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Gram Points in the Universality of the Dirichlet Series with Periodic Coefficients

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Abstract: Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic multiplicative sequence of complex numbers and $L(s;\mathfrak{a})$, $s = \sigma + it$ a Dirichlet series with coefficients a_m. In the paper, we obtain a theorem on the approximation of non-vanishing analytic functions defined in the strip $1/2 < \sigma < 1$ via discrete shifts $L(s + iht_k;\mathfrak{a})$, $h > 0$, $k \in \mathbb{N}$, where $\{t_k : k \in \mathbb{N}\}$ is the sequence of Gram points. We prove that the set of such shifts approximating a given analytic function is infinite. This result extends and covers that of [Korolev, M.; Laurinčikas, A. A new application of the Gram points. *Aequat. Math.* **2019**, *93*, 859–873]. For the proof, a limit theorem on weakly convergent probability measures in the space of analytic functions is applied.

Keywords: space of analytic functions; approximation of analytic functions; universality; weak convergence

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1. Introduction

Let, as usual, $\mathbb{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all prime, positive integers, integers, rational, real and complex numbers, respectively. The main object of the analytic number theory—Riemann zeta function $\zeta(s)$, $s = \sigma + it$ in the half-plane $\sigma > 1$ is defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

and has a meromorphic continuation to the whole complex plane. The point $s = 1$ is its simple pole with residue 1.

Let $\Gamma(s)$ denote the Euler gamma function. The Riemann zeta function satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Suppose that $\varphi(t)$, $t > 0$ is the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. Since the function $\varphi(t)$ is unbounded and monotonically increases for $t > t^*$ (it is well known that $t^* = 6.289836\ldots$ and $\varphi(t^*) = -3.530573\ldots$), the equation

$$\varphi(t) = (k-1)\pi, \quad k \in \mathbb{N}, \tag{1}$$

for $t > t^*$ has the unique solution t_k. Gram was the first investigator of the numbers t_k; therefore, they are now called Gram points. The Gram points are important in the analytic number theory because they are closely related with imaginary parts of non-trivial zeros of the function $\zeta(s)$. For more information, see [1–4].

Without other important properties, the function $\zeta(s)$ has the universality property obtained by Voronin [5]. In other words, this property means that a wide class of analytic functions uniformly on compact sets can be approximated by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. The initial Voronin theorem [5] states that if $0 < r < 1/4$, and $f(s)$ is a continuous and non-vanishing function on the disc $|s| \leqslant r$ and analytic in the interior of that disc, then, for all $\varepsilon > 0$, there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s|\leqslant r}\left|f(s) - \zeta\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon.$$

A lot of authors, among them Gonek, Bagchi, Matsumoto, Pańkowski, Steuding, Laurinčikas, Garunkštis, Macaitienė, Kačinskaitė, and others, improved and extended the above Voronin theorem. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and let $H_0(K)$, $K \in \mathcal{K}$, denote the class of continuous non-vanishing functions on K that are analytic in the interior of K and $\mathcal{M}A$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the modern version of the Voronin theorem, see, for example, refs. [6–8] and informative paper [9], says that if $K \in \mathcal{K}$, $f(s) \in H_0(K)$, then, for all $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathcal{M}\left\{\tau \in [0,T] : \sup_{s\in K}|f(s) - \zeta(s+i\tau)| < \varepsilon\right\} > 0.$$

The latter inequality means that the set of shifts $\zeta(s+i\tau)$ approximating a given function $f(s) \in H_0(K)$ has a positive lower density.

Now, we will define the Dirichlet series with periodic coefficients. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be the periodic sequence of complex numbers with minimal period $l \in \mathbb{N}$. The Dirichlet series with periodic coefficients $L(s;\mathfrak{a})$, for $\sigma > 1$ is defined by the series

$$L(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Since the sequence $\mathfrak{a}$ is periodic, we have

$$L(s;\mathfrak{a}) = \frac{1}{l^s}\sum_{q=1}^{l} a_q \zeta\left(s, \frac{q}{l}\right), \tag{2}$$

where $\zeta(s,\alpha)$ is the classical Hurwitz zeta function with parameter $\alpha \in (0,1]$, which has, as $\zeta(s)$, a meromorphic continuation to the whole complex plane with a unique simple pole at the point $s = 1$ with residue 1. Hence, the function $L(s;\mathfrak{a})$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$a \overset{\text{def}}{=} \frac{1}{l}\sum_{q=1}^{l} a_q.$$

If $a = 0$, $L(s;\mathfrak{a})$ is an entire function. If the sequence $\mathfrak{a}$ is multiplicative ($a_{mn} = a_m a_n$ for $(m,n) = 1$, and $a_1 = 1$), then, for $\sigma > 1$, the function $L(s;\mathfrak{a})$ has the Euler product

$$L(s;\mathfrak{a}) = \prod_{p\in\mathbb{P}}\left(1 + \sum_{k=1}^{\infty}\frac{a_{p^k}}{p^{ks}}\right).$$

Universality for the function $L(s;\mathfrak{a})$, i.e., approximation of a wide class of analytic functions by shifts $L(s+i\tau;\mathfrak{a})$, $\tau \in \mathbb{R}$, was investigated by various authors, among them, Bagchi [10], Steuding [8,11], and others. The universality of the function $L(s;\mathfrak{a})$ with a multiplicative sequence $\mathfrak{a}$ was obtained in [12].

Theorem 1 ([12])**.** *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and, for all $p \in \mathbb{P}$,*

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\alpha/2}} \leqslant c < 1. \tag{3}$$

Let $K \in \mathcal{K}$, and $f(s) \in H_0(K)$. Then, for all $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathcal{M}\left\{\tau \in [0,T] : \sup_{s\in K} |f(s) - L(s+i\tau;\mathfrak{a})| < \varepsilon\right\} > 0.$$

Note that the requirement (3) is technical and can be removed.

The universality of the Dirichlet series with periodic coefficients is a complicated problem. Kaczorowski in [13] observed that not all Dirichlet series with periodic coefficients are universal in the Voronin sense. He obtained the necessary and sufficient conditions of the universality for $L(s;\mathfrak{a})$ with prime period l.

Theorem 2 ([13])**.** *Let l be a prime number and let $\mathfrak{a} \neq 0$. The corresponding Dirichlet series with periodic coefficients are universal in the sense of Voronin if and only if one of the following possibilities holds:*

1. *Not all numbers $a_1, \dots, a_{l-1}$ are equal;*
2. *We have $a_1 = \cdots = a_{l-1} = 0$;*
3. *We have $a_1 = \cdots = a_{l-1} \neq 0$ and*

$$\left|1 - \frac{a_l}{a_1}\right| \leqslant \sqrt{l} \qquad \text{or} \qquad \left|1 - \frac{a_1}{a_l}\right| \geqslant l.$$

The discrete universality for zeta functions was proposed by Reich [14]. The first result on the approximation of analytic functions by discrete shifts $L(\sigma + ikh;\mathfrak{a})$, with a fixed number $h > 0$ such that $\exp\left\{\frac{2\pi k}{h}\right\}$ is rational for all $k \in \mathbb{Z}$, has been obtained in [15] and a more general result in [16]. Theorem 2 of [17] with $w(u) \equiv 1$ implies the following result. Let $\operatorname{Card} A$ denote the number of elements of the set A, and, for $h > 0$,

$$\mathfrak{L}(\mathbb{P},h,\pi) = \left\{(\log p : p \in \mathbb{P}), \frac{2\pi}{h}\right\}.$$

Theorem 3 ([17])**.** *Suppose that the set $\mathfrak{a}$ is multiplicative and the set $\mathfrak{L}(\mathbb{P},h,\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for all $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\operatorname{Card}\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - L(s+ikh;\mathfrak{a})| < \varepsilon\right\} > 0.$$

The universality of zeta functions is a very surprising and useful phenomenon which, in some sense, reduces a study of a class of analytic functions to that of a comparatively simple one and the same zeta function. Moreover, universality theorems are applied in a lot of number-theoretical problems as the functional independence, zero distribution, denseness, and the moment problems, etc. This is the motivation to study and extend the notion of universality for zeta functions. One of the ways in this direction is to prove the universality theorems for new classes of zeta functions. The Linnik–Ibragimov conjecture (or programme), see [8], Section 1.6, asserts that all functions given by the Dirichlet series, having analytic continuation and satisfying some natural growth conditions, are universal in the Voronin sense. On the other hand, there are examples of non-universal Dirichlet series; Theorem 2 confirms this, and there are also Dirichlet series in which universality is an open problem.

Using the generalized shifts, there is an another way to extend the universality for the Dirichlet series. This idea, for Dirichlet L-functions, was proposed by Pańkowski in [18] with function $\nu(\tau) = \tau^{\alpha}(\log\tau)^{\beta}$ with certain $\alpha, \beta \in \mathbb{R}$. In [19], the universality for the Dirichlet series with periodic coefficients with multiplicative sequence $\mathfrak{a}$ using generalized shifts $L(s + ia\gamma(\tau); \mathfrak{a})$, where $a \neq 0$ is a real number, and $\gamma(\tau)$ is increasing to an ∞ continuously differentiable function with monotonic derivative $\gamma'(\tau)$ on $[T_0, \infty)$, $T_0 > 0$, such that

$$\gamma(2\tau) \max_{\tau \leqslant u \leqslant 2\tau} \frac{1}{\gamma'(u)} \ll \tau, \quad \tau \to \infty,$$

was obtained. Now, let $\{\gamma_k : k \in \mathbb{N}\}$ be a sequence of imaginary parts of non-trivial zeros of the Riemann zeta function and let the hypothesis

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ |\gamma_k - \gamma_l| < c/\log T}} 1 \ll T \log T, \quad c > 0, \tag{4}$$

be satisfied. This estimate is the weak form of the Montgomery pair correlation conjecture [20]. Then, in [21], it was obtained that, under (4), for a fixed number $h > 0$, multiplicative sequence $\mathfrak{a}$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and all $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \mathrm{Card}\left\{1 \leqslant k \leqslant N : \sup_{s \in K} |L(s + ih\gamma_k; \mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

In [22], Korolev and Laurinčikas in approximating shifts of the Riemann zeta function involved the Gram points, and in [23], developed their result in short intervals. We notice that Gram points t_k are asymptotically connected to the numbers γ_k by the equality [22]

$$\lim_{k \to \infty} \frac{t_k}{\gamma_k} = 1.$$

Moreover, using the points t_k for generalized shifts is more convenient than γ_k because t_k has a continuous differentiable version t_u, $u \in \mathbb{R}$, see Lemma 4 bellow, and the analogue of (4) is not needed. The aim of this paper is to generalize the latter result for the Dirichlet series with periodic coefficients. The main result of the paper is the following theorem.

Theorem 4. *Suppose that the sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $h > 0$ be a fixed number. Then, for all $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N} \mathrm{Card}\left\{1 \leqslant k \leqslant N : \sup_{s \in K} |L(s + iht_k; \mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, "lim inf" can be replaced by "lim" for all but at most, countably many $\varepsilon > 0$.

Theorem 4 implies that the set of shifts $L(s + iht_k; \mathfrak{a})$ approximating a given function $f(s)$ with accuracy ε is infinite for every $K \in \mathcal{K}$.

The class of functions $L(s; \mathfrak{a})$ is sufficiently wide; for example, it includes all Dirichlet L-functions which are the main analytic tool for the investigation of prime numbers in arithmetic progressions. When $a_m \equiv 1$, we obtain the Riemann zeta function $\zeta(s)$. Therefore, Theorem 4 extends and covers the main result of [22].

Theorem 4 is theoretical, it is not connected to specific numerical calculations, and can be considered as an impact to the Linnik–Ibragimov programme.

Theorem 4 is stated in density terms (lower density and density), the expression $1/N\mathrm{Card}\{1 \leqslant k \leqslant N : \sup_{s \in K} |L(s + iht_k; \mathfrak{a}) - f(s)| < \varepsilon\}$ with respect to ε is a probabilistic distribution function. Therefore, for its proof, it is convenient to use a probabilistic approach.

More precisely, for the proof of Theorem 4, we apply a limit theorem on weakly convergent probability measures in the space of analytic functions with an explicitly given probability limit measure.

2. Limit Theorems

The space of analytic functions on D endowed with the topology of uniform convergence on compacta is denoted by $H(D)$, and the Borel σ-field of a topological space $\mathbb{T}$ is denoted by $\mathscr{B}(\mathbb{T})$. In this section, we will prove a theorem on the weak convergence of the measure

$$P_N(A) \stackrel{def}{=} \frac{1}{N}\mathrm{Card}\{1 \leqslant k \leqslant N : L(s+iht_k;\mathfrak{a}) \in A\}, \quad A \in \mathscr{B}(H(D)),$$

as $N \to \infty$.

Let $y = \{s \in \mathbb{C} : |s| = 1\}$. Define the set

$$\mathbb{Y} = \prod_{p\in\mathbb{P}} y_p,$$

where $y_p = y$ for all $p \in \mathbb{P}$. With the product topology and operation of pointwise multiplication, the torus $\mathbb{Y}$, in view of the classical Tikhonov theorem, see, for example, ref. [7], is a compact topological Abelian group. Therefore, on $(\mathbb{Y}, \mathscr{B}(\mathbb{Y}))$, we can define the probability Haar measure μ_H. Thus, we can construct the probability space $(\mathbb{Y}, \mathscr{B}(\mathbb{Y}), \mu_H)$. Denote by $y(p)$ the pth component of an element $y \in \mathbb{Y}$, $p \in \mathbb{P}$. Now, on the probability space $(\mathbb{Y}, \mathscr{B}(\mathbb{Y}), \mu_H)$, define the $H(D)$-valued random element

$$L(s, y; \mathfrak{a}) = \prod_{p\in\mathbb{P}} \left(1 + \sum_{\alpha=1}^{\infty} \frac{a_{p^\alpha} y^\alpha(p)}{p^{\alpha s}}\right),$$

and let P_L denote the distribution of $L(s, y; \mathfrak{a})$, i. e.,

$$P_L(A) = \mu_H\{y \in \mathbb{Y} : L(s, y; \mathfrak{a}) \in A\}, \quad A \in \mathscr{B}(H(D)).$$

We can now formulate the main theorem of this section.

Theorem 5. *The measure P_N converges weakly to P_L as $N \to \infty$.*

We divide the proof of Theorem 5 into separate lemmas. At first, for $A \in \mathscr{B}(\mathbb{Y})$, define

$$V_N(A) = \frac{1}{N}\mathrm{Card}\Big\{1 \leqslant k \leqslant N : \Big(p^{-iht_k} : p \in \mathbb{P}\Big) \in A\Big\}.$$

Lemma 1. *The measure V_N converges weakly to the Haar measure μ_H as $N \to \infty$.*

Proof. The character χ of the group $\mathbb{Y}$, see, for example, ref. [7], has the representation

$$\chi(y) = \prod_{p\in\mathbb{P}} y^{k_p}(p), \quad y \in \mathbb{Y},$$

where only a finite number of integers k_p are distinct from zero and does not depend on the sequence $\mathfrak{a}$. Therefore, the proof of the lemma applies the Fourier transform method and coincides with the proof of Lemma 3.2 from [22]. □

From Lemma 1, we can obtain a limit lemma in the space $H(D)$ for the absolutely convergent Dirichlet series. Let, for fixed $\beta > 1/2$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\beta\right\}, \quad m, n \in \mathbb{N},$$

and

$$L_n(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s}.$$

Since $v_n(m) \ll m^{-L/n^\beta}$ with all $L > 0$ and $a_m \ll 1$, the latter series is absolutely convergent for all $s \in \mathbb{C}$. Moreover, let

$$L_n(s, y;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m y(m) v_n(m)}{m^s},$$

where

$$y(m) = \prod_{\substack{p^k | m \\ p^{k+1} \nmid m}} y^k(p), \quad m \in \mathbb{N}.$$

Then, the latter series is also absolutely convergent for all $s \in \mathbb{C}$ because $|y(m)| = 1$. For $\mathscr{B}(H(D))$, define

$$T_{N,n}(A) = \frac{1}{N}\mathrm{Card}\{1 \leqslant k \leqslant N : L_n(s + iht_k;\mathfrak{a}) \in A\}.$$

Let $u_n : \mathbb{Y} \to H(D)$ be defined by the formula

$$u_n(y) = L_n(s, y;\mathfrak{a}).$$

For the proof of the limit lemma for the absolutely convergent Dirichlet series, we need a lemma on the preservation of probability measures under continuous mappings. Let P be a probability measure on $(\mathbb{T}, \mathscr{B}(\mathbb{T}))$ and $u : \mathbb{T} \to \mathbb{T}_1$ be a measurable mapping. Then, Pu^{-1} is defined, for $A \in \mathscr{B}(\mathbb{T}_1)$, by $Pu^{-1}(A) = P(u^{-1}A)$.

Lemma 2. *Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(\mathbb{T}_1, \mathscr{B}(\mathbb{T}_1))$, and $u : \mathbb{T}_1 \to \mathbb{T}_2$ be a continuous mapping. The measure $P_n u^{-1}$ converges weakly to Pu^{-1} as $n \to \infty$ if the measure P_n converges weakly to P as $n \to \infty$.*

The lemma is a partial case of the Theorem 2.7 from [24].

Lemma 3. *The measure $T_{N,n}$, as $N \to \infty$, converges weakly to a measure $T_n \overset{\text{def}}{=} \mu_H u_n^{-1}$.*

Proof. Seeing that the series for $L_n(s, y;\mathfrak{a})$ is absolutely convergent for $\sigma > 1/2$, then the function u_n is continuous. From the definitions of the measures V_N, $T_{N,n}$ and mapping u_n, we have $T_{N,n} = V_n u_n^{-1}$. Therefore, from Lemmas 1 and 2, we have the assertion of the lemma. □

The next step is the approximation of $L(s;\mathfrak{a})$ by $L_n(s;\mathfrak{a})$ in the mean. For the proof of this fact, we need the following lemmas. The first of them is devoted to asymptotics of the function t_u with arbitrary $u \geqslant 0$.

Lemma 4 ([3])**.** *Suppose that t_u, $u \geqslant 0$, denotes the unique solution of the equation*

$$\varphi(t_u) = (u-1)\pi$$

satisfying $\varphi'(t_u) > 0$ and that $u \to \infty$. Then

$$t_u = \frac{2\pi u}{\log u}\left(1 + \frac{\log\log u}{\log u}(1 + o(1))\right)$$

and

$$t'_u = \frac{2\pi}{\log u}\left(1 + \frac{\log\log u}{\log u}(1 + o(1))\right).$$

Also, we recall the Gallagher lemma. This lemma connects discrete and continuous second moments of some functions. For details, see, for example, Lemma 1.4 of [25].

Lemma 5 ([25])**.** *Let $T_0 \geqslant \delta$, $T \geqslant \delta$, $\delta > 0$, $\mathcal{T}$ be a non-empty finite set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$d_\delta(\tau) = \sum_{\substack{t \in \mathcal{T} \\ |t-\tau| < \delta}} 1, \quad \tau \in \mathcal{T}.$$

Moreover, let the function $\mathcal{F}(t)$ be continuous on $[T_0, T_0 + T]$, which takes complex values, and have a continuous derivative on $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} d_\delta^{-1}(t)|\mathcal{F}(t)|^2 \leqslant \frac{1}{\delta} \int\limits_{T_0}^{T_0+T} |\mathcal{F}(t)|^2 \, \mathrm{d}t + \left(\int\limits_{T_0}^{T_0+T} |\mathcal{F}(t)|^2 \, \mathrm{d}t \int\limits_{T_0}^{T_0+T} |\mathcal{F}'(t)|^2 \, \mathrm{d}t \right)^{1/2}.$$

On D, there exists a sequence $\{K_q : q \in \mathbb{N}\}$ of compact subsets such that

$$D = \bigcup_{q=1}^{\infty} K_q,$$

$K_q \subset K_{q+1}$, $\forall q \in \mathbb{N}$, and $K \subset K_q$ for some q if $K \subset D$ is a compact set. Then, the formula

$$\rho(w_1, w_2) = \sum_{q=1}^{\infty} 2^{-q} \frac{\sup_{s \in K_q} |w_1(s) - w_2(s)|}{1 + \sup_{s \in K_q} |w_1(s) - w_2(s)|}, \quad w_1, w_2 \in H(D),$$

gives a metric in $H(D)$. This metric induces its topology of uniform convergence on compacta. To move from the function $L_n(s; \mathfrak{a})$ to $L(s; \mathfrak{a})$, we need the following lemma.

Lemma 6. *For fixed $h > 0$, the following statement*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \rho(L(s + iht_k; \mathfrak{a}), L_n(s + iht_k; \mathfrak{a})) = 0$$

holds.

Proof. First, we obtain some discrete second moment estimates for the function $L(s; \mathfrak{a})$. For $\sigma > 1/2$, the bounds

$$\int\limits_0^T |L(\sigma + it; \mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma, \mathfrak{a}} T$$

and

$$\int\limits_0^T |L'(\sigma + it; \mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma, \mathfrak{a}} T$$

are well known. Hence, for $\sigma > 1/2$ and $t \in \mathbb{R}$, we have

$$\int\limits_0^T |L(\sigma + it + iht_u; \mathfrak{a})|^2 \, \mathrm{d}u \ll_{\sigma, \mathfrak{a}} T(1 + |t|) \tag{5}$$

and

$$\int_0^T |L'(\sigma+it+iht_u;\mathfrak{a})|^2\,\mathrm{d}u \ll_{\sigma,\mathfrak{a}} T(1+|t|). \tag{6}$$

Actually, in view of Lemma 4, the function t_u is increasing. Then, by the same lemma, for $X \geqslant 1$ and $\sigma > 1/2$,

$$\begin{aligned}
\int_X^{2X} |L(\sigma+it+iht_u;\mathfrak{a})|^2\,\mathrm{d}u &= \int_X^{2X} \frac{1}{t'_u}|L(\sigma+it+iht_u;\mathfrak{a})|^2\,\mathrm{d}t_u \\
&\ll_{h,\sigma,\mathfrak{a}} \max_{X\leqslant t\leqslant 2X}\frac{1}{t'_u}\int_X^{2X}\mathrm{d}\left(\int_1^{t+ht_u}|L(\sigma+iv;\mathfrak{a})|^2\,\mathrm{d}v\right) \\
&\ll_{h,\sigma,\mathfrak{a}} \max_{X\leqslant t\leqslant 2X}\frac{1}{t'_u}\int_1^{t+ht_u}|L(\sigma+iv;\mathfrak{a})|^2\,\mathrm{d}v\Big|_X^{2X} \\
&\ll_{h,\sigma,\mathfrak{a}} (t_{2X}+|t|)\max_{X\leqslant t\leqslant 2X}\frac{1}{t'_u} \\
&\ll_{h,\sigma,\mathfrak{a}} \frac{X}{\log X}\log X + |t|\log X \ll_{h,\sigma,\mathfrak{a}} X + |t|\log X \\
&\ll_{h,\sigma,\mathfrak{a}} X(1+|t|).
\end{aligned}$$

Now, taking $X = T2^{-l-1}$ and summing over $l = 0,1,\dots$, we obtain (5). Similarly, we obtain estimate (6).

Now, we will obtain the estimate for the discrete mean value of $L(s;\mathfrak{a})$ involving Gram points. For this, we apply Lemma 5. Let $\delta = 1$, $T_0 = 1$, $T = N$, and $\mathcal{T} = \{3/2, 2, 3, \dots, N, N+1/2\}$. In this case, $d_\delta(x) = 1$. In view of (5) and (6), for $t \in \mathbb{R}$, we find

$$\begin{aligned}
\sum_{k=1}^{N} |L(\sigma+it+iht_k;\mathfrak{a})|^2 \ll_{\sigma,h,\mathfrak{a}} &\int_{3/2}^{N+1/2} |L(\sigma+it+iht_u;\mathfrak{a})|^2\,\mathrm{d}u \\
&+\left(\int_{3/2}^{N+1/2}|L(\sigma+it+iht_u,;\mathfrak{a})|^2\,\mathrm{d}u\right. \\
&\times\left.\int_{3/2}^{N+1/2}|L'(\sigma+it+iht_u;\mathfrak{a})|^2\,\mathrm{d}u\right)^{1/2} \\
\ll_{\sigma,h,\mathfrak{a}}& N(1+|t|).
\end{aligned} \tag{7}$$

Let $\beta > 1/2$ be from the definition of $v_n(m)$, and, for $n \in \mathbb{N}$,

$$b_n(s) = \frac{s}{\beta}\Gamma\left(\frac{s}{\beta}\right)n^s.$$

Then, for $\sigma > 1/2$, by the Mellin formula, the function $L_n(s;\mathfrak{a})$ has the expression by the contour integral

$$\begin{aligned}
L_n(s;\mathfrak{a}) &= \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{a_m}{m^s}\int_{\beta-i\infty}^{\beta+i\infty}\frac{z}{\beta}\Gamma\left(\frac{z}{\beta}\right)\left(\frac{m}{n}\right)^{-z}\frac{\mathrm{d}z}{z} = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\left(\frac{b_n(z)}{z}\sum_{m=1}^{\infty}\frac{a_m}{m^{s+z}}\right)\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} L(s+z;\mathfrak{a})b_n(z)\frac{\mathrm{d}z}{z}.
\end{aligned} \tag{8}$$

Let $K \subset D$ be a fixed compact set and define $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for any point $s = \sigma + it \in K$. Then, for $s \in K$, we obtain

$$L_n(s;\mathfrak{a}) - L(s;\mathfrak{a}) = \frac{1}{2\pi i} \int_{-\theta - i\infty}^{-\theta + i\infty} L(s+z;\mathfrak{a}) b_n(z) \frac{\mathrm{d}z}{z} + \frac{a b_n(1-s)}{1-s},$$

where

$$\theta = \sigma - \varepsilon - \frac{1}{2}, \qquad \beta = \frac{1}{2} + \varepsilon.$$

Hence, we obtain the inequality

$$\begin{aligned} |L(s+iht_k;\mathfrak{a}) - L_n(s+iht_k;\mathfrak{a})| \leqslant & \frac{1}{2\pi} \int_{-\infty}^{+\infty} |L(s+iht_k - \theta + i\tau;\mathfrak{a})| \frac{|b_n(-\theta + it)|}{|-\theta + i\tau|} \mathrm{d}\tau \\ & + \frac{|a b_n(1-s-iht_k)|}{|1-s-iht_k|}. \end{aligned}$$

Then, shifting $t + \tau$ to τ, we have

$$\begin{aligned} & |L(s+iht_k;\mathfrak{a}) - L_n(s+iht_k;\mathfrak{a})| \\ & \leqslant \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| L\left(\frac{1}{2} + \varepsilon + i(\tau + ht_k);\mathfrak{a}\right) \right| \sup_{s \in K} \frac{|b_n(1/2 + \varepsilon - s + i\tau)|}{|1/2 + \varepsilon - s + i\tau|} \mathrm{d}\tau \\ & \quad + \frac{|a b_n(1-s-iht_k)|}{|1-s-iht_k|}. \end{aligned}$$

Summing over $1 \leqslant k \leqslant N$, we obtain

$$\frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |L(s+iht_k;\mathfrak{a}) - L_n(s+iht_k;\mathfrak{a})| \ll I_1 + I_2,$$

where

$$I_1 = \frac{1}{2\pi N} \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{N} \left| L\left(\frac{1}{2} + \varepsilon + i(\tau + ht_k);\mathfrak{a}\right) \right| \right) \sup_{s \in K} \frac{|b_n(1/2 + \varepsilon - s + i\tau)|}{|1/2 + \varepsilon - s + i\tau|} \mathrm{d}\tau,$$

and

$$I_2 = \frac{|a|}{N} \sum_{k=1}^{N} \sup_{s \in K} \frac{|b_n(1-s-iht_k)|}{|1-s-iht_k|}.$$

Using the well-known Stirling formula, uniformly for $0 \leqslant \xi \leqslant 1$, we obtain the estimate

$$|\Gamma(\xi + it)| \ll (|t| + 1)^{\xi - 1/2} \exp\left\{ -\frac{\pi |t|}{2} \right\}.$$

It implies the bound

$$\begin{aligned} \frac{|b_n(1/2 + \varepsilon - s + i\tau)|}{|1/2 + \varepsilon - s + i\tau|} &= \frac{n^{1/2 + \varepsilon - \sigma}}{\beta} \left| \Gamma\left(\frac{1/2 + \varepsilon - \sigma}{\beta} + \frac{i(t-\tau)}{\beta} \right) \right| \\ &\ll \frac{n^{-\varepsilon}}{\beta} \left(1 + \frac{|t-\tau|}{\beta} \right)^{(1/2 + \varepsilon - \sigma)/\beta - 1/2} \exp\left\{ -\frac{\pi}{2\beta} |t - \tau| \right\}. \end{aligned}$$

Define $t_0 = t_0(K) = \sup_{s \in K} |\mathrm{Im}\, s| + 1$. Then, $|t - \tau| \geqslant |\tau| - |t| \geqslant |\tau| - t_0$, and hence,

$$\frac{|b_n(1/2 + \varepsilon - s + i\tau)|}{|1/2 + \varepsilon - s + i\tau|} \ll \frac{n^{-\varepsilon}}{\beta} \exp\left\{\frac{\pi t_0}{2\beta}\right\} \exp\left\{-\frac{\pi|\tau|}{2\beta}\right\} \ll_{\beta,K} n^{-\varepsilon} \exp\left\{-\frac{\pi|\tau|}{2\beta}\right\}.$$

As above, we obtain

$$\frac{|b_n(1 - s - iht_k)|}{|1 - s - iht_k|} \ll_{\beta,K} n^{1-\sigma} \exp\left\{-\frac{\pi h t_k}{2\beta}\right\}.$$

To obtain the estimate for the sum I_2, we separate it into two parts I_{21} and I_{22}: over $1 \leqslant k \leqslant \log N$ and over $\log N \leqslant k \leqslant N$, respectively. It is easily seen that

$$I_{21} \ll_{\beta,K} \frac{\log N}{N} n^{1-\sigma},$$

and

$$I_{22} \ll_{\beta,K} \frac{n^{1-\sigma}}{N} \sum_{k \geqslant \log N} \exp\left\{-\frac{\pi h}{2\beta} \frac{2\pi k}{\log k}\right\} \ll_{\beta,K,h} \frac{n^{1-\sigma}}{N} \exp\left\{-\frac{\pi^2 h}{2\beta} \frac{\log N}{\log\log N}\right\} \ll \frac{n^{1-\sigma}}{N}.$$

Hence,

$$I_2 \ll_{\beta,K,h} \frac{\log N}{N} n^{1/2-2\varepsilon}.$$

Further, using the Cauchy inequality and (7), we have

$$\begin{aligned} \sum_{k=1}^{N} \left| L\left(\frac{1}{2} + \varepsilon + i(t + ht_k); \mathfrak{a}\right) \right| &\leqslant \left(N \sum_{k=1}^{N} \left| L\left(\frac{1}{2} + \varepsilon + i(t + ht_k); \mathfrak{a}\right) \right|^2 \right)^{1/2} \\ &\ll_{h,\mathfrak{a}} N(1 + |t|)^{1/2}, \end{aligned}$$

and

$$I_1 \ll_{\beta,K,h,\mathfrak{a}} n^{-\varepsilon} \int_{-\infty}^{+\infty} \frac{1}{N} N(1 + |\tau|)^{1/2} \exp\left\{-\frac{\pi|\tau|}{2\theta}\right\} \mathrm{d}\tau \ll_{\beta,K,h,\mathfrak{a}} n^{-\varepsilon}.$$

Using the above estimates for I_1 and I_2, we have

$$\frac{1}{N} \sum_{k+1}^{N} \sup_{s \in K} |L(s + iht_k; \mathfrak{a}) - L_n(s + iht_k; \mathfrak{a})| \ll_{\theta,K,h,\mathfrak{a}} n^{-\varepsilon} + \frac{\log N}{N} n^{1/2-2\varepsilon}.$$

Tending $N \to \infty$, and then $n \to \infty$, we obtain the assertion of the lemma using the definition of the metric ρ. $\square$

Now, we are ready to prove Theorem 5. For the proof, we will apply the following assertion, see, for example, [24], Theorem 3.2.

Lemma 7 ([24])**.** *Suppose that $(\mathbb{T}, \widehat{\rho})$ is the separable space and the $\mathbb{T}$-valued random elements Y_N and $X_{1n}, X_{2,n}, \ldots$ are defined on the same probability space with measure $\widetilde{P}$. Let, for all l,*

$$X_{ln} \xrightarrow[n \to \infty]{\mathcal{D}} X_l,$$

and

$$X_l \xrightarrow[l \to \infty]{\mathcal{D}} X.$$

If

$$\lim_{l \to \infty} \limsup_{n \to \infty} \widetilde{P}\{\widehat{\rho}(X_{ln}, Y_n) \geqslant \varepsilon\} = 0$$

for each $\varepsilon > 0$*, then*

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

Proof of Theorem 5. Let $(\widehat{\Omega}, \mathscr{B}(\widehat{\Omega}), \widetilde{P})$ be a certain probability space. Suppose that ξ_N is a random variable on the above probability space such that

$$\widetilde{P}\{\xi_N = ht_k\} = \frac{1}{N}, \quad k = 1, \ldots, N.$$

Using ξ_N, define two $H(D)$-valued random elements

$$X_{N,n} = X_{N,n}(s) = L_n(s + i\xi_N; \mathfrak{a})$$

and

$$X_N = X_N(s) = L(s + i\xi_N; \mathfrak{a}).$$

Let $X_n = X_n(s)$ be the $H(D)$-valued random element of which the distribution is T_n; here, T_n is the limit measure from Lemma 3. Then, by Lemma 3,

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n. \tag{9}$$

Using a standard method, see, for example, [7], we obtain that the family $\{T_n : n \in \mathbb{N}\}$ of probability measures is tight, i.e., for each $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon)$ of the set D such that

$$T_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. From this and the classical Prokhorov theorem, see, for example, Theorem 5.1 of [24], we determine that $\{T_n : n \in \mathbb{N}\}$ is relatively compact, i.e., each sequence of T_n contains a subsequence T_{n_l} which converges weakly to a certain probability measure P on $(H(D), \mathscr{B}(H(D)))$ as $l \to \infty$. Thus,

$$X_{n_l} \xrightarrow[l\to\infty]{\mathcal{D}} P. \tag{10}$$

Now, using Lemma 6, for all $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \limsup_{N\to\infty} \widetilde{P}\{\rho(X_N(s), X_{N,n}(s)) \geqslant \varepsilon\} = 0.$$

This relation, (9), (10), and Lemma 7 prove that

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P. \tag{11}$$

This means that, for $N \to \infty$, P_N converges weakly to P. Additionally, (11) shows that the measure P in (10) does not depend on the subsequence T_{n_l}. Therefore, we have that T_n converges weakly to P as $n \to \infty$.

Moreover, we need to identify the measure P. In [12], it was received that

$$\frac{1}{T}\mathcal{M}\{\tau \in [0, T] : L(s + i\tau; \mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $n \to \infty$, also converges weakly to the limit measure P of T_n, and P coincides with P_L. Thus, P_N converges weakly to P_L as $N \to \infty$ as well. □

3. Proof of Universality Theorem

For the proof of Theorem 4, we need the Mergelyan theorem on the approximation of analytic functions by polynomials, see [26].

Lemma 8. *Let $K \subset \mathbb{C}$ be a compact set with a connected complement and $h(s)$ be a continuous function on K and analytic in the interior of K. Then, for each $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s \in K} |h(s) - p_\varepsilon(s)| < \varepsilon.$$

Before the proof of Theorem 4, we recall some equivalents of the weak convergence of probability measures, see, for example, [24], Theorem 2.1. Recall that the set A is a continuity set of the measure $\widehat{P}$ if $\widehat{P}(\partial A) = 0$, where ∂A is the boundary of the set A.

Lemma 9. *Suppose that $\widehat{P}_n$, $n \in \mathbb{N}$, and $\widehat{P}$ are probability measures on $(\mathbb{T}, \mathscr{B}(\mathbb{T}))$. Then, the following statements are equivalent:*

(i) *P_n converges weakly to P as $n \to \infty$;*
(ii) *For all open sets $G \subset \mathbb{T}$,*

$$\liminf_{n \to \infty} \widehat{P}_n(G) \geqslant \widehat{P}(G);$$

(iii) *For all continuity sets A of $\widehat{P}$,*

$$\lim_{n \to \infty} \widehat{P}_n(A) = \widehat{P}(A).$$

Proof of Theorem 4. Define

$$\mathcal{S} = \{h(s) \in H(D) : h(s) \neq 0 \;\text{ or }\; h(s) \equiv 0\}.$$

In [7], it was obtained that the set $\mathcal{S}$ is the support of the measure P_L ($\mathcal{S}$ is a minimal closed subset of $H(D)$ such that $P_L(\mathcal{S}) = 1$). The set $\mathcal{S}$ consists of all $h(s) \in H(D)$ such that the inequality $P(\mathcal{H}) > 0$ is satisfied for all open neighborhoods $\mathcal{H}$ of $h(s)$.

Now, we define the set

$$\mathcal{H}_\varepsilon = \left\{ h(s) \in H(D) : \sup_{s \in K} \left| h(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\},$$

where $p(s)$ is some polynomial. Since $e^{p(s)} \neq 0$, $e^{p(s)}$ is an element of $\mathcal{S}$. Thus,

$$P_L(\mathcal{H}_\varepsilon) > 0.$$

Therefore, using Theorem 5 and 9 of Lemma 9, we have

$$\liminf_{N \to \infty} P_N(\mathcal{H}_\varepsilon) \geqslant P_L(\mathcal{H}_\varepsilon) > 0,$$

or

$$\liminf_{N \to \infty} \frac{1}{N} \mathrm{Card} \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} \left| L(s + iht_k; \mathfrak{a}) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\}. \tag{12}$$

In virtue of Lemma 8, we can choose the polynomial $p(s)$ such that

$$\sup_{s \in K} \left| f(s) - e^{p(s)} \right| < \frac{\varepsilon}{2}. \tag{13}$$

This and (12) gives the first assertion of the theorem.

Define one more set

$$\widetilde{\mathcal{H}}_\varepsilon = \left\{ h(s) \in H(D) : \sup_{s \in K} |h(s) - f(s)| < \varepsilon \right\}.$$

Then, from (13), we give that $\mathcal{H}_\varepsilon \subset \widetilde{\mathcal{H}}_\varepsilon$.

The boundary $\partial\widetilde{\mathcal{H}}_\varepsilon$ of the set $\widetilde{\mathcal{H}}_\varepsilon$ is the set

$$\left\{h(s) \in H(D) : \sup_{s\in K} |h(s) - f(s)| = \varepsilon\right\}.$$

Thus, $\partial\widetilde{\mathcal{H}}_{\varepsilon_1} \cap \partial\widetilde{\mathcal{H}}_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. Therefore, $P_L(\partial\widetilde{\mathcal{H}}_\varepsilon) > 0$ for, at most, countably many $\varepsilon > 0$. Hence, the set $\widetilde{\mathcal{H}}_\varepsilon$ is a continuity set ($\partial\widetilde{\mathcal{H}}_\varepsilon = 0$) of the measure P_L for all but at most, countably many $\varepsilon > 0$. Therefore, using Theorem 5 and 9 of Lemma 9, we have that

$$\lim_{N\to\infty} P_n(\widetilde{\mathcal{H}}_\varepsilon) = P_L(\widetilde{\mathcal{H}}_\varepsilon),$$

or

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Card}\left\{1 \leqslant k \leqslant N : \sup_{s\in K}|L(s + iht_k; \mathfrak{a}) - f(s)| < \varepsilon\right\} = P_L(\widetilde{\mathcal{H}}_\varepsilon)$$

for all but at most, countably many $\varepsilon > 0$. The last step of the proof is to show that $P_L(\widetilde{\mathcal{H}}_\varepsilon) > 0$. However, we have just seen that $\mathcal{H}_\varepsilon \subset \widetilde{\mathcal{H}}_\varepsilon$. Since $P_L(\mathcal{H}_\varepsilon) > 0$, we have that $P_L(\widetilde{\mathcal{H}}_\varepsilon) > 0$ as well, and we prove the theorem. □

4. Conclusions

In the paper, we consider the Dirichlet series

$$L(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad s = \sigma + it,\ \sigma > 1,$$

where $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ is a periodic multiplicative sequence of complex numbers with period $l \in \mathbb{N}$. For example, a_m can be a Dirichlet character modulo l.

We prove a theorem on the approximation of non-vanishing analytic functions defined on the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ by shifts

$$L(s + iht_k; \mathfrak{a}), \quad h > 0,$$

where $\{t_k\}$ is the sequence of Gram numbers, i. e., t_k is a solution of the equation

$$\varphi(t) = (k - 1)\pi, \quad k \in \mathbb{N},$$

and $\varphi(t)$ is the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$, and $\Gamma(s)$ is the Euler gamma function along the segment, connected the points $1/2$ and $1/2 + it$. We obtain that the set of the above shifts approximating a given analytic function has a positive lower density; thus, it is infinite. The problem of a positive density is also discussed. For the proof, a probabilistic approach based on the weak convergence of probability measures in the space of analytic functions is applied.

The results obtained cover those of [22].

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