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Relative Orderings of Modified Proportional Hazard Rate and Modified Proportional Reversed Hazard Rate Models

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Abstract: In this paper, we identify several relative ordering properties of the modified proportional hazard rate and modified proportional reversed hazard rate models. For this purpose, we use two well-known relative orderings, namely the relative hazard rate ordering and the relative reversed hazard rate ordering. The investigation is to see how a relative ordering between two possible base distributions for the response distributions in these models is preserved when the parameters of the underlying models are changed. We will give some examples to illustrate the results and the conditions under which they are obtained. Numerical simulation studies have also been provided to examine the examples presented.

Keywords: proportional hazard rates; proportional reversed hazard rates; relative hazard rate order; relative reversed hazard rate order; stochastic order; relative aging

MSC: 62N05; 90B25; 62E10; 65C20

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1. Introduction

The parameters of a distribution are typically considered to be real or perhaps vector values. In the literature, families of distributions are considered that are characterized by having a parameter that is itself a distribution function. These families are called semiparametric because they also contain a real parameter. Choosing the parameter that is first a distribution function is one possible way to use a semiparametric model. The underlying distribution is the formal name for this distribution function. In practice, the selection of an underlying distribution leads to the selection of a parametric model, but the selection is limited to families with the structure of the semiparametric model. Let F be a cumulative distribution function (cdf) of a lifetime unit with random life length X_0. In the context of reliability and survival analysis, this random variable usually stands as the random lifetime of a unit in a standard situation. Suppose that $d_{\boldsymbol{\theta}} : [0,1] \mapsto [0,1]$ is a non-decreasing function for which $d_{\boldsymbol{\theta}}(0) = 0$, $d_{\boldsymbol{\theta}}(1) = 1$ where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k) \in \Theta$ is a vector of unknown parameters, and Θ is the parameter space. Then, the random variable X_1 with the following cdf

$$F^*(t;\boldsymbol{\theta}) = d_{\boldsymbol{\theta}}(F(t)). \tag{1}$$

is said to have a semiparametric model where F is the baseline distribution. In reliability, the random variable X_1 denotes the random lifetime of a unit in a fresh environment. The underlying distribution F might already have one or more parameters, in which case a semiparametric family might provide a way to include a new parameter, extending the family from which F originates. One can imagine that the standard families of the Gamma and Weibull distributions are derived from the exponential distribution via semiparametric families that include a second parameter. The Weibull and Gamma families can both be found as special cases of a three-parameter family using the same technique. The study of semiparametric families is therefore advantageous for two reasons: it provides a new understanding of traditional distribution families, and it offers strategies for extending families to make data fitting more flexible (see, e.g., Marshall and Olkin [1]).

The concept of the hazard rate (hr) function and also the concept of the reversed hazard rate (rhr) function play an important role in reliability and survival analysis. These quantities measure the instantaneous risk of failure of an aging technical system from the perspective of probability theory. Let X be the random lifetime of an aging technical system; then let the hr function of X, denoted by h_X, be defined as

$$h_X(t) = \lim_{\delta \to 0^+} \frac{1}{\delta} P(X \leq t + \delta | X > t) = \frac{f_X(t)}{\bar{F}_X(t)}, \tag{2}$$

where f_X is the probability density function (pdf) and $\bar{F}_X$ is the survival function (sf) associated with the random lifetime X. From (2) it can be seen that $h_X(t)$ measures the risk of failure of a unit or a technical system (with lifetime X) of age t in a very short interval in the immediate vicinity of the time t, for example, $(t, t+\delta]$, in which δ is very close to zero. Therefore, the measure $h_X(t)$ as a function of t, the current age of a lifetime unit or technical system, is able to quantify the instantaneous risk of failure of the unit or technical system under consideration at any age (cf. Prentice and Kalbfleisch [2]). In some situations, a situation with previous failures can also be taken into account. For example, the actual inactivity time of a failed technical system at an inspection time t may not be known. In such cases, the rhr function is a useful measure to quantify the risk of past failures. The rhr of X, is denoted by $\widetilde{h}_X$, defined as

$$\widetilde{h}_X(t) = \lim_{\delta \to 0^+} \frac{1}{\delta} P(X > t - \delta | X \leq t) = \frac{f_X(t)}{F_X(t)}, \tag{3}$$

where F_X is the cdf of X. It follows from (3) that $\widetilde{h}_X(t)$ measures the risk of failure of a unit or technical system found to be inactive during an inspection at time t in a very short interval before time t, i.e., $(t-\delta, t]$, so that δ is almost zero. Consequently, the quantity $\widetilde{h}_X(t)$ is a function of t, the time at which the unit or technical system is inactive, can be used to measure the instantaneous risk of a past failure of the failed unit or technical system (see, for instance, Block et al. [3]).

Stochastic orders of random variables have long been a useful tool for making comparisons between probability distributions (see Müller and Stoyan [4], Shaked and Shanthikumar [5], Belzunce et al. [6], and Li and Li [7]). Basically, a stochastic order is a rule that defines the sense in which one stochastic variable is greater than or less than another. Some researchers used stochastic orders comparing distributions in terms of the magnitude of random variables to perform stochastic comparisons between semiparametric models, including those presented in Equations (5), (9) and (13) in Section 2. To this end, we quantify the effect of varying the parameters of the model on the variation of the response variables and, furthermore, the effect of changing the underlying distribution on changing the distribution of the response variables using several known stochastic orders. For example, in the context of the proportional hazard rate (PHR) model for the case where λ is a random variable (frailty), Gupta and Kirmani [8] and subsequently Xu and Li [9] identified some stochastic ordering properties of the model. The PHR model has attracted the attention of many researchers in applied probability and statistics, for instance, see Psarrakos and Sordo [10], Sankaran and Kumar [11], Zhang et al. [12], Arnold et al. [13] and Kochar [14]. Considering the proportional reversed hazard rates (PRHR) model, Di Crescenzo [15] made some stochastic comparisons between two candidate distributions of the model that differ in their parameters. Kirmani and Gupta [16] derived some stochastic ordering results for the proportional odds rates (POR) model.

Recently, however, many researchers have focused on stochastic orders that compare lifetime distributions according to aging behavior, namely, the faster-aging stochastic orders.

One of the key ideas in reliability theory and survival analysis is stochastic aging. It broadly outlines the pattern of aging/degradation of a system over time. Three different notions of aging are presented in the literature: positive aging, negative aging, and no aging. Positive aging implies a stochastically decreasing remaining lifetime of the system. Negative aging implies just the opposite. The system does not mature with time if there is no aging. To study different characteristics of system aging, various aging classes have been presented in the literature based on these three aging principles. The increasing failure rate (IFR), decreasing failure rate (DFR), increasing failure rate on average (IFRA), decreasing failure rate on average (DFRA), increasing likelihood ratio (ILR), and decreasing likelihood ratio (DLR) are among the frequently applied aging classes. The reader can consult Barlow and Proschan [17] and Lai and Xie [18] for further discussion on this topic. In addition to these ideas about aging, relative aging is a useful concept to use when studying system reliability. Relative aging is used to measure how a system changes over time relative to another system.

In real life, there are many situations where we deal with multiple systems of the same type (e.g., TVs from different manufacturers, CPUs from different brands, etc.). In these circumstances, we often encounter the following problem: how can we determine whether one system is aging faster than others over time? The idea of relative aging provides a compelling answer to this problem. When dealing with the crossover hazards/medium remaining life phenomena, another component of relative aging proves helpful. Many real-life situations involve this type of circumstance. For example, Pocock et al. [19] examined survival data on the effects of two different treatments on breast cancer patients and became aware of the phenomenon of crossover hazards. In addition, Champlin et al. [20] described several cases in which the superiority of one treatment over another lasted only for a short period of time. The above considerations suggest that increasing/decreasing hazard ratio models are a viable option in a variety of real-world scenarios. In fact, Kalashnikov and Rachev [21] have developed a concept of relative aging based on the monotonicity of the ratio of two hazard rate functions called the relative hazard rate order. This concept is known as faster hazard rate aging. Sengupta and Deshpande [22] presented another idea in a similar way based on the monotonicity of the ratio of two cumulative hazard rate functions. Rezaei et al. [23] proposed a relative order based on the ratio of the reversed hazard rates of two random lifetimes and called it the relative reversed hazard rate order. We will utilize the relative hazard rate order and, further, the relative reversed hazard rate order to compare models belonging to a recently introduced semiparametric model.

Let us assume that X_0 and Y_0 are two non-negative random variables with continuous distribution functions (cdfs) F and G, respectively. Let X_1 and Y_1 be two non-negative random variables whose cdfs are transformations of cdfs of X_0 and Y_0, respectively. In view of (1), suppose that X_1 and Y_1 follow cdfs $F^*(t;\boldsymbol{\theta}) = d_\theta(F(t))$ and $G^*(t;\boldsymbol{\theta}) = d_\theta(G(t))$, respectively. The function $d_{\boldsymbol{\theta}}$ is called the distortion function and the cdfs $F^*(t;\boldsymbol{\theta})$ and $G^*(t;\boldsymbol{\theta})$ are called distorted distribution functions. These kinds of distributions, called also semiparametric distributions, have attracted the attention of many researchers in statistics, economics, actuarial studies and reliability (see, e.g., Navarro et al. [24], Navarro and Águila [25], Lando and Bertoli-Barsotti [26], Kayid and Al-Shehri [27] and Navarro and Pellerey [28]). From theoretical perspectives, the distribution of order statistics and the distribution of record values arisen from a sample from F follows the foregoing semiparametric model (see, e.g., Izadkhah et al. [29]). In practical studies, the baseline cdfs $F(t)$ and $G(t)$ represent two models in a population under a standard (reference) situation, for example, the model of the lifetime of an item under a controlled environment. The distorted distributions $F^*(t;\boldsymbol{\theta})$ and $G^*(t;\boldsymbol{\theta})$ then specify two altered models in a fresh environment, for instance, the lifetime of an item in a critical situation. One of the problems that has attracted the attention of researchers in applied probability is two study conditions under which

$$X_0 \leq_o Y_0 \Longrightarrow X_1 \leq_o Y_1, \tag{4}$$

where $\leq_o$ denotes a certain stochastic order. The preservation property given in (4) states that a certain stochastic ordering relation between the baseline populations leads to the same stochastic ordering relation between the altered populations. In the context of many well-known semiparametric models, the implication in (4) has been studied in the literature (see, e.g., Di Crescenzo [15], Kirmani and Gupta [16] and Gupta and Gupta [30]). However, the studies conducted in this regard have so far only dealt with stochastic orders that take into account the magnitude of the stochastic order. The considered semiparametric families of distributions in the previously accomplished studies have also had a single parameter.

The aim of this work is to perform stochastic comparisons of two newly defined semiparametric models. Each of these models has two parameters. The stochastic orderings we consider here are two recent stochastic orderings that focus on the relative aging of two lifetime units, namely, the relative hazard rate ordering and the relative inverse hazard rate ordering. In particular, we compare the modified proportional hazard rate model (MPHR) and the modified proportional reversed hazard rate model (MPRHR), which correspond to the relative hazard rate ordering and the relative reverse hazard rate ordering. Specifically, the MPHR model is identified as a particular case of (1), such that $\boldsymbol{\theta} = (\alpha, \lambda)$, in which $\alpha > 0$, $\lambda > 0$ ($\bar{\alpha} = 1 - \alpha$) and

$$d_{\boldsymbol{\theta}}(u) = \frac{1 - (1-u)^{\lambda}}{1 - \bar{\alpha}(1-u)^{\lambda}}.$$

The MPRHR model is also identified as a specific case of (1), so that $\boldsymbol{\theta} = (\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$ and

$$d_{\boldsymbol{\theta}}(u) = \frac{\alpha u^{\beta}}{1 - \bar{\alpha} u^{\beta}}.$$

We consider random lifetimes X_1 and Y_1 with an MPHR distribution with hr functions h_{X_1} and h_{Y_1} and show that

$$\frac{h_{X_1}(t)}{h_{Y_1}(t)} \text{ is non-decreasing in } t \geq 0,$$

which shows that the unit or technical system with random life X_1 ages faster than the unit or technical system with random life Y_1. In parallel, we assume that if X_1 and Y_1 have an MPRHR distribution with rhr functions $\widetilde{h}_{X_1}$ and $\widetilde{h}_{Y_1}$, then under certain conditions,

$$\frac{\widetilde{h}_{X_1}(t)}{\widetilde{h}_{Y_1}(t)} \text{ is non-increasing in } t > 0,$$

indicating that the unit or technical system with random life X_1 ages faster compared to the unit or technical system with random life Y_1.

The rest of the paper is organized as follows. In Section 2, we give some advanced preliminary considerations and auxiliary results. In Section 3, we consider the modified proportional hazard rate model for comparison in terms of the relative hazard rate order. In this section, we further consider the modified proportional reversed hazard rate model to give some ordering properties according to the relative reversed hazard rate order. In Section 4, some examples with numerical simulation studies are provided to show that the theorems are fulfilled. In Section 5, we conclude the paper with a more detailed summary and provide an outlook on possible future studies. In Appendix A, we gather the proofs of the main theorems of the paper and also insert four tables to report the results of the simulation studies.

2. Preliminaries

In this section we give some mathematical definitions of the notions that will be utilized in this paper. This section contains a description of well-known facts from probability theory. For a detailed review on mathematical reliability theory, for example, the reader can see Barlow and Proschan [31], Rykov et al. [32] and Rykov et al. [33]. In the literature, many semiparametric families of distributions have been introduced and studied. Among these models, some of them find their applicability in the context of lifetime events. The Cox's PHR model is of the important and frequently used semiparametric family of distributions (see Cox [34]). For a review of the PHR model we refer the reader to Kumar and Klefsjö [35]. Let us consider the parameter $\lambda > 0$, called the frailty parameter. Then the PHR model is defined as

$$\bar{F}(t;\lambda) = \bar{F}^{\lambda}(t),\ t \geq 0, \tag{5}$$

where $\bar{F}(\cdot;\lambda)$ is the survival function (sf) of the response random variable and $\bar{F}(\cdot)$ is the baseline sf. Let X_0 have an absolutely cdf $F(\cdot)$ with probability density function (pdf) $f(\cdot)$. Then, the hr of X_0, an important reliability quantity in survival analysis, measures the instantaneous risk of failure of a device with lifetime X_0 at a certain age (t, say). The hr of X_0 for all $t \geq 0$ that fulfills $\bar{F}(t) > 0$ is defined as follows:

$$h(t) := -\frac{d}{dt}\ln(\bar{F}(t)) = \frac{f(t)}{\bar{F}(t)}. \tag{6}$$

It is well-known that h characterizes the underlying sf, $\bar{F}$, as follows:

$$\bar{F}(t) = \exp\{-\int_0^t h(x)\,dx\}. \tag{7}$$

Suppose that $h(\cdot;\lambda)$ is the hr function associated with the sf (5). Then, it is plainly seen that for every $t \geq 0$ for which $\min(F(\bar{t}), \bar{F}(t;\lambda)) > 0$,

$$h(t;\lambda) = \lambda \times h(t). \tag{8}$$

In contrast to the PHR model, the PRHR model was introduced by Gupta et al. [36]. We refer the reader to Gupta and Gupta [30] for further descriptions of the PRHR model. In the PRHR model, a positive parameter, β, called the resilience parameter, is considered. The PRHR model is then defined as

$$F(t;\beta) = F^{\beta}(t),\ t \geq 0, \tag{9}$$

in which $F(\cdot;\beta)$ is the cdf of the response random variable and $F(\cdot)$ is the baseline cdf or the underlying distribution function of the model. The rhr of X_0, another reliability quantity, measures the risk of failure of a device (with original lifetime X_0) in the past at a certain time point t at which the device is found to be inactive. The rhr of X_0 for all $t \geq 0$ when $F(t) > 0$ is derived via the following relation:

$$\widetilde{h}(t) := \frac{d}{dt}\ln(F(t)) = \frac{f(t)}{F(t)}. \tag{10}$$

It has been verified that $\widetilde{h}$ characterizes the underlying cdf, F, as below:

$$F(t) = \exp\{-\int_t^{+\infty} \widetilde{h}(x)\,dx\}. \tag{11}$$

Let us now assume that $\widetilde{h}(\cdot;\beta)$ is the rhr function of the distribution with the cdf (9). Then, it is readily realized for all $t \geq 0$ for which $\min\{F(t), F(t;\beta)\} > 0$ that

$$\widetilde{h}(t;\beta) = \beta \times \widetilde{h}(t). \tag{12}$$

Another reputable semiparametric family of distributions is the POR model (see, e.g., Marshall and Olkin [37]). This model is defined with cdf

$$F(t;\alpha) = \frac{F(t)}{1-\bar{\alpha}\bar{F}(t)};\ \ t,\alpha \in R^{+}, \bar{\alpha} = 1-\alpha, \tag{13}$$

In some situations the following model is alternatively utilized:

$$F(t;\alpha) = \frac{\alpha F(t)}{1-\bar{\alpha}F(t)};\ \ t,\alpha \in R^{+}, \bar{\alpha} = 1-\alpha. \tag{14}$$

The POR model, also known as the Marshall–Olkin model, has been considered by Rykov et al. [33] in the context of sensitivity analysis. The odds rate function of X_0 measures the relative odds of the event $\{X_0 > t\}$ in terms of the event $\{X_0 \leq t\}$, where t is some point in time. The odds rate function of X_0 for all $t \geq 0$ when $F(t) > 0$ is defined as follows:

$$OR_0(t) := \frac{\bar{F}(t)}{F(t)}. \tag{15}$$

We assume that $OR(t;\alpha) = \frac{\bar{F}(t;\alpha)}{F(t;\alpha)}$ is the odds rate function of the distribution with the cdf (13). Then, it is easily verified for all $t \geq 0$ for which $\min\{F(t), F(t;\alpha)\} > 0$ that

$$OR(t;\alpha) = \alpha \times OR_0(t). \tag{16}$$

Balakrishnan et al. [38] utilized the PHR (resp., PRHR) model as a baseline model in (13) (resp., (14)) to propose two new models, referred to as the MPHR and MPRHR models, respectively.

Suppose that X_0 is a baseline random variable with survival function $\bar{F}$. Let $X_{11}, \cdots, X_{1n}$ be independent and identically distributed (i.i.d.) lifetimes of n components of a system with a common distribution function $F(\cdot;\alpha,\lambda)$. Then, $X_{11}, \cdots, X_{1n}$ are said to follow the MPHR model with tilt parameter α, modified proportional hazard rate λ and baseline survival function $\bar{F}$ (denoted as $MPHR(\alpha;\lambda;\bar{F})$) if and only if,

$$F(x;\alpha,\lambda) = \frac{1-\bar{F}^{\lambda}(x)}{1-\bar{\alpha}\bar{F}^{\lambda}(x)};\ \ x,\lambda,\alpha \in R^{+}, \bar{\alpha} = 1-\alpha. \tag{17}$$

For the case of $\alpha = 1$, (17) simply reduces to the PHR model. The MPHR model in (17) includes some well-known distributions such as the extended exponential and extended Weibull distributions (Marshall and Olkin [1]), extended Pareto distribution (Ghitany [39]) and extended Lomax distribution (Ghitany et al. [40]).

On the other hand, suppose $X_1, \cdots, X_n$ are i.i.d. lifetimes of n components of a system with a common distribution functions F. Then, $X_1, \cdots, X_n$ are said to follow the MPRHR model with tilt parameter α, modified proportional reversed hazard rate β and baseline distribution function F (denoted as $MPRHR\ (\alpha;\beta;F)$) if and only if

$$F(x;\alpha,\beta) = \frac{\alpha F^{\beta}(x)}{1-\bar{\alpha}F^{\beta}(x)};\ \ x,\beta,\alpha \in R^{+}, \bar{\alpha} = 1-\alpha. \tag{18}$$

Note that the PRHR model is a sub-model of (18) when $\alpha = 1$. In Table 1, we shall give a summary of the semiparametric models that are used in this paper together with the sf, the cdf and the pdf of the models.

We assume that the random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1-F$ and $\bar{G} = 1-G$, density functions f and g, hazard rate functions $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$ and reversed hazard rate functions $\widetilde{h}_X = f/F$ and $\widetilde{h}_Y = g/G$, respectively. To compare the magnitude of random variables, some notions of stochastic orders are introduced below.

Definition 1. *Suppose that X and Y are two non-negative random variables that denote the lifetime of two systems. The random variable X is then said to be smaller than the random variable Y in the*

(i) *usual stochastic order (denoted by $X \preceq_{st} Y$) if*

$$\bar{F}(x) \leq \bar{G}(x) \text{ for all } x \geq 0;$$

(ii) *hazard rate order (denoted by* $X \preceq_{hr} Y$ *) if*

$$\frac{\bar{G}(x)}{\bar{F}(x)} \text{ is non-decreasing in } x \geq 0,$$

or equivalently, if $h_X(x) \geq h_Y(x)$ *for all* $x \geq 0$;

(iii) *reversed hazard rate order (denoted by* $X \preceq_{rh} Y$*) if*

$$\frac{G(x)}{F(x)} \text{ is non-decreasing in } x > 0,$$

or equivalently, if $\widetilde{h}_X(x) \leq \widetilde{h}_Y(x)$ *for all* $x > 0$;

(iv) *likelihood ratio order (denoted by* $X \preceq_{lr} Y$*) if*

$$g(x)/f(x) \text{ is non-decreasing in } x \geq 0;$$

(v) *relative hazard rate order (denoted by* $X \preceq_{c} Y$*) if*

$$\frac{h_X(x)}{h_Y(x)} \text{ is non-decreasing in } x \geq 0;$$

(vi) *relative reversed hazard rate order (denoted by* $X \preceq_{b} Y$*) if*

$$\frac{\widetilde{h}_X(x)}{\widetilde{h}_Y(x)} \text{ is non-increasing in } x \geq 0.$$

Some stochastic orders in Definition 1 are connected to each other. In this regard, $X \preceq_{lr} Y$ implies $X \preceq_{hr} Y$ and $X \preceq_{lr} Y$ also implies $X \preceq_{rh} Y$. Furthermore, $X \preceq_{hr} Y$ gives $X \preceq_{st} Y$ and $X \preceq_{rh} Y$ yields $X \preceq_{st} Y$. For further relations and properties of the stochastic orders $\preceq_{lr}$, $\preceq_{hr}$, $\preceq_{rh}$ and $\preceq_{st}$, we refer the reader to Shaked and Shanthikumar [5]. For more descriptions of the relative order $\preceq_c$, we refer the reader to Kalashnikov and Rachev [21] and Sengupta and Deshpande [22]. For further properties of the relative order $\preceq_b$, the reader can see Rezaei et al. [23].

Table 1. Summary of semiparametric models with their distributional characteristics.

Model	sf	cdf	pdf
POR	$\frac{\alpha\bar{F}(t)}{1-\bar{\alpha}\bar{F}(t)}$	$\frac{F(t)}{1-\bar{\alpha}\bar{F}(t)}$	$\frac{\alpha f(t)}{(1-\bar{\alpha}\bar{F}(t))^2}$
PHR	$\bar{F}^{\lambda}(t)$	$1-\bar{F}^{\lambda}(t)$	$\lambda\bar{F}^{\lambda-1}(t)f(t)$
PRHR	$1-F^{\beta}(t)$	$F^{\beta}(t)$	$\beta F^{\beta-1}(t)f(t)$
MPHR	$\frac{\alpha\bar{F}^{\lambda}(t)}{1-\bar{\alpha}\bar{F}^{\lambda}(t)}$	$\frac{1-\bar{F}^{\lambda}(t)}{1-\bar{\alpha}\bar{F}^{\lambda}(t)}$	$\frac{\lambda\alpha\bar{F}^{\lambda-1}(t)}{(1-\bar{\alpha}\bar{F}^{\lambda}(t))^2}f(t)$
MPRHR	$\frac{1-F^{\beta}(t)}{1-\bar{\alpha}F^{\beta}(t)}$	$\frac{\alpha F^{\beta}(t)}{1-\bar{\alpha}F^{\beta}(t)}$	$\frac{\beta\alpha F^{\beta-1}(t)}{(1-\bar{\alpha}F^{\beta}(t))^2}f(t)$

3. Main Results

In this section, we present the main results of the paper, which include two main preservation properties. The preservation property of the relative hazard rate order under the setting of the MPHR model is studied. The preservation property of the relative reversed hazard rate order under the setting of the MPRHR model is also investigated. The proofs of the main theorems are moved to Appendix A.

3.1. Results on Relative Orderings of MPHR Distributions

In this section, we obtain a relative ordering property in the MPHR model according to the relative hazard rate order. We will consider the MPHR model in two settings where two sets of parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, which are possibly different, are assigned and the possibly different baseline sfs $\bar{F}$ and $\bar{G}$ are also taken into account. Finding conditions on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and also conditions on $\bar{F}$ and $\bar{G}$ to establish the preservation of the relative hazard rate ordering property in the MPHR model is the main objective of this section.

Before stating the next result we introduce some notation. Let X_0 and Y_0 have pdfs f and g and sfs $\bar{F}$ and $\bar{G}$, respectively, and, further, $X_1 \sim MPHR(\alpha_1; \lambda_1; \bar{F})$ and $Y_1 \sim MPHR(\alpha_2; \lambda_2; \bar{G})$. Then, using (17), the sfs of X_1 and Y_1, which are denoted by $\bar{F}(x; \alpha_1, \lambda_1)$ and $\bar{G}(x; \alpha_2, \lambda_2)$, respectively, can be written as follows:

$$\bar{F}(x; \alpha_1, \lambda_1) = \frac{\alpha_1 \times \bar{F}^{\lambda_1}(x)}{1 - \bar{\alpha_1} \bar{F}^{\lambda_1}(x)} \text{ and } \bar{G}(x; \alpha_2, \lambda_2) = \frac{\alpha_2 \times \bar{G}^{\lambda_2}(x)}{1 - \bar{\alpha_2} \bar{G}^{\lambda_2}(x)}. \tag{19}$$

Now, let us denote by $f(\cdot; \alpha_1, \lambda_1)$ and $g(\cdot; \alpha_2, \lambda_2)$ the pdfs of X_1 and Y_1, which can be obtained by taking derivatives of the cdfs in (19) as follows:

$$f(x; \alpha_1, \lambda_1) = \frac{\lambda_1 \alpha_1 \bar{F}^{\lambda_1 - 1}(x)}{(1 - \bar{\alpha_1} \bar{F}^{\lambda_1}(x))^2} f(x) \text{ and } g(x; \alpha_2, \lambda_2) = \frac{\lambda_2 \alpha_2 \bar{G}^{\lambda_2 - 1}(x)}{(1 - \bar{\alpha_2} \bar{G}^{\lambda_2}(x))^2} g(x). \tag{20}$$

Appealing to (19) together with (20), the hazard rate function of X_1 and the hazard rate function of Y_1 are acquired as:

$$h(x; \alpha_1, \lambda_1) = \frac{h(x)}{\Phi(\bar{F}(x); \alpha_1, \lambda_1)} \text{ and } s(x; \alpha_2, \lambda_2) = \frac{s(x)}{\Phi(\bar{G}(x); \alpha_2, \lambda_2)} \tag{21}$$

where $h(\cdot)$ and $s(\cdot)$ are the hazard rate functions of X_0 and Y_0, respectively, and the function $\Phi(u; \alpha, \lambda)$ is given by

$$\Phi(u; \alpha, \lambda) = \frac{1}{\lambda}(1 - \bar{\alpha} u^{\lambda}), \quad u \in [0, 1].$$

We define here two measures of the relative hazard rates of Y_0 and X_0 with hazard rate functions $s(\cdot)$ and $h(\cdot)$, respectively. Let us denote two limiting points of the hazard rate ratio $\frac{s(t)}{h(t)}$ as follows:

$$\eta_0 := \lim_{t \to 0^+} \frac{s(t)}{h(t)} \text{ and } \eta_1 := \lim_{t \to +\infty} \frac{s(t)}{h(t)}.$$

Theorem 1. *Let X_0 and Y_0 have sfs $\bar{F}$ and $\bar{G}$, respectively. Let $X_1 \sim MPHR(\alpha_1; \lambda_1; \bar{F})$ and $Y_1 \sim MPHR(\alpha_2; \lambda_2; \bar{G})$, where $\alpha_i \in [0, 1]$ and $\lambda_i > 0$ for every $i = 1, 2$. Let $M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0) \geq 0$ be a function of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and η_0, such that*

$$M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0) := \sup_{u \in [0,1]} \left(\frac{u^{-\lambda_2 \eta_0} - \bar{\alpha}_2}{u^{-\lambda_1} - \bar{\alpha}_1} \right).$$

If

$$\frac{\bar{\alpha}_2}{\bar{\alpha}_1} . \frac{\lambda_2}{\lambda_1} \eta_1 \geq M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0)$$

then

$$X_0 \preceq_c Y_0 \;\Rightarrow\; X_1 \preceq_c Y_1. \tag{22}$$

Remark 1. *In the context of Theorem 1, the obtained result is immediately followed when $\alpha_2 \le 1$ and $\alpha_1 \ge 1$. To prove this claim, note that if $\alpha_2 \le 1$ and $\alpha_1 \ge 1$, then, for all $t \ge 0$ and for every $\lambda_i > 0, i = 1, 2$, one has*

$$\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)} \le 0 \ \ \textit{and} \ \ \bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)} \ge 0.$$

Therefore, the parenthetical statement on the right-hand side of the inequality given in (A1) is non-positive. Thus, it is straightforward that if $\alpha_2 \le 1$, $\alpha_1 \ge 1$ and $X_0 \preceq_c Y_0$, then $X_1 \preceq_c Y_1$. In this case, the additional supremum condition in Theorem 1 can be omitted.

The following theorem states another setting for the parameters of two MPHR distributions so that the result of Theorem 1 is obtained under a different condition.

Theorem 2. *Let X_0 and Y_0 have sfs $\bar{F}$ and $\bar{G}$, respectively. Let $X_1 \sim MPHR(\alpha_1;\lambda_1;\bar{F})$ and $Y_1 \sim MPHR(\alpha_2;\lambda_2;\bar{G})$, where $\alpha_i \in (1,+\infty)$ and $\lambda_i > 0$ for every $i = 1, 2$. Let $m(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_1) \ge 0$ be a function of $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)$, $\boldsymbol{\lambda} = (\lambda_1,\lambda_2)$ and also η_1 such that*

$$m(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_1) := \inf_{u\in[0,1]}\left(\frac{u^{-\lambda_2\eta_1} - \bar{\alpha}_2}{u^{-\lambda_1} - \bar{\alpha}_1}\right).$$

If

$$\frac{\alpha_2 - 1}{\alpha_1 - 1}.\frac{\lambda_2}{\lambda_1}\eta_0 \le m(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_1)$$

then

$$X_0 \preceq_c Y_0 \ \Rightarrow \ X_1 \preceq_c Y_1. \tag{23}$$

3.2. Results on Relative Orderings of MPRHR Distributions

In this section, we investigate the relative reversed hazard rate ordering property in two MPRHR models with possibly different sets of parameters $(\boldsymbol{\alpha},\boldsymbol{\beta})$, where $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)$ and $\boldsymbol{\beta} = (\beta_1,\beta_2)$, and also under possibly different baseline distributions F and G.

We start by introducing some notation. Let X_0 and Y_0 have pdfs f and g with underlying cdfs F and G, respectively. Let us assume that $X_1^\star \sim MPRHR(\alpha_1;\beta_1;F)$ and $Y_1^\star \sim MPRHR(\alpha_2;\beta_2;G)$ and denote them by $F^\star(\cdot;\alpha_1,\beta_1)$ and $G^\star(\cdot;\alpha_2,\beta_2)$. Then, using (18), we have

$$F^\star(x;\alpha_1,\beta_1) = \frac{\alpha_1 \times F^{\beta_1}(x)}{1 - \bar{\alpha_1}F^{\beta_1}(x)} \ \text{ and } \ G^\star(x;\alpha_2,\beta_2) = \frac{\alpha_2 \times G^{\beta_2}(x)}{1 - \bar{\alpha_2}G^{\beta_2}(x)}. \tag{24}$$

Using (24), the pdfs of $X_1^\star$ and $Y_1^\star$ (signified by $f^\star(\cdot;\alpha_1,\beta_1)$ and $g^\star(\cdot;\alpha_2,\beta_2)$) are acquired as below:

$$f^\star(x;\alpha_1,\beta_1) = \frac{\beta_1 \times \alpha_1 \times F^{\beta_1-1}(x)}{(1 - \bar{\alpha_1}F^{\beta_1}(x))^2}f(x) \ \text{ and } \ g^\star(x;\alpha_2,\beta_2) = \frac{\beta_2 \times \alpha_2 \times G^{\beta_2-1}(x)}{(1 - \bar{\alpha_2}G^{\beta_2}(x))^2}g(x). \tag{25}$$

By dividing the pdfs in (25) into the cdfs given in (24), the reversed hazard rate function of X_1 and the reversed hazard rate function of Y_1 are derived as follows:

$$\widetilde{h}(x;\alpha_1,\beta_1) = \frac{\widetilde{h}(x)}{\Psi(F(x);\alpha_1,\beta_1)} \ \text{ and } \ \widetilde{s}(x;\alpha_2,\beta_2) = \frac{\widetilde{s}(x)}{\Psi(G(x);\alpha_2,\beta_2)} \tag{26}$$

where $\widetilde{h}(\cdot)$ and $\widetilde{s}(\cdot)$ are the reversed hazard rate functions of X_0 and Y_0, respectively, and further, the function $\Psi(u;\alpha,\beta)$ is defined as

$$\Psi(u;\alpha,\beta) = \frac{1}{\beta}(1 - \bar{\alpha}u^{\beta}), \ \ u \in [0,1].$$

Now, let us define two measures of relative reversed hazard rates of Y_0 and X_0 having reverenced hazard rate functions $\widetilde{s}(\cdot)$ and $\widetilde{h}(\cdot)$, respectively. The limiting points of the reversed hazard rate ratio $\frac{\widetilde{h}(t)}{\widetilde{s}(t)}$ is as follows:

$$\eta_0^\star := \lim_{t\to 0^+} \frac{\widetilde{h}(t)}{\widetilde{s}(t)} \text{ and } \eta_1^\star := \lim_{t\to +\infty} \frac{\widetilde{h}(t)}{\widetilde{s}(t)}.$$

Theorem 3. *Let X_0 and Y_0 have cdfs F and G, respectively. Let $X_1^\star \sim MPRHR(\alpha_1;\beta_1;F)$ and $Y_1^\star \sim MPRHR(\alpha_2;\beta_2;G)$, where $\alpha_i \in [0,1]$ and $\beta_i > 0$ for every $i = 1,2$. Suppose that $m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star)$, which is a non-negative function of $(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star)$ is defined as*

$$m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star) := \inf_{u\in[0,1]} \left(\frac{u^{-\frac{\beta_1}{\eta_1^\star}} - \bar{\alpha}_1}{u^{-\beta_2} - \bar{\alpha}_2} \right).$$

If

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_2} \times \frac{\beta_1}{\beta_2} \times \eta_0^\star \le m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star)$$

then

$$X_0 \preceq_b Y_0 \;\Rightarrow\; X_1^\star \preceq_b Y_1^\star. \tag{27}$$

Remark 2. *In the setting of Theorem 3, the derived result can be acquired when $\alpha_2 \le 1$ and $\alpha_1 \ge 1$. To verify this claim, one needs to observe that if $\alpha_2 \le 1$ and $\alpha_1 \ge 1$, then, for all $t > 0$ and for $\beta_i > 0, i = 1,2$, we have*

$$F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} \ge 0 \;\; and \;\; G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)} \le 0.$$

Thus, the parenthetical statement on the right-hand side of the inequality given in (A7) is clearly non-negative. Hence, it is not hard to see that if $\alpha_2 \le 1$, $\alpha_1 \ge 1$ and $X_0 \preceq_b Y_0$, then $X_1^* \preceq_b Y_1^*$. In this setting, the additional infimum condition in Theorem 3 can be removed.

In the next theorem, we obtain the result of Theorem 3 under different conditions.

Theorem 4. *Let X_0 and Y_0 follow cdfs F and G, respectively. Suppose that $X_1^\star \sim MPRHR(\alpha_1;\beta_1;F)$ and $Y_1^\star \sim MPRHR(\alpha_2;\beta_2;G)$, where $\alpha_i \in (1,+\infty)$ and $\beta_i > 0$ for every $i = 1,2$. Consider $M^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_0^\star)$ as a non-negative function of $(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_0^\star)$ defined as*

$$M^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_0^\star) := \sup_{u\in[0,1]} \left(\frac{u^{-\beta_1\times\eta_0^\star} - \bar{\alpha}_1}{u^{-\beta_2} - \bar{\alpha}_2} \right).$$

If

$$\frac{\alpha_1 - 1}{\alpha_2 - 1} \times \frac{\beta_1}{\beta_2} \times \eta_1^\star \ge M^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_0^\star)$$

then

$$X_0 \preceq_b Y_0 \;\Rightarrow\; X_1^\star \preceq_b Y_1^\star. \tag{28}$$

4. Examples and Simulation Analysis

In this section, we provide examples to show that the results of Theorems 1–4 are fulfilled. We also present some numerical studies to verify that the theorems and examples are valid, as shown in a simulation analysis. The numerical computations are moved to Appendix A.

In the following example, we show that the result of Theorem 1 is applicable.

Example 1. *Let us write $X \sim W(c,d)$ when X follows the Weibull distribution with shape parameter c and scale parameter d, with $c,d > 0$, having sf $\bar{F}_X(t) = \exp(-(dt)^c), t \geq 0$. Suppose that $X_0 \sim W(3,1)$ and $Y_0 \sim W(3,2)$. Assume that $X_1 \sim MPHR(\alpha_1;\lambda_1;\bar{F})$ and $Y_1 \sim MPHR(\alpha_2;\lambda_2;\bar{G})$ with $\alpha_1 = 0.8, \alpha_2 = 0.1, \lambda_1 = 10$ and $\lambda_2 = 1$ and, further, $\bar{F}(t) = \exp(-t^3)$ and $\bar{G}(t) = \exp(-8t^3)$. We can observe that X_0 and Y_0 have hrs $h(t) = 3t^2$ and $s(t) = 24t^2$. Therefore,*

$$\frac{s(t)}{h(t)} = 8,\ \eta_0 = \eta_1 = 8.$$

Hence, $X_0 \preceq_c Y_0$. We can observe that

$$M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0) := \sup_{u \in [0,1]} \left(\frac{u^{-8} - 0.9}{u^{-10} - 0.2} \right) = 0.56812,$$

and, on the other hand, we can see that

$$\frac{\bar{\alpha}_2}{\bar{\alpha}_1} . \frac{\lambda_2}{\lambda_1} \eta_1 = 3.6.$$

Thus, obviously, $\frac{\bar{\alpha}_2}{\bar{\alpha}_1} . \frac{\lambda_2}{\lambda_1} > M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0)$, and using Theorem 1 we conclude that $X_1 \preceq_c Y_1$. In Figure 1, the graph of $\frac{s(t;\alpha_2,\lambda_2)}{s(t;\alpha_1,\lambda_1)}$ is plotted to exhibit that it is non-increasing in $t \in (0,4)$.

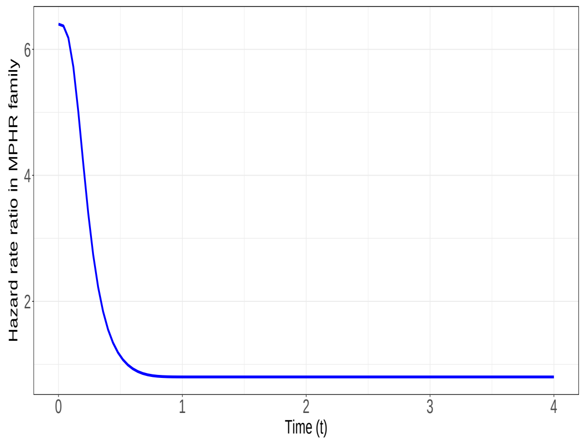

Figure 1. Plot of the hazard rate ratio $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ in Example 1 for $\alpha_1 = 0.8, \alpha_2 = 0.1, \lambda_1 = 10$ and $\lambda_2 = 1$ when $t \in (0,4)$.

We conduct here a simulation study to empirically ascertain that the result of Theorem 1 is valid. The parameters of the MPHR distributions are chosen exactly as in Example 1, i.e., $(\alpha_1, \lambda_1) = (0.8, 10)$ and $(\alpha_2, \lambda_2) = (0.1, 1)$, to fulfill the ordering result from the perspective of simulation analysis. We suppose that $X_0 \sim W(3,1)$ and $Y_0 \sim W(3,2)$, exactly as in Example 1. We assume X and Y follow MPHR distributions with cdfs $F(t;\alpha_1,\lambda_1)$ and $G(t;\alpha_2,\lambda_2)$ and right inverse functions $F^{-1}(u;\alpha_1,\lambda_1)$ and $G^{-1}(u;\alpha_2,\lambda_2)$, respectively. We also denote by F^{-1} and G^{-1}, the right inverse functions associated with F and G, the cdfs of X_0 and Y_0, respectively. On applying the runif function in R, we generate $u_1, u_2, \ldots, u_n \sim U(0,1)$, where $U(0,1)$ denotes the uniform distribution on $(0,1)$. We utilize the inverse transform technique to produce x_i as

$$\begin{aligned}
x_i &= F^{-1}(u_i;\alpha_1,\lambda_1) \\
&= F^{-1}\left(1-\sqrt[\lambda_1]{\frac{1-u_i}{\alpha_1+u_i\bar{\alpha}_1}}\right) \\
&= \sqrt[3]{-\frac{1}{\lambda_1}\times\ln\left(\frac{1-u_i}{\alpha_1+u_i\bar{\alpha}_1}\right)},\ i=1,2,\ldots,n,
\end{aligned}$$

by which n samples from $F(t;\alpha_1,\lambda_1)$ are generated when $(\alpha_1,\lambda_1)=(0.8,10)$, i.e., $x_i{\sim}MPHR$ $(0.8;10;\bar{F})$ with $\bar{F}(t)=\exp(-t^3)$. In a similar manner, one can produce y_i as

$$\begin{aligned}
y_i &= G^{-1}(u_i;\alpha_2,\lambda_2) \\
&= G^{-1}\left(1-\sqrt[\lambda_2]{\frac{1-u_i}{\alpha_2+u_i\bar{\alpha}_2}}\right) \\
&= \sqrt[3]{-\frac{8}{\lambda_2}\times\ln\left(\frac{1-u_i}{\alpha_2+u_i\bar{\alpha}_2}\right)},\ i=1,2,\ldots,n,
\end{aligned}$$

which provides n samples from $G(t;\alpha_2,\lambda_2)$ where $(\alpha_2,\lambda_2)=(0.1,1)$, i.e., $y_i{\sim}MPHR(0.1;1;\bar{G})$ in which $\bar{G}(t)=\exp(-8t^3)$. Using the produced samples, we estimate (α_1,λ_1) and (α_2,λ_2) using the maximum likelihood method to obtain a maximum likelihood estimation (MLE) of (α_i,λ_i), which is denoted by $(\widehat{\alpha}_i,\widehat{\lambda}_i), i=1,2$. The estimations will be acquired by solving the likelihood equations. The MLE $(\widehat{\alpha}_1,\widehat{\lambda}_1)$ is derived by solving the next system of equations in terms of (α_1,λ_1) :

$$\begin{cases}
\dfrac{n}{\lambda_1}-\displaystyle\sum_{i=1}^{n}x_i^3-2\bar{\alpha}_1\sum_{i=1}^{n}\frac{x_i^3e^{-\lambda_1x_i^3}}{(1-\bar{\alpha}_1\times e^{-\lambda_1x_i^3})}=0, \\
\dfrac{n}{\alpha_1}-2\displaystyle\sum_{i=1}^{n}\frac{e^{-\lambda_1x_i^3}}{(1-\bar{\alpha}_1\times e^{-\lambda_1x_i^3})}=0.
\end{cases} \tag{29}$$

The MLE $(\widehat{\alpha}_2,\widehat{\lambda}_2)$ is also obtained by solving the equations based on (α_2,λ_2):

$$\begin{cases}
\dfrac{n}{\lambda_2}-8\displaystyle\sum_{i=1}^{n}y_i^3-16\bar{\alpha}_2\sum_{i=1}^{n}\frac{y_i^3e^{-8\lambda_2y_i^3}}{(1-\bar{\alpha}_2\times e^{-8\lambda_2y_i^3})}=0, \\
\dfrac{n}{\alpha_2}-2\displaystyle\sum_{i=1}^{n}\frac{e^{-8\lambda_2y_i^3}}{(1-\bar{\alpha}_2\times e^{-8\lambda_2y_i^3})}=0.
\end{cases} \tag{30}$$

To solve the systems of equations given in (29) and in (30), we used the package nleqslv in R. In Table A1 in Appendix A, we report the MLEs of the parameters of the MPHR model numerically under different sample sizes and also derive the amounts of MLE of $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ for different selected ages t. It is shown that the estimated hazard rate ratio for the two choices made in Example 1 is non-increasing in t, as was expected.

The following example provides a situation where the result of Theorem 2 is applicable.

Example 2. *Suppose that X_0 follows a gamma distribution with sf $\bar{F}(t)=(1+3t)\exp(-3t), t\geq 0$ and Y_0 has sf $\bar{G}(t)=(1+3t)^2\exp(-6t), t\geq 0$. It is easily seen that the hrs of X_0 and Y_0 are $h(t)=\frac{9t}{1+3t}$ and $s(t)=\frac{18t}{1+3t}$, respectively. Therefore,*

$$\eta_0=\eta_1=\frac{s(t)}{h(t)}=2.$$

Now, since $\frac{s(t)}{h(t)}$ *is non-increasing in* t*, then* $X_0 \preceq_c Y_0$*. We assume that* $X_1 \sim MPHR(\alpha_1; \lambda_1; \bar{F})$ *and* $Y_1 \sim MPHR(\alpha_2; \lambda_2; \bar{G})$ *with* $\alpha_1 = 10, \alpha_2 = 3, \lambda_1 = 4$ *and* $\lambda_2 = 2$*. It is observable that*

$$m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1) := \inf_{u \in [0,1]} \left(\frac{u^{-4} + 2}{u^{-4} + 9} \right) = 0.3,$$

and, in parallel, it is seen that

$$\frac{\alpha_2 - 1}{\alpha_1 - 1} \times \frac{\lambda_2}{\lambda_1} \eta_0 = \frac{2}{9}.$$

Therefore, clearly, $\frac{\alpha_2 - 1}{\alpha_1 - 1} \times \frac{\lambda_2}{\lambda_1} \eta_0 < m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1)$*, and thus, an application of Theorem 2 concludes that* $X_1 \preceq_c Y_1$*. In Figure 2, the graph of* $\frac{s(t;\alpha_2,\lambda_2)}{s(t;\alpha_1,\lambda_1)}$ *is plotted to indicate that this ratio is non-increasing in* $t \in (0, 4)$*.*

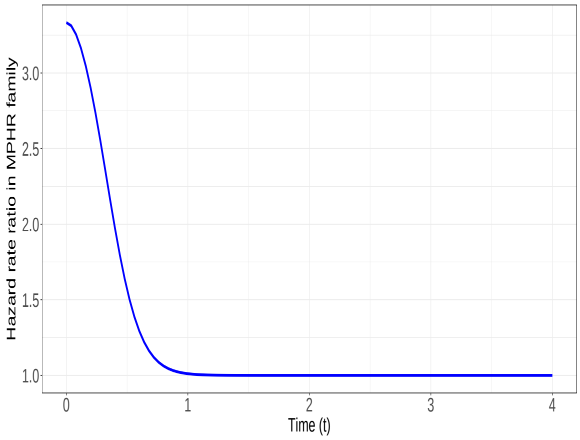

Figure 2. Plot of the hazard rate ratio $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ in Example 2 for $\alpha_1 = 10, \alpha_2 = 3, \lambda_1 = 4$ and $\lambda_2 = 2$ when $t \in (0, 4)$.

As reported in Table A2 in Appendix A, the values of the MLEs of (α_i, λ_i) are available under different sample sizes by simulating data from $MPHR(\alpha_1; \lambda_1; \bar{F})$ and $MPHR(\alpha_2; \lambda_2; \bar{G})$, with $(\alpha_i, \lambda_i), i = 1, 2$ and $\bar{F}$ and $\bar{G}$ exactly as chosen in Example 2. We additionally report the values of the MLE of $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ for some selected ages t. It is indicated that, for the two candidate models in Example 2, the estimated hazard rate ratio is non-increasing in t, as was claimed.

Next, we make use of Theorem 3 to show that the result is fulfilled.

Example 3. *Let us assume* $X \sim IW(c, d)$ *whenever* X *has an inverse Weibull distribution with shape parameter* c *and scale parameter* d*, where* $c > 0$ *and also* $d > 0$*. Then,* X *has cdf* $F_X(t) = \exp(-\left(\frac{d}{t}\right)^c)$ *for* $t > 0$*. We assume that* $X_0 \sim IW(2, 1)$ *and* $Y_0 \sim IW(2, 3)$*. Further, we suppose that* $X_1^* \sim MPRHR(\alpha_1; \beta_1; F)$ *and* $Y_1^* \sim MPRHR(\alpha_2; \beta_2; G)$ *with* $\alpha_1 = 0.25, \alpha_2 = 0.5, \beta_1 = 2$ *and* $\beta_2 = 18$ *so that* $F(t) = \exp(-(\frac{3}{t})^2)$ *is the cdf of* X_0 *and* $G(t) = \exp(-\left(\frac{1}{t}\right)^2)$*. It can be readily shown that* X_0 *and* Y_0 *have rhrs* $\widetilde{h}(t) = \frac{2}{t^3}$ *and* $\widetilde{s}(t) = \frac{18}{t^3}$*, respectively. Thus,*

$$\frac{\widetilde{h}(t)}{\widetilde{s}(t)} = \frac{1}{9}, \; \eta_0^\star = \eta_1^\star = \frac{1}{9}.$$

Consequently, $X_0 \preceq_b Y_0$*. One can easily check that*

$$m^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_1^\star) := \inf_{u \in [0,1]} \left(\frac{u^{-18} - 0.75}{u^{-18} - 0.5} \right) = 0.5,$$

and, simultaneously, one has

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_2} \times \frac{\beta_1}{\beta_2}\eta_0^* \simeq 0.00823.$$

Therefore, one realizes that $\frac{\bar{\alpha}_1}{\bar{\alpha}_2} \times \frac{\beta_1}{\beta_2}\eta_0^* < m^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_1^\star)$*, and using Theorem 3, we deduce that* $X_1^\star \preceq_b Y_1^\star$*. In Figure 3, the graph of* $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ *is exhibited to indicate that it is non-decreasing in* $t \in (0,4)$.

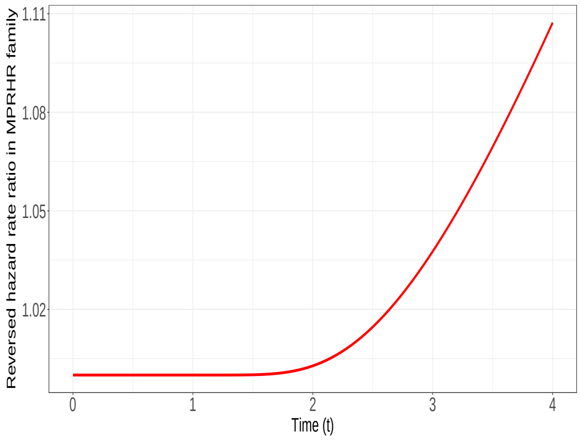

Figure 3. Plot of the reversed hazard rate ratio $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ in Example 3 for $\alpha_1 = 0.25, \alpha_2 = 0.5, \beta_1 = 2$ and $\beta_2 = 18$ when $t \in (0,4)$.

We proceed now with another simulation study to examine the correctness of the result of Theorem 3. The parameters of the MPRHR distributions are selected as $(\alpha_1, \beta_1) = (0.25, 2)$ and $(\alpha_2, \beta_2) = (0.5, 18)$ and also assume that $X_0 \sim IW(2,1)$ and $Y_0 \sim IW(2,3)$, exactly as in Example 3. It is supposed that X and Y follow cdfs $F^\star(t;\alpha_1,\beta_1)$ and $G^\star(t;\alpha_2,\beta_2)$ and denote their right inverse functions by $F^{\star,-1}(u;\alpha_1,\beta_1)$ and $G^{\star,-1}(u;\alpha_2,\beta_2)$, respectively. We generate $u_1, u_2, \ldots, u_n \sim U(0,1)$. The inverse transform technique is implemented to simulate x_i as

$$\begin{aligned} x_i &= F^{\star,-1}(u_i;\alpha_1,\beta_1) \\ &= F^{-1}\left(\sqrt[-\beta_1]{\frac{\alpha_1}{u} + \bar{\alpha}_1} \right) \\ &= \sqrt{\frac{9\beta_1}{\ln(\frac{\alpha_1}{u} + \bar{\alpha}_1)}}, \; i = 1, 2, \ldots, n, \end{aligned}$$

from which one obtains n samples from $F^\star(t;\alpha_1,\lambda_1)$ so that $(\alpha_1,\beta_1) = (0.25, 2)$, that is, $x_i \sim MPRHR(0.25; 2; F)$ with $F(t) = \exp(-\frac{9}{t^2})$. Analogously, to produce y_i, one has

$$\begin{aligned} y_i &= G^{\star,-1}(u_i;\alpha_2,\beta_2) \\ &= G^{-1}\left(\sqrt[-\beta_2]{\frac{\alpha_2}{u} + \bar{\alpha}_2} \right) \\ &= \sqrt{\frac{\beta_1}{\ln(\frac{\alpha_1}{u} + \bar{\alpha}_1)}}, \; i = 1, 2, \ldots, n, \end{aligned}$$

through which n samples from $G^\star(t;\alpha_2,\beta_2)$ are simulated in which $(\alpha_2,\beta_2) = (0.5, 18)$, i.e., $y_i \sim MPRHR(0.5; 18; G)$, where $G(t) = \exp(-\frac{1}{t^2})$. On the basis of the simulated samples, we want to find the MLEs of (α_1,β_1) and (α_2,β_2), which are denoted by $(\widehat{\alpha}_i, \widehat{\beta}_i), i = 1, 2$. The MLE $(\widehat{\alpha}_1, \widehat{\beta}_1)$ is derived by solving, with respect to (α_1,β_1), the system of equations:

$$\begin{cases} \dfrac{n}{\beta_1} - 9\sum_{i=1}^{n} \dfrac{1}{x_i^2} - 18\bar{\alpha}_1 \sum_{i=1}^{n} \dfrac{e^{-\frac{9\beta_1}{x_i^2}}}{(1-\bar{\alpha}_1 \times e^{-\frac{9\beta_1}{x_i^2}})} = 0, \\ \dfrac{n}{\alpha_1} - 2\sum_{i=1}^{n} \dfrac{e^{-\frac{9\beta_1}{x_i^2}}}{(1-\bar{\alpha}_1 \times e^{-\frac{9\beta_1}{x_i^2}})} = 0. \end{cases} \tag{31}$$

The MLE $(\widehat{\alpha}_2, \widehat{\beta}_2)$ is also acquired by solving, in terms of (α_2, β_2), the equations:

$$\begin{cases} \dfrac{n}{\beta_2} - \sum_{i=1}^{n} \dfrac{1}{y_i^2} - 2\bar{\alpha}_2 \sum_{i=1}^{n} \dfrac{e^{-\frac{\beta_2}{y_i^2}}}{(1-\bar{\alpha}_2 \times e^{-\frac{\beta_2}{y_i^2}})} = 0, \\ \dfrac{n}{\alpha_2} - 2\sum_{i=1}^{n} \dfrac{e^{-\frac{9\beta_2}{y_i^2}}}{(1-\bar{\alpha}_2 \times e^{-\frac{\beta_2}{y_i^2}})} = 0. \end{cases} \tag{32}$$

We gathered in Table A3 in Appendix A the values of the MLEs of the parameters of the MPRHR models under different sample sizes, and also obtain the MLE of $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ for some t. It is acknowledged that the estimated reversed hazard rate ratio in the context of Example 3 is non-decreasing in t, as shown theoretically.

The following example is provided to examine the result of Theorem 4.

Example 4. *Let X_0 have cdf $F(t) = (1-\exp(-\theta.t))^{\frac{1}{3}}, t \geq 0$ and let Y_0 have an exponential distribution with cdf $G(t) = 1-\exp(-\theta.t), t \geq 0$, where $\theta > 0$ is a common parameter in F and G. Note that $\widetilde{h}(t) = \frac{1}{3}.\widetilde{s}(t)$ for all $t > 0$, where $\widetilde{h}$ is the rhr of X_0 and $\widetilde{s}$ is the rhr of Y_0, respectively. Hence, $\eta_0^\star = \eta_1^* = \frac{1}{3}$, and also, clearly, $X_0 \preceq_b Y_0$. Suppose that $X_1^* \sim MPRHR(\alpha_1, \beta_1, F)$ and $Y_1^* \sim MPRHR(\alpha_2, \beta_2, G)$ such that $\alpha_1 = 5, \alpha_2 = 2, \beta_1 = 5$ and $\beta_2 = 2$. In view of the notations and definitions in Theorem 4, we have*

$$M^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^*) := \sup_{u\in[0,1]} \left(\frac{u^{-\frac{5}{3}} + 4}{u^{-2}+1} \right) = 2.5,$$

and on the other hand, one has

$$\frac{\alpha_1 - 1}{\alpha_2 - 1} \times \frac{\beta_1}{\beta_2} \times \eta_1^* = \frac{10}{3}.$$

Thus, it is obvious that $\frac{\alpha_1-1}{\alpha_2-1} \times \frac{\beta_1}{\beta_2} \times \eta_1^* > M^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^*)$. Therefore, Theorem 4 is applicable, which provides that $X_1^* \preceq_b Y_1^*$. In Figure 4, the curve of $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ when $\theta = 2$, is plotted to verify that it is non-decreasing in $t \in (0,4)$.

We have listed in Table A4 in Appendix A the values of the MLEs of $(\alpha_i, \beta_i), i = 1,2$ under various sample sizes. We simulated data from $MPRHR(\alpha_1;\beta_1;F)$ and $MPRHR(\alpha_2;\beta_2;G)$ so that the parameters $(\alpha_i, \beta_i), i = 1,2$ together with the baseline cdfs F and G are chosen exactly as in Example 4. Further, we report the values of the MLE of $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ for some selected times t. It is deduced that the estimated reversed hazard rate ratio of the two MPRHR distributions in Example 4 is non-decreasing in t. This proves the result of Theorem 4.

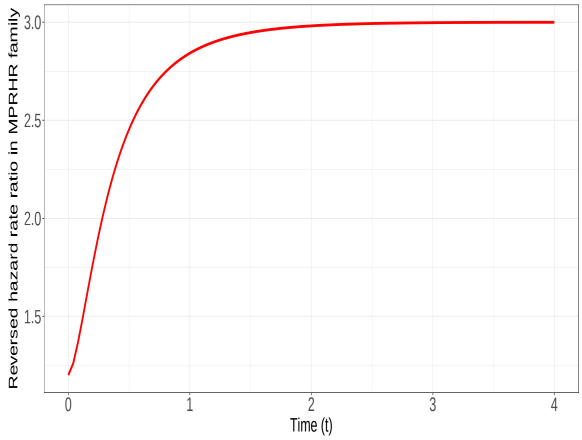

Figure 4. Plot of the reversed hazard rate ratio $\frac{\tilde{s}(t;\alpha_2,\beta_2)}{\tilde{h}(t;\alpha_1,\beta_1)}$ in Example 4 for $\alpha_1 = 5, \alpha_2 = 2, \beta_1 = 5$, $\beta_2 = 2$ and $\theta = 2$ when $t \in (0, 4)$.

Remark 3. *The examples presented in this section show that the results obtained apply exclusively to exponential laws. The exponential family of distributions is a very important class of distributions. For example, in Bayesian statistics a prior distribution is multiplied by a likelihood function and then normalised to produce a posterior distribution. In the case of a likelihood that belongs to an exponential family, there exists a conjugate prior, which is often also in an exponential family. There are many standard lifetime distributions that belong to this family of distributions. And, in general, the move to semiparametric models is nothing new in statistics (see, for example, Bayesian methods, which can provide more fundamental results in reliability theory from this point of view). However, the two semiparametric models, namely, the MPHR model and MPRHR model, introduced by Balakrishnan et al. [38], have been found to be applied in different contexts, including reliability and survival analysis. This is because these models encompass three reputable classes of models, namely, the Marshall–Olkin or POR model, the PHR model and the PRHR model. These models have so many applications in reliability and survival analysis (see, e.g., Carree [41]).*

5. Concluding Remarks

In this paper, we have examined two recently proposed semiparametric models, namely, the MPHR model and the MPRHR model. As shown by Balakrishnan et al. [38], these models include as special cases three important models in the literature, namely, the proportional hazard rate model, the proportional reversed hazard rate model and the proportional odds ratio model. Because these three models have found many applications in the literature so far and because they are available to the two newly defined semiparametric models, an analytical study of the latter models is needed because they cover and generalize the previous studies. The study of stochastic orderings for model comparisons has been carried out in the literature in various contexts, including reliability theory, survival analysis, actuarial analysis, risk theory, biostatistics and many other areas. Stochastic orderings are very useful potential tools for model analysis. For example, stochastic orders are very useful for detecting underestimation and overestimation problems in models. Stochastic orderings are usually recognized as tools for making inferences about models without data. The ordering properties of probability distributions reveal other aspects of the distribution or a family of distributions that can be used for various purposes.

The study conducted in this paper addresses situations in which there is a relative ordering property between two candidates from the MPHR family and, moreover, two candidates from the MPRHR family of semiparametric distributions. In general, the base distributions were assumed to be unknown but to satisfy a relative ordering property according to either the relative hazard rate order ($\preceq_c$) or the relative reversed hazard rate order ($\preceq_b$). It was assumed that the external parameters of the candidate models were generally different. Sufficient conditions were established for the conservation of the relative hazard rate order in the MPHR model and also for the conservation of the relative reversed hazard rate order in the MPRHR model. In the literature, for the preservation of the stochastic order in some scenarios, some stochastic orders are set as assumptions, which is a very strong condition. However, the conditions we found and presented in our work involve comparisons between two numbers, one of which is the supremum or infimum of a function and the other a function of the parameters of the models. With some examples we have shown that even very well-known standard statistical distributions that belong to the exponential family of distributions, such as the Weibull, Gamma or reversed Weibull distributions, can be used as the basic distribution in the MPHR and MPRHR model.

In many studies, different reliability models are considered with different intensities or hazard rate (reversed hazard rate) functions; moreover, there are even studies with compound and generalized intensities that have a discontinuity and atoms, as well as lattice distributions (see, for example, Kalimulina and Zverkina [42] and Kalimulina and Zverkina [43]). As can be seen from the graphs, the intensities considered in this paper are only continuous functions. This is a well-studied class of models (essentially exponential, generally a Weibull distribution). However, generalization of the results of this paper for more complicated intensities can be considered in future work.

In a future study, we can also consider stochastic comparisons in the MPHR and MPRHR models according to other stochastic orders, such as the likelihood ratio order ($\preceq_{lr}$), hazard rate order ($\preceq_{hr}$), reversed hazard rate order ($\preceq_{rh}$) and the usual stochastic order ($\preceq_{st}$). In the context of the MPHR model, in view of (20), when X_1 and Y_1 follow the pdfs $f(x;\alpha_1,\lambda_1)$ and $g(x;\alpha_2,\lambda_2)$, respectively, then $X_0 \preceq_{lr} Y_0$ implies $X_1 \preceq_{lr} Y_1$ if

$$\frac{\Phi_1(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi_1(\bar{F}(t);\alpha_1,\lambda_1)} \text{ is non-decreasing in } \; t \geq 0,$$

where $\Phi_1(u;\alpha,\lambda) := \frac{\lambda.\alpha.u^{\lambda-1}}{(1-\bar{\alpha}.u^{\lambda})^2}$. In addition, in the context of the MPHR model, when X_1 and Y_1 follow the sfs $\bar{F}(x;\alpha_1,\lambda_1)$ and $\bar{G}(x;\alpha_2,\lambda_2)$, respectively, as given in (19), then $X_0 \preceq_{st} Y_0$ implies $X_1 \preceq_{st} Y_1$ if

$$\Phi_2(u;\alpha_1,\lambda_1) \leq \Phi_2(u;\alpha_2,\lambda_2), \text{ for all } \; u \in [0,1],$$

where $\Phi_2(u;\alpha,\lambda) = \frac{\alpha.u^{\lambda}}{1-\bar{\alpha}.u^{\lambda}}$. In parallel, when the MPHR model is under consideration, as X_1 and Y_1 have hrs $h(x;\alpha_1,\lambda_1)$ and $s(x;\alpha_2,\lambda_2)$, respectively, as formulated in (21), then $X_0 \preceq_{hr} Y_0$ yields $X_1 \preceq_{hr} Y_1$ if

$$\inf_{t\geq 0}\left(\frac{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\right) \geq 1,$$

where Φ is defined as in (21). On the other hand, concerning the MPRHR model, by appealing to (25) and assuming that X_1^* and Y_1^* have pdfs $f^\star(x;\alpha_1,\beta_1)$ and $g^*(x;\alpha_2,\beta_2)$, respectively, then $X_0 \preceq_{lr} Y_0$ implies $X_1^* \preceq_{lr} Y_1^*$ if

$$\frac{\Psi_1(G(t);\alpha_2,\beta_2)}{\Psi_1(F(t);\alpha_1,\beta_1)} \text{ is non-decreasing in } \; t \geq 0,$$

where $\Psi_1(u;\alpha,\beta) := \frac{\beta.\alpha.u^{\beta-1}}{(1-\bar{\alpha}.u^{\beta})^2}$. Moreover, by considering the MPRHR model, as X_1^* and Y_1^* follow cdfs $F^*(x;\alpha_1,\beta_1)$ and $G^*(x;\alpha_2,\beta_2)$, respectively, as provided in (24), then $X_0 \preceq_{st} Y_0$ implies $X_1^* \preceq_{st} Y_1^*$ if

$$\Psi_2(u;\alpha_1,\beta_1) \geq \Psi_2(u;\alpha_2,\beta_2), \text{ for all } \; u \in [0,1],$$

where $\Psi_2(u;\alpha,\beta) = \frac{\alpha.u^{\beta}}{1-\bar{\alpha}.u^{\beta}}$. Furthermore, when the MPRHR model is regarded, so that $X_1^\star$ and Y_1^* have rhrs $\widetilde{h}(x;\alpha_1,\beta_1)$ and $\widetilde{s}(x;\alpha_2,\beta_2)$, respectively, as written in (26), then $X_0 \preceq_{rh} Y_0$ yields $X_1^* \preceq_{rh} Y_1^*$ if

$$\sup_{t\geq 0}\left(\frac{\Psi(G(t);\alpha_2,\beta_2)}{\Psi(F(t);\alpha_1,\beta_1)}\right)\geq 1,$$

in which Ψ is defined earlier in equations (26). The analogous study can also be carried out in the context of other stochastic orders such as the dispersive order, star order and super-additive order.

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Appendix A

Proof of Theorem 1. It suffices to prove that $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ is non-increasing in $t\geq 0$. Since

$$\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}=\frac{s(t)}{h(t)}\times\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}$$

and, by assumption, $\frac{s(t)}{h(t)}$ is non-increasing in $t\geq 0$, it is sufficient to show that $\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}$ is non-increasing in $t\geq 0$, which holds if and only if,

$$\frac{\partial}{\partial t}\ln\left(\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}\right)\leq 0,\quad\text{for all }\ t\geq 0.$$

Denote $\Phi'(u;\alpha,\lambda)=\frac{\partial}{\partial u}\Phi(u;\alpha,\lambda)$. We have:

$$\begin{aligned}\frac{\partial}{\partial t}\ln\left(\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}\right)&=g(t)\times\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}-f(t)\times\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\\&=s(t)\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}-h(t)\bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\\&\leq h(t)\times\left(\eta_1\times\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}-\bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\right),\end{aligned}\tag{A1}$$

where the last inequality follows from the fact that, for $\alpha_i\in[0,1]$ and $\lambda_i>0$,

$$\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}\leq 0,\quad\text{for all }\ t\geq 0,$$

and that $X_0\preceq_c Y_0$ yields

$$\frac{s(t)}{h(t)}\geq\lim_{t\to+\infty}\frac{s(t)}{h(t)}=\eta_1,$$

as it implies that $s(t)\geq\eta_1\times h(t)$ for all $t\geq 0$. The right-hand side of the inequality in (A1) is negative if and only if,

$$\eta_1\geq\frac{\gamma(\bar{F}(t);\alpha_1,\lambda_1)}{\gamma(\bar{G}(t);\alpha_2,\lambda_2)}\quad\text{for all }\ t\geq 0,\tag{A2}$$

in which

$$\gamma(u;\alpha,\lambda) := \frac{u\Phi'(u;\alpha,\lambda)}{\Phi(u;\alpha,\lambda)}$$
$$= \frac{(\alpha-1)\times\lambda}{u^{-\lambda}-\bar{\alpha}}, \quad \text{for all} \quad u \in [0,1].$$

The inequality in (A2) is satisfied if

$$\begin{aligned}
\eta_1 &\geq \sup_{t\geq 0}\left(\frac{\gamma(\bar{F}(t);\alpha_1,\lambda_1)}{\gamma(\bar{G}(t);\alpha_2,\lambda_2)}\right) \\
&= \sup_{t\geq 0}\left(\frac{\frac{(\alpha_1-1).\lambda_1}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}}{\frac{(\alpha_2-1)\times\lambda_2}{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}}\right) \\
&= \frac{\bar{\alpha}_1}{\bar{\alpha}_2}\times\frac{\lambda_1}{\lambda_2}\times\sup_{t\geq 0}\left(\frac{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}\right).
\end{aligned} \tag{A3}$$

On the other hand, since $X_0 \preceq_c Y_0$ further implies that

$$\frac{s(t)}{h(t)} \leq \lim_{t\to 0^+}\frac{s(t)}{h(t)} = \eta_0;$$

thus, $s(t) \leq \eta_0 h(t)$, for all $t \geq 0$. Hence, using (7),

$$\begin{aligned}
\bar{G}(t) &= \exp\{-\int_0^t s(x)dx\} \\
&\geq \exp\{-\eta_0\int_0^t h(x)dx\} = \bar{F}^{\eta_0}(t).
\end{aligned}$$

Thus, $\bar{G}^{-\lambda_2}(t) \leq \bar{F}^{-\lambda_2\eta_0}(t)$ for all $t \geq 0$, which further implies that

$$\frac{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1} \leq \frac{\bar{F}^{-\lambda_2\eta_0}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}, \quad \text{for all} \quad t \geq 0.$$

Therefore, the inequality in (A3) is satisfied if

$$\begin{aligned}
\eta_1 &\geq \frac{\bar{\alpha}_1}{\bar{\alpha}_2}\times\frac{\lambda_1}{\lambda_2}\times\sup_{t\geq 0}\left(\frac{\bar{F}^{-\lambda_2\eta_0}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}\right) \\
&= \frac{\bar{\alpha}_1}{\bar{\alpha}_2}\times\frac{\lambda_1}{\lambda_2}\times M(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_0),
\end{aligned}$$

or equivalently if

$$\frac{\bar{\alpha}_2}{\bar{\alpha}_1}\times\frac{\lambda_2}{\lambda_1}\times\eta_1 \geq M(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_0).$$

□

Proof of Theorem 2. Similarly, as in the proof of Theorem 1, we need to demonstrate that $\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}$ is non-increasing in $t \geq 0$, which holds if and only if,

$$\frac{\partial}{\partial t}\ln\left(\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}\right) \leq 0, \quad \text{for all} \quad t \geq 0.$$

Analogously, as in the proof of Theorem 1, one has

$$\frac{\partial}{\partial t}\ln\left(\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}\right) = s(t)\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)} - h(t)\bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}$$
$$\leq h(t)\times\left(\eta_0\times\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)} - \bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\right), \quad \text{(A4)}$$

in which the last inequality follows because, for $\alpha_i > 1$ and $\lambda_i > 0$,

$$\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)} \geq 0, \ \text{ for all } \ t \geq 0,$$

and, moreover, $X_0 \preceq_c Y_0$ gives

$$\frac{s(t)}{h(t)} \leq \lim_{t\to 0^+}\frac{s(t)}{h(t)} = \eta_0,$$

which implies that $s(t) \leq \eta_0.h(t)$ for all $t \geq 0$. The right-hand side of the inequality in (A4) is negative if and only if

$$\eta_0 \leq \frac{\gamma(\bar{F}(t);\alpha_1,\lambda_1)}{\gamma(\bar{G}(t);\alpha_2,\lambda_2)} \ \text{ for all } \ t \geq 0, \quad \text{(A5)}$$

in which $\gamma(u;\alpha,\lambda) = \frac{(\alpha-1)\times\lambda}{u^{-\lambda}-\bar{\alpha}}$. The inequality in (A5) stands valid if

$$\eta_0 \leq \inf_{t\geq 0}\left(\frac{\gamma(\bar{F}(t);\alpha_1,\lambda_1)}{\gamma(\bar{G}(t);\alpha_2,\lambda_2)}\right)$$
$$= \frac{\alpha_1-1}{\alpha_2-1}\times\frac{\lambda_1}{\lambda_2}\times\inf\left(\frac{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}\right). \quad \text{(A6)}$$

Moreover, since $X_0 \preceq_c Y_0$ provides that

$$\frac{s(t)}{h(t)} \geq \lim_{t\to+\infty}\frac{s(t)}{h(t)} = \eta_1,$$

then, consequently, $s(t) \geq \eta_1.h(t)$ for all $t \geq 0$. Therefore, using (7), we obtain

$$\bar{G}(t) = \exp\{-\int_0^t s(x)dx\}$$
$$\leq \exp\{-\eta_1\int_0^t h(x)dx\} = \bar{F}^{\eta_1}(t).$$

Thus, $\bar{G}^{-\lambda_2}(t) \geq \bar{F}_{-\lambda_2\eta_1}(t)$, which in turn gives

$$\frac{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1} \geq \frac{\bar{F}^{-\lambda_2\eta_1}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}, \ \text{ for all } \ t \geq 0.$$

Therefore, the inequality in (A6) is fulfilled if

$$\eta_0 \leq \frac{\alpha_1-1}{\alpha_2-1}\times\frac{\lambda_1}{\lambda_2}\times\inf_{t\geq 0}\left(\frac{\bar{F}^{-\lambda_2\eta_1}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}\right)$$
$$= \frac{\alpha_1-1}{\alpha_2-1}\times\frac{\lambda_1}{\lambda_2}\times m(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_1),$$

which holds if and only if

$$\frac{\alpha_2-1}{\alpha_1-1}\times\frac{\lambda_2}{\lambda_1}\times\eta_0\le m(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_1).$$

□

Proof of Theorem 3. To prove (27), it is sufficient to establish that $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ is non-decreasing in $t>0$. Following Equations (26), one has:

$$\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}=\frac{\widetilde{s}(t)}{\widetilde{h}(t)}\times\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}$$

and, due to assumption, $\frac{\widetilde{s}(t)}{\widetilde{h}(t)}$ is non-decreasing in $t>0$. Thus, it is enough to prove that $\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}$ is non-decreasing in $t>0$. The latter statement is valid if and only if

$$\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right)\ge 0,\quad\text{for all }\ t>0.$$

We use the notation $\Psi'(u;\alpha,\beta):=\frac{\partial}{\partial u}\Psi(u;\alpha,\beta)$. We obtain

$$\begin{aligned}\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right)&=f(t)\times\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)}-g(t)\times\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\\&=\widetilde{h}(t)F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)}-\widetilde{s}(t)G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\\&\ge\widetilde{s}(t)\times\left(\eta_0^\star\times F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)}-G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\right),\end{aligned}\tag{A7}$$

in which the last inequality is due to the fact that, for $\alpha_i\in[0,1]$ and $\beta_i>0$, whenever $i=1,2$,

$$F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)}\le 0,\quad\text{for all }\ t>0,$$

and, further, that $X_0\preceq_b Y_0$ provides that

$$\frac{\widetilde{h}(t)}{\widetilde{s}(t)}\le\lim_{t\to 0^+}\frac{\widetilde{h}(t)}{\widetilde{s}(t)}=\eta_0^+,$$

which further implies that $\widetilde{h}(t)\le\eta_0^\star.\widetilde{s}(t)$ for all $t>0$. Note that the right-hand side in (A7) is non-negative if and only if

$$\eta_0^\star\le\frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)}\ \text{ for all }\ t>0,\tag{A8}$$

where the function $\gamma^\star(\cdot;\alpha,\beta)$ is defined as below:

$$\begin{aligned}\gamma^\star(u;\alpha,\beta)&:=\frac{u\Psi'(u;\alpha,\beta)}{\Psi(u;\alpha,\beta)}\\&=\frac{(\alpha-1)\times\beta}{u^{-\beta}-\bar{\alpha}},\quad\text{for all }\ u\in[0,1].\end{aligned}$$

Now, it is sufficient to observe that the inequality in (A8) is fulfilled if

$$\begin{aligned}
\eta_0^\star &\le \inf_{t\ge 0}\left(\frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)}\right)\\
&= \inf_{t\ge 0}\left(\frac{\frac{(\alpha_2-1)\times\beta_2}{G^{-\beta_2}(t)-\bar{\alpha}_2}}{\frac{(\alpha_1-1)\times\beta_1}{F^{-\beta_1}(t)-\bar{\alpha}_1}}\right)\\
&= \frac{\bar{\alpha}_2}{\bar{\alpha}_1}\times\frac{\beta_2}{\beta_1}\times\inf_{t\ge 0}\left(\frac{F^{-\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}\right).
\end{aligned} \tag{A9}$$

Note that, since $X_0 \preceq_b Y_0$ yields

$$\frac{\widetilde{s}(t)}{\widetilde{h}(t)} \le \lim_{t\to+\infty}\frac{\widetilde{s}(t)}{\widetilde{h}(t)} = \eta_1^\star,$$

hence, $\widetilde{s}(t) \le \eta_1^\star \times \widetilde{h}(t)$, for all $t > 0$. Consequently, for all $t > 0$, using the characterization relation (11), one obtains:

$$\begin{aligned}
F(t) &= \exp\{-\int_t^{+\infty}\widetilde{h}(x)dx\}\\
&\le \exp\{-\frac{1}{\eta_1^\star}\times\int_t^{+\infty}\widetilde{s}(x)dx\} = G^{\frac{1}{\eta_1^\star}}(t).
\end{aligned}$$

Thus, $F^{-\beta_1}(t) \ge G^{-\frac{\beta_1}{\eta_1^\star}}(t)$ for all $t > 0$, which leads to

$$\frac{F^{-\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2} \ge \frac{G^{-\frac{\beta_1}{\eta_1^\star}}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2} \quad \text{for all} \quad t > 0.$$

As a result, the inequality in (A9) stands valid if

$$\begin{aligned}
\eta_0^\star &\le \frac{\bar{\alpha}_2}{\bar{\alpha}_1}\times\frac{\beta_2}{\beta_1}\times\inf_{t\ge 0}\left(\frac{G^{-\frac{\beta_1}{\eta_1^\star}}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}\right)\\
&= \frac{\bar{\alpha}_2}{\bar{\alpha}_1}\times\frac{\beta_2}{\beta_1}\times m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star),
\end{aligned}$$

or, equivalently, if

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_2}\times\frac{\beta_1}{\beta_2}\times\eta_0^\star \le m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star).$$

□

Proof of Theorem 4. In order to verify the implication in (28), as in the proof of Theorem 3, it suffices to show that

$$\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right) \ge 0, \quad \text{for all} \quad t > 0.$$

Analogously, as in the proof of Theorem 3, we can obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right) &= \widetilde{h}(t)F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} - \widetilde{s}(t)G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\\
&\ge \widetilde{s}(t)\times\left(\eta_1^\star\times F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} - G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\right),
\end{aligned} \tag{A10}$$

where the last inequality is due to the fact that, for $\alpha_i > 1$ and $\beta_i > 0$ for every $i = 1, 2$,

$$F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} \geq 0, \text{ for all } t > 0,$$

and, moreover, because $X_0 \preceq_b Y_0$,

$$\frac{\widetilde{h}(t)}{\widetilde{s}(t)} \geq \lim_{t\to+\infty} \frac{\widetilde{h}(t)}{\widetilde{s}(t)} = \eta_1^+,$$

from which one obtains $\widetilde{h}(t) \geq \eta_1^\star \times \widetilde{s}(t)$ for all $t > 0$. Now, one can see that the right-hand side in (A10) is non-negative if and only if

$$\eta_1^\star \geq \frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)} \text{ for all } t > 0, \tag{A11}$$

where the function $\gamma^*(\cdot;\alpha,\beta)$ is as defined in the proof of Theorem 3. It is now enough to see that the inequality in (A11) is satisfied when

$$\begin{aligned}\eta_1^\star &\geq \sup_{t\geq 0}\left(\frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)}\right)\\ &= \frac{\alpha_2-1}{\alpha_1-1}\times\frac{\beta_2}{\beta_1}\times\sup_{t\geq 0}\left(\frac{F^{-\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}\right).\end{aligned} \tag{A12}$$

Now observe that $X_0 \preceq_b Y_0$ gives

$$\frac{\widetilde{s}(t)}{\widetilde{h}(t)} \geq \lim_{t\to 0^+} \frac{\widetilde{s}(t)}{\widetilde{h}(t)} = \frac{1}{\eta_0^\star}.$$

Therefore, $\widetilde{s}(t) \geq (\eta_0^\star)^{-1} \times \widetilde{h}(t)$ for all $t > 0$. Hence, for all $t > 0$, by appealing to relationship (11) we can write:

$$\begin{aligned}F(t) &= \exp\{-\int_t^{+\infty} \widetilde{h}(x)dx\}\\ &\geq \exp\{-\eta_0^\star \times \int_t^{+\infty} \widetilde{s}(x)dx\} = G^{\eta_0^\star}(t).\end{aligned}$$

As a result, $F^{-\beta_1}(t) \leq G^{-\eta_0^\star\beta_1}(t)$ for all $t > 0$, providing that

$$\frac{F^{-\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2} \leq \frac{G^{-\eta_0^\star\times\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}, \text{ for all } t > 0.$$

The inequality in (A12) is, therefore, fulfilled if

$$\begin{aligned}\eta_1^\star &\geq \frac{\alpha_2-1}{\alpha_1-1}\times\frac{\beta_2}{\beta_1}\times\sup_{t\geq 0}\left(\frac{G^{-\eta_0^\star\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}\right)\\ &= \frac{\alpha_2-1}{\alpha_1-1}\times\frac{\beta_2}{\beta_1}\times M^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_0^\star),\end{aligned}$$

or, equivalently, when

$$\frac{\alpha_1-1}{\alpha_2-1}\times\frac{\beta_1}{\beta_2}\times\eta_1^\star \geq M^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_0^\star).$$

□

Table A1. The MLE of the hazard rate ratio of the MPHR distributions in Example 1 for different sample sizes $n = 5, 10, 30, 50, 100, 200$ and different ages $t = 0.05, 0.1, 0.3, 0.6, 0.8, 1$.

n	(α_1, λ_1)	(α_2, λ_2)	$(\hat{\alpha}_1, \hat{\lambda}_1)$	$(\hat{\alpha}_2, \hat{\lambda}_2)$	t	$\frac{s(t;\hat{\alpha}_2,\hat{\lambda}_2)}{h(t;\hat{\alpha}_1,\hat{\lambda}_1)}$
5	(0.8,10)	(0.1,1)	(0.723,9.455)	(0.123,0.875)	0.05	6.2539
					0.1	5.0954
					0.3	3.8754
					0.6	0.9874
					0.8	0.5466
					1	0.4658
10	(0.8,10)	(0.1,1)	(0.744,9.522)	(0.114,0.922)	0.05	6.2986
					0.1	5.1215
					0.3	3.9574
					0.6	0.9899
					0.8	0.5878
					1	0.4955
30	(0.8,10)	(0.1,1)	(0.759,9.665)	(0.112,0.937)	0.05	6.3344
					0.1	5.5514
					0.3	4.0128
					0.6	1.0236
					0.8	0.5991
					1	0.5033
50	(0.8,10)	(0.1,1)	(0.779,9.858)	(0.108,0.968)	0.05	6.4111
					0.1	5.8101
					0.3	4.1003
					0.6	1.2231
					0.8	0.6021
					1	0.5395
100	(0.8,10)	(0.1,1)	(0.791,9.911)	(0.103,0.986)	0.05	6.5499
					0.1	5.9888
					0.3	4.1099
					0.6	1.4268
					0.8	0.6411
					1	0.5895
200	(0.8,10)	(0.1,1)	(0.799,9.989)	(0.101,0.993)	0.05	6.5533
					0.1	5.9899
					0.3	4.1101
					0.6	1.4298
					0.8	0.6471
					1	0.5929

Table A2. The MLE of the hazard rate ratio of the MPHR distributions in Example 2 for sample sizes $n = 5, 10, 30, 50, 100, 200$ and ages $t = 0.05, 0.1, 0.3, 0.6, 0.8, 1$.

n	(α_1, λ_1)	(α_2, λ_2)	$(\hat{\alpha}_1, \hat{\lambda}_1)$	$(\hat{\alpha}_2, \hat{\lambda}_2)$	t	$\frac{s(t;\hat{\alpha}_2,\hat{\lambda}_2)}{h(t;\hat{\alpha}_1,\hat{\lambda}_1)}$
5	(10,4)	(3,2)	(9.232,4.452)	(2.185,3.111)	0.05	2.4978
					0.1	2.1305
					0.3	1.3858
					0.6	0.9022
					0.8	0.8914
					1	0.5778
10	(10,4)	(3,2)	(9.389,4.472)	(2.231,2.998)	0.05	2.6017
					0.1	2.2764
					0.3	1.4111
					0.6	0.9954
					0.8	0.9517
					1	0.6012

Table A2. *Cont.*

n	(α_1, λ_1)	(α_2, λ_2)	$(\hat{\alpha}_1, \hat{\lambda}_1)$	$(\hat{\alpha}_2, \hat{\lambda}_2)$	t	$\frac{s(t;\hat{\alpha}_2,\hat{\lambda}_2)}{h(t;\hat{\alpha}_1,\hat{\lambda}_1)}$
30	(10,4)	(3,2)	(9.589,4.325)	(2.541,2.6578)	0.05	2.7507
					0.1	2.3520
					0.3	1.5210
					0.6	1.1123
					0.8	0.9512
					1	0.7898
50	(10,4)	(3,2)	(9.698,4.239)	(2.661,2.451)	0.05	2.8514
					0.1	2.5012
					0.3	1.7102
					0.6	1.2054
					0.8	1.0021
					1	0.8395
100	(10,4)	(3,2)	(9.9614,4.0750)	(2.845,2.201)	0.05	2.9995
					0.1	2.8112
					0.3	1.9958
					0.6	1.3798
					0.8	1.2211
					1	0.9823
200	(10,4)	(3,2)	(9.9919,4.004)	(2.899,2.014)	0.05	3.3134
					0.1	2.9891
					0.3	2.1192
					0.6	1.4776
					0.8	1.2345
					1	1.0000

Table A3. The MLE of the reversed hazard rate ratio of the MPRHR distributions in Example 3 under different sample sizes $n = 5, 10, 30, 50, 100, 200$ for $t = 2, 2.5, 3, 3.5, 3.8, 4$.

n	(α_1, β_1)	(α_2, β_2)	$(\hat{\alpha}_1, \hat{\beta}_1)$	$(\hat{\alpha}_2, \hat{\beta}_2)$	t	$\frac{\tilde{s}(t;\hat{\alpha}_2,\hat{\beta}_2)}{\tilde{h}(t;\hat{\alpha}_1,\hat{\beta}_1)}$
5	(0.25,2)	(0.5,18)	(0.195,1.871)	(0.411,16.466)	2	0.6312
					2.5	0.6512
					3	0.7019
					3.5	0.8127
					3.8	0.8511
					4	0.8145
10	(0.25,2)	(0.5,18)	(0.204,1.912)	(0.429,17.012)	2	0.6987
					2.5	0.7018
					3	0.7155
					3.5	0.8321
					3.8	0.8843
					4	0.9181
30	(0.25,2)	(0.5,18)	(0.216,1.942)	(0.438,17.268)	2	0.8119
					2.5	0.8211
					3	0.8455
					3.5	0.8772
					3.8	0.9211
					4	0.9441
50	(0.25,2)	(0.5,18)	(0.224,1.956)	(0.459,17.611)	2	0.9655
					2.5	0.8845
					3	1.0145
					3.5	1.0349
					3.8	1.0011
					4	1.0097

Table A3. *Cont.*

n	(α_1, β_1)	(α_2, β_2)	$(\hat{\alpha}_1, \hat{\beta}_1)$	$(\hat{\alpha}_2, \hat{\beta}_2)$	t	$\frac{\widetilde{s}(t;\hat{\alpha}_2,\hat{\beta}_2)}{\widetilde{h}(t;\hat{\alpha}_1,\hat{\beta}_1)}$
100	(0.25,2)	(0.5,18)	(0.244,1.989)	(0.485,17.891)	2	0.9721
					2.5	0.9811
					3	1.0399
					3.5	1.0413
					3.8	1.0901
					4	1.1001
200	(0.25,2)	(0.5,18)	(0.249,1.991)	(0.491,17.992)	2	1.0041
					2.5	1.0138
					3	1.0415
					3.5	1.0719
					3.8	1.0989
					4	1.1014

Table A4. The MLE of the reversed hazard rate ratio of the MPRHR distributions in Example 4 with sample sizes $n = 5, 10, 30, 50, 100, 200$ and time points $t = 2, 2.5, 3, 3.5, 3.8, 4, 1$.

n	(α_1, β_1)	(α_2, β_2)	$(\hat{\alpha}_1, \hat{\beta}_1)$	$(\hat{\alpha}_2, \hat{\beta}_2)$	t	$\frac{\widetilde{s}(t;\hat{\alpha}_2,\hat{\beta}_2)}{\widetilde{h}(t;\hat{\alpha}_1,\hat{\beta}_1)}$
5	(5,5)	(2,2)	(4.111,4.231)	(1.671,1.523)	2	2.901
					2.5	2.921
					3	2.9267
					3.5	2.9453
					3.8	2.9566
					4	2.9667
10	(5,5)	(2,2)	(4.129,4.361)	(1.675,1.536)	2	2.9111
					2.5	2.9252
					3	2.9410
					3.5	2.9555
					3.8	2.9661
					4	2.9746
30	(5,5)	(2,2)	(4.392,4.436)	(1.749,1.666)	2	2.9231
					2.5	2.9351
					3	2.9448
					3.5	2.9655
					3.8	2.9742
					4	2.9814
50	(5,5)	(2,2)	(4.601,4.635)	(1.891,1.798)	2	2.9612
					2.5	2.9732
					3	2.9774
					3.5	2.9796
					3.8	2.9799
					4	2.9831
100	(5,5)	(2,2)	(4.892,4.895)	(1.946,1.992)	2	2.9712
					2.5	2.9832
					3	2.9875
					3.5	2.9890
					3.8	2.9893
					4	2.9897
200	(5,5)	(2,2)	(4.912,4.992)	(1.999,2)	2	2.9812
					2.5	2.9932
					3	2.9975
					3.5	2.9990
					3.8	2.9994
					4	2.9996

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