^2dx'dx_3\le((\ddot{\mathbf{u}}^\varepsilon_1(t),\ddot{\mathbf{u}}^\varepsilon_2(t)),(\mathbf{v}_1,\mathbf{v}_2))$$
$$+\sum_{1\le l\le 2}a(\mathbf{u}^\varepsilon_l(t),\mathbf{v}_l)+J^\varepsilon(\mathbf{v}_1,\mathbf{v}_2)+\sum_{l=1}^{2}\int_{\Omega^\varepsilon_l}\mathbf{f}^\varepsilon_l(\dot{\mathbf{u}}^\varepsilon_l(t)-\mathbf{v}_l)dx'dx_3 \tag{22}$$

For $r\in[0,t]$, by integration, we obtain

$$\left(\|\dot{\mathbf{u}}^\varepsilon_1(t)\|^2_{L^2(\Omega^\varepsilon_1)}+\|\dot{\mathbf{u}}^\varepsilon_2(t)\|^2_{L^2(\Omega^\varepsilon_2)}+\sum_{1\le l\le 2}a(\mathbf{u}^\varepsilon_l(t),\mathbf{u}^\varepsilon_l(t))\right)$$
$$+2\sum_{l=1}^{2}\delta^\varepsilon_l\int_0^t\int_{\Omega^\varepsilon_l}g_l|\dot{\mathbf{u}}^\varepsilon_l(r)|^2dxdr$$
$$\le\|\left(\mathbf{u}^1_1,\mathbf{u}^1_2\right)\|^2_{0,\Omega_1\times\Omega_2}+3\sqrt{3}M\|\left(\nabla\mathbf{u}^0_1,\nabla\mathbf{u}^0_2\right)\|^2_{0,\Omega^\varepsilon_1\times\Omega^\varepsilon_2} \tag{23}$$
$$+2\int_0^t((\ddot{\mathbf{u}}^\varepsilon_1(r),\ddot{\mathbf{u}}^\varepsilon_2(r)),(\mathbf{v}_1,\mathbf{v}_2))dr$$
$$+2\sum_{1\le l\le 2}\int_0^t a(\mathbf{u}^\varepsilon_l(r),\mathbf{v}_l)dr+2TJ^\varepsilon(\mathbf{v}_1,\mathbf{v}_2)$$
$$+2\sum_{l=1}^{2}\int_0^t\int_{\Omega^\varepsilon_l}\mathbf{f}^\varepsilon_l(r)(\dot{\mathbf{u}}^\varepsilon_l(r)-\mathbf{v}_l)dxdr,$$

where $M=\max_{1\le i,j,p,q\le 3}\left\|\mathcal{E}^l_{ijpq}\right\|_{L^\infty\left(\Omega^\varepsilon_l\right)}$, $1\le l\le 2$.

We use Korn's inequality and hypotheses $(H_1)-(H_5)$. There exists a constant $C_k>0$ independent of $\varepsilon$, such that

$$\sum_{1\le l\le 2}a(\mathbf{u}^\varepsilon_l(t),\mathbf{u}^\varepsilon_l(t))\ge 2\mu_1C_K\|\nabla\mathbf{u}^\varepsilon_1\|^2_{0,\Omega^\varepsilon_1}+2\mu_2C_K\|\nabla\mathbf{u}^\varepsilon_2\|^2_{0,\Omega^\varepsilon_2}. \tag{24}$$

On the other hand, when we apply the Young's inequality

$$ab\le\eta^2\frac{a^2}{2}+\eta^{-2}\frac{b^2}{2},\quad \textit{for every}\quad \eta>0,$$

in (13) for $\eta = \sqrt{\dfrac{\mu_l C_K}{2}}$, we find

$$\left|\sum_{1\leq l\leq 2} a(\mathbf{u}_l^\varepsilon(t), \mathbf{v}_l(t))\right| \leq \frac{C_K}{4}\left(\mu_1\|\nabla \mathbf{u}_1^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + \mu_2\|\nabla \mathbf{u}_2^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)}\right) + \frac{M}{C_K}\left(\mu_1\|\nabla \mathbf{v}_1(t)\|^2_{L^2(\Omega_1^\varepsilon)} + \mu_2\|\nabla \mathbf{v}_2(t)\|^2_{L^2(\Omega_2^\varepsilon)}\right)$$

By integration of the last inequality between 0 and $t$, we have

$$\begin{aligned}2\sum_{1\leq l\leq 2}\int_0^t a(\mathbf{u}_l^\varepsilon(\theta), \mathbf{v}_l(\theta))d\theta \leq{}& \frac{2TM}{C_K}\left(\mu_1\|\nabla \mathbf{v}_1(t)\|^2_{L^2(\Omega_1^\varepsilon)} + \mu_2\|\nabla \mathbf{v}_2(t)\|^2_{L^2(\Omega_2^\varepsilon)}\right)\\ &+\frac{C_K}{2}\int_0^t\left(\mu_1\|\nabla \mathbf{u}_1^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + \mu_2\|\nabla \mathbf{u}_2^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)}\right)dr\\ &+2\int_0^t((\ddot{\mathbf{u}}_1^\varepsilon(r), \ddot{\mathbf{u}}_2^\varepsilon(r)), (\mathbf{v}_1, \mathbf{v}_2))dr\\ &\leq \frac{1}{2}\|(\dot{\mathbf{u}}_1^\varepsilon(t), \dot{\mathbf{u}}_2^\varepsilon(t))\|^2_{0,\Omega_1\times\Omega_2}\\ &+\frac{1}{2}\|\left(\mathbf{u}_1^1, \mathbf{u}_2^1\right)\|^2_{0,\Omega_1\times\Omega_2} + \|(\mathbf{v}_1, \mathbf{v}_2)\|^2_{0,\Omega_1\times\Omega_2}\end{aligned} \tag{25}$$

as

$$2\int_0^t(\mathbf{f}_l^\varepsilon(r), \dot{\mathbf{u}}_l^\varepsilon(r))dr = 2(f_l^\varepsilon(t), \mathbf{u}_l^\varepsilon(t)) - 2(f_l^\varepsilon(0), \mathbf{u}_l^0(0)) - 2\int_0^t\left(\frac{\partial \mathbf{f}_l^\varepsilon}{\partial t}(r), \mathbf{u}_l^\varepsilon(r)\right)dr, l = 1,2 \tag{26}$$

Using Poincaré's inequality [1],

$$\|\mathbf{u}^\varepsilon\|_{0,\Omega_l^\varepsilon} \leq \varepsilon h^\star\|\nabla \mathbf{u}^\varepsilon\|_{0,\Omega_l^\varepsilon}, \quad l = 1,2$$

one has

$$\begin{aligned}\left|\int_0^t(\mathbf{f}_1^\varepsilon(r), \dot{\mathbf{u}}_1^\varepsilon(r))dr + \int_0^t(\mathbf{f}_2^\varepsilon(r), \dot{\mathbf{u}}_2^\varepsilon(r))dr\right| \leq{}& \frac{\mu_1 C_K}{2}\|\nabla \mathbf{u}_1^\varepsilon(t)\|^2_{0,\Omega_1^\varepsilon} + \frac{\mu_2 C_K}{2}\|\nabla \mathbf{u}_2^\varepsilon(t)\|^2_{0,\Omega_2^\varepsilon}\\ &+\frac{1}{2}\frac{(\varepsilon h^\star)^2}{C_K}\sum_{l=1}^{2}\frac{1}{\mu_l}\|\mathbf{f}_l^\varepsilon(t)\|^2_{0,\Omega_l^\varepsilon} + \frac{(\varepsilon h^\star)^2}{2}\|(\mathbf{f}_1^\varepsilon(0), \mathbf{f}_2^\varepsilon(0))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}\\ &+\frac{1}{2}\|\left(\nabla \mathbf{u}_1^0, \nabla \mathbf{u}_2^0\right)\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon} + \frac{C_K}{4}\sum_{l=1}^{2}\mu_l\int_0^t\|\nabla \mathbf{u}_\mathbf{l}{}^\varepsilon(r)\|^2_{0,\Omega_l^\varepsilon}dr\\ &+\frac{(\varepsilon h^\star)^2}{C_K}\sum_{l=1}^{2}\frac{1}{\mu_l}\int_0^t\left\|\frac{\partial \mathbf{f}_l^\varepsilon}{\partial t}(r)\right\|^2_{0,\Omega_l}dr.\end{aligned} \tag{27}$$

Likewise, via Poincaré's inequality, we give:

$$\left|-\int_0^t(\mathbf{f}_1^\varepsilon(r), \mathbf{v}_1^\varepsilon)dr - \int_0^t(\mathbf{f}_2^\varepsilon(r), \mathbf{v}_2^\varepsilon)dr\right| \leq \frac{(\varepsilon h^\star)^2}{2C_K}\sum_{l=1}^{2}\int_0^t\frac{1}{\mu_l}\|\mathbf{f}_l^\varepsilon(r)\|^2_{0,\Omega_l^\varepsilon}dr + \frac{T}{2}\|(\nabla \mathbf{v}_1, \nabla \mathbf{v}_2)\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon} \tag{28}$$

Substituting Formula (24)–(28) into (23), we find:

$$\left(\frac{1}{2}\|(\dot{\mathbf{u}}_1^\varepsilon(t),\dot{\mathbf{u}}_2^\varepsilon(t))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\sum_{1\le l\le 2}\mu_l C_K\|\nabla\mathbf{u}_l^\varepsilon(t)\|^2_{0,\Omega_l^\varepsilon}\right)+2\sum_{l=1}^{2}\delta_l^\varepsilon\int_0^t\int_{\Omega_l^\varepsilon}g_l|\dot{\mathbf{u}}_l^\varepsilon(r)|^2dxdr$$
$$\le\frac{3}{2}\|\left(\mathbf{u}_1^1,\mathbf{u}_2^1\right)\|^2_{0,\Omega_1\times\Omega_2}+2T|J^\varepsilon(\mathbf{v}_1,\mathbf{v}_2)|+(\varepsilon h^\star)^2\|(\mathbf{f}_1^\varepsilon(0),\mathbf{f}_2^\varepsilon(0))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}$$
$$+\left(1+3\sqrt{3}M\right)\|\left(\nabla\mathbf{u}_1^0,\nabla\mathbf{u}_2^0\right)\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\left(1+\frac{2M\mu^*}{C_K}\right)T\|(\nabla\mathbf{v}_1,\nabla\mathbf{v}_2)\|^2_{0,\Omega_1\times\Omega_2}$$
$$+\frac{(\varepsilon h^\star)^2}{C_K}\sum_{l=1}^{2}\frac{1}{\mu_l}\|\mathbf{f}_l^\varepsilon(t)\|^2_{0,\Omega_l^\varepsilon}+\frac{2(\varepsilon h^\star)^2}{C_K}\sum_{l=1}^{2}\frac{1}{\mu_l}\int_0^t\left(\left\|\frac{\partial\mathbf{f}_l^\varepsilon}{\partial t}(r)\right\|^2_{0,\Omega_l}+\|\mathbf{f}_l^\varepsilon(r)\|^2_{0,\Omega_l^\varepsilon}dr\right)dr$$
$$+\int_0^t\left(\frac{1}{2}\|(\dot{\mathbf{u}}_1^\varepsilon(r),\dot{\mathbf{u}}_2^\varepsilon(r))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\sum_{l=1}^{2}\mu_l C_K\|\nabla\mathbf{u}_l^\varepsilon(r)\|^2_{0,\Omega_l^\varepsilon}\right)dr. \quad (29)$$

By simple calculations of the change in scale with respect to the third component given by Formula (17), we give

$\varepsilon^2\|\mathbf{f_l}^\varepsilon\|^2_{0,\Omega_l^\varepsilon}=\varepsilon^{-1}\|\hat{\mathbf{f}}_\mathbf{l}\|^2_{0,\Omega_l^\varepsilon},\, l=1,2$ and $J^\varepsilon(\mathbf{v}_1,\mathbf{v}_2)=\varepsilon^{-1}\hat{J}(\mathbf{v}_1,\mathbf{v}_2)$.

Then, multiplying (29) by $\varepsilon$, we obtain:

$$\varepsilon\left(\frac{1}{2}\|(\dot{\mathbf{u}}_1^\varepsilon(t),\dot{\mathbf{u}}_2^\varepsilon(t))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\sum_{1\le l\le 2}\mu_l C_K\|\nabla\mathbf{u}_l^\varepsilon(t)\|^2_{0,\Omega_l^\varepsilon}\right)$$
$$2\varepsilon\sum_{l=1}^{2}\delta_l^\varepsilon\int_0^t\int_{\Omega_l^\varepsilon}g_l|\dot{\mathbf{u}}_l^\varepsilon(\theta)|^2dxd\theta\le B+ \quad (30)$$
$$\int_0^t\varepsilon\left(\frac{1}{2}\|(\dot{\mathbf{u}}_1^\varepsilon(r),\dot{\mathbf{u}}_2^\varepsilon(r))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\sum_{1\le l\le 2}\mu_l C_K\|\nabla\mathbf{u}_l^\varepsilon(r)\|^2_{0,\Omega_l^\varepsilon}\right)dr,$$

where $\mu_*=\min(\mu_1,\mu_2)$, $\mu^*=\max(\mu_1,\mu_2)$ and $B$ does not depend on $\varepsilon$

$$\begin{aligned}B\ =\ &\frac{3}{2}\|\left(\hat{\mathbf{u}}_1^1,\hat{\mathbf{u}}_2^1\right)\|^2_{0,\Omega_1\times\Omega_2}+\left(1+3\sqrt{3}M\right)\|\left(\nabla\hat{\mathbf{u}}_1^0,\nabla\hat{\mathbf{u}}_2^0\right)\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}\\&+\left(1+\frac{2M\mu^*}{C_K}\right)T\|(\nabla\hat{\mathbf{v}}_1,\nabla\hat{\mathbf{v}}_2)\|^2_{0,\Omega_1\times\Omega_2}+2T\left|\hat{J}(\hat{\mathbf{v}}_1,\hat{\mathbf{v}}_2)\right|+\\&(h^\star)^2\|\left(\hat{\mathbf{f}}_1(0),\hat{\mathbf{f}}_2(0)\right)\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\frac{(h^\star)^2}{C_K\mu_*}\|\left(\hat{\mathbf{f}}_1(t),\hat{\mathbf{f}}_1(t)\right)\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}\\&+\frac{2(\varepsilon h^\star)^2}{C_K\mu_*}\left\|\left(\frac{\partial\hat{\mathbf{f}}_1}{\partial t},\frac{\partial\hat{\mathbf{f}}_2}{\partial t}\right)\right\|^2_{L^2(0,T;L^2(\Omega_1)^3\times L^2(\Omega_2)^3)}\\&+\frac{2(\varepsilon h^\star)^2}{C_K\mu_*}\|\left(\hat{\mathbf{f}}_1,\hat{\mathbf{f}}_2\right)\|^2_{L^\infty(0,T;L^2(\Omega_1)^3\times L^2(\Omega_2)^3)}.\end{aligned}$$

Using Gronwall's Lemma, we obtain (19) and (20).

The proof of (21) is based on the techniques used in the proof of inequalities (19)–(20). Indeed, in a first step, we derive the associated approximate problem (15) with respect to $t$. Then, we choose $(\mathbf{v}_1,\mathbf{v}_2)=\left(\dot{\mathbf{u}}_{1\zeta}^\varepsilon(t),\dot{\mathbf{u}}_{2\zeta}^\varepsilon(t)\right)$ in the expression found, and, by applying hypotheses (1)–(3) of the dissipative terms $g_l$ and Korn's inequality, we obtain the analogue of (30). Finally, Gronwall's Lemma assures the existence of a constant $C$ that is independent of $\varepsilon$ and satisfies (21). The proof of Theorem 3 is complete. □

## 5. Convergence Results and Limit Problem

**Theorem 4.** *If the hypotheses of Theorem 2 hold, then there exists* $(\mathbf{u}_1^\star, \mathbf{u}_2^\star) = (u_{1\alpha}^\star, u_{2\alpha}^\star)$ *in* $L^2(0,T,H_z) \cap L^\infty(0,T,H_z)$, $\alpha = 1,2$, *such that*

$$\left.\begin{array}{l} (\hat{u}_{1\alpha}^\varepsilon, \hat{u}_{2\alpha}^\varepsilon) \rightharpoonup (u_{1\alpha}^\star, u_{2\alpha}^\star), \\ \left(\frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial t}, \frac{\partial \hat{u}_{2\alpha}^\varepsilon}{\partial t}\right) \rightharpoonup \left(\dot{u}_{1\alpha}^\star, \dot{u}_{2\alpha}^\star\right) \end{array}\right\} \begin{array}{l} \textit{weakly in } L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \\ \textit{weakly} \star \textit{ in } L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \end{array} \tag{31}$$

$$\left.\begin{array}{l} \varepsilon\left(\hat{e}_{\alpha\beta}(\hat{u}_1^\varepsilon), \hat{e}_{\alpha\beta}(\hat{u}_2^\varepsilon)\right) \rightharpoonup (0,0) \\ \varepsilon\left(\frac{\partial}{\partial t}\hat{e}_{\alpha\beta}(\hat{u}_1^\varepsilon), \frac{\partial}{\partial t}\hat{e}_{\alpha\beta}(\hat{u}_2^\varepsilon)\right) \rightharpoonup (0,0) \end{array}\right\} \begin{array}{l} \textit{weakly in } L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \\ \textit{weakly} \star \textit{ in } L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \end{array} \tag{32}$$

$$\left.\begin{array}{l} \varepsilon\left(\frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial t}, \frac{\partial \hat{u}_{2\alpha}^\varepsilon}{\partial t}\right) \rightharpoonup (0,0) \\ \varepsilon\left(\frac{\partial^2 \hat{u}_{1\alpha}^\varepsilon}{\partial t^2}, \frac{\partial^2 \hat{u}_{2\alpha}^\varepsilon}{\partial t^2}\right) \rightharpoonup (0,0) \end{array}\right\} \begin{array}{l} \textit{weakly in } L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \\ \textit{weakly} \star \textit{ in } L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \end{array} \tag{33}$$

$$\left.\begin{array}{l} \varepsilon^2\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial x_\alpha}, \frac{\partial \hat{u}_{23}^\varepsilon}{\partial x_\alpha}\right) \rightharpoonup (0,0) \\ \varepsilon^2\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial t}, \frac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right) \rightharpoonup (0,0) \\ \varepsilon^2\left(\frac{\partial^2 \hat{u}_{13}^\varepsilon}{\partial x_\alpha \partial t}, \frac{\partial^2 \hat{u}_{23}^\varepsilon}{\partial x_\alpha \partial t}\right) \rightharpoonup (0,0) \\ \varepsilon\left(\hat{e}_{33}(\hat{u}_1^\varepsilon), \hat{e}_{33}(\hat{u}_2^\varepsilon)\right) \rightharpoonup (0,0) \\ \varepsilon\left(\frac{\partial}{\partial t}\hat{e}_{33}(\hat{u}_1^\varepsilon), \frac{\partial}{\partial t}\hat{e}_{33}(\hat{u}_2^\varepsilon)\right) \rightharpoonup (0,0) \\ \varepsilon^2\left(\frac{\partial^2 \hat{u}_{13}^\varepsilon}{\partial t^2}, \frac{\partial^2 \hat{u}_{23}^\varepsilon}{\partial t^2}\right) \rightharpoonup (0,0) \end{array}\right\} \begin{array}{l} \textit{weakly in } L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \\ \textit{weakly} \star \textit{ in } L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \end{array} \tag{34}$$

$$\left.\begin{array}{l} \left(g_{1\alpha}\frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial t}, g_{2\alpha}\frac{\partial \hat{u}_{2\alpha}^\varepsilon}{\partial t}\right) \rightharpoonup \left(g_{1\alpha}\frac{\partial u_{1\alpha}^\star}{\partial t}, g_{2\alpha}\frac{\partial u_{2\alpha}^\star}{\partial t}\right) \\ \varepsilon^2\left(g_{13}\frac{\partial \hat{u}_{13}^\varepsilon}{\partial t}, g_{23}\frac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right) \rightharpoonup (0,0) \end{array}\right\} \begin{array}{l} \textit{weakly in } L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \\ \textit{weakly} \star \textit{ in } L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2)) \end{array} \tag{35}$$

**Proof.** Using estimates (19)–(21) for $\alpha = 1,2$, we obtain

$$\left\|\left(\frac{\partial \hat{u}_{1\alpha}^\varepsilon}{\partial z}, \frac{\partial \hat{u}_{2\alpha}^\varepsilon}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 \le C, \quad \left\|\left(\frac{\partial^2 \hat{u}_{1\alpha}^\varepsilon}{\partial z \partial t}, \frac{\partial^2 \hat{u}_{2\alpha}^\varepsilon}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 \le C\,. \tag{36}$$

We apply Poincaré's inequality in $(\Omega_1 \cup \Omega_2) \times (0,T)$, with a simple comparison of the two estimates given in (36). We deduce

$$\begin{array}{rcl} \|\hat{u}_{l\alpha}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_l))} & \le & h^*\left\|\frac{\partial \hat{u}_{l\alpha}^\varepsilon}{\partial z}\right\|_{L^\infty(0,T;L^2(\Omega_l))} \le h^* C\,,\ l = 1,2 \\ \left\|\frac{\partial \hat{u}_{l\alpha}^\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_l))} & \le & h^*\left\|\frac{\partial^2 \hat{u}_{l\alpha}}{\partial z \partial t}\right\|_{L^2(0,T;L^2(\Omega_l))} \le h^* C\,,\ l = 1,2 \end{array}$$

Since $\left(\hat{u}_{l\alpha}^\varepsilon\right)_{l,\alpha=1,2}$ is bounded in $W^{1,2}(0,T;H_z) \cap L^\infty(0,T;H_z)$, by the injection $W^{1,2}(0,T;H_z) \hookrightarrow C(0,T;H_z)$ as in ([6], Lemma 2.2), we obtain convergence (31). Finally, by the expressions of $\hat{e}_{i,j}(\hat{u}_l^{\varepsilon})$ given in $\mathrm{Def}_{\hat{e}_{ij}(\hat{u}_l^{\varepsilon})}$, weak convergences (32)–(35) follow from (19)–(21) and (31). □

**Theorem 5.** *If the hypotheses of Theorem 2 hold, the solution* $\mathbf{u}^{\star} = (\mathbf{u}_1^{\star}, \mathbf{u}_2^{\star})$ *satisfies the limit variational problem:*

$$\left.\begin{aligned} &\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_1}\overset{*}{\mathcal{E}}{}^{,1}\left(\frac{\partial u_{1\alpha}^{\star}}{\partial z}\right).\frac{\partial}{\partial z}(\hat{v}_{1\alpha}-\frac{\partial u_{1\alpha}^{\star}}{\partial t})dx+\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_2}\overset{*}{\mathcal{E}}{}^{,2}\left(\frac{\partial u_{2\alpha}^{\star}}{\partial z}\right).\frac{\partial}{\partial z}(\hat{v}_{2\alpha}-\frac{\partial u_{2\alpha}^{\star}}{\partial t})dx \\ &+\sum_{\alpha,l=1}^{2}\hat{\delta}_l\int_{\Omega_l}\hat{g}_{l\alpha}\left(\frac{\partial u_{l\alpha}^{\star}}{\partial t}\right)\left(\hat{v}_{l\alpha}-\frac{\partial u_{l\alpha}^{\star}}{\partial t}\right)dx+\int_{\omega}\hat{\kappa}|\hat{\mathbf{v}}_{1\tau}-\hat{\mathbf{v}}_{2\tau}-s|dx'-\int_{\omega}\hat{\kappa}|\dot{\mathbf{u}}_{1\tau}^{\star}-\dot{\mathbf{u}}_{1\tau}^{\star}-s|dx' \\ &\geq\sum_{\alpha=1}^{2}\int_{\Omega_1}\hat{f}_{1\alpha}(\hat{v}_{1\alpha}-\frac{\partial u_{1\alpha}^{\star}}{\partial t})dx+\sum_{\alpha=1}^{2}\int_{\Omega_2}\hat{f}_{2\alpha}(\hat{v}_{2\alpha}-\frac{\partial u_{2\alpha}^{\star}}{\partial t})dx,\quad\forall(\hat{\mathbf{v}}_1,\hat{\mathbf{v}}_2)\in\Pi(W),\forall t\in[0,T], \end{aligned}\right\} \tag{37}$$

*and the limit problem:*

$$\left.\begin{aligned} &\left(-\frac{1}{2}\frac{\partial}{\partial z}\left[\overset{*}{\mathcal{E}}{}^{,1}\frac{\partial\mathbf{u}_1^{\star}}{\partial z}\right]+\hat{\delta}_1\hat{g}_1\left(\frac{\partial\mathbf{u}_1^{\star}}{\partial t}\right),-\frac{1}{2}\frac{\partial}{\partial z}\left[\overset{*}{\mathcal{E}}{}^{,2}\frac{\partial\mathbf{u}_2^{\star}}{\partial z}\right]+\hat{\delta}_2\hat{g}_2\left(\frac{\partial\mathbf{u}_2^{\star}}{\partial t}\right)\right)=(\hat{\mathbf{f}}_1,\hat{\mathbf{f}}_2) \\ &\qquad in\ L^2\left(0,T;L^2(\Omega_1)^2\times(L^2(\Omega_2))^2\right), \end{aligned}\right\} \tag{38}$$

$$\mathbf{u}^{\star}(0)=(\mathbf{u}_1^{\star}(0),\mathbf{u}_2^{\star}(0))=\underline{\widehat{\mathbf{u}}}(0)=(\underline{\widehat{\mathbf{u}}}_1^0,\underline{\widehat{\mathbf{u}}}_2^0). \tag{39}$$

**Remark 2.** *By convergences (31)–(35), the matrix's* $\widehat{\mathcal{E}}^{l}$, $l=1,2$ *converges (for* $\varepsilon\to 0$*) to*

$$\overset{*}{\mathcal{E}}{}^{,l}=\begin{pmatrix}\overset{*}{\mathcal{E}}{}^{,l}_{1313} & \overset{*}{\mathcal{E}}{}^{,l}_{1323}\\ \overset{*}{\mathcal{E}}{}^{,l}_{2313} & \overset{*}{\mathcal{E}}{}^{,l}_{2323}\end{pmatrix}.$$

**Proof of Theorem 5.** By passage to the limit, when $\varepsilon\to 0$ in the variational inequality (18) and using the convergence results of Theorem 4 with the fact that $\hat{J}$ is convex and lower semi-continuous, we obtain directly Formula (37).

Now, for the proof of (38), we choose ([15]): $\hat{v}_{1\alpha}=\dfrac{\partial u_{1\alpha}^{\star}}{\partial t}\pm\psi_{1\alpha}$, $\hat{v}_{2\alpha}=\dfrac{\partial u_{2\alpha}^{\star}}{\partial t}\pm\psi_{2\alpha}$, where $(\psi_{1\alpha},\psi_{2\alpha})\in H_0^1(\Omega_1)\times H_0^1(\Omega_2)$, $\forall t\in]0,T[$, $\alpha=1,2$, we find

$$\begin{aligned} &\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_1}\overset{*}{\mathcal{E}}{}^{,1}\left(\frac{\partial u_{1\alpha}^{\star}}{\partial z}\right).\frac{\partial\psi_{1\alpha}}{\partial z}dx+\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_2}\overset{*}{\mathcal{E}}{}^{,2}\left(\frac{\partial u_{2\alpha}^{\star}}{\partial z}\right).\frac{\partial\psi_{2\alpha}}{\partial z}dx \\ &+\sum_{\alpha=1}^{2}\hat{\delta}_1\int_{\Omega_1}\hat{g}_{1\alpha}\left(\frac{\partial u_{1\alpha}^{\star}}{\partial t}\right)\psi_{1\alpha}dx+\sum_{\alpha=1}^{2}\hat{\delta}_2\int_{\Omega_2}\hat{g}_{2\alpha}\left(\frac{\partial u_{2\alpha}^{\star}}{\partial t}\right)\psi_{2\alpha}dx \\ =\ &\sum_{\alpha=1}^{2}\int_{\Omega_1}\hat{f}_{1\alpha}\psi_{1\alpha}dx+\sum_{\alpha=1}^{2}\int_{\Omega_2}\hat{f}_{2\alpha}\psi_{2\alpha}dx. \end{aligned}$$

Using Green's formula, we find

$$\begin{aligned} &-\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_1}\frac{\partial}{\partial z}\left(\overset{*}{\mathcal{E}}{}^{,1}\left(\frac{\partial u_{1\alpha}^{\star}}{\partial z}\right)\right).\psi_{1\alpha}dx-\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_2}\frac{\partial}{\partial z}\left(\overset{*}{\mathcal{E}}{}^{,2}\left(\frac{\partial u_{2\alpha}^{\star}}{\partial z}\right)\right).\psi_{2\alpha}dx \\ &+\sum_{\alpha=1}^{2}\hat{\delta}_1\int_{\Omega_1}\hat{g}_{1\alpha}\left(\frac{\partial u_{1\alpha}^{\star}}{\partial t}\right)\psi_{1\alpha}dx+\sum_{\alpha=1}^{2}\hat{\delta}_2\int_{\Omega_2}\hat{g}_{2\alpha}\left(\frac{\partial u_{2\alpha}^{\star}}{\partial t}\right)\psi_{2\alpha}dx \\ =\ &\sum_{\alpha=1}^{2}\int_{\Omega_1}\hat{f}_{1\alpha}\psi_{1\alpha}dx+\sum_{\alpha=1}^{2}\int_{\Omega_2}\hat{f}_{2\alpha}\psi_{2\alpha}dx,\quad\forall(\psi_{1\alpha},\psi_{2\alpha})\in H_0^1(\Omega_1)\times H_0^1(\Omega_2),\forall t\in[0,T] \end{aligned}$$

Therefore, we deduce

$$\left(-\frac{1}{2}\frac{\partial}{\partial z}\left[\overset{*,1}{\mathcal{E}}\frac{\partial \mathbf{u}^{\star}_{1\alpha}}{\partial z}\right]+\hat{\delta}_1\hat{g}_{1\alpha}\left(\frac{\partial \mathbf{u}^{\star}_{1\alpha}}{\partial t}\right),-\frac{1}{2}\frac{\partial}{\partial z}\left[\overset{*,2}{\mathcal{E}}\frac{\partial \mathbf{u}^{\star}_{2\alpha}}{\partial z}\right]+\hat{\delta}_2\hat{g}_{2\alpha}\left(\frac{\partial \mathbf{u}^{\star}_{2\alpha}}{\partial t}\right)\right) \tag{40}$$
$$=(\hat{\mathbf{f}}_{1\alpha},\hat{\mathbf{f}}_{2\alpha}) \text{ in } L^2\Big(0,T;H^{-1}(\Omega_1)\times H^{-1}(\Omega_2)\Big),$$

We know that if $\left(\hat{f}_{1\alpha},\hat{f}_{2\alpha}\right)\in L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2))$, then (40) is true in $L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2))$. Condition (39) is an immediate consequence of the second equation of (18) and (31). □

**Theorem 6.** *If the hypotheses of Theorem 2 hold, we have the following equality*

$$\theta^{\star}_1(x',t)=\theta^{\star}_2(x',t) \ \ in \ L^2(\omega)^2, \quad \forall t\in\,]0,T[ \tag{41}$$

$$\int_{\omega}\hat{\kappa}\left(\left|\phi_1-\phi_2+\frac{\partial q^{\star}_1}{\partial t}-\frac{\partial q^{\star}_2}{\partial t}\right|-\left|\frac{\partial q^{\star}_1}{\partial t}-\frac{\partial q^{\star}_2}{\partial t}-s\right|\right)dx'$$
$$-\frac{1}{2}\int_{\omega}(\theta^{\star}_1\phi_1-\theta^{\star}_2\phi_2)dx'\geq 0,\ \forall\phi_1,\phi_2\in\Big(L^2(\omega)\Big)^2,\forall t\in\,]0,T[ \tag{42}$$

$$\begin{cases} |\theta^{\star}_l|<\hat{\kappa}\Rightarrow \dfrac{\partial q^{\star}_1}{\partial t}-\dfrac{\partial q^{\star}_2}{\partial t}=s, \\ |\theta^{\star}_l|=\hat{\kappa}\Rightarrow \exists\lambda\geq 0,\ such\ that\ \dfrac{\partial q^{\star}_1}{\partial t}-\dfrac{\partial q^{\star}_2}{\partial t}=s+\lambda\theta^{\star}_l, \end{cases} \quad on\ \omega\times\,]0,T[. \tag{43}$$

*where*

$$q^{\star}_l(x',t)=\hat{\mathbf{u}}^{\star}_l(x',0,t),\qquad \theta^{\star}_l(x',t)=\overset{*,l}{\mathcal{E}}(x',0)\frac{\partial \hat{\mathbf{u}}^{\star}_l}{\partial z}(x',0,t),\quad l=1,2.$$

**Proof.** Choosing in (37), $\hat{v}_{1i}=\dfrac{\partial u^{\star}_{1i}}{\partial t}+\phi_{1i},\hat{v}_{2i}=\dfrac{\partial u^{\star}_{2i}}{\partial t}+\phi_{2i}$ for $i=1,2$, with $(\phi_{1i},\phi_{2i})\in H^1_{\Gamma_1\cup\Gamma_{L_1}(\Omega_1)}\times H^1_{\Gamma_2\cup\Gamma_{L_2}(\Omega_2)}$, then, we pass to the limit and applying Green's formula. We obtain

$$-\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_1}\frac{\partial}{\partial z}\left(\overset{*,1}{\mathcal{E}}(x',z,t)\ \frac{\partial u^{\star}_{1\alpha}(x',t)}{\partial z}\right)\phi_{1\alpha}(x',t)dx'dz$$
$$-\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_2}\frac{\partial}{\partial z}\left(\overset{*,2}{\mathcal{E}}(x',z,t)\ \frac{\partial u^{\star}_{2\alpha}(x',t)}{\partial z}\right)\phi_{2\alpha}(x',t)dx'dz$$
$$+\frac{1}{2}\int_{\omega}\left(\overset{*,1}{\mathcal{E}}(x',0)\ \frac{\partial u^{\star}_{1\alpha}(x',t)}{\partial z}\phi_1(x',0)-\overset{*,2}{\mathcal{E}}(x',0)\ \frac{\partial u^{\star}_{2\alpha}(x',t)}{\partial z}\phi_2(x',0)\right)dx'$$
$$+\int_{\omega}\hat{\kappa}\left(\left|\psi_1-\psi_2+\frac{\partial q^{\star}_1}{\partial t}-\frac{\partial q^{\star}_2}{\partial t}\right|-\left|\frac{\partial q^{\star}_1}{\partial t}-\frac{\partial q^{\star}_2}{\partial t}-s\right|\right)dx'$$
$$+\sum_{\alpha=1}^{2}\hat{\delta}_1\int_{\Omega_1}\hat{g}_{1\alpha}\left(\frac{\partial u^{\star}_{1\alpha}(x',t)}{\partial t}\right)\phi_{1\alpha}(x',t)dx'dz+\sum_{\alpha=1}^{2}\hat{\delta}_2\int_{\Omega_2}\hat{g}_{2\alpha}\left(\frac{\partial u^{\star}_{2\alpha}(x',t)}{\partial t}\right)\phi_{2\alpha}(x',t)dx'dz$$
$$\geq\ \sum_{\alpha=1}^{2}\int_{\Omega_1}\hat{f}_{1\alpha}(x',t)\phi_{1\alpha}(x',t)dx'dz+\sum_{\alpha=1}^{2}\int_{\Omega_2}\hat{f}_{2\alpha}(x',t)\phi_{2\alpha}(x',t)dx'dz,$$

On the other hand, from (38), we deduce that:

$$\int_\omega \hat{\kappa}\left(\left|\phi_1 - \phi_2 + \frac{\partial q_1^\star}{\partial t} - \frac{\partial q_2^\star}{\partial t}\right| - \left|\frac{\partial q_1^\star}{\partial t} - \frac{\partial q_2^\star}{\partial t} - s\right|\right)dx'$$
$$-\frac{1}{2}\int_\omega \big(\theta_1^\star(x',t).\phi_1(x',0) - \theta_2^\star(x',t).\phi_2(x',0)\big)dx' \geq 0.$$

This inequality remains valid for any $\phi_1, \phi_2 \in (D(\omega))^2$, and, by the density of $D(\omega)^2$ in $L^2(\omega)^2$, we have

$$\int_\omega \hat{\kappa}\left(\left|\phi_1 - \phi_2 + \frac{\partial q_1^\star}{\partial t} - \frac{\partial q_2^\star}{\partial t}\right| - \left|\frac{\partial q_1^\star}{\partial t} - \frac{\partial q_2^\star}{\partial t} - s\right|\right)dx'$$
$$-\frac{1}{2}\int_\omega \big(\theta_1^\star(x',t).\phi_1(x',0) - \theta_2^\star(x',t).\phi_2(x',0)\big)dx' \geq 0, \text{ for all } \phi_1, \phi_2 \in (L^2(\omega))^2$$

In the particular case of $\phi_1 = \phi_2 = \pm\phi$, we obtain

$$\int_\omega \big(\theta_1^\star(x',t) - \theta_2^\star(x',t)).\phi(x',0)\big)dx' = 0, \forall\, \phi \in (L^2(\omega))^2.$$

which gives Formulas (41)–(42). For the proof of (43), we follow the same techniques as in the fluids problem (as in [1]). □

**Theorem 7.** *If the components* $\overset{*,1}{\mathcal{E}}_{\alpha 3\beta 3}$ *and* $\overset{*,2}{\mathcal{E}}_{\alpha 3\beta 3}$ *are independent at the variable* $z$ *for all* $\alpha, \beta = 1,2$, *then the initial problem converges toward the following weak form:*

$$\int_\omega \left(\int_0^h \big[F_1(x',z,t) + F_2(x',-z,t)\big]dz - h\big[F_1(x',h,t) + F_2(x',-h,t)\big]\right).\nabla\psi(x')dx'$$
$$+\int_\omega \left(h\big[G_1(x',h,t) + G_2(x',-h,t)\big] - \int_0^h \big[G_1(x',z,t) + G_2(x',-z,t)\big]dz+\right).\nabla\psi(x')dx'$$
$$+\frac{1}{2}\int_\omega \left(\int_0^h \overset{*,1}{\mathcal{E}}(x')\hat{\mathbf{u}}_1^\star(x',z,t)dz + \int_{-h}^0 \overset{*,2}{\mathcal{E}}(x')\hat{\mathbf{u}}_2^\star(x',z,t)dz\right).\nabla\psi(x')dx' = 0,$$
$$\text{for all } \psi(x') \in H^1(\omega), \tag{44}$$

*where*

$$\begin{aligned} F_1(x',z,t) &= \int_0^z \int_0^\zeta \hat{\mathbf{f}}_1(x',\theta,t)d\theta d\zeta, \quad G_1(x',z,t) = \hat{\delta}_1 \int_0^z \int_0^\zeta \hat{g}_1\left(\frac{\partial u_1^\star(x',t)}{\partial t}\right)d\theta d\zeta \\ F_2(x',z,t) &= \int_z^0 \int_\zeta^0 \hat{\mathbf{f}}_2(x',\theta,t)d\theta d\zeta, \quad G_2(x',z,t) = \hat{\delta}_2 \int_z^0 \int_\zeta^0 \hat{g}_2\left(\frac{\partial u_2^\star(x',t)}{\partial t}\right)d\theta d\zeta \end{aligned}$$

**Proof.** By integrating twice the first equation of (38) over $[0,z]$ and the second between $z$ and 0, and taking into account $\overset{*,l}{\mathcal{E}}_{\alpha 3\beta 3}$, $l = 1,2$ depending only on $x'$, we infer

$$-\frac{1}{2}\overset{*,1}{\mathcal{E}}(x')\hat{\mathbf{u}}_1^\star(x',z,t) + \frac{1}{2}\overset{*,1}{\mathcal{E}}(x')q_1^\star(x',t) + \frac{1}{2}z\theta_1^\star(x',t) + G_1(x',z,t)$$
$$= F_1(x',z,t), \tag{45}$$
$$-\frac{1}{2}\overset{*,2}{\mathcal{E}}(x')\hat{\mathbf{u}}_2^\star(x',z,t) + \frac{1}{2}\overset{*,2}{\mathcal{E}}(x')q_2^\star(x',t) + \frac{1}{2}z\theta_2^\star(x',t) + G_2(x',z,t)$$
$$= F_2(x',z,t). \tag{46}$$

Now, for $t \in [0,T]$, by setting $z = h(x')$ in (45) and $z = -h(x')$ in (46), and as $\hat{\mathbf{u}}_1^\star(x', h(x'), t) = \hat{\mathbf{u}}_2^\star(x', -h(x'), t) = 0$, we find

$$\frac{1}{2}\overset{*,1}{\mathcal{E}}(x')q_1^\star(x',t) + \frac{1}{2}h\theta_1^\star(x',t) + G_1(x',h(x'),t) = F_1(x',h(x'),t), \quad (47)$$

$$\frac{1}{2}\overset{*,2}{\mathcal{E}}(x')q_2^\star(x',t) - \frac{1}{2}h\theta_2^\star(x',t) + G_2(x',-h(x'),t) = F_2(x',-h(x'),t). \quad (48)$$

Using (41), we have

$$\frac{1}{2}\left(\overset{*,1}{\mathcal{E}}(x')q_1^\star(x',t) + \overset{*,2}{\mathcal{E}}(x')q_2^\star(x',t)\right) = -G_1(x',h(x'),t) - G_2(x',-h(x'),t) + F_1(x',h(x'),t) + F_2(x',-h(x'),t) \quad (49)$$

Now, we integrate (45) between 0 and $h(x')$ and (46) between $-h(x')$ and 0, we obtain

$$\frac{1}{2}\overset{*,1}{\mathcal{E}}(x')\int_0^{h(x')}\hat{\mathbf{u}}_1^\star(x',z,t)dz = \frac{h(x')}{2}\overset{*,1}{\mathcal{E}}(x')q_1^\star(x',t) + \frac{h^2(x')}{4}\theta_1^\star(x',t) + \int_0^{h(x')}G_1(x',z,t)dz + \int_0^{h(x')}F_1(x',z,t)dz, \quad (50)$$

$$\frac{1}{2}\overset{*,2}{\mathcal{E}}(x')\int_{-h(x')}^0\hat{\mathbf{u}}_2^\star(x',z,t)dz = \frac{h(x')}{2}\overset{*,2}{\mathcal{E}}(x')q_2^\star(x',t) - \frac{h^2(x')}{4}\theta_2^\star(x',t) + \int_{-h(x')}^0 G_2(x',z,t)dz + \int_{-h(x')}^0 F_2(x',z,t)dz, \quad (51)$$

From (50)–(51) and (49), we derive relation (44), which was needed. □

**Theorem 8.** *Suppose that the assumptions of the previous theorem hold; then the solution* $\mathbf{u}^\star = (\mathbf{u}_1^\star, \mathbf{u}_2^\star)$ *of the limit problems (37)–(39) is unique in* $L^2(0,T,H_z) \cap L^\infty(0,T,H_z)$.

**Proof.** Suppose that for $t \in [0,T]$, problems (37)–(39) have two different solutions: $\mathbf{u}^\star = (\mathbf{u}_1^\star, \mathbf{u}_2^\star)$ and $\phi^\star = (\phi_1^\star, \phi_2^\star)$. Taking $\mathbf{v}^\star = \left(\frac{\partial \phi_1^\star}{\partial t}, \frac{\partial \phi_2^\star}{\partial t}\right)$ in (37) and then $\mathbf{v}^\star = \left(\frac{\partial \mathbf{u}_1^\star}{\partial t}, \frac{\partial \mathbf{u}_2^\star}{\partial t}\right)$ in the same inequality and summing the two new forms, we deduce for $\mathbb{T}_1 = \mathbf{u}_1^\star - \phi_1^\star$ and $\mathbb{T}_2 = \mathbf{u}_2^\star - \phi_2^\star$ :

$$\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_1}\overset{*,1}{\mathcal{E}}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}\mathbb{T}_1(t)\right).\frac{\partial}{\partial z}\mathbb{T}_1 dx + \frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega_2}\overset{*,2}{\mathcal{E}}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}\mathbb{T}_2(t)\right).\frac{\partial}{\partial z}\mathbb{T}_2 dx + \sum_{\alpha,l=1}^{2}\hat{\delta}_l\int_{\Omega_l}\left[\hat{g}_{l\alpha}\left(\frac{\partial u_{l\alpha}^\star}{\partial t}\right) - \hat{g}_{l\alpha}\left(\frac{\partial \phi_{l\alpha}^\star}{\partial t}\right)\right]\left[\frac{\partial u_{l\alpha}^\star}{\partial t} - \frac{\partial \phi_{l\alpha}^\star}{\partial t}\right]dx \le 0. \quad (52)$$

Now, using the assumption that $\hat{g}_l$ is monotonous, we obtain

$$\sum_{\alpha=1}^{2}\int_{\Omega_1}\overset{*,1}{\mathcal{E}}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}\mathbb{T}_1(t)\right).\frac{\partial}{\partial z}\mathbb{T}_1(t)dx + \sum_{\alpha=1}^{2}\int_{\Omega_2}\overset{*,2}{\mathcal{E}}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}\mathbb{T}_2(t)\right).\frac{\partial}{\partial z}\mathbb{T}_2(t)dx \le 0 \quad (53)$$

Since $\mathbb{T}_1(0) = \mathbb{T}_2(0) = 0$, we integrate (53) between 0 and $t$. We find

$$\sum_{\alpha=1}^{2}\left\langle\overset{*,1}{\mathcal{E}}\frac{\partial}{\partial z}\mathbb{T}_1(t), \frac{\partial}{\partial z}\mathbb{T}_1(t)\right\rangle_{\Omega_1} + \sum_{\alpha=1}^{2}\left\langle\overset{*,2}{\mathcal{E}}\frac{\partial}{\partial z}\mathbb{T}_2(t), \frac{\partial}{\partial z}\mathbb{T}_2(t)\right\rangle_{\Omega_2} \le 0 \quad (54)$$

We must now check the ellipticity of the matrix's $\overset{*}{\mathcal{E}}^{,l}$ (hypothesis (H$_3$)). Let $\eta = (\eta_\alpha)_{\alpha=1,2} \in \mathbb{R}^2$. We return, now, to hypotheses (2) and (H$_3$). By choosing symmetric tensors, $e = (e_{\alpha\beta})$ is given by $e = \begin{pmatrix} 0 & 0 & \eta_1 \\ 0 & 0 & \eta_2 \\ \eta_1 & \eta_2 & 0 \end{pmatrix}$, and we will obtain

$$\begin{aligned} \widehat{\mathcal{E}}^l_{ijkl} e_{kl} e_{ij} &= 2\widehat{\mathcal{E}}^l_{\alpha 3\beta 3}(e_{\beta 3})(e_{\alpha 3}) + 2\widehat{\mathcal{E}}^l_{\alpha 333}(e_{33})(e_{\alpha 3}) + 2\widehat{\mathcal{E}}^l_{33\alpha 3}(e_{\alpha 3})(e_{33}) + \widehat{\mathcal{E}}^l_{3333}(e_{33})(e_{33}) \\ &= \overset{*}{\mathcal{E}}^{,l}_{\alpha\beta\eta_\beta\eta_\alpha}, \text{ for } \alpha, \beta, l = 1, 2. \end{aligned}$$

Consequently, as $|e|^2 = 2|\eta|^2$, we obtain

$$\overset{*}{\mathcal{E}}^{,l} \eta.\eta \geq 2m_{\mathcal{E}^l}|\eta|^2, \quad \text{for all } l = 1,2,\ \eta = (\eta_1, \eta_2) \in \mathbb{R}^2.$$

Hence, inequality (54) becomes

$$2m_{\mathcal{E}^1}\left\|\frac{\partial}{\partial z}\mathbb{T}_1(t)\right\|^2_{0,\Omega_1} + 2m_{\mathcal{E}^2}\left\|\frac{\partial}{\partial z}\mathbb{T}_2(t)\right\|^2_{0,\Omega_2} \leq 0.$$

As $\mathbb{T}_1(0) = \mathbb{T}_2(0) = 0$, and $m_{\mathcal{E}^1}$, $m_{\mathcal{E}^2} > 0$, we have

$$\left\|\frac{\partial}{\partial z}\mathbb{T}_1(t)\right\|^2_{0,\Omega_1} = \left\|\frac{\partial}{\partial z}\mathbb{T}_2(t)\right\|^2_{0,\Omega_2} = 0.$$

Using Poincaré's inequality, we obtain

$$\|(\mathbb{T}_1(t), \mathbb{T}_2(t))\|^2_{L^2(0,T;H_z)} = \|(\mathbb{T}_1(t), \mathbb{T}_2(t))\|^2_{L^\infty(0,T;H_z)} = 0.$$

where we give $(\mathbf{u}_1^\star, \mathbf{u}_2^\star) = (\phi_1^\star, \phi_2^\star)$ in $L^2(0,T;H_z) \cap L^\infty(0,T,H_z)$, which concludes the uniqueness of problems (37)–(39). □

## 6. Conclusions

The subject of this article falls within the framework of the study of a transmission problem with friction law and increasing continuous terms in a thin layer. To obtain the desired goal, and after the variational formulation of each problem using the change in scale and new unknowns to conduct the study on a domain does not depend on ε. Then, we demonstrate different estimates of the displacement and the source term independently of ε. Finally, by passing to the limit, we obtain the limit problem and the generalized weak equation of the problem considered.

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## References


1. Bayada, G.; Lhalouani, K. Asymptotic and numerical analysis for unilateral contact problem with Coulomb's friction between an elastic body and a thin elastic soft layer. *Asymptot. Anal.* **2001**, *25*, 329–362.
2. Irago, H.; Viano, J.M.; Rodrguez-Areos, A. Asymptotic derivation of frictionless contact models for elastic rods on a foundation with normal compliance. *Nonlinear Anal.* **2013**, *14*, 852–866. [CrossRef]
3. Viano, J.M.; Rodrguez-Areos, A.; Sofonea, M. Asymptotic derivation of quasistatic frictional contact models with wear for elastic rods. *J. Math. Anal. Appl.* **2013**, *401*, 641–653. [CrossRef]
4. Monneau, R.; Murat, F.; Sili, A. Error Estimate for the Transition 3d-1d in Anisotropic Heterogeneous Linearized Elasticity. Preprint. 2002. Available online: http://cermics.enpc.fr/~monneau/home.html (accessed on 1 October 2023).
5. Murat, F.; Sili, A. Asymptotic behavior of solutions of the anisotropic heterogeneous linearized elasticity system in thin cylinders. *C.R. Acad. Sci. Paris Serie I* **1999**, *328*, 179–184 [CrossRef]
6. Sofonea, M.; Matei, A. Mathematical Models in Contact Mechanics, *London Mathematical Society Lecture Note Series*; Cambridge University Press: Cambridge, UK, 2012.
7. Akram, M.; Khan, A.; Dilshad, M. Convergence of some iterative algorithms for system of generalized set-valued variational inequalities. *J. Funct. Spaces* **2021**, *2021*, 6674349. [CrossRef]
8. Alghamdi, A.M.; Gala, S.; Ragusa, M.A. Global regularity for the 3D micropolar fluid flows. *Filomat* **2022**, *36*, 1967–1970. [CrossRef]
9. Dai, X.Q.; Li, W.K. Non-global solution for visco-elastic dynamical system with nonlinear source term in control problem. *Electron. Res. Arch.* **2021**, *29*, 4087–4098. [CrossRef]
10. Benseridi, H.; Dilmi, M. Some inequalities and asymptotic behavior of dynamic problem of linear elasticity. *Georgian Math. J.* **2013**, *20*, 25–41. [CrossRef]
11. Saadallah, A.; Benseridi, H.; Dilmi, M.; Drabla, S. Estimates for the asymptotic convergence of a non-isothermal linear elasticity with friction. *Georgian Math. J.* **2016**, *23*, 435–446. [CrossRef]
12. Bayada, G.; Boukouche, M. On a free boundary problem for Reynolds equation derived from the Stokes system with Tresca boundary conditions. *J. Math. Anal. Appl.* **2003**, *382*, 212–231. [CrossRef]
13. Benseridi, H.; Letoufa, Y.; Dilmi, M. On the Asymptotic Behavior of an interface Problem in a Thin Domain. *M. Proc. Natl. Acad. Sci. India Sect. A Phys. Sci.* **2019**, *89*, 1–10 [CrossRef]
14. Boukrouche, M.; El Mir, R. Asymptotic analysis of non-Newtonian fluid in a thin domain with Tresca law. *Nonlinear Anal. Theory Methods Appl.* **2004**, *59*, 85–105. [CrossRef]
15. Boukrouche, M.; Lukaszewicz, G. On a lubrication problem with Fourier and Tresca boundary conditions. *Math. Models Methods Appl. Sci.* **2004**, *14*, 913–941. [CrossRef]
16. Manaa, S.; Benseridi, H.; Dilmi, M. 3D-2D asymptotic analysis of an interface problem with a dissipative term in a dynamic regime. *Bol. Soc. Mat. Mex.* **2021**, *27*, 10. [CrossRef]
17. Benseghir, A.; Benseridi, H.; Dilmi, M. On the asymptotic study of transmission problem a thin domain. *J. Inv. Ill-Posed. Prob.* **2019**, *27*, 53–67. [CrossRef]
18. Hemici, N.; Matei, A. A frictionless contact problem with adhesion between two elastic bodies. *An. Univ. Cairova Math. Comp. Sci. Ser.* **2003**, *30*, 90–99.
19. Kim, T.; Kim, D.S. Degenerate r-Whitney numbers and degenerate r-Dowling polynomials via boson operators. *Adv. Appl. Math.* **2022**, *140*, 102394. [CrossRef]
20. Kim, T.; Kim, D.S.; Kim, H.K. Normal ordering of degenerate integral powers of number operator and its applications. *Appl. Math. Sci. Eng.* **2022**, *30*, 440–447. [CrossRef]
21. Li, Y.; Shen, F.; Ke, L. Multi-physics electrical contact analysis considering the electrical resistance and Joule heating. *Int. J. Solids Struct.* **2022**, *256*, 111975. [CrossRef]
22. Shen, F.; Li, Y.; Ke, L. On the size distribution of truncation areas for fractal surfaces. *Int. J. Mech. Sci.* **2022**, *237*, 107789. [CrossRef]
23. Shen, F.; Li, Y.; Ke, L. A novel fractal contact model based on size distribution law. *Int. J. Mech. Sci.* **2023**, *249*, 108255. [CrossRef]
24. Duvant, G.; Lions, J.L. *Les Inéquations en Mécanique et en Physique*; Dunod: Paris, France, 1972.
25. Lions, J.L. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*; Dunod: Paris, France, 1969.