



Quasihomeomorphisms and Some Topological Properties

Khedidja Dourari ¹, Alaa M. Abd El-latif ², Sami Lazaar ³, Abdelwaheb Mhemdi ⁴ and Tareq M. Al-shami ^{5,*}

¹ Department of Mathematics, Faculty of Exact Sciences, Frères Mentouri University, Constantine 1, Constantine 25000, Algeria; maramdourari@gmail.com

² Mathematics Department, Faculty of Arts and Science, Northern Border University, Rafha 91911, Saudi Arabia; alaa.ali@nbu.edu.sa

³ Department of Mathematics, Faculty of Sciences, University of Taibah, Madina 41311, Saudi Arabia

⁴ Department of Mathematics, College of Sciences and Humanities in Aflaj, Prince Sattam Bin Abdulaziz University, Riyadh 16273, Saudi Arabia; mhemdiabd@gmail.com

⁵ Department of Mathematics, Sana'a University, Sana'a P.O. Box 1247, Yemen

* Correspondence: t.alshami@su.edu.ye or tareqalshami83@gmail.com

Abstract: In this paper, we study the properties of topological spaces preserved by quasihomeomorphisms. Particularly, we show that quasihomeomorphisms preserve Whyburn, weakly Whyburn, submaximal and door properties. Moreover, we offer necessary conditions on continuous map $q : X \rightarrow Y$ where Y is Whyburn (resp., weakly Whyburn) in order to render X Whyburn (resp., weakly Whyburn). Also, we prove that if $q : X \rightarrow Y$ is a one-to-one continuous map and Y is submaximal (resp., door), then X is submaximal (resp., door). Finally, we close this paper by studying the relation between quasihomeomorphisms and k -primal spaces.

Keywords: quasihomeomorphism; Alexandroff space; Whyburn space; submaximal space; primal space; factorisation systems

MSC: 54F05; 54A10; 54D80



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1. Introduction and Preliminaries

The concept of quasihomeomorphism was first introduced by the Grothendieck school [1,2]. This notion is mainly used in algebraic geometry, and it was shown that this concept arises naturally in the theory of some foliations associated to closed connected manifolds [1].

We recall that subset L of topological space X is called locally closed if it is an intersection of an open set and a closed set of X . We take $\mathcal{O}(X)$, $\mathcal{F}(X)$, and $\mathcal{L}(X)$ as the families of all open, closed, and locally closed subsets of X , respectively; we call a continuous map $q : X \rightarrow Y$ a quasihomeomorphism if $A \mapsto q^{-1}(A)$ represents a bijection from $\mathcal{O}(Y)$ (resp., $\mathcal{F}(Y)$, $\mathcal{L}(Y)$) to $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$, $\mathcal{L}(X)$).

Topological space X is called Whyburn [3] if for every non-closed subset A of X and for every $x \in \bar{A} \setminus A$ there exists $B \subseteq A$ such that $\bar{B} \setminus A = \{x\}$ or, equivalently, there exists $B \subseteq A$ such that $\bar{B} \setminus B = \{x\}$. It is called weakly Whyburn [4] if for every non-closed subset A of X there exists $B \subseteq A$ such that $|\bar{B} \setminus A| = 1$ or, equivalently, there exists $B \subseteq A$ such that $|\bar{B} \setminus B| = 1$. It was illustrated that Whyburn space is weakly Whyburn, whereas the converse side is not always true; see [5], Theorem 3.8.

A door space is a topological space in which every subset is either open or closed. By a submaximal space, we mean a topological space in which every subset is locally closed or, equivalently, every dense subset is open.

A principal space, which is also recognized as Alexandroff space, is a topological space in which any intersection of open sets is open. The most fundamental property of Alexandroff spaces is that the category of Alexandroff spaces is isomorphic to the category of qosets. We let $\mathbf{A}X$ denote the category of Alexandroff spaces and \mathbf{Qos} the category of

quasi-ordered sets. We let $\phi : \mathbf{Qos} \rightarrow \mathbf{Alx}$ be the map defined by $\phi((X, \leq)) = (X, \tau_{\leq})$ where τ_{\leq} is the specialization topology defined by choosing, $\mathcal{B} = \{(x \uparrow) \mid x \in X\}$ as a basis of τ_{\leq} , where $(x \uparrow) = \{y \in X; x \leq y\}$ is called the upset determined by x . In this context, the closure $\overline{\{x\}}^{\tau_{\leq}}$ is exactly the downset $(\downarrow x) = \{y \in X : y \leq x\}$. Therefore, it is clear that $\phi((X, \leq))$ is an Alexandroff space. Similarly, we define $\psi : \mathbf{Alx} \rightarrow \mathbf{Qos}$ by $\psi((X, \tau)) = (X, \leq_{\tau})$ where \leq_{τ} , called corresponding specialization quasi-order, is defined by for any $x, y \in X; x \leq_{\tau} y$ if and only if $x \in \overline{\{y\}}^{\tau}$. Clearly, ϕ and ψ are inverse maps one of the other, which means that considering an Alexandroff space is equivalent to considering a qoset.

In this paper, we detect topological properties preserved by quasihomomorphisms. We show that the Whyburn (resp., weakly Whyburn, Submaximal, door) property is preserved by quasihomomorphisms. Furthermore, we lay bare the necessary conditions on continuous map $q : X \rightarrow Y$ where Y is Whyburn (resp., weakly Whyburn) to ensure that X is Whyburn (resp., weakly Whyburn).

This paper is organized as follows. In Section 2, we show that quasihomomorphisms preserve the properties of being Whyburn, weakly Whyburn, submaximal, and door spaces. We devote Section 3 to revealing the necessary conditions on continuous map $q : X \rightarrow Y$ where Y is Whyburn (resp., weakly Whyburn, submaximal, door) in order to assure that X is Whyburn (resp., weakly Whyburn, submaximal, door). Finally, in Section 4, we prove that given a quasihomomorphism, $q : X \rightarrow Y$ between two Alexandroff spaces. Then, if q is an onto (resp., one-to-one) and X (resp., Y) is strongly primal, so is Y (resp., X).

2. Quasihomomorphisms and Some Topological Properties

In this section, we detect topological properties preserved by quasihomomorphisms. Firstly, let us start by recalling the following results.

Lemma 1 ([6]). *We let $q : X \rightarrow Y$ be a continuous onto map. Then, the following statements are equivalent:*

1. q is a quasihomomorphism;
2. q is open and equality $q^{-1}(q(U)) = U$ holds for any open set U in X ;
3. q is closed and equality $q^{-1}(q(F)) = F$ holds for any closed set F in X .

Lemma 2 ([7]). *If $q : X \rightarrow Y$ is a quasihomomorphism, then the following statements are equivalent:*

1. q is onto;
2. $q^{-1}(\overline{A}) = \overline{q^{-1}(A)}$ for any set A in Y .

Theorem 1. *We let $q : X \rightarrow Y$ be an onto quasihomomorphism. If X is Whyburn, then Y is Whyburn.*

Proof. We let A be a non-closed subset of Y and $y \in \overline{A} \setminus A$. Since q is an onto, there exists $x \in X$ such that $y = q(x) \in \overline{A} \setminus A$. It can be seen easily that $x \in q^{-1}(\overline{A}) \setminus q^{-1}(A)$. Using the fact that X is Whyburn, we let B be a subset of $q^{-1}(A)$ that satisfies $\overline{B} \setminus B = \{x\}$. Let us show that $\overline{q(B)} \setminus q(B) = \{y\}$. Since q is continuous and closed, we obtain $q(\overline{B}) = \overline{q(B)}$ and thus $y = q(x) \in \overline{q(B)}$. Now, we suppose that $y = q(x) \in q(B)$; then, $x \in q^{-1}(q(B)) \subseteq q^{-1}(q(q^{-1}(A))) = q^{-1}(A)$ is a contradiction, so that $y \in \overline{q(B)} \setminus q(B)$.

Conversely, let $t' = q(t) \in \overline{q(B)} \setminus q(B)$, then $t \in q^{-1}(\overline{q(B)}) = q^{-1}(q(\overline{B})) = \overline{B}$. Now $t \notin q^{-1}(q(B)) \supseteq B$, so $t \in \overline{B} \setminus B$. Finally $t \in \overline{B} \setminus B = \{x\}$ and consequently $t' = q(t) = q(x) = y$. \square

In the following example, we show that the surjectivity of quasihomomorphism q is necessary to conclude that Y is Whyburn.

Example 1. *We let $X = \{0, 1, 2\}$ be a finite space whose open sets are $\emptyset, X, \{0, 1\}, \{0\}$ and $\{1\}$ and let $Y = \{0, 1, 2, 3\}$ equipped with topology $\{\emptyset, Y, \{0, 1, 3\}, \{1\}, \{0, 3\}\}$. Then, X is a*

Whyburn space. In contrast, $\{3\}$ is not a closed subset of Y and $2 \in \overline{\{3\}} \setminus \{3\} = \{0, 2\}$. We notice that subset B of $\{3\}$ satisfying $\overline{B} \setminus B = \{2\}$ does not exist, so Y is not Whyburn. Now, we let $q : X \rightarrow Y$ be the canonical injection. Clearly, q is a quasihomeomorphism that is not an onto.

The following example shows that if in Theorem 1, q is not a quasihomeomorphism, then Y need not be a Whyburn space.

Example 2. We let $X = \{0, 1, 2, 3, 4\}$, equipped with the topology whose basis of open sets is

$$\{\{3\}, \{4\}, \{0, 2\}, \{1, 3\}\}.$$

We let $Y = \{a, b, c, d\}$ be a space whose basis of open sets is

$$\{\{c\}, \{d\}, \{b, d\}, \{a, b, d\}\}.$$

Then, X is a Whyburn space and Y is not Whyburn. We let $q : X \rightarrow Y$ be defined by $q(0) = q(2) = d, q(1) = a, q(3) = b$ and $q(4) = c$. Hence, q is an onto continuous map and $q^{-1}(O) \neq \{3\}$ whenever O is an open set in Y . Thus, q is not an quasihomeomorphism.

Next, we show that the property weakly Whyburn is preserved by onto quasihomeomorphisms.

Theorem 2. We let $q : X \rightarrow Y$ be an onto quasihomeomorphism. If X is weakly Whyburn, then Y is weakly Whyburn.

Proof. We let A be a non-closed subset of Y . Since q is an onto quasihomeomorphism, then $q^{-1}(A)$ is a non-closed subset of X . Regarding X being weakly Whyburn, we consider set $B \subseteq q^{-1}(A)$ satisfying $|\overline{B} \setminus q^{-1}(A)| = 1$. Hence,

$$\begin{aligned} | \overline{q(B)} \setminus A | &= | q(q^{-1}(\overline{q(B)} \setminus A)) | \\ &= | q(q^{-1}(\overline{q(B)} \setminus A)) | \\ &= | q(q^{-1}(\overline{q(B)}) \setminus q^{-1}(A)) | \\ &= | q(\overline{B} \setminus q^{-1}(A)) | \\ &= 1. \end{aligned}$$

Therefore, Y is weakly Whyburn. \square

Remark 1. (1) The fact that q is a quasihomeomorphism in Theorem 2 is necessary as shown by Example 2. Of course, X is weakly Whyburn. However, Y is not. Indeed, $|\overline{\{d\}} \setminus \{d\}| \neq 1$
 (2) The condition of surjectivity in Theorem 2 is necessary as shown by Example 1.

Theorem 3. We let $q : X \rightarrow Y$ be an onto quasihomeomorphism. If X is submaximal, then Y is submaximal.

Proof. We let B be a dense subset of Y . Then, $q^{-1}(\overline{B}) = q^{-1}(Y) = X$. Since q is an onto quasihomeomorphism, we obtain $\overline{q^{-1}(B)} = X$. Hence, $q^{-1}(B)$ is an open subset of X and $B = q(q^{-1}(B))$ is open in Y . Therefore, Y is a submaximal space. \square

Example 3. We let $X = \{0, 1\}$ be equipped with topology $\{\emptyset, X, \{0\}\}$. Then, the dense subset $\{0\}$ is open in X and thus X is submaximal. Now, we let $Y = \{0, 1, 2\}$ be a finite space whose open sets are \emptyset, Y and $\{0\}$. Then, we have $\{0, 1\}$ as a dense subset of Y and it is not open. We let $q : X \rightarrow Y$ be the canonical injection. It can be seen easily that q is a quasihomeomorphism that is not an onto.

Example 4. We let $X = \{0, 1, 2\}$ be a space whose open sets are $\emptyset, X, \{1, 2\}, \{2\}$ and $\{1\}$. Then, every dense subset of X is open and thus X is submaximal. We let $Y = \{0, 1, 2\}$ be equipped

with topology $\{\emptyset, Y, \{0\}\}$. It can easily be seen that Y is not a submaximal space. Now, we let $q : X \rightarrow Y$ be the identity map. Thus, q is an onto that is not quasihomeomorphism.

Theorem 4. We let $q : X \rightarrow Y$ be an onto quasihomeomorphism. If X is door, then Y is door.

Proof. We let A be a subset of Y . Since X is door, we obtain $q^{-1}(A)$ that is either open or closed. Hence $A = q(q^{-1}(A))$ is either open or closed in Y . \square

Example 5. We let $X = \{1, 2, 3\}$ be a space whose open sets are $\emptyset, X, \{1, 3\}, \{1\}, \{3\}$ and let $Y = \{1, 2, 3, 4\}$ be quipped with topology $\{\emptyset, Y, \{1, 3\}, \{1\}, \{3\}\}$. It can be seen easily that X is door and Y is not a door space. We let $q : X \rightarrow Y$ be the canonical injection. Hence, q is a quasihomeomorphism that is not an onto.

Example 6. We let $X = \{1, 2, 3\}$ be a space whose open sets are $\emptyset, X, \{1, 3\}, \{1\}, \{3\}$ and let $Y = \{1, 2\}$ be equipped with indiscrete topology. Then, X is door and Y is not. We let $q : X \rightarrow Y$ be defined by $q(1) = q(3) = 1$ and $q(2) = 2$. Thus, the continuous map q is an onto but not a quasihomeomorphism.

Remark 2. We remark that in Theorem 4, it is enough to suppose that q is an onto closed continuous map.

3. Inverse Preserved Topological Properties

We let $q : X \rightarrow Y$ be a continuous map. In this section, we are interested in additional conditions to X in order to satisfy properties satisfied by Y . Hence, we introduce the following definition:

Definition 1. Continuous map $q : X \rightarrow Y$ is said to be satisfying the complement closed property if and only if $q^{-1}(A)$ is not closed in X whenever A is not closed in Y .

We recall that Alexandroff spaces are topological spaces in which arbitrary intersections of open sets are open. It is clear that if space X is Alexandroff, then $\overline{A} = \bigcup\{\overline{\{x\}} : x \in A\}$, for every subset A of X .

Theorem 5. We let $q : X \rightarrow Y$ be a one-to-one continuous map from an arbitrary topological space X to a Whyburn space Y . Then, the following properties hold:

- (1) If X is Alexandroff, then it is Whyburn.
- (2) If q satisfies the complement closed property, then it is Whyburn.

Proof. (1) We let A be a non-closed subset of X and $x \in \overline{A} \setminus A$. Then, there exists $y \in A$ such that $y \neq x$ and $x \in \overline{\{y\}}$. We suppose that there exists $z \in X$ such that $z \neq y$, $z \neq x$ and $z \in \overline{\{y\}}$. Then, $q(x) \in \overline{\{q(y)\}}$ and $q(z) \in \overline{\{q(y)\}}$. Since Y is Whyburn, we have $|\overline{\{q(y)\}}| \leq 2$ and so a contradiction. Hence, $\overline{\{y\}} \setminus \{y\} = \{x\}$. Therefore, X is a Whyburn space.

(2) We let A be a non-closed subset of X and $x \in \overline{A} \setminus A$. Then, $q(x) \in \overline{q(A)} \setminus q(A)$. Since Y is a Whyburn space, there exists $B \subseteq q(A)$ such that $\overline{B} \setminus B = \{q(x)\}$. Hence, $q^{-1}(\overline{B}) \setminus q^{-1}(B) = \{x\}$; and since $q^{-1}(B)$ is not a closed subset of X , we obtain $q^{-1}(B) \neq \overline{q^{-1}(B)} \subseteq q^{-1}(\overline{B})$. Thus, $q^{-1}(\overline{B}) \setminus q^{-1}(B) = \{x\}$. Therefore, X is Whyburn. \square

Example 7. We let $X = \{1, 2, 3\}$ and $Y = \{\alpha, \beta\}$ both be equipped with indiscrete topology. Then, X is not Whyburn and Y is a Whyburn space. We let $q : X \rightarrow Y$ be defined by $q(1) = q(3) = \alpha$ and $q(2) = \beta$. It may be checked easily that q is continuous and q is not one-to-one.

Now, we study the weakly Whyburn property.

Theorem 6. We let $q : X \rightarrow Y$ be a one-to-one continuous map from an arbitrary topological space X to a weakly Whyburn space Y . Then, the following properties hold:

- (1) If X is Alexandrov, then it is weakly Whyburn.
- (2) If q satisfies the complement closed property, then it is weakly Whyburn.

Proof. (1) We let A be a non-closed subset of X , then there exists $x \in \overline{A} \setminus A$. Then, $x \in \overline{\{y\}}$, for some $y \in X$ such that $y \neq x$. We suppose that there exists $z \neq x$ and $z \neq y$ such that $z \in \{y\}$. Thus, $q(x), q(z) \in \overline{\{q(y)\}}$. Since Y is weakly Whyburn, we obtain $|\overline{\{q(y)\}} \setminus \{q(y)\}| = 1$ and so a contradiction.
 (2) We let A be a non-closed subset of X ; then, there exists $x \in \overline{A} \setminus A$ and so $q(x) \in \overline{q(A)} \setminus q(A)$. Since Y is weakly Whyburn, there exists $B \subseteq q(A)$ such that $\overline{B} \setminus B = \{z\}$ for some $z \in q(A)$. Hence, $q^{-1}(\overline{B}) \setminus q^{-1}(B) = q^{-1}(z)$. Now, we suppose that $q^{-1}(z) = \emptyset$, then $q^{-1}(\overline{B}) \subseteq q^{-1}(B) \subseteq q^{-1}(z)$, which is a contradiction. Thus, $q^{-1}(\overline{B}) \setminus q^{-1}(B) = \{q^{-1}(z)\}$.
 □

Theorem 7. We let $q : X \rightarrow Y$ be a one-to-one continuous map. Then, the following properties hold:

- (1) If Y is submaximal, then X is submaximal.
- (2) If Y is door, then X is door.

Proof. (1) We let A be a dense subset of X . Then, $q(A)$ is a dense subset of $q(X)$. But it can easily be seen that $q(A) \cup (Y \setminus q(X))$ is a dense subset of Y and since Y is submaximal, $q(A) \cup (Y \setminus q(X))$ is open and thus $A = q^{-1}(q(A) \cup (Y \setminus q(X)))$ is open. Therefore, X is submaximal.
 (2) We let A be a subset of X ; then, $A = q^{-1}(q(A))$, which is open or closed.
 □

Example 8. We let $X = \{0, 1, 2\}$ be a space whose open sets are \emptyset, X and $\{0\}$. We let $Y = \{0, 1\}$ equipped with topology $\{\emptyset, \{0, 1\}, \{0\}\}$. Subset $\{0, 2\}$ is a dense subset of X and it is not open, and thus X is non-submaximal. On the other hand, it can be seen easily that Y is a submaximal space. Now, we let $q : X \rightarrow Y$ be defined by $q(0) = 0, q(1) = q(2) = 1$. Then, q is a continuous map and q is non-one-to-one.

4. Quasihomomorphisms and k -Primal Spaces

Ayatollah Zadeh Shirazi and Golestani [8] on the one hand and Echi [9] on the other hand, working independently, explicitly introduced a class of Alexandroff topologies on X called by Echi primal spaces and by Shirazi and Golestani functional Alexandroff spaces. In this paper, we use the terminology of primal spaces. Given map $f : X \rightarrow X$, we define the Alexandroff topology on X by taking the closure of point $x \in X$ in this topology orbit $\{f^n(x) : n \in \mathbb{N}\}$, where \mathbb{N} is the set of all natural numbers including 0. Therefore, subset A in this topology is closed if and only if it is an f -invariant set, that is, $f(A) \subseteq A$. Equivalently, in the language of qosets, we define (X, \leq_f) by for any $x, y \in X, y \leq_f x$ if and only if $y = f^n(x)$ for $n \in \mathbb{N}$. Such a topology is denoted by $(X, \mathcal{P}(f)) = (X, \leq_f)$.

Topology τ on set X is primal if it is $\mathcal{P}(f)$ for some $f : X \rightarrow X$. Since their recent introduction, functional Alexandroff topologies, were further investigated in [10–17].

The equivalence between Alexandroff spaces and quasi-ordered sets is the motivation to introduce the following definition.

In [9], Echi introduced the notion of primal spaces as follows:

Definition 2. We let X be a non-empty set and f be a map from X to itself. Subset A of X is said f -invariant if $f(A) \subseteq A$. The family of all f -invariant sets forms closed sets of a topology on X called primal topology and denoted by $(X, \mathcal{P}(f))$.

In [11], the authors showed that when giving a quasihomomorphism q from X to Y , if q is onto and X is primal, then Y is primal, and if q is one-to-one and Y is primal, then so is X . Some examples showing the importance of conditions of quasihomomorphism, onto and one-to-one, were given (for more information, see [11,18]).

On the other hand, in [19], the authors introduced the notion of k -primal spaces as follows:

Definition 3. For any non-zero positive integer k and any family $\{f_i : 1 \leq i \leq k\}$ of functions from a given set X to itself, we define the k -primal space $(X, \mathcal{P}(f_1, \dots, f_k)) = (X, \leq_{f_1, \dots, f_k})$ as the intersection of primal spaces $(X, \mathcal{P}(f_i)) = (X, \leq_{f_i})$; $1 \leq i \leq k$. That is, for any $x, y \in X$, $x \leq_{f_1, \dots, f_k} y$ if and only if $x \leq_{f_i} y$ for every $1 \leq i \leq k$.

Hence, the following definition is immediate:

Definition 4. An Alexandroff space is called a strongly primal space if and only if it is a k -primal space for some positive integer k .

The characterization of weakly primal spaces, in finite cases, is given by [19].

Theorem 8. We let (X, \leq) be a finite qoset. Then, (X, \leq) is a k -primal space for some $k \in \mathbb{N}$, $k \neq 0$ if and only if for every cyclic point a and every $x, y \in X$ we have

$$\begin{cases} a \leq y \\ x \leq y \end{cases} \implies a \leq x.$$

In this section, we study the relation between strongly primal spaces and quasihomomorphisms in the class of finite Alexandroff spaces.

Before offering the main result of this section, we need the following lemma:

Lemma 3. We let $q : X \rightarrow Y$ be quasihomomorphisms between two Alexandroff spaces. Then, for every $x, y \in X$, $x \leq y \iff q(x) \leq q(y)$.

Proof. If $x \leq y$, then $x \in \overline{\{y\}}$, which is included in $\overline{\{q(y)\}}$ by continuity of q and thus $q(x) \leq q(y)$.

Conversely, we let $x, y \in X$ such that $q(x) \leq q(y)$. Since q is a quasihomomorphism, there exists a unique closed subset F of Y satisfying $\{y\} = q^{-1}(F)$. Now, using $q(y) \in F$ and $q(x) \leq q(y)$, we obtain $q(x) \in F$ and thus $x \in \{y\} = q^{-1}(F)$ which means that $x \leq y$. \square

Now, we are in a position to cite and prove our main theorem in this section.

Theorem 9. We let $q : X \rightarrow Y$ be quasihomomorphisms between two finite Alexandroff spaces. Then, the following properties hold:

- (1) If q is onto and X is strongly primal, then so is Y .
- (2) If q is one-to-one and Y is strongly primal, then so is X .

Proof. (1) We let b be a cyclic point in Y , x' and y' two points such that $b \leq y'$ and $x' \leq y'$. Since q is an onto, there exist $a, x, y \in X$ with $b = q(a)$, $x' = q(x)$ and $y' = q(y)$. It can easily be seen that a is a cyclic point in X such that $a \leq y$ and $x \leq y$. Now, since X is strongly primal, $a \leq x$ and consequently, $b = q(a) \leq q(x)$ which complete the proof.

(2) Now, we consider cyclic point a in X and $x, y \in X$ with $a \leq y$ and $x \leq y$. Since q is one-to-one and a is cyclic, then $q(a)$ is a cyclic point in Y such that $q(a) \leq q(y)$ and $q(x) \leq q(y)$. Hence, by the fact that Y is strongly primal, we obtain $q(a) \leq q(x)$. Therefore, $a \leq x$ and X is strongly primal.

\square

- Example 9.** (1) We consider $X = \{0, 1\}$ equipped with the indiscrete topology and Y is an infinite set equipped with the indiscrete topology. We let q from X to a Y be a constant map. Clearly, q is a non-onto quasihomomorphism. However, X is primal and thus a strongly primal space but not Y (see [19], Proposition 2.2). This example shows that the surjectivity of q in Theorem 9 is necessary.
- (2) We consider X an infinite set equipped with the indiscrete topology and $Y = \{0, 1\}$ equipped with the indiscrete topology. We let q from X to a Y be a map which associates zero to a fixed point $x \in X$ and one to the complement of $\{x\}$. Clearly, q is a non-one-to-one quasihomomorphism. However, Y is primal and thus a strongly primal space, but not X . This example shows that the injectivity of q in Theorem 9 is necessary.

5. Conclusions

In this paper, we investigated some topological properties that are preserved by surjective quasihomomorphisms such as Whyburn, weakly Whyburn, submaximal and door spaces. On the other hand, we demonstrated the necessary conditions that guarantee to keep these properties under the pre-image continuous. In the end, we discussed the relationships between quasihomomorphisms and k -primal spaces.

As it is well known that topology has been exploited to address many real-life problems via different fields [20–22], we hope the results obtained herein open the way to conduct further studies on both sides theoretical and application. It will be interesting to investigate the current concepts in the frame of soft settings and examine their properties and characterizations.

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References

- Grothendieck, A.; Dieudonné, J.A. *Eléments de Géométrie Algébrique Grundlehren der Mathematischen Wissenschaften*; Springer: New York, NY, USA, 1971; Volume 166.
- Grothendieck, A.; Dieudonné, J.A. *Eléments de Géométrie Algébrique. I. Le langage des Schéémas*; Publications Mathématiques de l'Institut des Hautes Etudes: Bures-sur-Yvette, France, 1960.
- Pultr, A.; Tozzi, A. Equationally closed subframes and representation of quotient spaces. *Cah. Topol. Géom. Différ. Catég.* **1993**, *34*, 167–183.
- Simon, P. On accumulation points. *Cah. Topol. Géom. Différ. Catég.* **1994**, *35*, 321–327.
- Bella, A.; Yaschenko, I.V. On AP and WAP spaces. *Comment. Math. Univ. Carolin* **1999**, *40*, 531–536.
- Echi, O. Quasihomomorphisms, Goldspectral spaces and Jacspectral spaces. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* **2003**, *8*, 489–507.
- Dridi, L.; Lazaar, S.; Turki, T. F -Door spaces and F -Submaximal spaces. *Appl. Gen. Topol.* **2013**, *14*, 97–113. [[CrossRef](#)]
- Shirazi, F.A.Z.; Golestani, N. Functional Alexandroff spaces. *Hacet. J. Math. Stat.* **2011**, *40*, 515–522.
- Echi, O. The categories of flows of Set and Top. *Topol. Its Appl.* **2012**, *159*, 2357–2366. [[CrossRef](#)]
- Echi, O.; Turki, T. Spectral primal spaces. *J. Algebra Appl.* **2019**, *18*, 1950030. [[CrossRef](#)]
- Haouati, A.; Lazaar, S. Primal spaces and quasihomomorphisms. *Appl. Gen. Topol.* **2015**, *16*, 109–118. [[CrossRef](#)]
- Shirazi, F.A.Z.; Golestani, N. More about functional Alexandroff topological spaces. *Sci. Magna.* **2010**, *6*, 64–69.
- Alexandroff, P. Diskrete Räume. *Mat. Sbornik.* **1937**, *44*, 501–519.
- Alexandroff, P. *Combinatorial Topology*; Dover: New York, NY, USA, 2011.
- Alexandroff, P.; Hopf, H. *Topologie, Grundlehren der Mathematischen Wissenschaften*; Springer: Berlin/Heidelberg, Germany, 1935.
- Echi, O. On the product of primal spaces. *Quaest. Math.* **2020**, *45*, 1–10. [[CrossRef](#)]
- Guale, A.; Mejia, F.; Vielm, J. Paths in prinal spaces and the collatz conjecture. *Quaest. Math.* **2020**, *44*, 1485–1491. [[CrossRef](#)]

18. Yip, K.W. Quasi-homeomorphisms and lattice-equivalences of topological spaces. *J. Austral. Math. Soc.* **1972**, *14*, 41–44.
19. Amor, A.B.; Lazaar, S.; Richmond, T.; Sabri, H. k -primal spaces. *Topol. Its Appl.* **2022**, *309*, 107907. [[CrossRef](#)]
20. Al-shami, T.M. Topological approach to generate new rough set models. *Complex Intell. Syst.* **2022**, *8*, 4101–4113. [[CrossRef](#)]
21. Al-shami, T.M.; Mhemdi, A. Approximation operators and accuracy measures of rough sets from an infra-topology view. *Soft Comput.* **2023**, *27*, 1317–1330. [[CrossRef](#)] [[PubMed](#)]
22. Al-shami, T.M.; Alshammari, I. Rough sets models inspired by supra-topology structures. *Artif. Intell. Rev.* **2023**, *56*, 6855–6883. [[CrossRef](#)] [[PubMed](#)]

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