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A Novel Class of Separation Axioms, Compactness, and Continuity via C-Open Sets

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Abstract: In this paper, we originate a new class of open sets, namely C-open sets, and we review its important properties. Then, some separation axioms of C-open sets are introduced and investigated. In addition, we define the so-called C-compact and C′-compact spaces via C-open sets, and the theorems based on them are discussed with counterexamples. Moreover, we entitle the C-continuous and C′-continuous functions by applying C-open sets. In particular, several inferred properties of them and their connection with the other topological spaces are studied theoretically. Many examples are given to explain the concepts lucidly. The results established in this research paper are new in the field of topology.

Keywords: C-open set; C-regular space; C-normal space; C-compact space; C′-compact space; C-continuous function; C′-continuous function

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1. Introduction

In topology and its applications, the concept of an open set is fundamental. Many researchers have defined classes of open sets, (see [1–5]); through them, new definitions of compactness and continuity have been found, see [3,6]. From this point of view, I have defined a new class of open set, namely a C-open set, for a topological space as follows: an open subset A of a topological space X is called a C-open set if $cl(\mathtt{A})\setminus\mathtt{A}$ is a countable set, and, through it, created new concepts of the separation axioms, compactness, and continuity. Undoubtedly, many classic topological concepts, such as the different classes of open sets, have been used in soft topology, which has achieved many impressive and valuable results, for example, S. Al-Ghour [7] used the notion of the ω-open set [8] on soft topology. In 2013, B. Chen [9] provided the idea of soft semi-open sets inspired by the class of semi-open sets [3]. In [1,2,6,9,10], many researchers utilized the notions of open, pre-open [4], b-open [11], α-open [12], β-open [13], somewhere dense sets [14], and others to obtain new notions of soft topology.

The motivations for writing this paper are, firstly, to study a new class of open sets in topological spaces, which is an active path of research; secondly, to introduce a new framework to contribute in the near future to produce soft topological concepts, such as soft operators and soft continuity, which are inspired by classical topologies. Certainly, the researchers can explore other notions like covering properties and separation axioms of the proposed class of C-open and C-closed sets; and finally, it plays a crucial role in the development of modern concepts such as soft and fuzzy topology.

The arrangement of this article is as follows: In Section 2, we define a class of C-open and C-closed sets and establish their master properties. In Section 3, we study the concepts of C-regular, C-normal, $\mathtt{C}\text{–}\mathtt{T}_i$ for $i \in \{0, 1, 2, 3, 4\}$, C-compact, and C′-compact spaces via

$\mathtt{C}$-open sets, and the theorems based on them are discussed with blue counterexamples. The concepts of $\mathtt{C}$-continuous, $\mathtt{C}'$-continuous, $\mathtt{C}$-homeomorphism, and $\mathtt{C}'$-homeomorphism functions via $\mathtt{C}$-open sets are introduced and probed in Section 4. In addition, we review some of their important properties with $\mathtt{C}$-compact and $\mathtt{C}'$-compact space, and many examples are given to explain the concepts lucidly. Finally, some conclusions and the possible upcoming works are given in Section 5.

Throughout this paper, let $(\mathtt{X}, \Im)$ be a topological space and let $\mathtt{B}$ be a subset of X. We denote the complement of $\mathtt{B}$ in $\mathtt{X}$ by $\mathtt{X}\backslash B$, the interior of $\mathtt{B}$ in $\mathtt{X}$ by $int\mathtt{B}$, and the closure of $\mathtt{B}$ in $\mathtt{X}$ by $cl(\mathtt{B})$. Also, we denote the set of positive integer numbers by $\mathbb{N}$, the set of integer numbers by $\mathbb{Z}$, the set of rational numbers by $\mathbb{Q}$, the set of irrational numbers by $\mathbb{I}$, the set of real numbers by $\mathbb{R}$, and the usual topology in $\mathbb{R}$ by $\mathtt{U}$ [15]. Unless or otherwise mentioned, $\mathtt{X}$ stands for the topological space $(\mathtt{X}, \Im)$. We do not assume $\mathtt{T}_2$ in the definition of compactness. A $\mathtt{T}_3$ space is regular and a $\mathtt{T}_1$ space, where a $\mathtt{T}_4$ space is also normal and a $\mathtt{T}_1$ space. A subset $\mathtt{B}$ of space $\mathtt{X}$ is said to be regularly-open and also called open domain if it is the interior of its own closure [16]. A subset $\mathtt{B}$ is said to be regularly-closed and also called closed domain if it is the closure of its own interior or if its complement is an open domain.

2. Main Properties of $\mathtt{C}$-Openness and $\mathtt{C}$-Closedness

This part presents the definitions of $\mathtt{C}$-open and $\mathtt{C}$-closed sets, and the theorems and properties based on them are discussed with counterexamples.

Definition 1. *An open subset $\mathtt{A}$ of a topological space $(\mathtt{X}, \Im)$ is called $\mathtt{C}$-open set if $cl(\mathtt{A})\backslash\mathtt{A}$ is a countable set. That is, $\mathtt{A}$ is an open set and the frontier of $\mathtt{A}$ is a countable set.*

Definition 2. *A closed subset $\mathtt{A}$ of a topological space $(\mathtt{X}, \Im)$ is called $\mathtt{C}$-closed set if $A\backslash int(A)$ is a countable set. That is, $\mathtt{A}$ is a closed set and the frontier of $\mathtt{A}$ is a countable set.*

We denoted for the collection of all $\mathtt{C}$-open (resp., $\mathtt{C}$-closed) subsets of a topological space $(\mathtt{X}, \Im)$ by $\mathtt{CO}(\mathtt{X})$ (resp., $\mathtt{CC}(\mathtt{X})$).

Theorem 1. *The complement of any $\mathtt{C}$-open (resp., $\mathtt{C}$-closed) subset of a topological space $(\mathtt{X}, \Im)$ is a $\mathtt{C}$-closed (resp., $\mathtt{C}$-open) set.*

Proof. Let $\mathtt{A}$ be any $\mathtt{C}$-open subset of a topological space $(\mathtt{X}, \Im)$. Then $\mathtt{X}\backslash\mathtt{A}$ is closed and $(\mathtt{X}\backslash\mathtt{A})\backslash int(\mathtt{X}\backslash\mathtt{A}) = (\mathtt{X}\backslash\mathtt{A})\backslash(\mathtt{X}\backslash cl(A)) = (\mathtt{X}\backslash\mathtt{A}) \cap cl(A) = cl(A)\backslash(A)$ is countable, because $\mathtt{A}$ is $\mathtt{C}$-open. Therefore, $\mathtt{X}\backslash\mathtt{A}$ is a $\mathtt{C}$-closed set. On the other hand, suppose that $\mathtt{A}$ be any $\mathtt{C}$-closed subset of a topological space $(\mathtt{X}, \Im)$. Then $\mathtt{X}\backslash\mathtt{A}$ is open and $cl(\mathtt{X}\backslash\mathtt{A}) \setminus (\mathtt{X}\backslash\mathtt{A}) = ((X)\backslash\mathtt{X}\backslash\mathtt{A}) \cap cl(\mathtt{X}\backslash\mathtt{A}) = (A) \cap cl(\mathtt{X}\backslash\mathtt{A}) = (A)\backslash((X)\backslash cl(\mathtt{X}\backslash\mathtt{A})) = (A)\backslash int((A))$ is countable because $\mathtt{A}$ is $\mathtt{C}$-closed. Therefore, $\mathtt{X}\backslash\mathtt{A}$ is a $\mathtt{C}$-open set. □

Recall that an open (resp., closed) subset $\mathtt{A}$ of a topological space $(\mathtt{X}, \Im)$ is called an $\mathtt{F}$-*open* (resp., $\mathtt{F}$-closed) set if $cl(\mathtt{A})\backslash\mathtt{A}$ (resp., $\mathtt{A}\backslash int(\mathtt{A})$) is a finite set [17].

Obviously, from the definitions, it is clear that any $\mathtt{F}$-open (resp., $\mathtt{F}$-closed) subset of a topological space $(\mathtt{X}, \Im)$ is a $\mathtt{C}$-open (resp., $\mathtt{C}$-closed) set. However, the converse always is not true. For example, the subset $\mathbb{R}\backslash\mathbb{Z}$ is a $\mathtt{C}$-open subset of the usual topological space $(\mathbb{R}, \mathtt{U})$, which is not $\mathtt{F}$-open. Also, by Theorem 1, any complement of the $\mathtt{C}$-open set is $\mathtt{C}$-closed, then $\mathbb{Z}$ is a $\mathtt{C}$-closed subset of the usual topological space $(\mathbb{R}, \mathtt{U})$, which is not $\mathtt{F}$-closed because $\mathbb{Z}\backslash int(\mathbb{Z}) = \mathbb{Z}\backslash\emptyset = \mathbb{Z}$ is not finite. It is clear by the definitions that, any clopen (closed-and-open) subset of a topological space $(\mathtt{X}, \Im)$ is a $\mathtt{C}$-clopen set. Also, any countable closed set is $\mathtt{C}$-closed. However, any countable open set may not be $\mathtt{C}$-open. For example, $\mathbb{R}$ with a particular point topology at 1. We have that $\mathbb{N}$ is a countable open set, but $cl(\mathbb{N})\backslash\mathbb{N} = \mathbb{R}\backslash\mathbb{N}$ is an uncountable set.

In $(\mathbb{R}, \mathtt{U})$, any open interval is a $\mathtt{C}$-open set. Also, any closed interval is a $\mathtt{C}$-closed set. It is clear by Definitions 1 and 2 that every $\mathtt{C}$-open and $\mathtt{C}$-closed sets are open and closed sets, respectively. However, the converse always is not true. Here is an example of an open (resp., closed) set which is not $\mathtt{C}$-open (resp., $\mathtt{C}$-closed).

Example 1. *Let $(\mathbb{R}, \Im_{\mathbb{I}})$ be the excluded set topological space on $\mathbb{R}$ by $\mathbb{I}$. Then $\mathbb{Q}$ is open in $(\mathbb{R}, \Im_{\mathbb{I}})$. But $cl(\mathbb{Q})\backslash\mathbb{Q} = \mathbb{R}\backslash\mathbb{Q} = \mathbb{I}$ is an uncountable set. Hence, $\mathbb{Q}$ is not a $\mathtt{C}$-open set. Also, $\mathbb{I}$ is an example of a closed set which is not a $\mathtt{C}$-closed set.*

There is an example of a $\mathtt{C}$-open (resp., $\mathtt{C}$-closed) set which is not an open (resp., closed) domain set.

Example 2. *Let $\mathtt{A} = (2,5) \cup (5,9)$ be a $\mathtt{C}$-open subset in $(\mathbb{R}, \mathtt{U})$. However, $\mathtt{A}$ is not an open domain. Moreover, $\mathbb{R}\backslash\mathtt{A}$ is a $\mathtt{C}$-closed set, but $\mathbb{R}\backslash\mathtt{A}$ is not a closed domain.*

In general, $\mathtt{C}$-open (resp., $\mathtt{C}$-closed) sets and open (resp., closed) domain sets are not comparable as shown by the following example.

Example 3. *By Example 1, let $\mathtt{H} = \{1,2,3\}$, then $int(cl(\mathtt{H})) = int(cl(\{1,2,3\})) = int(\{1,2,3\} \cup\mathbb{I}) = \{1,2,3\} = \mathtt{H}$ is an open domain. However, $cl(\mathtt{H})\backslash\mathtt{H} = cl(\{1,2,3\})\backslash\{1,2,3\} = (\{1,2,3\} \cup \mathbb{I})\backslash\{1,2,3\} = \mathbb{I}$ is an uncountable set, where $\mathtt{H}$ is no longer a $\mathtt{C}$-open set. Moreover, let $\mathtt{K} = \mathbb{R}\backslash\{1,2,3\}$, then $cl(int(\mathtt{K})) = cl(int(\mathbb{R}\backslash\{1,2,3\})) = cl(\mathbb{Q} \setminus \{1,2,3\}) = \mathbb{I} \cup (\mathbb{Q} \setminus \{1,2,3\}) = \mathbb{R}\backslash\{1,2,3\} = \mathtt{K}$ is a closed domain. However, $\mathtt{K}\backslash int(\mathtt{K}) = (\mathbb{R}\backslash\{1,2,3\})\backslash int(\mathbb{R}\backslash\{1,2,3\}) = (\mathbb{R} \setminus \{1,2,3\})\backslash(\mathbb{Q}\backslash\{1,2,3\}) = \mathbb{I}$ is an uncountable set, where $\mathtt{K}$ is no longer a $\mathtt{C}$-closed set.*

From the definitions of open, closed, $\mathtt{C}$-open, and $\mathtt{C}$-closed sets, the following diagram is obtained:

$$\begin{array}{c} \mathtt{F}\text{-open set} \longrightarrow \mathtt{C}\text{-open set} \longrightarrow \text{open set} \\ \mathtt{F}\text{-closed set} \longrightarrow \mathtt{C}\text{-closed set} \longrightarrow \text{closed set} \end{array}$$

Diagram (a)

None of the above implications are reversible.

Theorem 2. *Finite unions of $\mathtt{C}$-open sets is $\mathtt{C}$-open.*

Proof. Suppose that $\mathtt{K}_i$ be a $\mathtt{C}$-open set for all $i \in \{1,2,3,\ldots,n\}$. Then $\mathtt{K}_i$ is an open set and $cl(\mathtt{K}_i)\backslash\mathtt{K}_i$ is countable for all i. Since $\bigcup_{i=1}^{n} \mathtt{K}_i$ is open, then we need to show the other condition of the $\mathtt{C}$-open set. Now, we have,

$$cl(\bigcup_{i=1}^{n} \mathtt{K}_i) \setminus \bigcup_{i=1}^{n} \mathtt{K}_i = \bigcup_{i=1}^{n} (cl(\mathtt{K}_i)\backslash\mathtt{K}_i).$$

Since the finite union of countable sets is countable, then $cl(\bigcup_{i=1}^{n} \mathtt{K}_i)\setminus \bigcup_{i=1}^{n} \mathtt{K}_i$ is countable. Therefore, $\bigcup_{i=1}^{n} \mathtt{K}_i$ is $\mathtt{C}$-open. □

In the above Theorem, a countable union of $\mathtt{C}$-open sets is $\mathtt{C}$-open if a countable union of closure sets equals the closure of a countable union of sets. In general, this is not true for infinite cases as shown in the following example:

Example 4. *Let $X_i = \{\{a_i, b_i\} : a_i \neq b_i, i \in I\}$ be a family of non-empty pairwise disjoint spaces, where I is an uncountable index set. Let $\Im_i$ be a particular point topology on X_i at a_i. Let the set $\Im = \{V \subseteq \coprod_{i\in I} X_i : V \cap X_i \subseteq X_i$ open for all $i\}$ be a topology on the disjoint union $X = \coprod_{i\in I} X_i$. We call $(X, \Im)$ the topological sum of the X_i. For all $i \in I$, pick $B_i = \{a_i\} \subseteq X$, where $B_i = \{a_i\}$ for i and $B_i = \emptyset$ for all $j \neq i$ in I; hence, B_i is open in X and $cl(B_i)\backslash B_i$ is finite; thus, B_i is $\mathtt{C}$-open in X for all $i \in I$. However, $cl(\bigcup_{i\in I} B_i)\setminus \bigcup_{i\in I} B_i$ is uncountable. Therefore, $\bigcup_{i\in I} B_i$ is not $\mathtt{C}$-open in X.*

Corollary 1. *Finite intersections of $\mathtt{C}$-closed sets is $\mathtt{C}$-closed.*

Proof. Obvious by Theorems 1 and 2 and by Morgan's Laws. □

Theorem 3. *Finite union of* $\mathtt{C}$*-closed sets is* $\mathtt{C}$*-closed.*

Proof. Suppose that $\mathtt{K}_i$ be a $\mathtt{C}$-closed set for all $i \in \{1,2,3,\ldots,n\}$, then $\mathtt{K}_i$ is a closed set and $\mathtt{K}_i \backslash int(\mathtt{K}_i)$ is a countable set for all i. Since $\bigcup_{i=1}^{n} \mathtt{K}_i$ is closed, then we need to show the other condition of the $\mathtt{C}$-closed set.

Claim:

$$\bigcup_{i=1}^{n} \mathtt{K}_i \backslash int(\bigcup_{i=1}^{n} \mathtt{K}_i) \subseteq \bigcup_{i=1}^{n} (\mathtt{K}_i \backslash int(\mathtt{K}_i)).$$

Let $x \in \bigcup_{i=1}^{n} \mathtt{K}_i \backslash int(\bigcup_{i=1}^{n} \mathtt{K}_i)$ be arbitrary. Since $\bigcup_{i=1}^{n} int(\mathtt{K}_i) \subseteq int(\bigcup_{i=1}^{n} \mathtt{K}_i)$, then there exists $i_1 \in \{1,2,3,\ldots,n\}$ such that $\mathtt{x} \in \mathtt{K}_{i_1}$ and $\mathtt{x} \notin int(\mathtt{K}_i)$ for all $i \in \{1,2,3,\ldots,n\}$. Then $\mathtt{x} \in (\mathtt{K}_{i_1}) \backslash int(\mathtt{K}_{i_1})$, then $\mathtt{x} \in \bigcup_{i=1}^{n} (\mathtt{K}_i \backslash int(\mathtt{K}_i))$. The claim is proven.

Since the finite union of countable sets is countable, then, $\bigcup_{i=1}^{n} \mathtt{K}_i \backslash int(\bigcup_{i=1}^{n} \mathtt{K}_i)$ is a countable set. Therefore, $\bigcup_{i=1}^{n} \mathtt{K}_i$ is $\mathtt{C}$-closed. □

By Theorems 1 and 3 and by Morgan's Laws, we have the following Corollary:

Corollary 2. *Finite intersections of* $\mathtt{C}$*-open sets is* $\mathtt{C}$*-open.*

3. $\mathtt{C}$–$\mathtt{T_i}$-Spaces and $\mathtt{C}$-Compact Spaces

This part provides the definitions of $\mathtt{C}$-regular space, $\mathtt{C}$-normal space, $\mathtt{C}$–$\mathtt{T_i}$-space for $i \in \{0,1,2,3,4\}$, $\mathtt{C}$-compact space, and $\mathtt{C}'$-compact space via $\mathtt{C}$-open sets and investigates their main properties.

Definition 3. *Let* $(\mathtt{X}, \Im)$ *be a topological space. Then*

(i) $(\mathtt{X}, \Im)$ *is called a* $\mathtt{C}$*-regular space if and only if for each closed subset* $\mathtt{F} \subset \mathtt{X}$ *and each point* $\mathtt{x} \notin \mathtt{F}$*, there exist disjoint* $\mathtt{C}$*-open sets* $\mathtt{U}$ *and* $\mathtt{V}$ *such that* $\mathtt{x} \in \mathtt{U}$ *and* $\mathtt{F} \subseteq \mathtt{V}$*.*

(ii) $(\mathtt{X}, \Im)$ *is called a* $\mathtt{C}$*-normal space if and only if for each pair of closed disjoint subsets* $\mathtt{F}_1$ *and* $\mathtt{F}_2$ *of* $\mathtt{X}$*, there exist disjoint* $\mathtt{C}$*-open sets* $\mathtt{U}$ *and* $\mathtt{V}$ *such that* $\mathtt{F}_1 \subseteq \mathtt{U}$ *and* $\mathtt{F}_2 \subseteq \mathtt{V}$*.*

Obviously, from the definitions, any $\mathtt{C}$-regular space is regular and any $\mathtt{C}$-normal space is normal. However, the converse may not be true as shown by the following two examples.

Example 5 (The Niemytzki Plane [15]). *Let* $\mathbb{L} = \{(\mathtt{x},\mathtt{y}) \in \mathbb{R} : \mathtt{y} \geq 0\}$*, the upper half plane with the* $\mathtt{X}$*-axis. Let* $\mathtt{L}_1 = \{(\mathtt{x}, 0) :\in \mathbb{R}\}$*, i.e., the* $\mathtt{X}$*-axis. Let* $\mathtt{L}_2 = \mathbb{L} \backslash \mathtt{L}_1$*. For every* $(\mathtt{x}, 0) \in \mathtt{L}_1$ *and* $r \in \mathbb{R}, r > 0$*, let* $\mathtt{U}((\mathtt{x}, 0), r)$ *be the set of all points of* $\mathbb{L}$ *inside the circle of radius r tangent to* $\mathtt{L}_1$ *at* $(\mathtt{x}, 0)$ *and let* $\mathtt{U}_i((\mathtt{x}, 0)) = \mathtt{U}((\mathtt{x}, 0), \frac{1}{i}) \cup \{(\mathtt{x}, 0)\}$ *for* $i \in \mathbb{N}$*. For every* $(\mathtt{x},\mathtt{y}) \in \mathtt{L}_2$ *and* $r > 0$*, let* $\mathtt{U}((\mathtt{x},\mathtt{y}), r)$ *be the set of all points of* $\mathbb{L}$ *inside the circle of radius r and centered at* $(\mathtt{x},\mathtt{y})$ *and let* $\mathtt{U}_i((\mathtt{x},\mathtt{y})) = \mathtt{U}((\mathtt{x},\mathtt{y}), \frac{1}{i})$ *for* $i \in \mathbb{N}$*. The Niemytzki Plane is regular. Let* $\mathtt{A} = cl(\mathtt{U}((2,2), \frac{1}{2}))$ *be a closed set, as* $(-5, 2) \notin \mathtt{A}$ *cannot be separated by two disjoint* $\mathtt{C}$*-open sets because the smallest* $\mathtt{C}$*-open set containing* $\mathtt{A}$ *is* $\mathtt{L}_2$*. Hence, the Niemytzki Plane is not a* $\mathtt{C}$*-regular space.*

Example 6. *Consider* $(\mathbb{R}, \mathtt{U})$*, where* $\mathtt{U}$ *is the usual Topology on* $\mathbb{R}$ *[15]. Hence,* $(\mathbb{R} \times \mathbb{R}, \mathtt{U} \times \mathtt{U})$ *is a normal space. However,* $\mathtt{A} = [1,2] \times [1,2]$ *and* $\mathtt{B} = [-2,-1] \times [1,2]$ *are two disjoint closed sets which cannot be separated by two disjoint* $\mathtt{C}$*-open sets. Therefore,* $(\mathbb{R} \times \mathbb{R}, \mathtt{U} \times \mathtt{U})$ *is a normal space, but is not a* $\mathtt{C}$*-normal space.*

Definition 4. *Let* $(\mathtt{X}, \Im)$ *be a topology space, where we say that* $\mathtt{X}$ *is a* $\mathtt{C}$–$\mathtt{T}_0$*-space if and only if, given* $\mathtt{x}_1, \mathtt{x}_2 \in \mathtt{X}$*,* $\mathtt{x}_1 \neq \mathtt{x}_2$*, then there is either a* $\mathtt{C}$*-open set containing* $\mathtt{x}_1$ *but not* $\mathtt{x}_2$*, or a* $\mathtt{C}$*-open set containing* $\mathtt{x}_2$ *but not* $\mathtt{x}_1$*.*

Definition 5. *Let* $(\mathtt{X}, \Im)$ *be a topology space, where we say that* $\mathtt{X}$ *is a* $\mathtt{C}$–$\mathtt{T}_1$*-space if and only if, given* $\mathtt{x}_1, \mathtt{x}_2 \in \mathtt{X}$, $\mathtt{x}_1 \neq \mathtt{x}_2$, *then there is two* $\mathtt{C}$*-open subsets* $\mathtt{U}_1$ *and* $\mathtt{U}_2$ *of* $\mathtt{X}$ *such that* $\mathtt{x}_1 \in \mathtt{U}_1$, $\mathtt{x}_2 \notin \mathtt{U}_1$ *and* $\mathtt{x}_1 \notin \mathtt{U}_2$, $\mathtt{x}_2 \in \mathtt{U}_2$.

Definition 6. *Let* $(\mathtt{X}, \Im)$ *be a topology space, where we say that* $\mathtt{X}$ *is a* $\mathtt{C}$–$\mathtt{T}_2$*-space "*$\mathtt{C}$*-Hausdorff space" if and only if, given* $\mathtt{x}_1, \mathtt{x}_2 \in \mathtt{X}$, $\mathtt{x}_1 \neq \mathtt{x}_2$, *then there is two disjoint* $\mathtt{C}$*-open subsets* $\mathtt{U}_1$ *and* $\mathtt{U}_2$ *of* $\mathtt{X}$*, such that* $\mathtt{x}_1 \in \mathtt{U}_1$ *and* $\mathtt{x}_2 \in \mathtt{U}_2$.

Definition 7. *The space* $(\mathtt{X}, \Im)$ *is called a* $\mathtt{C}$–$\mathtt{T}_3$*-space if and only if it is a* $\mathtt{C}$–$\mathtt{T}_1$*-space and* $\mathtt{C}$*-regular. Moreover, the space* $(\mathtt{X}, \Im)$ *is called a* $\mathtt{C}$–$\mathtt{T}_4$*-space if and only if it is a* $\mathtt{C}$–$\mathtt{T}_1$*-space and* $\mathtt{C}$*-normal.*

It is clear from the definitions that any $\mathtt{C}$–$\mathtt{T}_\mathtt{i}$-space is a $\mathtt{T}_\mathtt{i}$-space, for $i \in \{0, 1, 2\}$. But, the converse may not be true as shown by the following examples.

Example 7. *From Example 5, the Niemytzki Plane is a* $\mathtt{T}_3$*-space, but it is not a* $\mathtt{C}$–$\mathtt{T}_0$*-space.*

Theorem 4. *Every* $\mathtt{C}$–$\mathtt{T}_4$*-space is a* $\mathtt{C}$–$\mathtt{T}_3$*-space.*

Proof. Let $(\mathtt{X}, \Im)$ be a $\mathtt{C}$–$\mathtt{T}_4$-space. Then $(\mathtt{X}, \Im)$ is $\mathtt{C}$-normal and a $\mathtt{C}$–$\mathtt{T}_1$-space. Suppose that F is any closed subset of X and x be any element of X with $x \notin F$. Since any $\mathtt{C}$–$\mathtt{T}_1$-space is a $\mathtt{T}_1$-space, then $(\mathtt{X}, \Im)$ is a $\mathtt{T}_1$-space; hence, $\{x\}$ is closed. Set F and $\{x\}$ are closed and disjoint, and by the $\mathtt{C}$-normality of $(\mathtt{X}, \Im)$, there exists two disjoint C-open sets U and V containing F and $\{x\}$, respectively. Therefore, $(\mathtt{X}, \Im)$ is a $\mathtt{C}$–$\mathtt{T}_3$-space. □

From the previous Theorems and Examples, the following diagram is obtained:

$$\begin{array}{ccccccccc}
\mathtt{C}\text{–}\mathtt{T}_4\text{-space} & \rightarrow & \mathtt{C}\text{–}\mathtt{T}_3\text{-space} & \rightarrow & \mathtt{C}\text{–}\mathtt{T}_2\text{-space} & \rightarrow & \mathtt{C}\text{–}\mathtt{T}_1\text{-space} & \rightarrow & \mathtt{C}\text{–}\mathtt{T}_0\text{-space}. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathtt{T}_4\text{-space} & \longrightarrow & \mathtt{T}_3\text{-space} & \longrightarrow & \mathtt{T}_2\text{-space} & \longrightarrow & \mathtt{T}_1\text{-space} & \longrightarrow & \mathtt{T}_0\text{-space}.
\end{array}$$

$Diagram(b)$

None of the above implications are reversible.

Definition 8. *Let* $(\mathtt{X}, \Im)$ *be a topological space, then* $\mathtt{X}$ *is* $\mathtt{C}$*-compact (resp.,* $\mathtt{C}'$*-compact) if and only if any open (resp.,* $\mathtt{C}$*-open) cover of* $\mathtt{X}$ *has a finite subcover of* $\mathtt{C}$*-open (resp., open) sets.*

Theorem 5. *Any* $\mathtt{C}$*-compact space is compact.*

Proof. Since any $\mathtt{C}$-open set is open, then from the definitions, any $\mathtt{C}$-compact space is compact. □

The converse is not always true. Here is an example of a compact space which is not $\mathtt{C}$-compact.

Example 8. *Overlapping Interval Topology [15]. On the set* $\mathtt{X} = [-1, 1]$, *we generate a topology from sets of the form* $[-1, b)$ *for* $b > 0$ *and* $(a, 1]$ *for* $a < 0$*. Then all sets of the form* (a, b) *are also open. Hence,* $\mathtt{X}$ *is a compact space, since in any open covering, the two sets which include* 1 *and* -1 *will cover* $\mathtt{X}$*. The space* $\mathtt{X}$ *is not a* $\mathtt{C}$*-compact space because there exists* $\{[-1, 0.5), (-0.5, 1]\}$*, which is an open cover for* $\mathtt{X}$ *but has no finite subcover of* $\mathtt{C}$*-open sets because* $[-1, 0.5)$ *and* $(-0.5, 1]$ *are not* $\mathtt{C}$*-open sets, (*$cl[-1, 0.5) \backslash [-1, 0.5) = [-1, 1] \backslash [-1, 0.5) = [0.5, 1]$ *is not countable, and* $cl(-0.5, 1] \backslash (-0.5, 1] = [-1, 1] \backslash (-0.5, 1] = [-1, -0.5]$ *is not a countable set).*

Theorem 6. *Any compact space is* $\mathtt{C}'$*-compact.*

Proof. Obviously, from the definitions, any compact space is $\mathtt{C}'$-compact. □

In general, the converse is not true. Here is an example of a $\mathtt{C}'$-compact space which is not compact.

Example 9. *Consider $(\mathbb{R}, \Im_r)$, where $\Im_r$ is the right order topology on $\mathbb{R}$ [15]. Hence, $(\mathbb{R}, \Im_r)$ is not a compact space. However, the family of the $\mathtt{C}$-open set in $(\mathbb{R}, \Im_r)$ is $\{\emptyset, \mathbb{R}\}$ only. Therefore, $(\mathbb{R}, \Im_r)$ is $\mathtt{C}'$-compact.*

From Theorems 5 and 6 and Examples 8 and 9, the following diagram is obtained:

$$\mathtt{C}\text{-compactness} \longrightarrow \text{compactness} \longrightarrow \mathtt{C}'\text{-compactness}.$$
$$Diagram(C)$$

None of the above implications are reversible.

A subset $\mathtt{A}$ of a space $\mathtt{X}$ is $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact) if and only if any open (resp., $\mathtt{C}$-open) cover of $\mathtt{A}$ has a finite subcover of $\mathtt{C}$-open (resp., open) sets. The $\mathtt{C}'$-compactness is not hereditary, for example:

Example 10. *Let $(\mathbb{R}, \Im_2)$ be the included point topological space on $\mathbb{R}$ by 2 (see [15]). Then $(\mathbb{R}, \Im_2)$ is a $\mathtt{C}'$-compact space because $\mathbb{R}$ is a $\mathtt{C}$-open finite subcover for any open cover of $\mathbb{R}$. However, $\mathbb{R}\backslash\{2\}$ is an infinite discrete subspace. Hence, $\mathbb{R}\backslash\{2\}$ with a discrete topology is not $\mathtt{C}'$-compact.*

The $\mathtt{C}'$-compactness is hereditary with respect to a $\mathtt{C}$-closed subspace as shown by the following theorem.

Theorem 7. *If $\mathtt{X}$ is $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact) and $\mathtt{K} \subseteq \mathtt{X}$ is $\mathtt{C}$-closed, then $\mathtt{K}$ is $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact).*

Proof. Let $\Psi = \{\mathtt{V}_\beta : \beta \in \Lambda\}$ be any open (resp., $\mathtt{C}$-open) cover of $\mathtt{K}$, where $\mathtt{V}_\beta \subseteq \mathtt{X}$ is open (resp., $\mathtt{C}$-open) in $\mathtt{X}$ for each $\beta \in \Lambda$. Since $\mathtt{K}$ is $\mathtt{C}$-closed in $\mathtt{X}$, then $\mathtt{X}\backslash\mathtt{K}$ is $\mathtt{C}$-open in $\mathtt{X}$. However, if any $\mathtt{C}$-open set is open, then $\mathtt{X}\backslash\mathtt{K}$ is open in $\mathtt{X}$. Thus, $\{\mathtt{V}_\beta : \beta \in \Lambda\} \cup (\mathtt{X}\backslash\mathtt{K})$ is an open (resp., $\mathtt{C}$-open) cover for $\mathtt{X}$. Since $\mathtt{X}$ is $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact), then there exists $\beta_1, \beta_2, \ldots, \beta_n$ such that $\mathtt{X} \subseteq \bigcup_{i=1}^n \mathtt{V}_i \cup (\mathtt{X}\backslash\mathtt{K})$. Thus, $\mathtt{K} \subseteq \bigcup_{i=1}^n \mathtt{V}_i$. Therefore, $\mathtt{K}$ is $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact). □

Since any $\mathtt{C}$-compact space is compact, then we have the following corollary:

Corollary 3. *If $\mathtt{X}$ is $\mathtt{C}$-compact and $\mathtt{K} \subseteq \mathtt{X}$ is closed, then $\mathtt{K}$ is compact.*

Theorem 8. *A $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact) subset of a $\mathtt{C}$-Hausdorff space is $\mathtt{C}$-closed.*

Proof. Let $\mathtt{A}$ be a $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact) subset of $\mathtt{C}$-Hausdorff space $\mathtt{X}$. If $\mathtt{A} = \mathtt{X}$, we are done because $\mathtt{X}$ is closed and $\mathtt{X}\backslash int(\mathtt{X}) = \emptyset$ is a countable set. Hence, $\mathtt{X}$ is $\mathtt{C}$-closed. Assume that $\mathtt{A} \neq \mathtt{X}$. Pick an arbitrary $\mathtt{x} \in \mathtt{X}\backslash\mathtt{A}, \mathtt{y} \in \mathtt{A}$ and $\mathtt{x} \neq \mathtt{y}$. Then there are two disjoint $\mathtt{C}$-open subsets $\mathtt{U}_\mathtt{y}$ and $\mathtt{V}_\mathtt{x}$ of $\mathtt{X}$ such that $\mathtt{x} \in \mathtt{U}_y$, $\mathtt{y} \in \mathtt{V}_\mathtt{x}$ and $\mathtt{U}_\mathtt{y} \cap \mathtt{V}_\mathtt{x} = \emptyset$. Let $\Psi = \{\mathtt{W}_\mathtt{y} : \mathtt{y} \in \Lambda\}$, then Ψ is an open (resp., $\mathtt{C}$-open) cover of $\mathtt{A}$. Since $\mathtt{A}$ is a $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact) subset of $\mathtt{X}$, then there exists $\mathtt{y}_1, \mathtt{y}_2, \ldots, \mathtt{y}_\mathtt{n}$ such that $\mathtt{A} \subseteq \mathtt{y}_\mathtt{i} \bigcup_{i=1}^n \mathtt{W}_{\mathtt{y}_\mathtt{i}} = \mathtt{W}$. Let $\mathtt{V} = \bigcap_{i=1}^n \mathtt{V}_{\mathtt{x}_\mathtt{i}}$, then by Corollary 2, $\mathtt{V}$ is a $\mathtt{C}$-open subset of $\mathtt{X}$ containing $\mathtt{X}$, clearly showing $\mathtt{V}$ does not intersect $\mathtt{W}$, so that $\mathtt{V} \subseteq \mathtt{X}\backslash\mathtt{A}$; hence, $\mathtt{X}$ is a $\mathtt{C}$-interior point of $\mathtt{X}\backslash\mathtt{A}$, so $\mathtt{X}\backslash\mathtt{A}$ is a $\mathtt{C}$-open subset in $\mathtt{X}$. Therefore, $\mathtt{A}$ is a $\mathtt{C}$-closed subset in $\mathtt{X}$. □

Theorem 9. *Let $(\mathtt{X}, \Im)$ be a $\mathtt{C}$-compact $\mathtt{C}$-Hausdorff space. Then $(\mathtt{X}, \Im)$ is $\mathtt{C}$-regular.*

Proof. Let $\mathtt{x} \in \mathtt{X}$ and let $\mathtt{K} \subseteq \mathtt{X}$ be a $\mathtt{C}$-closed set not containing $\mathtt{x}$. By $\mathtt{C}$-Hausdorffness, for each $\mathtt{a} \in \mathtt{K}$ and $\mathtt{x} \in \mathtt{X}\backslash\mathtt{K}$, there are two disjoint $\mathtt{C}$-open sets $\mathtt{V}_\mathtt{a}$ and $\mathtt{U}_\mathtt{x}$ containing $\mathtt{a}$ and $\mathtt{x}$, respectively. Then $\Psi = \{\mathtt{V}_\mathtt{a} : \mathtt{a} \in \mathtt{K}\}$ is a $\mathtt{C}$-open cover of $\mathtt{K}$. By Theorem 8, $\mathtt{K}$ is $\mathtt{C}$-compact; therefore, there is a $\mathtt{C}$-open finite subcover $\{\mathtt{V}_1, \mathtt{V}_2, \ldots, \mathtt{V}_\mathtt{n}\} \subseteq \Psi$. However,

$\mathtt{V} := \mathtt{V}_{\mathtt{a}_1} \cup \mathtt{V}_{\mathtt{a}_2} \cup \ldots \cup \mathtt{V}_{\mathtt{a}_n}$ and $\mathtt{U} := \mathtt{U}_{\mathtt{a}_1} \cap \mathtt{U}_{\mathtt{a}_2} \cap \ldots \cap \mathtt{U}_{\mathtt{a}_n}$ are disjoint $\mathtt{C}$-open sets containing $\mathtt{a}$ and $\mathtt{x}$, as required. □

Corollary 4. *Let $(\mathtt{X}, \Im)$ be a $\mathtt{C}$-compact $\mathtt{C}$-Hausdorff space. Then $(\mathtt{X}, \Im)$ is a $\mathtt{C}$–$\mathtt{T}_3$-space.*

Theorem 10. *Let $(\mathtt{X}, \Im)$ be a $\mathtt{C}'$-compact $\mathtt{C}$-Hausdorff space. Then $(\mathtt{X}, \Im)$ is regular.*

Proof. The same as Theorem 9. □

Corollary 5. *Let $(\mathtt{X}, \Im)$ be a $\mathtt{C}'$-compact $\mathtt{C}$-Hausdorff space. Then $(\mathtt{X}, \Im)$ is a $\mathtt{T}_3$-space.*

Theorem 11. *Let $(\mathtt{X}, \Im)$ be a $\mathtt{C}$-compact (resp., $\mathtt{C}'$-compact) $\mathtt{C}$-Hausdorff space. Then $(\mathtt{X}, \Im)$ is $\mathtt{C}$-normal (resp., normal).*

Proof. The same as Theorem 9. □

Corollary 6. *Let $(\mathtt{X}, \Im)$ be a $\mathtt{C}$-compact $\mathtt{C}$-Hausdorff space. Then $(\mathtt{X}, \Im)$ is a $\mathtt{C}$–$\mathtt{T}_3$-space.*

Since every $\mathtt{C}$-compact space is compact, the proof of the following theorem is omitted.

Theorem 12. *Suppose that $(\mathtt{X}_\mathtt{i}, \Im_i)$ is a topological space for each $i \in \{1, 2, \ldots, n\}$. The product space $\prod_{i=1}^{n} (\mathtt{X}_\mathtt{i}, \Im_i)$ is $\mathtt{C}$-compact if and only if $(\mathtt{X}_\mathtt{i}, \Im_i)$ is $\mathtt{C}$-compact for each $i \in \{1, 2, \ldots, n\}$.*

4. $\mathtt{C}$-Continuous and $\mathtt{C}'$-Continuous Functions and Other Results

This section presents the definitions of the $\mathtt{C}$-continuous, $\mathtt{C}'$-continuous, $\mathtt{C}$-homeomorphism, and $\mathtt{C}'$-homeomorphism function via the concept of $\mathtt{C}$-open sets and reviews some of their important properties with $\mathtt{C}$-compact and $\mathtt{C}'$-compact spaces.

Definition 9. *A function $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ is said to be $\mathtt{C}$-continuous (resp., $\mathtt{C}'$-continuous) if $\mathtt{h}^{-1}(\mathtt{U})$ is $\mathtt{C}$-open (resp., open) in $\mathtt{X}$ for any open (resp., $\mathtt{C}$-open) subset $\mathtt{U}$ in $\mathtt{Y}$.*

Obviously, from the definitions, every $\mathtt{C}$-continuous function is continuous and every continuous function is $\mathtt{C}'$-continuous. However, the converse may not be true as shown by the following two examples.

Example 11. *Let $(\mathbb{R}, \Im_{\mathbb{I}})$ be the excluded set topological space on $\mathbb{R}$ by $\mathbb{I}$. Then the identity function $\mathtt{I} : (\mathbb{R}, \Im_{\mathbb{I}}) \to (\mathbb{R}, \Im_{\mathbb{I}})$ is a continuous function, which is not $\mathtt{C}$-continuous because $\mathbb{Q} \in \Im_{\mathbb{I}}$ is open, and $\mathtt{I}^{-1}(\mathbb{Q}) = \mathbb{Q}$ is not $\mathtt{C}$-open because $cl(\mathbb{Q}) \backslash \mathbb{Q} = \mathbb{R} \backslash \mathbb{Q} = \mathbb{I}$ is an uncountable set.*

Example 12. *In Example 9, $(\mathbb{R}, \Im_r)$ is the right order topology on the set of all real numbers $\mathbb{R}$. The family of all $\mathtt{C}$-open sets in $(\mathbb{R}, \Im_r)$ is $\{\varnothing, \mathbb{R}\}$ only. Consider $(\mathbb{R}, \mathtt{F}_{cof})$, where $\mathtt{F}_{cof}$ is the finite complement topology on the set of all real numbers $\mathbb{R}$ [15]. Then the identity function $\mathtt{I} : (\mathbb{R}, \mathtt{F}_{cof}) \to (\mathbb{R}, \Im_r)$ is a $\mathtt{C}'$-continuous function, which is not continuous,*

From the definitions of continuous, $\mathtt{C}$-continuous, and $\mathtt{C}'$-continuous functions, and from Examples 11 and 12, the following diagram is obtained:

$$\mathtt{C}\text{-continuity} \longrightarrow \text{continuity} \longrightarrow \mathtt{C}'\text{-continuity}.$$
$$Diagram(d)$$

None of the above implications is reversible.

Definition 10. *Let $(\mathtt{X}, \Im)$ and $(\mathtt{Y}, \Im')$ be two topological spaces and $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$. Then $\mathtt{h}$ is said to be a $\mathtt{C}$-open function if and only if for any $\mathtt{C}$-open subset $\mathtt{U} \subseteq \mathtt{X}$, we have $\mathtt{h}(\mathtt{U})$ be a open subset in $\mathtt{Y}$. Moreover, $\mathtt{h}$ is said to be a $\mathtt{C}$-closed function if and only if for any $\mathtt{C}$-closed subset $\mathtt{V} \subseteq \mathtt{X}$, we have $\mathtt{h}(\mathtt{V})$ be a closed subset in $\mathtt{Y}$.*

Theorem 13. *Let* $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ *be a* C*-continuous (resp.,* C′*-continuous) onto function and* $(\mathtt{X}, \Im)$ *be* C′*-compact, then* Y *is* C′*-compact.*

Proof. Let $\{\mathtt{V}_\alpha : \alpha \in \Lambda\}$ be any C-open cover of Y. Since h is C-continuous (resp., C′-continuous) and any C-open set is open, then $h^{-1}(\mathtt{V}_\alpha)$ is C-open (resp., open) in X for each $\alpha \in \Lambda$. Since $\mathtt{Y} \subseteq \bigcup_{\alpha\in\Lambda} \mathtt{V}_\alpha$, then $\mathtt{X} = \mathtt{h}^{-1}(\mathtt{Y}) \subseteq \mathtt{h}^{-1}(\bigcup_{\alpha\in\Lambda} \mathtt{V}_\alpha) = \bigcup_{\alpha\in\Lambda} h^{-1}(V_\alpha)$; that is, $\{h^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a C-open cover of x. Then, by the C′-compactness of X, there exists $\alpha_1, \alpha_2, \cdots \alpha_n \in \Lambda$ such that $\mathtt{h}^{-1}(\mathtt{V}_{\alpha_1}) \cup \mathtt{h}^{-1}(\mathtt{V}_{\alpha_2}) \cup \cdots \cup \mathtt{h}^{-1}(\mathtt{V}_{\alpha_n}) = \mathtt{X}$, then $\mathtt{h}[\mathtt{h}^{-1}(\mathtt{V}_{\alpha_1}) \cup \mathtt{h}^{-1}(\mathtt{V}_{\alpha_2}) \cup \cdots \cup \mathtt{h}^{-1}(\mathtt{V}_{\alpha_n})] = \mathtt{h}(\mathtt{X})$, then $h(h^{-1}(V_{\alpha_1})) \cup h(h^{-1}(V_{\alpha_2})) \cup \cdots \cup h(\mathtt{h}^{-1}(\mathtt{V}_{\alpha_n})) = \mathtt{Y}$, then $\mathtt{V}_{\alpha_1} \cup \mathtt{V}_{\alpha_2} \cup \cdots \cup \mathtt{V}_{\alpha_n} = \mathtt{Y}$. Hence, $\{\mathtt{V}_{\alpha_1} \cup \mathtt{V}_{\alpha_2} \cup \cdots \cup \mathtt{V}_{\alpha_n}\}$ is a finite subcover of open sets for Y. Therefore, $(\mathtt{Y}, \Im')$ is a C′-compact space. □

Corollary 7. C′*-compactness is a topological property.*

From the previous theorem and the Diagram (c), we have the following corollary:

Corollary 8.
- *(i)* C*-continuous image of compact (resp.,* C*-compact,* C′*-compact) is compact (resp.,* C*-compact,* C′*-compact);*
- *(ii)* C′*-continuous image of compact (resp.,* C*-compact,* C′*-compact) is compact (resp.,* C*-compact,* C′*-compact);*
- *(iii) Continuous image of* C*-compact is* C*-compact.*

Theorem 14. *Let* $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ *be an onto* C*-continuous function and* $(\mathtt{X}, \Im)$ *be* C′*-compact, then* Y *is compact.*

Proof. Same proof as Theorem 13. □

Theorem 15. *Let* $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ *be* C*-continuous,* $(\mathtt{X}, \Im)$ *be* C′*-compact, and* $(\mathtt{Y}, \Im')$ *be a* $\mathtt{C}\text{–}\mathtt{T}_2$*-space, then* h *is a* C*-closed function.*

Proof. Let A be a C-closed subset in X. Since X is C′-compact, then from Theorem 7, A is C′-compact, since the image of a C′-compact space is C′-compact under a C-continuous function (see Corollary 8). Hence, h(A) is C′-compact. Since every C′-compact subspace of a $\mathtt{C}\text{–}\mathtt{T}_2$-space is C-closed (see Theorem 8), this implies that h(A) is C-closed. But, any C-closed set is closed. Therefore, h is a C-closed function. □

From Theorem 15 and Diagram (d), we have the following corollaries:

Corollary 9. *Let* $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ *be* C*-continuous,* $(\mathtt{X}, \Im)$ *be* C*-compact (resp., compact), and* $(\mathtt{Y}, \Im')$ *be a* $\mathtt{C}\text{–}\mathtt{T}_2$*-space, then* h *is a* C*-closed function.*

Corollary 10. *Let* $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ *be continuous,* $(\mathtt{X}, \Im)$ *be* C*-compact, and* $(\mathtt{Y}, \Im')$ *be a* $\mathtt{C}\text{–}\mathtt{T}_2$*-space, then* h *is a* C*-closed function.*

Definition 11. *A bijection function* $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ *is said to be* C*-homeomorphism (resp.,* C′*-homeomorphism) if and only if* h *and* h^{-1} *are* C*-continuous (resp.,* C′*-continuous).*

From the definitions, every C-homeomorphism function is homeomorphism and every homeomorphism function is C′-homeomorphism. However, the converse may not be true as shown by the following two examples.

Example 13. *See Example 11.*

Example 14. *Consider* $(\mathbb{R}, \Im_r)$ *is the right order topology on the set of all real numbers* $\mathbb{R}$ *and* $(\mathbb{R}, \Im_{\mathtt{ID}})$*, where* $\Im_{\mathtt{ID}}$ *is the indiscrete topology on the set of all real numbers* $\mathbb{R}$ *[15]. Then the identity function* $\mathtt{I} : (\mathbb{R}, \Im_{\mathtt{ID}}) \to (\mathbb{R}, \Im_r)$ *is a* $\mathtt{C}'$*-homeomorphism function, which is not continuous because the family of all* $\mathtt{C}$*-open sets in* $(\mathbb{R}, \Im_r)$ *and* $(\mathbb{R}, \Im_{\mathtt{ID}})$ *are* $\{\emptyset, \mathbb{R}\}$ *only.*

Theorem 16. *Let* $(\mathtt{X}, \Im)$ *be a* $\mathtt{C}$*-compact topological space and let* $(\mathtt{Y}, \Im')$ *be a* $\mathtt{C}$*-Hausdorff topological space. Then any* $\mathtt{C}'$*-continuous bijection* $\mathtt{h} : (\mathtt{X}, \Im) \to (\mathtt{Y}, \Im')$ *is a* $\mathtt{C}'$*-homeomorphism.*

Proof. Let $\mathtt{K} \subseteq \mathtt{X}$ be a $\mathtt{C}$-closed set. By Theorem 7, $\mathtt{K}$ is $\mathtt{C}$-compact; therefore, $\mathtt{h}(\mathtt{K})$ is $\mathtt{C}$-compact by Corollary 8. By Theorem 8, we have that $\mathtt{h}(\mathtt{K})$ is $\mathtt{C}$-closed, as required. □

5. Concluding Remarks and Further Work

This article contributes to expanding the literature on new topological properties with a new class of open sets. The obtained results show that some topological properties can be generalized, such as compression, connectivity, and others, to obtain new examples and properties that help to understand topological spaces more broadly.

In this paper, we have displayed the notions of $\mathtt{C}$-open and $\mathtt{C}$-closed sets and discussed their master properties. Then, we introduced some separation axioms of $\mathtt{C}$-open sets. In addition, we have defined the so-called $\mathtt{C}$-compact and $\mathtt{C}'$-compact spaces via $\mathtt{C}$-open sets, and the theorems based on them are discussed with counterexamples. Moreover, we have entitled the $\mathtt{C}$-continuous and $\mathtt{C}'$-continuous functions by applying $\mathtt{C}$-open sets. In particular, several inferred properties of them and their connection with the other topological spaces are studied theoretically. Many examples are given to explain the concepts lucidly. This work considers an auspicious line for future work; for example, we will complete the introduction of the master topological concepts using $\mathtt{C}$-open and $\mathtt{C}$-closed sets such as $\mathtt{C}$-paracompact, $\mathtt{C}$-connected spaces, soft $\mathtt{C}$-open, and soft $\mathtt{C}$-closed sets. Our roadmap for research also consists of the examination of the notions and results that began herein using other results of compactness and continuity in a topological space. Moreover, we will present new kinds of regularity and normality in a topological space using $\mathtt{C}$-open and $\mathtt{C}$-closed sets and apply them to develop the accuracy measures of sets, see [18,19].

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