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Well-Posedness and Stability Results for Lord Shulman Swelling Porous Thermo-Elastic Soils with Microtemperature and Distributed Delay

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Abstract: The Lord Shulman swelling porous thermo-elastic soil system with the effects of microtemperature, temperatures and distributed delay terms is considered in this study. The well-posedness result is established by the Lumer–Phillips corollary applied to the Hille–Yosida theorem. The exponential stability result is proven by the energy method under suitable assumptions.

Keywords: Lord Shulman; mathematical operators; swelling porous system; partial differential equations; general decay; distributed delay term

MSC: 35B40; 35L70; 74D05; 93D20



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1. Introduction and Preliminaries

The first theory that included a viscous liquid, solid and gas mixture was proposed by Eringen [1]. The field equations were obtained by investigating this heat-resistant combination [2]. The porous media theory, which investigates this type of issue, has also been used to classify expansive (swelling) soils. Due to numerous investigations aimed at mitigating the adverse effects of expansive soils, especially within the fields of architecture and civil engineering, this subject appears promising for further research exploration. For additional information, visit [3–9]. From the linear theory of swelling porous elastic soils, the fundamental field equations are

$$\begin{aligned}\rho_u u_{tt} &= P_{1x} + G_1 + \mathcal{H}_1, \\ \rho_\vartheta \vartheta_{tt} &= P_{2x} + G_2 + \mathcal{H}_2,\end{aligned}\quad (1)$$

in which $\rho_\vartheta, \rho_u > 0$ are the densities of the elastic solid material and fluid, while their respective displacements are denoted by u, ϑ . Furthermore, $(P_1, G_1, \mathcal{H}_1)$ represent the partial tension, internal forces of body and external forces acting on the displacement. $(P_2, G_2, \mathcal{H}_2)$ are similar, but applied to the elastic solid. Also, the constitutive equations for partial tensions are provided by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1, a_2 \\ a_2, a_3 \end{pmatrix}}_A \cdot \begin{pmatrix} u_x \\ \vartheta_x \end{pmatrix}, \quad (2)$$

where $a_1, a_3 > 0$, and $a_2 \neq 0$ is a real number. A is matrix positive definite with $a_1 a_3 > a_2^2$. Quintanilla [9] studied (1) by considering

$$G_1 = G_2 = \xi(u_t - \vartheta_t), \quad \mathcal{H}_1 = a_3 u_{xxt}, \quad \mathcal{H}_2 = 0,$$

where $\xi > 0$; the exponential stability can be achieved. Also, in [10], the researchers considered (1) by taking different conditions:

$$G_1 = G_2 = 0, \quad \mathcal{H}_1 = -\rho_u \gamma(x) u_t, \quad \mathcal{H}_2 = 0,$$

where the internal viscous damping function $\gamma(x)$ has a positive mean. They were able to determine the exponential stability using the spectral approach. To discover more, read [9–16]. Time delays are of significant importance in the majority of natural phenomena and industrial systems, as they have the potential to induce instability and should be treated with utmost consideration. Additionally, there are numerous works that have examined this category of issues, including [17–23].

Numerous researchers worked on similar problems in the literature from different perspectives [24–28]. In recent times, there has been a substantial surge of interest among scientists in Lord Shulman’s thermo-elasticity, leading to an extensive collection of contributions aimed at elucidating this theory. This theoretical framework encompasses the examination of a system comprising four hyperbolic equations coupled with heat transfer dynamics. Moreover, Lord Shulman thermo-elastic theory was introduced to usher in a more robust heat conduction law, as it concerns thermo-elastic materials exhibiting elastic vibrations. Notably, the heat equation within this context is itself hyperbolic and parallels the equation initially formulated by Fourier’s law. To delve deeper into the specifics and gain a comprehensive understanding of this theory, it is recommended that the reader consult the following papers: [29,30]. The core evolutionary equations governing one-dimensional models of porous thermo-elasticity, incorporating both microtemperature and temperature effects [31–34], can be expressed as follows:

$$\begin{aligned} \rho_u u_{tt} &= T_x, \\ \rho_\varphi \vartheta_{tt} &= \mathcal{H}_x + G, \\ \rho T_0 \eta_t &= q_x, \\ \rho E_t &= P_x^* + q - Q. \end{aligned} \tag{3}$$

In the context provided, the symbols $T, T_0, \mathcal{H}, E, \eta, q, G, Q$ and P^* denote the stress, reference temperature, equilibrated stress, first energy moment, entropy, heat flux vector, equilibrated body force, mean heat flux and first heat flux moment, respectively. For simplicity in computations, we set T_0 to be equal to 1.

This paper addresses the inherent counterpart of microtemperatures within the Lord Shulman theory. In this scenario, it becomes possible to adapt the constitutive equations in the subsequent manner:

$$\begin{aligned} T &= P_1 + G_1 + \mathcal{H}_1 & P^* &= -k_2 \mathfrak{R}_x, \\ \mathcal{H} &= P_2 + P_3 & \rho \eta &= \gamma_0 u_x + \gamma_1 \vartheta + \beta_1 (\kappa \theta_t + \theta), \\ G &= G_2 + \mathcal{H}_2 & Q &= (k_1 - k_3) \mathfrak{R} + (k - k_1) \theta_x, \\ q &= k \theta_x + k_1 \mathfrak{R} & \rho E &= -\beta_2 (\kappa \mathfrak{R}_t + \mathfrak{R}) - \gamma_2 \vartheta_x, \end{aligned} \tag{4}$$

in which the microtemperature vector is indicated by \mathfrak{R} , $\kappa > 0$ is the relaxation parameter and $\rho_u, \rho_\vartheta, a_1, a_2, a_3, \beta_1, \beta_2 > 0$. The coefficients γ_0, k, γ_1 denote the coupling between the temperature and displacement, the thermal conductivity, the coupling between the volume fraction and the temperature, respectively.

Taking $a_2 \neq 0$ and the coefficients $k_1, k_2, k_3, \gamma_2 > 0$ satisfies the inequalities

$$a = a_3 - \frac{a_2^2}{a_1} > 0, \tag{5}$$

$$k_1^2 < kk_3. \tag{6}$$

In the current work, we focus on the thermal effects, which is why we make the assumption $\beta_1, \beta_2 > 0$ for heat capacity. To add interest to the problem, we also add a distributed delay term to the second equation, creating a new case that differs from earlier research. Under the right assumptions, the system is shown to be well posed, and we use the energy method to demonstrate the result of the exponential stability.

In this work, the following are taken into account:

$$\begin{aligned} G_1 = G_2 = 0, \quad P_3 = -\gamma_2(\kappa\mathfrak{R}_t + \mathfrak{R}), \\ \mathcal{H}_1 = -\gamma_0(\kappa\theta_t + \theta), \\ \mathcal{H}_2 = \gamma_1(\kappa\theta_t + \theta) - \omega_1\vartheta_t - \int_{v_1}^{v_2} \omega_2(s)\vartheta_t(x, t - s)ds. \end{aligned} \tag{7}$$

By substituting (4)–(7) into (3), we have

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \vartheta_{xx} + \gamma_0(\kappa\theta_t + \theta)_x = 0, \\ \rho_\vartheta \vartheta_{tt} - a_3 \vartheta_{xx} - a_2 u_{xx} - \gamma_1(\kappa\theta_t + \theta) + \gamma_2(\kappa\mathfrak{R}_t + \mathfrak{R})_x \\ \quad + \omega_1 \vartheta_t + \int_{v_1}^{v_2} \omega_2(s)\vartheta_t(x, t - s)ds = 0, \\ \beta_1(\kappa\theta_t + \theta)_t + \gamma_0 u_{tx} + \gamma_1 \vartheta_t - k\theta_{xx} - k_1 \mathfrak{R}_x = 0, \\ \beta_2(\kappa\mathfrak{R}_t + \mathfrak{R})_t - k_2 \mathfrak{R}_{xx} + \gamma_2 \vartheta_{tx} + k_3 \mathfrak{R} + k_1 \theta_x = 0, \end{cases}$$

where

$$(x, s, t) \in \mathbb{H} = (0, 1) \times (v_1, v_2) \times (0, \infty),$$

and

$$\begin{aligned} u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), \\ \vartheta(x, 0) = \vartheta_0(x), \vartheta_t(x, 0) = \vartheta_1(x), \theta_t(x, 0) = \theta_1(x), \\ \mathfrak{R}(x, 0) = \mathfrak{R}_0(x), \mathfrak{R}_t(x, 0) = \mathfrak{R}_1(x), \quad x \in (0, 1), \\ u(0, t) = u(1, t) = \vartheta(0, t) = \vartheta(1, t) = 0, \quad 0 \leq t, \\ \vartheta_t(x, -t) = f_0(x, t), \quad (x, t) \in (0, 1) \times (0, v_2), \\ \theta(0, t) = \theta(1, t) = \mathfrak{R}(0, t) = \mathfrak{R}(1, t) = 0, \quad 0 \leq t. \end{aligned} \tag{8}$$

Next, we introduce a new variable, as mentioned in [23]:

$$\mathcal{Y}(x, \rho, s, t) = \vartheta_t(x, t - s\rho),$$

thus, the following is obtained:

$$\begin{cases} s\mathcal{Y}_t(x, \rho, s, t) + \mathcal{Y}_\rho(x, \rho, s, t) = 0, \\ \mathcal{Y}(x, 0, s, t) = \vartheta_t(x, t). \end{cases}$$

Our problem can be expressed in the following form:

$$\begin{cases} \rho_u u_{tt} - a_1 u_{xx} - a_2 \vartheta_{xx} + \gamma_0(\kappa\theta_t + \theta)_x = 0, \\ \rho_\vartheta \vartheta_{tt} - a_3 \vartheta_{xx} - a_2 u_{xx} - \gamma_1(\kappa\theta_t + \theta) + \gamma_2(\kappa\mathfrak{R}_t + \mathfrak{R})_x \\ \quad + \omega_1 \vartheta_t + \int_{v_1}^{v_2} \omega_2(s)\mathcal{Y}(x, 1, s, t)ds = 0, \\ \beta_1(\kappa\theta_t + \theta)_t + \gamma_0 u_{tx} + \gamma_1 \vartheta_t - k\theta_{xx} - k_1 \mathfrak{R}_x = 0, \\ \beta_2(\kappa\mathfrak{R}_t + \mathfrak{R})_t - k_2 \mathfrak{R}_{xx} + \gamma_2 \vartheta_{tx} + k_3 \mathfrak{R} + k_1 \theta_x = 0, \\ s\mathcal{Y}_t(x, \rho, s, t) + \mathcal{Y}_\rho(x, \rho, s, t) = 0, \end{cases} \tag{9}$$

in which

$$(x, \rho, s, t) \in (0, 1) \times \mathbb{H},$$

with

$$\begin{cases} u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), \\ \vartheta(x, 0) = \vartheta_0(x), \vartheta_t(x, 0) = \vartheta_1(x), \theta_t(x, 0) = \theta_1(x), \\ \mathfrak{R}(x, 0) = \mathfrak{R}_0(x), \mathfrak{R}_t(x, 0) = \mathfrak{R}_1(x), \quad x \in (0, 1), \\ \mathcal{Y}(x, \rho, s, 0) = f_0(x, s\rho), \quad (x, \rho, s) \in (0, 1) \times (0, 1) \times (0, \nu_2), \end{cases} \tag{10}$$

and

$$\begin{aligned} u(0, t) = u(1, t) = \vartheta(0, t) = \vartheta(1, t) = 0, \\ \theta(0, t) = \theta(1, t) = \mathfrak{R}(0, t) = \mathfrak{R}(1, t) = 0, \quad 0 \leq t. \end{aligned} \tag{11}$$

In this context, the integrals denote the presence of distributed delay components, where ν_1 and ν_2 —both greater than zero—represent time delays. The functions ω_1 and ω_2 are L^∞ functions and must adhere to the following conditions.

Hypothesis 1. $\omega_2 : [\nu_1, \nu_2] \rightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\nu_1}^{\nu_2} |\omega_2(s)| ds < \omega_1. \tag{12}$$

In this investigation, we delve into the realm of the Lord Shulman model for swelling porous thermo-elastic soils, incorporating the influence of microtemperature, temperatures and distributed delay components. Our focus lies in demonstrating the system’s well-posedness and examining the outcomes related to its exponential stability. This work is structured as follows: in Section 2, the well-posedness is illustrated, and the exponential stability is demonstrated in Section 3. We state that $c > 0$ in each of the sentences that follow.

2. Well-Posedness

Here, we will establish the well-posedness of the system (9)–(11). The following vector function is first introduced:

$$X = (u, u_t, \vartheta, \vartheta_t, \theta, \theta_t, \mathfrak{R}, \mathfrak{R}_t, \mathcal{Y})^T,$$

where variables $v = u_t, \varphi = \vartheta_t, \chi = \theta_t, \Sigma = \mathfrak{R}_t$; then, the system (9) is written as follows:

$$\begin{cases} X_t = \mathcal{T}X \\ X(0) = X_0 = (u_0, u_1, \vartheta_0, \vartheta_1, \theta_0, \theta_1, \mathfrak{R}_0, \mathfrak{R}_1, f_0)^T, \end{cases} \tag{13}$$

where $\mathcal{T} : \mathcal{S}(\mathcal{T}) \subset \mathfrak{V} \rightarrow \mathfrak{V}$ is a linear operator given by

$$\mathcal{T}X = \begin{pmatrix} v \\ \frac{1}{\rho_u} [a_1 u_{xx} + a_2 \vartheta_{xx} - \gamma_0 (\kappa \chi + \theta)_x] \\ \varphi \\ \frac{1}{\rho_\vartheta} [a_3 \vartheta_{xx} + a_2 u_{xx} + \gamma_1 (\kappa \chi + \theta) - \gamma_2 (\kappa \Sigma + \mathfrak{R})_x \\ - \omega_1 \varphi - \int_{\nu_1}^{\nu_2} \omega_2(s) \mathcal{Y}(x, 1, s, t) ds] \\ \chi \\ -\frac{1}{\kappa \beta_1} [\gamma_0 v_x + \gamma_1 \varphi - k \theta_{xx} + \beta_1 \chi - k_1 \mathfrak{R}_x] \\ \Sigma \\ -\frac{1}{\kappa \beta_2} [-k_2 \mathfrak{R}_{xx} + \gamma_2 \varphi_x + k_3 \mathfrak{R} + k_1 \theta_x + \beta_2 \Sigma] \\ -\frac{1}{s} \mathcal{Y}_\rho \end{pmatrix}, \tag{14}$$

in which energy space is denoted by \mathfrak{V} , such that

$$\mathfrak{V} = \mathcal{H}_0^1(0,1) \times L^2(0,1) \times \mathcal{H}_0^1(0,1) \times L^2(0,1) \times \mathcal{H}_0^1(0,1) \times L^2(0,1) \\ \times \mathcal{H}_0^1(0,1) \times L^2(0,1) \times L^2((0,1) \times (0,1) \times (v_1, v_2)),$$

for any

$$X = (u, v, \vartheta, \varphi, \theta, \chi, \mathfrak{R}, \Sigma, \mathcal{Y})^T \in \mathfrak{V}, \\ \widehat{X} = (\widehat{u}, \widehat{v}, \widehat{\vartheta}, \widehat{\varphi}, \widehat{\theta}, \widehat{\chi}, \widehat{\mathfrak{R}}, \widehat{\Sigma}, \widehat{\mathcal{Y}})^T \in \mathfrak{V},$$

with the following inner product:

$$\langle X, \widehat{X} \rangle_{\mathfrak{V}} = \rho_u \int_0^1 v \widehat{v} dx + a_1 \int_0^1 u_x \widehat{u}_x dx + \rho_{\vartheta} \int_0^1 \varphi \widehat{\varphi} dx + a_3 \int_0^1 \vartheta_x \widehat{\vartheta}_x dx \\ + a_2 \int_0^1 (u_x \widehat{\vartheta}_x + \widehat{u}_x \vartheta_x) dx + k_1 \kappa \int_0^1 (\theta_x \widehat{\mathfrak{R}} + \mathfrak{R} \widehat{\theta}_x) dx \\ + \beta_1 \int_0^1 (\kappa \chi + \theta)(\kappa \widehat{\chi} + \widehat{\theta}) dx + \beta_2 \int_0^1 (\kappa \Sigma + \mathfrak{R})(\kappa \widehat{\Sigma} + \widehat{\mathfrak{R}}) dx \\ + k \kappa \int_0^1 \theta_x \widehat{\theta}_x dx + k_2 \kappa \int_0^1 \mathfrak{R}_x \widehat{\mathfrak{R}}_x dx + k_3 \kappa \int_0^1 \mathfrak{R} \widehat{\mathfrak{R}} dx \\ + \int_0^1 \int_0^1 \int_{v_1}^{v_2} s |\omega_2(s)| \mathcal{Y} \widehat{\mathcal{Y}} ds d\rho dx. \tag{15}$$

The domain of \mathcal{T} is given by

$$\mathcal{S}(\mathcal{T}) = \left\{ \begin{array}{l} X \in \mathfrak{V} / u, \vartheta, \theta, \mathfrak{R} \in \mathcal{H}^2(0,1) \cap \mathcal{H}_0^1(0,1), v, \varphi, \chi, \Sigma \in \mathcal{H}_0^1(0,1), \\ \mathcal{Y}, \mathcal{Y}_{\rho} \in L^2((0,1) \times (0,1) \times (v_1, v_2)), \mathcal{Y}(x, 0, s, t) = \varphi \end{array} \right\}.$$

Clearly, $\mathcal{S}(\mathcal{T})$ is dense in \mathfrak{V} .

Theorem 1. Let $X_0 \in \mathfrak{V}$ and consider that (5), (6) and (12) hold. Then, a unique solution $X \in \mathcal{C}(\mathbb{R}_+, \mathfrak{V})$ for the problem (13) exists. Additionally, if $X_0 \in \mathcal{S}(\mathcal{T})$, then

$$X \in \mathcal{C}(\mathbb{R}_+, \mathcal{S}(\mathcal{T})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathfrak{V}).$$

Proof. First, we show that \mathcal{T} is a dissipative operator. For any $X_0 \in \mathcal{S}(\mathcal{T})$ and by utilizing (15), we achieve

$$\langle \mathcal{T}X, X \rangle_{\mathfrak{V}} = -\omega_1 \int_0^1 \varphi^2 dx - \int_0^1 \int_{v_1}^{v_2} \omega_2(s) \varphi \mathcal{Y}(x, 1, s, t) ds dx \\ - k \int_0^1 \theta_x^2 dx - k_3 \int_0^1 \mathfrak{R}^2 dx - 2k_1 \int_0^1 \mathfrak{R} \theta_x dx \\ - k_2 \int_0^1 \mathfrak{R}_x^2 dx - \int_0^1 \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}_{\rho} \mathcal{Y} ds d\rho dx. \tag{16}$$

For the last term of the RHS of (16), we have

$$- \int_0^1 \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}_{\rho} \mathcal{Y} ds d\rho dx = -\frac{1}{2} \int_0^1 \int_{v_1}^{v_2} \int_0^1 |\omega_2(s)| \frac{d}{d\rho} \mathcal{Y}^2 d\rho ds dx \\ = -\frac{1}{2} \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx \\ + \frac{1}{2} \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 0, s, t) ds dx. \tag{17}$$

Applying the inequality of Young, we have

$$\begin{aligned}
 - \int_0^1 \int_{v_1}^{v_2} \omega_2(s) \varphi \mathcal{Y}(x, 1, s, t) ds dx &\leq \frac{1}{2} \left(\int_{v_1}^{v_2} |\omega_2(s)| ds \right) \int_0^1 \varphi^2 dx \\
 &+ \frac{1}{2} \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx. \tag{18}
 \end{aligned}$$

Substituting (17) and (18) into (16) and utilizing $\mathcal{Y}(x, 0, s, t) = \varphi(x, t)$ and (12), the following is obtained:

$$\begin{aligned}
 \langle \mathcal{T}X, X \rangle_{\mathfrak{X}} &\leq -k \int_0^1 \theta_x^2 dx - k_3 \int_0^1 \mathfrak{R}^2 dx - 2k_1 \int_0^1 \mathfrak{R} \theta_x dx \\
 &- \eta_0 \int_0^1 \varphi^2 dx - k_2 \int_0^1 \mathfrak{R}_x^2 dx, \tag{19}
 \end{aligned}$$

where $\eta_0 = (\omega_1 - \int_{v_1}^{v_2} |\omega_2(s)| ds) > 0$. Additionally, by (6), we have

$$-k\theta_x^2 - k_3\mathfrak{R}^2 - 2k_1\mathfrak{R}\theta_x < 0. \tag{20}$$

Therefore, the operator \mathcal{T} is dissipative. Now, we show that the operator \mathcal{T} is a maximal. It is enough to prove that $(\lambda I - \mathcal{T})$ is a surjective operator. In fact, we demonstrate that a unique $X = (u, v, \vartheta, \varphi, \theta, \chi, \mathfrak{R}, \Sigma, \mathcal{Y})^T \in \mathcal{S}(\mathcal{T})$ exists for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathfrak{Y}$, such that

$$(\lambda I - \mathcal{T})X = F. \tag{21}$$

That is,

$$\left\{ \begin{aligned}
 \lambda u - v &= f_1 \in \mathcal{H}_0^1(0, 1) \\
 \rho_u \lambda v - a_1 u_{xx} - a_2 \vartheta_{xx} + \gamma_0 (\kappa \chi + \theta)_x &= \rho_u f_2 \in L^2(0, 1) \\
 \lambda \vartheta - \varphi &= f_3 \in \mathcal{H}_0^1(0, 1) \\
 \rho_\vartheta \lambda \varphi - a_3 \vartheta_{xx} - a_2 u_{xx} - \gamma_1 (\kappa \chi + \theta) + \gamma_2 (\kappa \Sigma + \mathfrak{R})_x + \omega_1 \varphi \\
 &+ \int_{v_1}^{v_2} \omega_2(s) \mathcal{Y}(x, 1, s, t) ds = \rho_\vartheta f_4 \in L^2 \\
 \lambda \theta - \chi &= f_5 \in \mathcal{H}_0^1(0, 1) \\
 \beta_1 \kappa \lambda \chi + \gamma_0 v_x + \gamma_1 \varphi - k \theta_{xx} - k_1 \mathfrak{R}_x + \beta_1 \chi &= \beta_1 \kappa f_6 \in L^2(0, 1) \\
 \lambda \mathfrak{R} - \Sigma &= f_7 \in \mathcal{H}_0^1(0, 1) \\
 \beta_2 \kappa \lambda \Sigma - k_2 \mathfrak{R}_{xx} + \gamma_2 \varphi_x + k_3 \mathfrak{R} + k_1 \theta_x + \beta_2 \Sigma &= \beta_2 \kappa f_8 \in L^2(0, 1) \\
 s \lambda \mathcal{Y}_t(x, \rho, s, t) + \mathcal{Y}_\rho(x, \rho, s, t) &= s f_9 \in L^2((0, 1) \times (0, 1) \times (v_1, v_2)).
 \end{aligned} \right. \tag{22}$$

It can be noticed that there exists a unique solution of (22)₉ with $\mathcal{Y}(x, 0, s, t) = \varphi(x, t)$, which is given by

$$\mathcal{Y}(x, \rho, s, t) = e^{-\lambda \rho s} \varphi + s e^{s \rho \lambda} \int_0^\rho e^{\lambda s \varrho} f_9(x, \varrho, s, t) d\varrho, \tag{23}$$

then,

$$\mathcal{Y}(x, 1, s, t) = e^{-\lambda s} \varphi + s e^{\lambda s} \int_0^1 e^{\lambda s \varrho} f_9(x, \varrho, s, t) d\varrho, \tag{24}$$

thus, we obtain

$$v = \lambda u - f_1, \quad \varphi = \lambda \vartheta - f_3, \quad \chi = \lambda \theta - f_5, \quad \Sigma = \lambda \mathfrak{R} - f_7. \tag{25}$$

Inserting (24) and (25) into (22)₂, (22)₄, (22)₆ and (22)₈, we obtain

$$\begin{cases} \rho_u \lambda^2 u - a_1 u_{xx} - a_2 \vartheta_{xx} + \gamma_0(\kappa\lambda + 1)\theta_x = h_1 \\ \omega_3 \vartheta - a_3 \vartheta_{xx} - a_2 u_{xx} - \gamma_1(\kappa\lambda + 1)\theta + \gamma_2(\kappa\lambda + 1)\mathfrak{R}_x = h_2 \\ \omega_4 \theta + \gamma_0 \lambda u_x + \gamma_1 \lambda \vartheta - k\theta_{xx} - k_1 \mathfrak{R}_x = h_3 \\ \omega_5 \mathfrak{R} - k_2 \mathfrak{R}_{xx} + \gamma_2 \lambda \vartheta_x + k_1 \theta_x = h_4, \end{cases} \tag{26}$$

where

$$\begin{cases} h_1 = \rho_u(\lambda f_1 + f_2) + \gamma_0 \kappa f_{5x}, \\ h_2 = \rho_\vartheta f_4 + (\rho_\vartheta \lambda + \omega_1 + \int_{v_1}^{v_2} |\omega_2(s)| e^{-s\lambda} ds) f_3 - \gamma_1 \kappa f_5 + \gamma_2 \kappa f_{7x}, \\ \quad - \int_{v_1}^{v_2} s |\omega_2(s)| e^{s\lambda} \int_0^1 e^{\lambda s \varrho} f_9(x, \varrho, s, t) d\varrho ds, \\ h_3 = \kappa \beta_1 f_6 + \gamma_0 f_{1x} + \gamma_1 f_3 + \beta_1(1 + \kappa\lambda) f_5, \\ h_4 = \kappa \beta_2 f_8 + \gamma_2 f_{3x} + \beta_2(1 + \kappa\lambda) f_7, \\ \omega_3 = \rho_\vartheta \lambda^2 + \omega_1 \lambda + \lambda \int_{v_1}^{v_2} |\omega_2(s)| e^{-\lambda s} ds, \\ \omega_4 = \beta_1 \lambda(\kappa\lambda + 1), \\ \omega_5 = \beta_2 \lambda(\kappa\lambda + 1) + k_3. \end{cases} \tag{27}$$

Multiplying (26) by $\hat{u}, \hat{\vartheta}, \hat{\theta}, \hat{\mathfrak{R}}$, and integrating the sum over $(0, 1)$, we have

$$B((u, \vartheta, \theta, \mathfrak{R}), (\hat{u}, \hat{\vartheta}, \hat{\theta}, \hat{\mathfrak{R}})) = \Gamma(\hat{u}, \hat{\vartheta}, \hat{\theta}, \hat{\mathfrak{R}}), \tag{28}$$

where

$$B : (\mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1))^2 \rightarrow \mathbb{R},$$

is the bilinear form given by

$$\begin{aligned} B((u, \vartheta, \theta, \mathfrak{R}), (\hat{u}, \hat{\vartheta}, \hat{\theta}, \hat{\mathfrak{R}})) &= \rho_u \lambda^2 \int_0^1 u \hat{u} dx + a_1 \int_0^1 u_x \hat{u}_x dx \\ &+ a_2 \int_0^1 \vartheta_x \hat{u}_x dx + \gamma_0(\kappa\lambda + 1) \int_0^1 \theta_x \hat{u} dx \\ &+ \omega_3 \int_0^1 \vartheta \hat{\vartheta} dx + a_3 \int_0^1 \vartheta_x \hat{\vartheta}_x dx \\ &+ a_2 \int_0^1 u_x \hat{\vartheta}_x dx - \gamma_1(\kappa\lambda + 1) \int_0^1 \theta \hat{\vartheta} dx \\ &+ \frac{k_1(\kappa\lambda + 1)}{\lambda} \int_0^1 \mathfrak{R}_x \hat{\vartheta} dx + \gamma_2(\kappa\lambda + 1) \int_0^1 \mathfrak{R}_x \hat{\theta} dx \\ &+ \frac{\omega_4(\kappa\lambda + 1)}{\lambda} \int_0^1 \theta \hat{\theta} dx + \gamma_0(\kappa\lambda + 1) \int_0^1 u_x \hat{\theta} dx \\ &+ \gamma_1(\kappa\lambda + 1) \int_0^1 \vartheta \hat{\theta} dx + \frac{k(\kappa\lambda + 1)}{\lambda} \int_0^1 \theta_x \hat{\theta}_x dx, \\ &+ \frac{\omega_5(\kappa\lambda + 1)}{\lambda} \int_0^1 \mathfrak{R} \hat{\mathfrak{R}} dx + \gamma_2(\kappa\lambda + 1) \int_0^1 \vartheta_x \hat{\mathfrak{R}} dx \\ &+ \frac{k_1(\kappa\lambda + 1)}{\lambda} \int_0^1 \theta_x \hat{\mathfrak{R}} dx + \frac{k_2(\kappa\lambda + 1)}{\lambda} \int_0^1 \mathfrak{R}_x \hat{\mathfrak{R}}_x dx, \end{aligned} \tag{29}$$

and

$$\Gamma : (\mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1), \times \mathcal{H}_0^1(0, 1)) \rightarrow \mathbb{R},$$

is the linear functional defined as

$$\begin{aligned} \Gamma(\hat{u}, \hat{\vartheta}, \hat{\theta}, \hat{\mathfrak{R}}) &= \int_0^1 h_1 \hat{u} dx + \int_0^1 h_2 \hat{\vartheta} dx + \frac{(\kappa\lambda + 1)}{\lambda} \int_0^1 h_3 \hat{\theta} dx \\ &+ \frac{(\kappa\lambda + 1)}{\lambda} \int_0^1 h_4 \hat{\mathfrak{R}} dx. \end{aligned} \tag{30}$$

Now, for $V = \mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1) \times \mathcal{H}_0^1(0, 1)$, with the norm

$$\|(u, \vartheta, \theta, \mathfrak{R})\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\vartheta\|_2^2 + \|\vartheta_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2 + \|\mathfrak{R}\|_2^2 + \|\mathfrak{R}_x\|_2^2,$$

we obtain

$$\begin{aligned} B((u, \vartheta, \theta, \mathfrak{R}), (u, \vartheta, \theta, \mathfrak{R})) &= \rho_u \lambda^2 \int_0^1 u^2 dx + a_1 \int_0^1 u_x^2 dx + \omega_3 \int_0^1 \vartheta^2 dx \\ &+ a_3 \int_0^1 \vartheta_x^2 dx + \frac{\omega_4(\kappa\lambda + 1)}{\lambda} \int_0^1 \theta^2 dx \\ &+ 2a_2 \int_0^1 u_x \vartheta_x dx + \frac{k(\kappa\lambda + 1)}{\lambda} \int_0^1 \theta_x^2 dx \\ &+ \frac{\omega_5(\kappa\lambda + 1)}{\lambda} \int_0^1 \mathfrak{R}^2 dx + \frac{k_2(\kappa\lambda + 1)}{\lambda} \int_0^1 \mathfrak{R}_x^2 dx. \end{aligned} \tag{31}$$

Also, we can write

$$\begin{aligned} a_1 u_x^2 + 2a_2 u_x \vartheta_x + a_3 \vartheta_x^2 &= \frac{1}{2} \left[a_1 \left(u_x + \frac{a_2}{a_3} \vartheta_x \right)^2 + a_3 \left(\vartheta_x + \frac{a_2}{a_1} u_x \right)^2 \right. \\ &\left. + u_x^2 \left(a_1 - \frac{a_2^2}{a_3} \right) + \vartheta_x^2 \left(a_3 - \frac{a_2^2}{a_1} \right) \right]. \end{aligned}$$

Because of (5), we deduce that

$$a_1 u_x^2 + 2a_2 u_x \vartheta_x + a_3 \vartheta_x^2 > \frac{1}{2} \left[u_x^2 \left(a_1 - \frac{a_2^2}{a_3} \right) + \vartheta_x^2 \left(a_3 - \frac{a_2^2}{a_1} \right) \right], \tag{32}$$

then, for some $M_0 > 0$,

$$B((u, \vartheta, \theta, \mathfrak{R}), (u, \vartheta, \theta, \mathfrak{R})) \geq M_0 \|(u, \vartheta, \theta, \mathfrak{R})\|_V^2. \tag{33}$$

B is, hence, coercive. As a result, we determine that (28) has a unique solution using the Lax–Milgram theorem:

$$u, \vartheta, \theta, \mathfrak{R} \in \mathcal{H}_0^1(0, 1). \tag{34}$$

Putting u, ϑ, θ , and \mathfrak{R} in (22)_{1,3,5,7}, we have

$$v, \varphi, \chi, \Sigma \in \mathcal{H}_0^1(0, 1).$$

In the same way, the compensation of φ in (23) with (22)₉ implies that

$$\mathcal{Y}, \mathcal{Y}_\rho \in L^2((0, 1) \times (0, 1) \times (v_1, v_2)).$$

Additionally, if we assume $\widehat{u} = \widehat{\vartheta} = \widehat{\mathfrak{R}} = 0$ in (29), we obtain

$$\begin{aligned} &a_3 \int_0^1 \vartheta_x \widehat{\vartheta}_x dx + \omega_3 \int_0^1 \vartheta \widehat{\vartheta} dx + a_2 \int_0^1 u_x \widehat{\vartheta}_x dx \\ &- \gamma_1(\kappa\lambda + 1) \int_0^1 \theta \widehat{\vartheta} dx + \gamma_2(\kappa\lambda + 1) \int_0^1 \mathfrak{R}_x \widehat{\vartheta} dx = \int_0^1 h_2 \widehat{\vartheta} dx, \quad \forall \widehat{\vartheta} \in \mathcal{H}_0^1(0, 1), \end{aligned}$$

which implies

$$a_3 \int_0^1 \vartheta_x \widehat{\vartheta}_x dx = \int_0^1 \left(h_2 - \omega_3 \vartheta + a_2 u_{xx} + (\kappa\lambda + 1)(\gamma_1 \theta - \gamma_2 \mathfrak{R}_x) \right) \widehat{\vartheta} dx, \quad \forall \widehat{\vartheta} \in \mathcal{H}_0^1,$$

that is,

$$a_3\vartheta_{xx} + a_2u_{xx} = \omega_3\vartheta - (\kappa\lambda + 1)(\gamma_1\theta - \gamma_2\mathfrak{R}_x) - h_2 \in L^2(0, 1). \tag{35}$$

Similarly, if we take $\widehat{\vartheta} = \widehat{\theta} = \widehat{\mathfrak{R}} = 0$ in (29), we obtain

$$a_2\vartheta_{xx} + a_1u_{xx} = \rho_u\lambda^2u + \gamma_0(\kappa\lambda + 1)\theta_x - h_1 \in L^2(0, 1). \tag{36}$$

Combining (35) and (36), and by using (5), we achieve that

$$u, \vartheta \in \mathcal{H}^2(0, 1) \cap \mathcal{H}_0^1(0, 1). \tag{37}$$

In the same way, if we let $\widehat{u} = \widehat{\vartheta} = \widehat{\mathfrak{R}} = 0$ in (29), we obtain

$$\begin{aligned} & \frac{\omega_4(\kappa\lambda + 1)}{\lambda} \int_0^1 \theta \widehat{\theta} dx + \frac{k(\kappa\lambda + 1)}{\lambda} \int_0^1 \theta_x \widehat{\theta}_x dx + \gamma_0(\kappa\lambda + 1) \int_0^1 u_x \widehat{\theta} dx \\ & + \gamma_1(\kappa\lambda + 1) \int_0^1 \vartheta \widehat{\theta} dx + \frac{k_1(\kappa\lambda + 1)}{\lambda} \int_0^1 \mathfrak{R}_x \widehat{\theta} dx = \frac{(\kappa\lambda + 1)}{\lambda} \int_0^1 h_3 \widehat{\theta} dx, \quad \forall \widehat{\theta} \in \mathcal{H}_0^1, \end{aligned}$$

which implies

$$k\theta_{xx} = \omega_4\theta + \gamma_0\lambda u_x + \gamma_1\lambda\vartheta + k_1\mathfrak{R}_x - h_3 \in L^2(0, 1).$$

Consequently,

$$\theta \in \mathcal{H}^2(0, 1) \cap \mathcal{H}_0^1(0, 1).$$

In a similar way, if we take $\widehat{\vartheta} = \widehat{\theta} = \widehat{u} = 0$ in (29), we find

$$k_2\mathfrak{R}_{xx} = \omega_5\mathfrak{R} + \gamma_2\vartheta_x + k_1\theta_x - h_4 \in L^2(0, 1).$$

Hence,

$$\mathfrak{R} \in \mathcal{H}^2(0, 1) \cap \mathcal{H}_0^1(0, 1).$$

Ultimately, leveraging the principles of regularity theory for linear elliptic equations guarantees the presence of a singular $X \in \mathcal{S}(\mathcal{T})$, which satisfies Equation (21) uniquely. In light of this, we deduce that \mathcal{T} is a maximal dissipative operator. The well-posedness finding is obtained as a result of the Lumer–Philips theorem [35]. \square

3. Exponential Decay

In this part, we demonstrate the system (9)–(11) stability result. The following lemmas apply to this.

Lemma 1. *The energy functional E , defined as*

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^1 \left[\rho_u u_t^2 + a_1 u_x^2 + \rho_\vartheta \vartheta_t^2 + a_3 \vartheta_x^2 + 2a_2 u_x \vartheta_x + \beta_1 (\kappa\theta_t + \theta)^2 \right] dx \\ &+ \frac{1}{2} \int_0^1 \left[\beta_2 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 + k_2 \kappa \mathfrak{R}_x^2 + k_3 \kappa \mathfrak{R}^2 + k\kappa \theta_x^2 + 2k_1 \kappa \mathfrak{R} \theta_x \right] dx \\ &+ \frac{1}{2} \int_0^1 \int_0^1 \int_{v_1}^{v_2} s |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx, \end{aligned} \tag{38}$$

satisfies

$$E'(t) \leq -k_4 \int_0^1 \theta_x^2 dx - k_2 \int_0^1 \mathfrak{R}_x^2 dx - k_5 \int_0^1 \mathfrak{R}^2 dx - \eta_0 \int_0^1 \vartheta_t^2 dx \leq 0, \tag{39}$$

where $\eta_0 = \omega_1 - \int_{v_1}^{v_2} |\omega_2(s)| ds > 0, k_4 = \frac{1}{2}(k - \frac{k_1^2}{k_3}) > 0, k_5 = \frac{1}{2}(k_3 - \frac{k_1^2}{k}) > 0.$

Proof. First, we multiply Equation (9)_{1,2,3,4} by $u_t, \vartheta_t, (\kappa\theta_t + \theta)$ and $(\kappa\mathfrak{R}_t + \mathfrak{R})$. In addition to this, applying (11), we have the following:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \left[\rho_u u_t^2 + a_1 u_x^2 + \rho_\vartheta \vartheta_t^2 + a_3 \vartheta_x^2 + 2a_2 u_x \vartheta_x + \beta_1 (\kappa\theta_t + \theta)^2 \right] dx \\ & \frac{1}{2} \frac{d}{dt} \int_0^1 \left[\beta_2 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 + k_2 \kappa \mathfrak{R}_x^2 + k_3 \kappa \mathfrak{R}^2 + k \kappa \theta_x^2 + 2k_1 \kappa \mathfrak{R} \theta_x \right] dx \\ & + \omega_1 \int_0^1 \vartheta_t^2 dx + \int_0^1 \vartheta_t \int_{v_1}^{v_2} \omega_2(s) \mathcal{Y}(x, 1, s, t) ds dx \\ & + k \int_0^1 \theta_x^2 dx + k_3 \int_0^1 \mathfrak{R}^2 dx + k_2 \int_0^1 \mathfrak{R}_x^2 dx + 2k_1 \int_0^1 \mathfrak{R} \theta_x dx = 0. \end{aligned} \tag{40}$$

Next, multiplying (9)₅ by $\mathcal{Y}|\omega_2(s)|$ and integrating, we have

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int_0^1 \int_0^1 \int_{v_1}^{v_2} s |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx \\ & = - \int_0^1 \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y} \mathcal{Y}_\rho(x, \rho, s, t) ds d\rho dx \\ & = - \frac{1}{2} \int_0^1 \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \frac{d}{d\rho} \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx \\ & = \frac{1}{2} \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| (\mathcal{Y}^2(x, 0, s, t) - \mathcal{Y}^2(x, 1, s, t)) ds dx \\ & = \frac{1}{2} \int_{v_1}^{v_2} |\omega_2(s)| ds \int_0^1 \vartheta_t^2 dx - \frac{1}{2} \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx. \end{aligned} \tag{41}$$

Putting (41) into (40) and utilizing Young’s inequality, the following is obtained:

$$\begin{aligned} E'(t) \leq & -k \int_0^1 \theta_x^2 dx - k_3 \int_0^1 \mathfrak{R}^2 dx - k_2 \int_0^1 \mathfrak{R}_x^2 dx \\ & - 2k_1 \int_0^1 \mathfrak{R} \theta_x dx - \left(\omega_1 - \int_{v_1}^{v_2} |\omega_2(s)| ds \right) \int_0^1 \vartheta_t^2 dx, \end{aligned}$$

and we have the following inequality:

$$k\theta_x^2 + k_3\mathfrak{R}^2 + 2k_1\mathfrak{R}\theta_x > \frac{1}{2} \left[\theta_x^2 \left(k - \frac{k_1^2}{k_3} \right) + \mathfrak{R}^2 \left(k_3 - \frac{k_1^2}{k} \right) \right]. \tag{42}$$

By (6), we obtain

$$k\theta_x^2 + k_3\mathfrak{R}^2 + 2k_1\mathfrak{R}\theta_x > k_4\theta_x^2 + k_5\mathfrak{R}^2, \tag{43}$$

where

$$k_4 = \frac{1}{2} \left(k - \frac{k_1^2}{k_3} \right) > 0, \quad k_5 = \frac{1}{2} \left(k_3 - \frac{k_1^2}{k} \right) > 0;$$

then, by (12), $\exists \eta_0 = \omega_1 - \int_{v_1}^{v_2} |\omega_2(s)| ds > 0$, so that

$$E'(t) \leq -k_4 \int_0^1 \theta_x^2 dx - k_5 \int_0^1 \mathfrak{R}^2 dx - k_2 \int_0^1 \mathfrak{R}_x^2 dx - \eta_0 \int_0^1 \vartheta_t^2 dx. \tag{44}$$

Thus, we achieve (39) (E is a non-increasing function). \square

Remark 1. Using (5), (6) and (32), we deduce that $E(t)$ fulfills

$$\begin{aligned}
 E(t) > & \frac{1}{2} \int_0^1 \left[\rho_u u_t^2 + a_4 u_x^2 + \rho_\theta \vartheta_t^2 + a_5 \vartheta_x^2 + \beta_1 (\kappa \theta_t + \theta)^2 \right] dx \\
 & + \frac{1}{2} \int_0^1 \left[\beta_2 (\kappa \mathfrak{R}_t + \mathfrak{R})^2 + k_2 \kappa \mathfrak{R}_x^2 + k_4 \kappa \theta_x^2 + k_5 \kappa \mathfrak{R}^2 \right] dx \\
 & + \frac{1}{2} \int_0^1 \int_0^1 \int_{v_1}^{v_2} s |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds dp dx,
 \end{aligned}$$

where

$$a_4 = \frac{1}{2} \left(a_1 - \frac{a_2^2}{a_3} \right) > 0, \quad a_5 = \frac{1}{2} \left(a_3 - \frac{a_2^2}{a_1} \right) > 0. \tag{45}$$

Thus, the non-negativity of the function $E(t)$ is obtained.

Lemma 2. The functional

$$D_1(t) := \rho_\theta \int_0^1 \vartheta_t \vartheta dx - \frac{a_2}{a_1} \rho_u \int_0^1 \vartheta u_t dx + \frac{\omega_1}{2} \int_0^1 \vartheta^2 dx, \tag{46}$$

satisfies, for any $\varepsilon_1 > 0$,

$$\begin{aligned}
 D'_1(t) \leq & -\frac{a}{2} \int_0^1 \vartheta_x^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + c \left(1 + \frac{1}{\varepsilon_1} \right) \int_0^1 \vartheta_t^2 dx \\
 & + c \int_0^1 (\kappa \theta_t + \theta)^2 dx + c \int_0^1 (\kappa \mathfrak{R}_t + \mathfrak{R})^2 dx \\
 & + c \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx.
 \end{aligned} \tag{47}$$

Proof. Via direct calculation, utilizing integration by parts, we obtain that

$$\begin{aligned}
 D'_1(t) &= -a_3 \int_0^1 \vartheta_x^2 dx + \rho_\theta \int_0^1 \vartheta_t^2 dx + \frac{a_2^2}{a_1} \int_0^1 \vartheta_x^2 dx - \frac{a_2}{a_1} \rho_u \int_0^1 \vartheta_t u_t dx \\
 &+ \gamma_1 \int_0^1 (\kappa \theta_t + \theta) \vartheta dx + \frac{a_2 \gamma_0}{a_1} \int_0^1 (\kappa \theta_t + \theta)_x \vartheta dx \\
 &- \gamma_2 \int_0^1 (\kappa \mathfrak{R}_t + \mathfrak{R})_x \vartheta dx - \int_0^1 \vartheta \int_{v_1}^{v_2} \omega_2(s) \mathcal{Y}(x, 1, s, t) ds dx \\
 &= -\left(a_3 - \frac{a_2^2}{a_1} \right) \int_0^1 \vartheta_x^2 dx + \rho_\theta \int_0^1 \vartheta_t^2 dx - \frac{a_2}{a_1} \rho_u \int_0^1 \vartheta_t u_t dx \\
 &+ \gamma_1 \int_0^1 (\kappa \theta_t + \theta) \vartheta dx - \frac{a_2 \gamma_0}{a_1} \int_0^1 (\kappa \theta_t + \theta) \vartheta_x dx \\
 &+ \gamma_2 \int_0^1 (\kappa \mathfrak{R}_t + \mathfrak{R}) \vartheta_x dx - \int_0^1 \vartheta \int_{v_1}^{v_2} \omega_2(s) \mathcal{Y}(x, 1, s, t) ds dx.
 \end{aligned} \tag{48}$$

Using Poincaré’s and Young’s inequalities, for $\delta_1, \delta_2, \delta_3, \varepsilon_1 > 0$, we obtain that

$$\begin{aligned}
 D'_1(t) \leq & -\left(\left(a_3 - \frac{a_2^2}{a_1} \right) - (\omega_1 c \delta_1 + c \delta_2 + 2 \delta_3) \right) \int_0^1 \vartheta_x^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx \\
 & + c \left(1 + \frac{1}{\varepsilon_1} \right) \int_0^1 \vartheta_t^2 dx + \left(\frac{c}{\delta_2} + \frac{c}{\delta_3} \right) \int_0^1 (\kappa \theta_t + \theta)^2 dx \\
 & + \frac{c}{\delta_3} \int_0^1 (\kappa \mathfrak{R}_t + \mathfrak{R})^2 dx + \frac{c}{\delta_1} \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx.
 \end{aligned} \tag{49}$$

Bearing in mind (5) and letting $\delta_1 = \frac{a}{6\omega_1 c}, \delta_2 = \frac{a}{6c}, \delta_3 = \frac{a}{12}$, we obtain estimate (47). \square

Lemma 3. *The functional*

$$D_2(t) := a_2 \left(\int_0^1 \vartheta_t u dx - \int_0^1 \vartheta u_t dx \right),$$

satisfies,

$$D_2'(t) \leq -\frac{a_2^2}{2\rho_\vartheta} \int_0^1 u_x^2 dx + c \int_0^1 \vartheta_x^2 dx + c \int_0^1 \vartheta_t^2 dx + c \int_0^1 (\kappa\theta_t + \theta)^2 dx + c \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 dx + c \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx. \tag{50}$$

Proof. By differentiating D_2 , then utilizing (9), integration by parts and (11), we obtain

$$D_2'(t) = -\frac{a_2^2}{\rho_\vartheta} \int_0^1 u_x^2 dx + \frac{a_2^2}{\rho_u} \int_0^1 \vartheta_x^2 dx - \left(\frac{a_2 a_3}{\rho_\vartheta} - \frac{a_1 a_2}{\rho_u} \right) \int_0^1 \vartheta_x u_x dx - \frac{a_2 \gamma_0}{\rho_u} \int_0^1 (\kappa\theta_t + \theta) \vartheta_x dx + \frac{a_2 \gamma_1}{\rho_\vartheta} \int_0^1 (\kappa\theta_t + \theta) u dx + \frac{a_2 \gamma_2}{\rho_\vartheta} \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R}) u_x dx - \frac{a_2 \omega_1}{\rho_\vartheta} \int_0^1 u \vartheta_t dx - \frac{a_2}{\rho_\vartheta} \int_0^1 u \int_{v_1}^{v_2} \omega_2(s) \mathcal{Y}(x, 1, s, t) ds dx. \tag{51}$$

We now evaluate the final six terms in the right hand side of (51), utilizing Poincaré’s and Young’s inequalities. For $\delta_4, \delta_5 > 0$, we have

$$-\left(\frac{a_2 a_3}{\rho_\vartheta} - \frac{a_1 a_2}{\rho_u} \right) \int_0^1 \vartheta_x u_x dx \leq \delta_4 \int_0^1 u_x^2 dx + \frac{c}{\delta_4} \int_0^1 \vartheta^2 dx,$$

$$-\frac{a_2 \omega_1}{\rho_\vartheta} \int_0^1 u \vartheta_t dx \leq c \delta_5 \int_0^1 u_x^2 dx + \frac{c}{\delta_5} \int_0^1 \vartheta_t^2 dx,$$

$$\frac{a_2 \gamma_1}{\rho_\vartheta} \int_0^1 (\kappa\theta_t + \theta) u dx \leq c \delta_5 \int_0^1 u_x^2 dx + \frac{c}{\delta_5} \int_0^1 (\kappa\theta_t + \theta)^2 dx,$$

$$\frac{a_2 \gamma_2}{\rho_\vartheta} \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R}) u_x dx \leq \delta_4 \int_0^1 u_x^2 dx + \frac{c}{\delta_4} \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 dx,$$

and

$$-\frac{a_2}{\rho_\vartheta} \int_0^1 u \int_{v_1}^{v_2} \omega_2(s) \mathcal{Y}(x, 1, s, t) ds dx \leq c \delta_5 \int_0^1 u_x^2 dx + \frac{c}{\delta_5} \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx.$$

By letting $\delta_4 = \frac{a_2}{8\rho_\vartheta}, \delta_5 = \frac{a_2}{12c\rho_\vartheta}$ and putting these into (51), we obtain (50). \square

Lemma 4. *The functional*

$$D_3(t) := -\rho_u \int_0^1 u_t u dx,$$

satisfies

$$D'_3(t) \leq -\rho_u \int_0^1 u_t^2 dx + 3a_1 \int_0^1 u_x^2 dx + \frac{a_3}{4} \int_0^1 \vartheta_x^2 dx + \frac{\gamma_0^2}{4a_1} \int_0^1 (\kappa\theta_t + \theta)^2 dx. \tag{52}$$

Proof. Direct computations give

$$D'_3(t) = -\rho_u \int_0^1 u_t^2 dx + a_1 \int_0^1 u_x^2 dx + a_2 \int_0^1 u_x \vartheta_x dx - \gamma_0 \int_0^1 u_x (\kappa\theta_t + \theta) dx.$$

Estimate (52) easily follows by utilizing Young’s inequality. \square

Lemma 5. *The functional*

$$D_4(t) := -\beta_1 \kappa^2 \int_0^1 \theta_t \theta dx - \frac{\beta_1 \kappa}{2} \int_0^1 \theta^2 dx - \beta_2 \kappa^2 \int_0^1 \mathfrak{R}_t \mathfrak{R} dx - \frac{\beta_2 \kappa}{2} \int_0^1 \mathfrak{R}^2 dx,$$

satisfies, for any $\varepsilon_2 > 0$,

$$D'_4(t) \leq -\frac{\beta_1}{2} \int_0^1 (\kappa\theta_t + \theta)^2 dx - \frac{\beta_2}{2} \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 dx + \varepsilon_2 \int_0^1 u_t^2 dx + c \int_0^1 \vartheta_t^2 dx + c(1 + \frac{1}{\varepsilon_2}) \int_0^1 \theta_x^2 dx + c \int_0^1 \mathfrak{R}_x^2 dx + c \int_0^1 \mathfrak{R}^2 dx. \tag{53}$$

Proof. Direct computations give

$$\begin{aligned} D'_4(t) &= -\beta_1 \kappa \int_0^1 (\kappa\theta_t + \theta)_t \theta dx - \beta_1 \kappa^2 \int_0^1 \theta_t^2 dx \\ &\quad - \beta_2 \kappa \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})_t \mathfrak{R} dx - \beta_2 \kappa^2 \int_0^1 \mathfrak{R}_t^2 dx \\ &= \gamma_0 \kappa \int_0^1 \theta u_{tx} dx + \gamma_1 \kappa \int_0^1 \vartheta_t \theta dx + k\kappa \int_0^1 \theta_x^2 dx - \beta_1 \kappa^2 \int_0^1 \theta_t^2 dx \\ &\quad + 2k_1 \kappa \int_0^1 \mathfrak{R} \theta_x dx + k_2 \kappa \int_0^1 \mathfrak{R}_x^2 dx + k_3 \kappa \int_0^1 \mathfrak{R}^2 dx \\ &\quad - \gamma_2 \kappa \int_0^1 \vartheta_t \mathfrak{R}_x dx - \beta_2 \kappa^2 \int_0^1 \mathfrak{R}_t^2 dx. \end{aligned}$$

Further simplification of (53) leads us to

$$\begin{aligned} -\int_0^1 (\kappa\theta_t)^2 dx &\leq -\frac{1}{2} \int_0^1 (\kappa\theta_t + \theta)^2 dx + \int_0^1 \theta^2 dx, \\ -\int_0^1 (\kappa\mathfrak{R}_t)^2 dx &\leq -\frac{1}{2} \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 dx + \int_0^1 \mathfrak{R}^2 dx. \end{aligned} \tag{54}$$

\square

Now, we introduce a functional stated in Lemma 6.

Lemma 6. *The functional*

$$D_5(t) := \int_0^1 \int_0^1 \int_{v_1}^{v_2} se^{-s\rho} |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx,$$

satisfies

$$\begin{aligned} D_5'(t) \leq & -\eta_1 \int_0^1 \int_0^1 \int_{v_1}^{v_2} s |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx + \omega_1 \int_0^1 \vartheta_t^2 dx \\ & -\eta_1 \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx, \end{aligned} \tag{55}$$

where $\eta_1 > 0$.

Proof. By differentiating D_5 with respect to t and utilizing the last equation in (9), we obtain

$$\begin{aligned} D_5'(t) &= -2 \int_0^1 \int_0^1 \int_{v_1}^{v_2} e^{-s\rho} |\omega_2(s)| \mathcal{Y} \mathcal{Y}_\rho(x, \rho, s, t) ds d\rho dx \\ &= - \int_0^1 \int_0^1 \int_{v_1}^{v_2} se^{-s\rho} |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx \\ &\quad - \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| [e^{-s} \mathcal{Y}^2(x, 1, s, t) - \mathcal{Y}^2(x, 0, s, t)] ds dx. \end{aligned}$$

Utilizing that $\mathcal{Y}(x, 0, s, t) = \vartheta_t(x, t)$ and $e^{-s} \leq e^{-s\rho} \leq 1, \forall 0 < \rho < 1$, we achieve that

$$\begin{aligned} D_5'(t) \leq & - \int_0^1 \int_0^1 \int_{v_1}^{v_2} se^{-s} |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx \\ & - \int_0^1 \int_{v_1}^{v_2} e^{-s} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx + \left(\int_{v_1}^{v_2} |\omega_2(s)| ds \right) \int_0^1 \vartheta_t^2 dx. \end{aligned}$$

□

As $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-v_2}$, for all $s \in [v_1, v_2]$. By setting $\eta_1 = e^{-v_2}$ and remembering (12), we discover (55). We can now proceed to proving the primary finding.

Theorem 2. *Let (5), (6) and (12) hold. Then, there exist $\zeta_1, \zeta_2 > 0$ such that the energy functional provide by (38) holds:*

$$E(t) \leq \zeta_1 e^{-\zeta_2 t}, \quad \forall t \geq 0. \tag{56}$$

Proof. To prove the required result, we introduce the Lyapunov functional as follows:

$$\mathcal{P}(t) := NE(t) + N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t) + N_5 D_5(t), \tag{57}$$

where $N, N_1, N_2, N_4, N_5 > 0$; we assign them later.

By differentiating (57) and using (39), (47), (50), (52), (53) and (55), we have

$$\begin{aligned}
 \mathcal{P}'(t) \leq & -\left[\frac{aN_1}{2} - cN_2 - \frac{a_3}{4}\right] \int_0^1 \vartheta_x^2 dx - [\rho_u - \varepsilon_1 N_1 - \varepsilon_2 N_4] \int_0^1 u_t^2 dx \\
 & -\left[\frac{a_2^2 N_2}{2\rho_\vartheta} - 3a_1\right] \int_0^1 u_x^2 dx - \left[k_4 N - c\left(1 + \frac{1}{\varepsilon_2}\right)N_4\right] \int_0^1 \theta_x^2 dx \\
 & -\left[\eta_0 N - cN_1\left(1 + \frac{1}{\varepsilon_1}\right) - cN_2 - cN_4 - \omega_1 N_5\right] \int_0^1 \vartheta_t^2 dx \\
 & -\left[\frac{\beta_1}{2}N_4 - cN_1 - cN_2 - \frac{\gamma_0}{4a_1}\right] \int_0^1 (\kappa\theta_t + \theta)^2 dx \\
 & -\left[\frac{\beta_2}{2}N_4 - cN_1 - cN_2\right] \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 dx \\
 & -[k_2 N - cN_4] \int_0^1 \mathfrak{R}_x^2 dx - [k_5 N - cN_4] \int_0^1 \mathfrak{R}^2 dx \\
 & -[N_5\eta_1 - cN_1 - cN_2] \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx \\
 & -N_5\eta_1 \int_0^1 \int_0^1 \int_{v_1}^{v_2} s|\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds dp dx.
 \end{aligned}$$

by setting

$$\varepsilon_1 = \frac{\rho_u}{4N_1}, \quad \varepsilon_2 = \frac{\rho_u}{4N_4},$$

we obtain

$$\begin{aligned}
 \mathcal{P}'(t) \leq & -\left[\frac{aN_1}{2} - cN_2 - \frac{a_3}{4}\right] \int_0^1 \vartheta_x^2 dx - \left[\frac{\rho_u}{2}\right] \int_0^1 u_t^2 dx \\
 & -\left[\frac{a_2^2 N_2}{2\rho_\vartheta} - 3a_1\right] \int_0^1 u_x^2 dx - [k_4 N - cN_4(1 + N_4)] \int_0^1 \theta_x^2 dx \\
 & -[\eta_0 N - cN_1(1 + N_1) - N_2 c - N_4 c - \omega_1 N_5] \int_0^1 \vartheta_t^2 dx \\
 & -\left[\frac{\beta_1}{2}N_4 - cN_1 - cN_2 - \frac{\gamma_0^2}{4a_1}\right] \int_0^1 (\kappa\theta_t + \theta)^2 dx \\
 & -\left[\frac{\beta_2}{2}N_4 - cN_1 - cN_2\right] \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 dx \\
 & -[k_2 N - cN_4] \int_0^1 \mathfrak{R}_x^2 dx - [k_5 N - cN_4] \int_0^1 \mathfrak{R}^2 dx \\
 & -[N_5\eta_1 - cN_1 - cN_2] \int_0^1 \int_{v_1}^{v_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx \\
 & -N_5\eta_1 \int_0^1 \int_0^1 \int_{v_1}^{v_2} s|\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds dp dx.
 \end{aligned}$$

Now, we choose our constants.

We take N_2 large enough, such that

$$\alpha_1 = \frac{a_2^2 N_2}{2\rho_\vartheta} - 3a_1 > 0;$$

then, we pick N_1 large enough, in such a way that

$$\alpha_2 = \frac{aN_1}{2} - cN_2 - \frac{a_3}{4} > 0.$$

Then, we pick N_4 and N_5 large enough, in such a way that

$$\begin{aligned} \alpha_3 &= \frac{\beta_1}{2}N_4 - cN_1 - cN_2 - \frac{\gamma_0^2}{4a_1} > 0, \\ \alpha_4 &= \frac{\beta_2}{2}N_4 - cN_1 - cN_2 > 0, \\ \alpha_5 &= N_5\eta_1 - cN_1 - cN_2 > 0. \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} \mathcal{P}'(t) \leq & -\alpha_2 \int_0^1 \vartheta_x^2 dx - \frac{\rho u}{2} \int_0^1 u_t^2 dx - \alpha_1 \int_0^1 u_x^2 dx - [\eta_0 N - c] \int_0^1 \vartheta_t^2 dx \\ & - [k_4 N - c] \int_0^1 \vartheta_x^2 dx - \alpha_3 \int_0^1 (\kappa\vartheta_t + \vartheta)^2 dx - \alpha_4 \int_0^1 (\kappa\mathfrak{R}_t + \mathfrak{R})^2 dx \\ & - \alpha_5 \int_0^1 \int_{\nu_1}^{\nu_2} |\omega_2(s)| \mathcal{Y}^2(x, 1, s, t) ds dx - [k_2 N - c] \int_0^1 \mathfrak{R}_x^2 dx \\ & - [k_5 N - c] \int_0^1 \mathfrak{R}^2 dx - \alpha_6 \int_0^1 \int_0^1 \int_{\nu_1}^{\nu_2} s |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx, \end{aligned} \tag{58}$$

where $\alpha_6 = \eta_1 N_5 > 0$.

Similarly, if we assume

$$\mathcal{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t) + N_5 D_5(t),$$

then

$$\begin{aligned} |\mathcal{L}(t)| \leq & N_1 \rho_\vartheta \int_0^1 |\vartheta \vartheta_t| dx + N_1 \frac{a_2}{a_1} \rho_u \int_0^1 |\vartheta u_t| dx + N_1 \frac{\omega_1}{2} \int_0^1 \vartheta^2 dx \\ & + N_2 a_2 \int_0^1 |\vartheta u_t - u \vartheta_t| dx + \rho_u \int_0^1 |u_t u| dx + N_4 \beta_1 \kappa^2 \int_0^1 |\vartheta_t \vartheta| dx \\ & + N_4 \frac{\beta_1 \kappa}{2} \int_0^1 \vartheta^2 dx + N_4 \beta_2 \kappa^2 \int_0^1 |\mathfrak{R}_t \mathfrak{R}| dx + N_4 \frac{\beta_2 \kappa}{2} \int_0^1 \mathfrak{R}^2 dx \\ & + N_5 \int_0^1 \int_0^1 \int_{\nu_1}^{\nu_2} s e^{-s\rho} |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

According to the Cauchy–Schwartz, Poincaré’s and Young’s inequalities, we find

$$\begin{aligned} |\mathcal{L}(t)| \leq & c \int_0^1 (u_t^2 + \vartheta_t^2 + \vartheta_x^2 + u_x^2 + (\kappa\vartheta_t + \vartheta)^2 + \vartheta_x^2) dx \\ & + c \int_0^1 (\mathfrak{R}^2 + \mathfrak{R}_x^2 + (\kappa\mathfrak{R}_t + \mathfrak{R})^2) dx \\ & + c \int_0^1 \int_0^1 \int_{\nu_1}^{\nu_2} s |\omega_2(s)| \mathcal{Y}^2(x, \rho, s, t) ds d\rho. \end{aligned}$$

On the other hand, by (32), we have

$$a_1 u_x^2 + 2a_2 \vartheta_x u_x + a_3 \vartheta_x^2 > \frac{1}{2} \left[(a_1 - \frac{a_2^2}{a_3}) u_x^2 + (a_3 - \frac{a_2^2}{a_1}) \vartheta_x^2 \right].$$

Similarly, by $(k k_3 > k_1^2)$, we have

$$k \vartheta_x^2 + 2k_1 \mathfrak{R} \vartheta_x + k_3 \mathfrak{R}^2 > \frac{1}{2} \left[(k - \frac{k_1^2}{k_3}) \vartheta_x^2 + (k_3 - \frac{k_1^2}{k}) \mathfrak{R}^2 \right].$$

Hence, we obtain

$$|\mathcal{L}(t)| = |\mathcal{P}(t) - NE(t)| \leq cE(t),$$

that is,

$$(N - c)E(t) \leq \mathcal{P}(t) \leq (N + c)E(t). \tag{59}$$

At this point, we choose N large enough, such that

$$N - c > 0, \quad N\eta_0 - c > 0, \quad Nk_4 - c > 0 > 0, \quad Nk_2 - c > 0, \quad Nk_5 - c > 0 > 0.$$

Simplification of (38), (58) and (59) leads us to

$$c_1E(t) \leq \mathcal{P}(t) \leq c_2E(t), \forall t \geq 0, \tag{60}$$

and

$$\mathcal{P}'(t) \leq -d_1E(t), \forall t \geq 0, \tag{61}$$

for some $d_1, c_1, c_2 > 0$.

Consequently, for some $\zeta_2 > 0$, we find

$$\mathcal{P}'(t) \leq -\zeta_2\mathcal{P}(t), \forall t \geq 0. \tag{62}$$

From further simplification of (62), we have the following:

$$\mathcal{P}(t) \leq \mathcal{P}(0)e^{-\zeta_2 t}, \forall t \geq 0. \tag{63}$$

Hence, (56) is achieved by (60) and (63). \square

4. Conclusions

This work studies a swelling porous elastic system coupled with thermo-elasticity of the Lord Shulman type, microtemperature and distributed delay, an approach which is more general than classical thermo-elasticity. Furthermore, the problem circumvents the absurd situation of the infinite propagation of the effect of a thermal or mechanical disturbance in the medium. We established the well-posedness of our problem using the semigroup method. Additionally, we used the energy method to prove the stability result for the system. It is intriguing to know that the result was obtained independently of the wave velocities of the system or any form of interactions between coefficients of the system other than hypotheses (5), (6) and (12), which guarantees the positivity of the energy of the system. The present result contributes significantly to the existing literature on swelling porous elastic problems. In future work, we will investigate the system with some damping and source terms.

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