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Measurable Sensitivity for Semi-Flows

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Abstract: Sensitive dependence on initial conditions is a crucial characteristic of chaos. The concept of measurable sensitivity (MS) was introduced as a measure-theoretic version of sensitive dependence on initial conditions. Their research demonstrated that MS arises from light mixing, indicates a finite number of eigenvalues for a transformation, and is not present in the case of infinite measure preservation. Unlike the traditional understanding of sensitivity, MS carries up to account for isomorphism in the sense of measure theory, which ignores the function's behavior on null sets and eliminates dependence on the chosen metric. Inspired by the results of James on MS, this paper generalizes some of the concepts (including MS) that they used in their study of MS for conformal transformations to semi-flows, and generalizes their main results in this regard to semi-flows.

Keywords: semi-flows; measure-preserving transformation; measurable sensitivity

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1. Introduction

Sensitive dependence on initial conditions, introduced by Guckenheimer in [1], is widely recognized as a fundamental concept in chaos theory [2–5]. However, it should be noted that sensitivity is a topological concept rather than a measurable one. For a transformation T on a metric space (X, d), we say that T is sensitive with respect to d if there exists $\delta > 0$ such that for any $\varepsilon > 0$ and any $x \in X$, there exists an integer $n \geq 0$ and a point $y \in B_\varepsilon(x)$ satisfying $d(T^n(x), T^n(y)) > \delta$. The concept of sensitivity has been extensively investigated in the literature [6–12]. The relationship between measure-theoretic notions, such as weak mixing, and sensitive dependence was studied in [13–16]. The notion of strong sensitivity was introduced by [2]. A transformation T on a metric space (X, d) is said to be strongly sensitive with respect to d if there exists a $\delta > 0$ such that, for any given $\varepsilon > 0$ and any point $x \in X$, there is an integer $m \geq 0$ such that $d(T^n(x), T^n(y)) > \delta$ for all integers $n \geq m$ and some $y \in B_\varepsilon(x)$. It is evident that sensitivity and strong sensitivity are topological notions, dependent on both the metric selected and the transformation's behavior on null sets. In their paper [14], the authors introduced the notions of MS and weak measurable sensitivity (wMS) as ergodic-theoretic versions to strong sensitive dependence and sensitive dependence, respectively. Additionally, they provided a sufficient condition for a nonsingular transformation to exhibit wMS (resp., MS), and identified necessary conditions for an ergodic nonsingular and MS transformation. Moreover, they established an ergodic, finite measure-preserving, and MS transformation, demonstrating that MS does not imply weak mixing. Furthermore, they proved that every ergodic infinite measure-preserving transformation cannot be MS (although it can be wMS), implying that MS and wMS are distinct.

Inspired by the Refs. [4,14], we attempt in this paper to generalize the relevant concepts and main results of the Ref. [14] to semi-flows. It is shown that, for a finite measure semi-flow that is both weakly mixing and non-lightly mixing, there are two measurable sets that meet all six properties simultaneously (refer to Theorem 1). Furthermore, it is proven that such a semi-flow is not categorized as MS (as stated in Theorem 2). For an MS semi-flow defined within a Lebesgue space of finite measure, if there exists a positive real number $t \in \mathbb{R}^+$ for which φ_t is ergodic, then there is a positive integer n_t satisfying the existence of n_t invariant sets with positive measures under φ_t. These invariant sets nearly encompass the entirety of the base space and are subject to the lightly mixing constraint imposed by φ_t (refer to Theorem 3). Additionally, this research unveils a class of spaces wherein no finite measure-preserving MS semi-flows exists.

Section 2 demonstrates that a doubly ergodic nonsingular semi-flow exhibits wMS. Moreover, a lightly mixing nonsingular semi-flow, including a mixing finite measure-preserving semi-flow, demonstrates MS. It should be noted that for a semi-flow, the presence of MS does not imply weak mixing. Moving on to Section 3, we establish that if an ergodic nonsingular semi-flow, denoted as φ, showcases measurable sensitivity, then for any $t \geq 0$, the existence of a positive integer $n_t > 0$ guarantees the occurrence of n_t invariant subsets for $\varphi_t^{n_t}$. It further follows that the restriction of $\varphi_t^{n_t}$t on each of these subsets exhibits weak mixing characteristics. The subsequent Section 4 provides proof that an ergodic finite measure-preserving semi-flow, designated as φ, demonstrates MS. Under this condition, for any $t \geq 0$, there exists a $n_t > 0$ where $\varphi_t^{n_t}$ possesses n_t invariant sets of positive measure covering X almost everywhere. Furthermore, the restriction of $\varphi_t^{n_t}$ on each of these sets showcases light mixing attributes. Lastly, in the final section, we establish that an ergodic infinite measure-preserving semi-flow cannot display MS. It is crucial to mention that it can exhibit wMS, as outlined in Section 2.

Here and in the following, all spaces are Lebesgue spaces with a probability or a σ-finite measure defined on them, and all measures are regular. Throughout the paper, we suppose $(X, S(X), \mu)$ is a Lebesgue space X with a positive, finite or σ-finite non-atomic measure μ, and $S(X)$ is the collection of μ-measurable subsets of X. It is well known that any two such spaces are isomorphic under a nonsingular isomorphism [14]. A metric d on X is good if all nonempty open sets have positive measure [14]. When X has a good metric we suppose that the measures defined on X are regular. Let (X, d) be a metric space. For any two nonempty subsets A and B of X, we define $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

Definition 1. *Let φ and ψ be two semi-flows on $(X, S(X), \mu)$ and $(Y, S(Y), \nu)$, respectively. The semi-flow φ is said to be measure-theoretically isomorphic to ψ if for any $t \geq 0$, (X, μ, φ_t) is measure-theoretically isomorphic to (Y, ν, ψ_t).*

Definition 2. *A semi-flow φ on $(X, S(X), \mu)$ is said to be nonsingular if for any $t \geq 0$, (X, μ, φ_t) is nonsingular.*

Definition 3. *A nonsingular semi-flow φ on $(X, S(X), \mu)$ is said to be MS if whenever a semi-flow ψ on $(Y, S(Y), \nu)$ is measure-theoretically isomorphic to φ and d is a good metric on Y, then there exists a $\delta > 0$ such that for all $y \in Y$ and all $\varepsilon > 0$ there exists an $a \geq 0$ such that for all $t \geq a$,*

$$\nu(\{x \in B_\varepsilon(y) : d(\psi_t(x), \psi_t(y)) > \delta\}) > 0.$$

Definition 4. *A nonsingular semi-flow φ on $(X, S(X), \mu)$ is said to be wMS if whenever a semi-flow ψ on $(Y, S(Y), \nu)$ is measure-theoretically isomorphic to φ and d is a good metric on Y, then there exists a $\delta > 0$ such that for all $y \in Y$ and $\varepsilon > 0$ there exists an $a \geq 0$ with*

$$\nu(\{x \in B_\varepsilon(y) : d(\psi_a(x), \psi_a(y)) > \delta\}) > 0.$$

Here, δ will be referred to as a constant of sensitivity.

Proposition 1. *Assume that X is an interval of finite length in R, and d is the standard Euclidean metric on X. If a continuous semi-flow $\varphi : R^+ \times X \to X$ is sensitive with respect to d, then it is strongly sensitive with respect to d.*

Proof. Suppose that φ is sensitive with sensitivity constant δ, and $I_1, I_2, \cdots, I_n$ is disjoint (except at endpoints) intervals with closed or open endpoints which cover X and each has length shorter than $\frac{\delta}{2}$. It is easily seen that each interval of length at least δ must contain one of these intervals. Since φ is sensitive, for each $j \in \{1, 2, \cdots, n\}$ there exists a $t_j \geq 0$ such that $\varphi_{t_j}(I)$ has length at least δ, where I is a nonempty interval in X. Consequently, for any interval I with length at least δ, and any $n \in \{1, 2, \cdots\}$, $\varphi_n(I)$ contains one of $\varphi_t(I_i)$ where $1 \leq i \leq n$ and $0 \leq t \leq t_i$. Let δ' be one third of the minimum of the lengths of these intervals. Then $\delta' > 0$. Since φ is sensitive, for any $x \in X$ and any $\varepsilon > 0$ there exists some t_0 satisfying that $\varphi_{t_0}(B_\varepsilon(x))$ contains an interval of length δ, and consequently, any for $t > t_0$, $\varphi_t(B_\varepsilon(x))$ contains an interval of the length at least $3\delta'$ and hence contains a point whose distance is at least $\frac{3}{2}\delta'$ from $\varphi_t(x)$. So, φ is strongly sensitive with strong sensitivity constant $\frac{3}{2}\delta'$. Thus, the proof is finished. □

2. Extension of Definitions for Semi-Flows

In this section, we initially present several basic concepts and demonstrate that wMS is established by double ergodicity for a nonsingular semi-flow. A nonsingular transformation T on $(X, S(X), \mu)$ is said to be doubly ergodic if for all sets A and B of positive measure there exists an integer $n > 0$ with $\mu(T^n(A) \bigcap A) > 0$ and $\mu(T^n(A) \bigcap B) > 0$. Its corresponding concept to semi-flows is given as follows.

Definition 5. *A nonsingular semi-flow φ on $(X, S(X), \mu)$ is said to be doubly ergodic if for all sets A and B of positive measure there exists a $t_0 \geq 0$ with $\mu(\varphi_{t_0}(A) \bigcap A) > 0$ and $\mu(\varphi_{t_0}(A) \bigcap B) > 0$.*

Definition 6 ([14])**.** *A measure-preserving semi-flow φ on $(X, S(X), \mu)$ is said to be weakly mixing if for any $A, B \in \mathcal{B}(X)$,*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mid \mu(A \bigcap \varphi_s^{-1}(B)) - \mu(A)\mu(B) \mid ds = 0.$$

It can be demonstrated effortlessly that a finite measure-preserving semi-flow φ on $(X, S(X), \mu)$ is considered weakly mixing if and only if $\varphi \times \varphi$ also satisfies this condition.

The reference [14] has established that for a nonsingular transformation, the presence of double ergodicity leads to wMS. Specifically, finite measure-preserving transformations that are weakly mixing exhibit wMS. Correspondingly, a similar result to Proposition 2.1 from [14] can be derived for a nonsingular semi-flow.

Proposition 2. *If φ is a nonsingular, doubly ergodic semi-flow on $(X, S(X), \mu)$, then it is wMS. In particular, a weakly mixing, finite measure-preserving semi-flow on $(X, S(X), \mu)$ is wMS.*

Proof. Assume that ψ is a semi-flow on $(X_1, S(X_1), \mu_1)$ which is measure-theoretically isomorphic to φ, and d is a good metric on X_1. So, by the definition, there exist sets $A, C \subset X_1$ of positive measure with $d(A, C) > 0$. Take $0 < \delta < \frac{d(A,C)}{2}$. Therefore, for any $\varepsilon > 0$ and a fixed $x \in X_1$, by double ergodicity of ψ there exists a $t_0 \geq 0$ with $\mu_1(\psi_{t_0}^{-1}(C) \bigcap B_\varepsilon(x)) > 0$ and $\mu_1(\psi_{t_0}^{-1}(A) \bigcap B_\varepsilon(x)) > 0$. This implies that $\mu_1(\{y \in B_\varepsilon(x) : \psi_{t_0}(y) \in A\}) > 0$ and $\mu_1(\{y \in B_\varepsilon(x) : \psi_{t_0}(y) \in C\}) > 0$. As $\psi_{t_0}(x)$ cannot be within δ of both A and C, $\mu_1(\{y \in B_\varepsilon(x) : d(\psi_{t_0}(x), \psi_{t_0}(y)) > \delta\}) > 0$. By the definition, φ is measurably sensitive. Thus, the proof is finished. □

We now establish the relationship between MS and light mixing. A system (X, μ, T) on a finite measure space is said to be lightly mixing if

$$\liminf_{n\to\infty} \mu(T^{-n}(A) \bigcap B) > 0$$

for any two subsets $A, B \subset X$ of positive measure. Similarly, one can give the following definition.

Definition 7. *A semi-flow φ on a finite measure space $(X, S(X), \mu)$ is said to be lightly mixing if*

$$\liminf_{t\to\infty} \mu(\varphi_t^{-1}(A) \bigcap B) > 0$$

for any two subsets $A, B \subset X$ of positive measure.

In [14] the authors proved that if (X, μ, T) is a nonsingular, lightly mixing dynamical system, then T is MS. However, for semi-flows we have the following result.

Proposition 3. *If φ is a nonsingular, lightly mixing semi-flow on a finite measure space $(X, S(X), \mu)$, then it is MS.*

Proof. Suppose that ψ is a semi-flow on $(X_1, S(X_1), \mu_1)$ which is measure-theoretically isomorphic to φ, and d is a good metric on X_1. Therefore, by the definition, there exist subsets $A, C \subset X_1$ of positive measure with $d(A, C) > 0$. Take $0 < \delta < \frac{d(A,C)}{2}$. Therefore, for any $\varepsilon > 0$ and a fixed $x \in X_1$, by light mixing of ψ there exists a $t_0 \geq 0$ with $\mu_1(\psi_t^{-1}(C) \bigcap B_\varepsilon(x)) > 0$ and $\mu_1(\psi_t^{-1}(A) \bigcap B_\varepsilon(x)) > 0$ for any $t \geq t_0$. This implies that $\mu_1(\{y \in B_\varepsilon(x) : \psi_t(y) \in A\}) > 0$ and $\mu_1(\{y \in B_\varepsilon(x) : \psi_t(y) \in C\}) > 0$ for any $t \geq t_0$. Since $\psi_t(x)$ cannot be within δ of both A and C for any $t \geq t_0$, $\mu_1(\{y \in B_\varepsilon(x) : d(\psi_t(x), \psi_t(y)) > \delta\}) > 0$ for any $t \geq t_0$. By the definition, φ is MS. Thus, the proof is finished. □

Let S be a Lebesgue measurable set of R^+. Its upper and lower densities are defined, respectively, by

$$\bar{d}(S) := \limsup_{t\to\infty} \frac{1}{t} l(S \cap [0, t])$$

and

$$\underline{d}(S) := \liminf_{t\to\infty} \frac{1}{t} l(S \cap [0, t]),$$

where $l(S)$ is the Lebesgue measure of S ([15]), and its density is defined by

$$d(S) := \lim_{t\to\infty} \frac{1}{t} l(S \cap [0, t])$$

and if it exists.

3. Measurable Sensitivity for Semi-Flows and Eigenvalues

In this section, we shall prove that for an ergodic nonsingular semi-flow φ on a measure space $(X, S(X), \mu)$ is MS, if there is some $t \geq 0$ such that φ_t is ergodic then φ_t can have only finitely many eigenvalues. Recall that λ is an (L^∞) eigenvalue of a given transformation T if there is a nonzero a.e. $f \in L^\infty$ such that $f \circ T = \lambda f$ a.e. Also, it is well known that if a given transformation T is ergodic and finite measure-preserving, then its L^2 eigenfunctions are in L^∞; and all (L^∞) eigenvalues of ergodic transformations lie on the unit circle. This is needed to give a further characterization of MS transformations. An eigenvalue is rational if it is of finite order and irrational if it is not [14].

Definition 8. *A semi-flow φ on a measure space $(X, S(X), \mu)$ is said to be ergodic if for any $A, B \in S(X)$,*

$$\frac{1}{t}\int_0^t \mu(\varphi_r^{-1}(A)\bigcap B)dr \to \mu(A)\mu(B).$$

The following lemmas are needed.

Lemma 1. *Let φ be an ergodic nonsingular semi-flow on a measure space $(X, S(X), \mu)$. Assume that for some $t \geq 0$, φ_t is ergodic and has an eigenfunction f_t with an eigenvalue that is of the form $\exp(2\pi i q_t)$ with q_t irrational, with $|f_t| = 1$. Then for any measurable set $A \in S^1$ of positive Haar measure, the backwards orbit of the set $f_t^{-1}(A)$ equals X mod μ.*

Proof. We define $h_t : S^1 \to S^1$ by $h_t(x) = xe^{2\pi i q_t}$ for any $x \in S^1$. Since h_t is an irrational rotation for any $t \geq 0$, for each $t \geq 0$ the pushed measure f_t^{-1} that is invariant under h_t must be the Haar measure. Then, for every $t \geq 0$ we have

$$\bigcup_{n=0}^{\infty}(\varphi_t)^{-n}(f_t^{-1}(A)) = \bigcup_{n=0}^{\infty} f_t^{-1}(h_t^n(A)) = f_t^{-1}(S^1) = X.$$

Thus, the proof is completed. □

Lemma 2. *Let φ be an ergodic nonsingular semi-flow on a measure space $(X, S(X), \mu)$. Assume that for some $t \geq 0$, φ_t is ergodic and has an eigenfunction f_t with an eigenvalue that is of the form $\exp(2\pi i q_t)$ with q_t irrational. Then φ is not MS.*

Proof. By the definition, it is clear that if φ is MS then φ_t is MS for any $t \geq 0$. By Lemma 3.2 in [14] and the definition, the conclusion of Lemma 2 is true. Thus, the proof is complete. □

Lemma 3. *Let φ be a nonsingular semi-flow on a measure space $(X, S(X), \mu)$. If there is some $t \geq 0$ such that φ_t is an ergodic nonsingular transformation on a Lebesgue space X with infinitely many rational eigenvalues, then φ is not MS.*

Proof. By the definition and Lemma 3.3 in [14], φ is not MS. This completes the proof. □

Corollary 1. *For any ergodic, nonsingular, MS semi-flow φ on a measure space $(X, S(X), \mu)$, if there is some $t \geq 0$ such that φ_t is ergodic, then φ_t has finitely many eigenvalues.*

Proof. It follows from the definition and Lemma 3. □

Proposition 4. *For an ergodic, nonsingular and MS semi-flow $\varphi : R^+ \times X \to X$, if there is some $t \geq 0$ such that φ_t is ergodic, then there exists some $n_t \in \mathbb{N}$ such that $\varphi_t^{n_t}$ has n_t invariant subsets and the restriction of $\varphi_t^{n_t}$ to each of these subsets is weakly mixing.*

Proof. By the definition and Proposition 3.5 in [14], the result in Proposition 4 is true. □

Corollary 2. *Let φ be a MS semi-flow on a measure space $(X, S(X), \mu)$. If there is some $t \geq 0$ such that φ_t is totally ergodic, then φ_t is weakly mixing.*

Proof. It follows from the definitions and Corollary 3.6 in [14]. □

4. Measurable Sensitivity for Finite Measure-Preserving Semi-Flows

This section focuses on evaluating the MS of measure-preserving semi-flows on finite measure spaces. It is assumed that the spaces under consideration have a total measure of 1.

Lemma 4. *Suppose $\varphi : R^+ \times X \to X$ is a measure-preserving and not lightly mixing semi-flow. Then there exist sets C_1 and D_1 of positive measure and an infinite, increasing sequence $\{t_k\}_{k=1}^{\infty}$ such that $\varphi_{t_k}(C_1) \bigcap D_1 = \emptyset$ for all $k \in \mathbb{N}$.*

Proof. By the definition of lightly mixing, there are sets C and D of positive measure with

$$\liminf_{t\to\infty} \mu(C \bigcap \varphi_t^{-1}(D)) = 0.$$

So, we can choose an increasing sequence of distinct real numbers $\{t_k\}_{k=1}^{\infty}$ such that

$$\mu(C \bigcap \varphi_{t_k}^{-1}(D)) \leq 2^{-k-1}\mu(C)$$

for each $k \geq 1$. Let

$$C_1 = C \backslash \bigcup_{k=1}^{\infty} (C \bigcap \varphi_{t_k}^{-1}(D))$$

and $D_1 = D$. Then we have $\mu(C_1) > \frac{1}{2}\mu(C) > 0$ and $\varphi_{t_k}(C_1) \bigcap D_1 = \emptyset$ for every $k \geq 1$. □

Theorem 1. *Let $\varphi : R^+ \times X \to X$ be a finite measure-preserving, weakly mixing and not lightly mixing semi-flow. Then there exist sequences of measurable sets $\{C_i\}_{i=1}^{\infty}$ and $\{D_i\}_{i=1}^{\infty}$ satisfying the following properties.*

(1) $\mu(C_i) > 0$ *and* $\mu(D_i) > 0$ *for* $i \geq 1$;
(2) $C_i \subset C_{i-1}$ *for* $i > 1$;
(3) $D_{i-1} \subset D_i$ *for* $i > 1$;
(4) $\lim_{i\to\infty} \mu(D_i) = 1$;
(5) $\lim_{i\to\infty} \mu(C_i) = 0$;
(6) *There is a sequence* $\{t_k\}_{k=1}^{\infty}$ *such that* $\varphi_{t_k}(C_i) \bigcap D_i = \emptyset$ *for* $i \geq 1$ *and* $k \geq 1$.

Proof. Let C_1, D_1, and $\{t_k\}_{k=1}^{\infty}$ be as defined in Lemma 4. It is evident that they clearly satisfy properties 1–3 and property 6. For the inductive step, we assume that C_i and D_i have been chosen to satisfy properties 1-6 for all $i \leq j$. By the definition of weak mixing, there is a zero density subset $E_1 \subset R^+$ such that

$$\lim_{t\to\infty, t\notin E_1} \mu(\varphi_t^{-1}(C_j) \bigcap C_j) = (\mu(C_j))^2.$$

So, the set $\{t \in R^+ : \mu(\varphi_t^{-1}(C_j) \bigcap C_j) > 0\}$ has density 1. Similarly, there exists a zero density subset $E_2 \subset R^+$ such that

$$\lim_{t\to\infty, t\notin E_1} \mu(\varphi_t^{-1}(D_j) \bigcap D_j) = (\mu(D_j))^2.$$

Hence, there is a $r_j \in R^+$ with $\mu(\varphi_{r_j}^{-1}(C_j) \bigcap C_j) > 0$ and

$$\mu(\varphi_{r_j}^{-1}(D_j) \bigcap D_j) < \frac{1}{2}((\mu(D_j))^2 + \mu(D_j)).$$

This implies that

$$\mu(\varphi_{r_j}^{-1}(D_j) \backslash D_j) > \frac{1}{2}(\mu(D_j) - (\mu(D_j))^2).$$

For integers $j \geq 1$, let $C_{j+1} = \varphi_{r_j}^{-1}(C_j) \bigcap C_j$ and $D_{j+1} = D_j \bigcup \varphi_{r_j}^{-1}(D_j)$. By the definitions of C_{j+1} and D_{j+1} properties 2 and 3 are satisfied. Property 1 for D_{j+1} follows from the fact that D_1 has positive measure and $D_i \subset D_{i+1}$ for each $1 \leq i \leq j$. Property 1 for C_{j+1} follows from the fact that t_j satisfies $\mu(C_{j+1}) > 0$. Applying Lemma 4.2 in [10] to the function

$f(x) = \frac{3}{2}x - \frac{1}{2}x^2$ and by the lower bound for the measure of D_{i+1} in terms of D_i, property 4 holds. To prove property 6, it is enough to see that

$$\varphi_{t_k}(\varphi_{r_j}^{-1}(C_j) \bigcap C_j) \bigcap (D_j \bigcup \varphi_{r_j}^{-1}(D_j)) \subset \varphi_{r_j}^{-1}(\varphi_{t_k}(C_j) \bigcap D_j) \bigcup (\varphi_{t_k}(C_j) \bigcap D_j) = \emptyset.$$

From property 4 and the fact that $\varphi_{t_k}^{-1}(D_i)$ and C_i must be disjoint, we know that property 5 is true. Thus, the entire proof is complete. □

Theorem 2. *If $\varphi : R^+ \times X \to X$ is a weakly mixing, finite measure-preserving and not lightly mixing semi-flow, then it is not MS.*

Proof. Let $C_0 = X$ and $D_0 = \emptyset$ and C_i and D_i be as in Theorem 1. Then the space X can be decomposed as

$$X = \left(\bigcup_{i,j=0}^{\infty} (C_i \backslash C_{i+1}) \bigcap (D_{j+1} \backslash D_j) \right) \quad mod\ \mu.$$

Let $g_{i,j}$ be a nonsingular isomorphism from $(C_i \backslash C_{i+1}) \bigcap (D_{j+1} \backslash D_j)$ to $(2^{-j} + 2^{-i-j-1}, 2^{-j} + 2^{-i-j})$ with Lebesgue measure whenever $(C_i \backslash C_{i+1}) \bigcap (D_{j+1} \backslash D_j)$ has positive measure. Let N denote the backwards orbit of the points where no $g_{i,j}$ is defined. Then this set has measure zero. Therefore, by the definition the restriction of φ to $R^+ \times (X \backslash N)$ is isomorphic to φ. Let φ' denote this restriction. Define a function g on $X \backslash N$ by $g(x) = g_{i,j}(x)$ for any $x \in (X \backslash N)$. Let $d(x, y) = |g(x) - g(y)|$ for any $x \in (X \backslash N)$. Then d is a metric on $X \backslash N$. Since each of the maps $g_{i,j}$ is nonsingular, every ball around a point $x \in (X \backslash N)$ must have positive measure. Hence the metric d is good. It is clear that $g(D_j) \subset (2^{-j}, 1)$ and $g(X \backslash D_j) \subset (0, 2^{-j})$. Take $x \in (C_i \backslash N) \bigcap D_j$ for some j where such a point exists, and choose $\varepsilon > 0$ such that $B_\varepsilon^d(x) \subset C_i \bigcup D_j$. By property 6 of Theorem 1, there is a sequence $\{t_k\}_{k=1}^{\infty} \subset R^+$ with

$$g(\varphi'_{t_k}(B_\varepsilon^d(x))) \subset g^{-1}((0, 2^{-i})).$$

As any sensitivity constant $\delta > 0$ must be smaller than 2^{-i} for each integer $i \geq 1$, there is no possible sensitivity constant. So, φ' does not exhibit strong sensitive dependence for any good metric d. Consequently, φ is not MS. Thus, the proof is finished. □

Lemma 5. *Suppose $\varphi : R^+ \times X \to X$ is a MS semi-flow. If there is a $t \in R^+$ such that $\varphi_t : X \to X$ has finitely many invariant subsets of positive measure $A_1, A_2, \cdots, A_{n_t}$, then the restriction of φ_t to each subset is MS.*

Proof. It is clear that if $\varphi : R^+ \times X \to X$ is MS then for any $t \in R^+$ $\varphi_t : X \to X$ is MS by the definitions. By Lemma 4.5 in [14], the result in Lemma 5 holds. □

Theorem 3. *Suppose $\varphi : R^+ \times X \to X$ is a MS semi-flow on a finite measure Lebesgue space X. If there is a $t \in R^+$ such that $\varphi_t : X \to X$ is ergodic, there is some integer n_t such that φ_t has n_t invariant sets of positive measure which cover almost all of X, and the restriction of φ_t to each of the sets is lightly mixing.*

Proof. Since $\varphi : R^+ \times X \to X$ is MS, for any $t \in R^+$ $\varphi_t : X \to X$ is MS by the definitions. By Theorem 4.6 in [14], the result in Theorem 3 holds. □

5. Semi-Flows on Infinite Measure Spacess

There is a lack of corresponding light mixing semi-flow for the infinite measure-preserving case, although the existence of lightly mixing finite measure-preserving semi-flows implies the existence of finite measure-preserving, MS semi-flows. The authors have demonstrated the absence of ergodic, infinite measure-preserving, MS transformations. In a similar vein, we present the following conclusion.

Theorem 4. *Let X be a σ-finite measure space with infinite measure, and let $X = \bigcup_{i=1}^{\infty} A_i$ where each set A_i has positive finite measure. Then there are no infinite measure-preserving, MS semi-flows on the space X.*

Proof. Let φ be a measure-preserving semi-flow on the σ-finite measure space X with infinite measure and $D_i = \bigcup_{j=1}^{i} A_j$. Since both A_{i+1} and D_i have finite measure,

$$\liminf_{t\to\infty} \mu(\varphi_t^{-1}(D_i) \bigcap A_{i+1}) = 0.$$

So, we can choose an increasing sequence $\{t_{i,k}\}_{k=1}^{\infty} \subset R^+$ with

$$\mu(\varphi_{t_{i,k}}^{-1}(D_i) \bigcap A_{i+1}) < 2^{-k-1}\mu(A_{i+1}).$$

Let

$$C_i = A_{i+1} \backslash \left(\bigcup_{k=1}^{\infty} \varphi_{t_{i,k}}^{-1}(D_i) \right).$$

Then the set C_i has positive measure and satisfies that $C_i \subset D_{i+1}$ and $\varphi_{t_{i,k}}(C_i) \bigcap D_i = \emptyset$ for every $k \geq 1$. Let $g_i : C_i \to (2^{-2i}, 2^{-2i+1})$ be a nonsingular isomorphism with Lebesgue measure and $h_i : A_{i+1} \backslash C_i \to (2^{-2i+1}, 2^{-2i+2})$ a nonsingular isomorphism with Lebesgue measure whenever $A_{i+1} \backslash C_i$ has positive measure. Let N be the set where none of the functions h_i and g_i are defined as well as their backwards orbits. Then this set must have measure zero due to the nonsingularity of φ. So, by the definition of φ', the restriction of φ to $R^+ \times (X \backslash N)$, is measurably isomorphic to φ. Let $g : X \backslash N \to (0, 1)$ be equal to whichever of h_i and g_i is defined. It is obvious that $g(D_i) \subset (0, 2^{-2i})$. We define metric d on $X \backslash N$ by $d(x, y) = |g(x) - g(y)|$ for any $x, y \in (X \backslash N)$. Since each isomorphism is nonsingular, every ball in metric d around any point in $X \backslash N$ must have positive measure. Therefore, this metric is good. Let $x \in (C_i \backslash N)$. Then, for sufficiently small $\varepsilon > 0$, we have $B_\varepsilon^d(x) \subset C_i$. So, we obtain $\varphi'_{t_{i,k}}(B_\varepsilon^d(x)) \subset (X \backslash D_i) \backslash N$. Since any two points in $(X \backslash D_i) \backslash N$ have a maximum distance of 2^{-2i+1} between them, every MS constant must be at most 2^{-2i+1}. Consequently, φ' is not MS in this metric. This means that φ is not MS as the metric is good. Thus, the proof is completed. □

6. Conclusions

In this paper, we present the notion of MS in relation to a semi-flow and obtain the semi-flow version of the relevant findings outlined in [14]. Specifically, for a finite measurable weak Mixing and non lightly mixing semi-flow, there exist two measurable sets that satisfy six properties simultaneously (see Theorem 1), and such a semiflow is not MS (see Theorem 2). For an MS semi-flow defined in a finite measure Lebesgue space, if there exists $t \in \mathbb{R}^+$ such that φ_t is ergodic, then there exists a positive integer n_t such that φ_t has n_t invariant sets with positive measures, which almost cover the entire base space, and the constraint of φ_t on these invariant sets is lightly mixing (see Theorem 3). In addition, this research presents a class of spaces on which there does not exist any finite measure-preserving MS semi-flows. Our future research will be devoted to exploring the MS as well as other dynamic characteristics of certain fuzzy sets and systems. To familiarize readers with the concepts and symbols used in fuzzy sets and systems, please refer to [17,18]. We will also explore the application of MS to some important mathematical models, e.g., on the model in [19,20].

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