mathematics

Article

Modeling of Nonlinear Sea Wave Modulation in the Presence of Ice Coverage

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Abstract: A model accounting for the influence of ice coverage on the propagation of surface sea waves is suggested. The model includes higher-order linear and nonlinear terms in the equation of wave motion. The asymptotic solution is obtained to account for nonlinear modulated wave propagation and attenuation. Two kinds of attenuation are revealed. The influence of the higher-order nonlinear, dispersion, and dissipative terms on the shape and velocity of the modulated nonlinear wave is studied. Despite the presence of higher-order terms in the original equation, the modulated solitary wave solution contains free parameters, which is important for the possible generation of such waves.

Keywords: nonlinear equation; ice; asymptotic solution; modulated wave

MSC: 35C20; 35C07; 35Q55

Citation: Porubov, A.V.; Krivtsov, A.M. Modeling of Nonlinear Sea Wave Modulation in the Presence of Ice Coverage. *Mathematics* **2023**, *11*, 4805. https://doi.org/10.3390/math11234805

Academic Editor: Nikolai A. Kudryashov

Received: 22 October 2023
Revised: 24 November 2023
Accepted: 27 November 2023
Published: 28 November 2023

1. Introduction

The influence of ice on sea waves' propagation and attenuation attracts considerable interest [1–8]. Conventional modeling includes use of the Navier–Stokes equations and various models for elastic and visco-elastic sheets to describe the dynamics of the ice on the water surface. As a result, an equation for the water surface variations is obtained.

Description of more realistic sea ice dynamics results in more complex modeling by adding so-called higher-order linear and nonlinear terms in the governing equation [9–12]. Higher order means a higher-order derivative for the linear terms and a higher-order power together with a higher-order derivative for nonlinear terms. The higher-order linear terms frequently have a dispersion nature; they describe the dependence of the wave velocity on its length. Dissipative terms can be included using the generalization of the Maxwell and Focht models [4–8,13].

Analysis of these generalized models faces serious problems. An important class of the solutions is a nonlinear localized traveling wave solution or a traveling solitary wave solution. Such waves transfer considerable localized energy. These solutions appear as a result of a balance of nonlinearity, dispersion, and dissipation. In particular, the known traveling solitary wave solution to the Korteweg-de Vries (KdV) equation appears as a result of a balance between nonlinearity and dispersion. It requires a specific initial condition in the form of the single solitary wave solution at $t = 0$. It is known in the KdV case that a more general input splits into a sequence of solitary waves of different amplitude and velocity; however, each of them can be described by the single solitary wave solution. This happens since the single solution contains a free parameter allowing various amplitudes of the wave.

Like the KdV equation, many nonlinear wave equations admit a single traveling solitary wave solution. However, the amplitude and/or the velocity of the wave are usually fixed by the coefficients of equation, and no waves with different amplitudes propagate. Then, an arbitrary localized input cannot produce a sequence of solitary waves with different amplitudes, while specific initial conditions are unlikely to realize in practice.

A possible way to overcome this problem is to consider another class of the solutions, the so-called modulated waves [14–20]. An asymptotic procedure is applied to transform the problem of obtaining the original model equation solution to the problem for finding the solution to the equation for the amplitude part of the solution. The familiar modulation equation is the nonlinear Schrödinger equation (NLSE) and its generalizations [21]. One can note only a few papers where modulation NLSE is used for ice–sea dynamics [22,23].

The advantage of the approach in this paper is based on the simultaneous inclusion of the higher-order terms in the model and looking for the solutions in the form of a modulated nonlinear wave. For this purpose, a phenomenological model is suggested. The asymptotic procedure is applied to derive the model equation for the wave amplitude, whose solution contains free parameters. Then, we examine the role of higher-order nonlinear, dispersion, and dissipative terms on the modulated nonlinear wave behavior using the asymptotic solution to the modulation equation. The presence of free parameters in the traveling wave solution allows us to anticipate its arising in unsteady processes. A disadvantage could be connected with the restricted class of the solution; however, universal consideration of nonlinear waves is impossible as a rule.

2. Model Equation

We consider the model for the surface waves under an ice sheet including nonlinear, dispersive, and dissipative factors. The influence of these factors is described by the so-called higher-order terms.

Among the variety of the models of the ice–sea interaction, one can note nonlinear long wavelength models. These models generalize the nonlinear Korteweg–de Vries (KdV) equation to account for surface sea wave variations $u(x,t)$ due to the influence of an ice sheet. The simplest generalization concerns an addition of the higher-order dispersive term [9,10],

$$u_t + 6u\, u_x + u_{xxx} + \gamma\, u_{5x} \;=\; 0, \tag{1}$$

where γ is a constant accounting for the influence of higher-order dispersion.

This equation admits a particular exact traveling solitary wave solution [24]. The solution has fixed amplitude, and a numerical study revealed another localized wave solution [25]. This solution cannot be expressed analytically, and no relationship between the wave parameters and the equation coefficients (physical factors) was obtained.

It is known that localized waves can propagate keeping their shape and velocity due to a balance between different factors, often between nonlinearity and dispersion. Then, the generalized models of the ice–sea waves include, in addition to linear, higher-order nonlinear terms. In particular, the model was developed in [11]

$$u_t + 6u\, u_x + u_{xxx} + \alpha u\, u_{xxx} + \beta\, u_x\, u_{xx} + \gamma\, u_{5x} + \delta u^2\, u_x \;=\; 0, \tag{2}$$

where α, β, and δ are constants describing the influence of the higher-order nonlinear terms. Further generalization was suggested in [12]. The known exact localized traveling wave solutions of the generalized equations also do not possess free parameters.

As noted before, attenuation of the waves under an ice sheet is an important problem. Its solution requires the addition of the so-called dissipative terms to the model equation. One way is to use a combination of the Maxwell and Focht models. They can be applied for the linear strains [4,13]. These dissipative models are generalized for the nonlinear case in [5]. The obvious generalization takes into account the fact that the models establish a connection between stress and the derivatives of strain in the ice sheet. In the linear one-dimensional case, strain V is linearly proportional to the spacial derivative of the displacement U, $V = U_x$, and its derivatives. In the nonlinear case, $V = U_x + 1/2U_x^2$, and this is called geometrical nonlinearity. Looking at the linear models [4,13] and taking into account geometrical nonlinearity, one can suggest following equation

$$\begin{aligned} u_t + 6u\, u_x + u_{xxx} + \alpha u\, u_{xxx} + \beta\, u_x\, u_{xx} + \gamma\, u_{5x} + \delta u^2\, u_x + \\ + \kappa_1\, u - \eta_1\, u_{xx} + \kappa_2\, u^2 - \eta_2\, (u^2)_{xx} \;=\; 0, \end{aligned} \tag{3}$$

where κ_1, κ_2, η_1, η_2 are positive constants, where the terms with the coefficients η_i comes from the Focht model, while those with the coefficients κ_i follow from the Maxwell model. We do not know the exact traveling wave solution to this equation. However, the presence of dissipative terms usually makes such solutions more problematic and requires more restrictions on the coefficients of equation.

3. Wave Modulation and Attenuation

We obtain asymptotic solution to Equation (3) accounting for nonlinear wave modulation and attenuation. First, the model equation is transformed to another one using the multiple-scale method. Then, the solutions to this equation are obtained and analyzed.

3.1. Derivation of the Governing Nonlinear Modulation Equation

We introduce a small parameter ε. The small parameter accounts for weak nonlinearity and describes small values of u. The solution is sought in the form

$$u = \varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \ldots.$$

We consider the case when some of dissipative terms are small, $\eta_1 = \varepsilon^2 \tilde{\eta}_1$, $\eta_2 = \varepsilon^2 \tilde{\eta}_2$. Then, we consider dissipative factors mainly described by the Maxwell model. Moreover, we introduce fast and slow variables, so that $u_i = u_i(x, t, T, X, \tau)$, where $T = \varepsilon t$, $X = \varepsilon x$, $\tau = \varepsilon^2 t$.

It follows from Equation (3) at order ε

$$u_{0,t} + u_{0,xxx} + \gamma\, u_{0,5x} \; + \kappa_1\, u_0 \;=\; 0. \tag{4}$$

The solution to Equation (4) is sought in the form

$$u_0 = A(X, T, \tau) \exp(\imath(k\, x - \omega t)) + (^*), \tag{5}$$

where $(^*)$ is the complex conjugate. Substitution of Equation (5) into Equation (4) results in the equation for the frequency ω,

$$\imath(\omega + k^3 - \gamma k^5) - \kappa_1 \;=\; 0,$$

whose solution is sought as a sum of the real and imaginary parts, $\omega = \omega_r + \imath\, \omega_i$, where

$$\omega_r = k^3(\gamma\, k^2 - 1), \qquad \omega_i = -\kappa_1. \tag{6}$$

The presence of the negative imaginary part in the frequency ω means that the harmonic wave (5) propagates with the phase velocity ω_r/k, while its amplitude decreases as $\exp(-\omega_i\, t)$.

The equation at order ε^2 is

$$\begin{aligned} u_{1,t} + u_{1,xxx} + \gamma\, u_{1,5x} + \kappa_1\, u_1 = \\ = -u_{0,T} - 3u_{0,xxX} - 5\gamma u_{0,xxxxX} - 6u_0\, u_x - \alpha u_0\, u_{0,xxx} - \beta u_{0,x} u_{0,xx} - \kappa_2\, u_0^2. \end{aligned} \tag{7}$$

The terms at $\exp(\imath(k\, x - \omega\, t))$ in the right hand side of Equation (7) give rise to the secular terms. Then, we have to assume that

$$A_T - (3 - 5\gamma k^2)\, k^2 A_X \;=\; 0. \tag{8}$$

This means that $A = A(\xi)$, $\xi = X - W\,T$,

$$W = (3 - 5\gamma k^2)\,k^2.$$

In the absence of the secular terms, the solution to Equation (7) can be obtained in the form

$$u_1 = B(X,T,\tau)\,A^2 \exp(2\imath(kx - \omega t)) + (^*), \tag{9}$$

where $B = B_r + \imath\,B_i$,

$$B_r = \frac{\kappa_1\,\kappa_2 + 6k^4(5\gamma k^2 - 1)((\alpha+\beta)k^2 - 6)}{\kappa_1^2 + 36k^6(5\gamma k^2 - 1)^2}$$

$$B_i = \frac{\kappa_2\,k^3(5\gamma k^2 - 1) - \kappa_1\,k\,((\alpha+\beta)k^2 - 6)}{\kappa_1^2 + 36k^6(5\gamma k^2 - 1)^2}.$$

Next, order (ε^3) equation is

$$\begin{gathered} u_{2,t} + u_{2,xxx} + \gamma\,u_{2,5x} + \kappa_1\,u_2 = \\ = -u_{0,\tau} - u_{1,T} - 3u_{1,xxX} - 3u_{0,xXX} - 5\gamma u_{1,xxxxX} - 10\gamma u_{0,xxxXX} - \\ -6(u_0\,u_1)_x - 6u_0\,u_{0,X} - \delta u_0^2\,u_{0,x} - \alpha u_0\,u_{1,xxx} - \alpha u_1\,u_{0,xxx} - 3\alpha u_0\,u_{0,xxX} - \\ -\beta u_{0,X} u_{0,xx} - 2\beta u_{0,x} u_{0,xX} - \beta(u_{0,X} u_{1,x})_x - 2\kappa_2\,u_0\,u_1 + \tilde{\eta}_1\,u_{0,xx}. \end{gathered} \tag{10}$$

Equating to zero, the terms at $\exp(\imath(kx - \omega t))$ in the right hand side of Equation (10) result in the equation for the amplitude A, required to suppress the secular terms,

$$\imath A_\tau + pA^2A^* - k\,(3 + 10\gamma\,k^2)\,A_{\xi\xi} + \imath\,\tilde{\eta}_1 k^2\,A = 0, \tag{11}$$

where

$$\begin{gathered} p = p_r + \imath p_i, \\ p_r = B_r\,k\,(7\alpha\,k^2 - 6) - 2B_i(\beta\,k^3 - \kappa_2) - \delta\,k, \\ p_i = B_i\,k\,(k^2(7\alpha - 2\beta) - 6) - 2\kappa_2\,B_r. \end{gathered}$$

This equation generalizes the integrable Nonlinear Schrödinger equation (NLSE) [14–17]; however, it differs from the Ginzburg–Landau equation (another generalization of NLSE) due to the absence of an imaginary part in the coefficient at $A_{\xi\xi}$.

3.2. Attenuation of Modulated Waves

The solutions to Equation (11) are sensitive to the sign of the product of the coefficients at A^2A^* and $A_{\xi\xi}$. The sign, in turn, depends on the coefficients at the higher-order terms in Equation (3).

We begin with the case of the absence of dissipation, $\kappa_i = 0$, $\eta_i = 0$. Equation (3) becomes the NLSE. The analytical solution is sought by separating the real and imaginary parts of the equation using the following representation:

$$A = \Phi(\tau,\xi)\exp(\imath\varphi(\tau,\xi)),$$

where Φ and φ are real functions. Then, the real and imaginary parts of Equation (11) are

$$p_r\Phi^3 - k\,(3 + 10\gamma\,k^2)(\Phi_{\xi\xi} - \Phi\,\varphi_\xi^2) - \Phi\,\varphi_\tau = 0, \tag{12}$$

$$\Phi_\tau - k\,(3 + 10\gamma\,k^2)(\Phi\varphi_{\xi\xi} + 2\Phi_\xi\,\varphi_\xi) = 0. \tag{13}$$

The known NLSE exact solutions can be obtained if we assume that $\Phi = \Phi(\chi)$, $\varphi = \varphi(\tau, \chi)$, where $\chi_\xi = 1$, $\chi_\tau = -w$. Then, the solution to Equation (13) is

$$\varphi = s_1 \tau + s_2 \chi, \tag{14}$$

where s_1, s_2 are constants, provided that

$$w = -2k\,(3 + 10\gamma\,k^2)\,s_2.$$

Then, Equation (12) is

$$p_r \Phi^3 - k\,(3 + 10\gamma\,k^2)\,\Phi_{\chi\chi} - \left(s_1 + k\,(3 + 10\gamma\,k^2) s_2^2\right)\Phi = 0,$$

and the so-called bright solitary wave solution [14] is

$$\Phi = \sqrt{\frac{-2k\,(3 + 10\gamma\,k^2)}{p_r}}\, m\,\mathrm{sech}(m\,\chi), \tag{15}$$

provided that

$$s_1 = -k\,(3 + 10\gamma\,k^2)(s_2^2 + m^2), \tag{16}$$

where m is a free parameter. This localized wave solution exists at $2k\,(3 + 10\gamma\,k^2)/p_r < 0$. It always happens in the absence of higher-order terms, $\gamma = 0,\ \alpha = 0,\ \beta = 0,\ \delta = 0$. When only the linear higher-order term is taken into account, $\alpha = 0,\ \beta = 0,\ \delta = 0$, the absolute value of p_r always decreases with an increase in γ. The coefficient γ also affects the value of the coefficient at dispersion term $A_{\xi\xi}$ in Equation (11). The variation in the value of the coefficients at higher-order nonlinear terms, $\alpha,\ \beta,\ \delta$, also influence both the value and the sign of the ratio $2k\,(3 + 10\gamma\,k^2)/p_r$.

At $2k\,(3 + 10\gamma\,k^2)/p_r > 0$, there exists another known so-called dark solitary wave solution expressed through the *tanh* function that does not vanish at infinity [14]. That is why the sign of the ratio of coefficients is important.

Assume that the coefficients at dissipative terms in Equation (11), p_I and η_1 are small and introduce a small parameter ε_d to account for this smallness, $p_I = \varepsilon_d \tilde{p}_I$, $\tilde{\eta}_1 = \varepsilon_d \tilde{\tilde{\eta}}_1$,

$$\imath A_\tau + p_r\,A^2 A^* - k\,(3 + 10\gamma\,k^2)\,A_{\xi\xi} = -\imath\,\varepsilon_d(\tilde{p}_i A^2 A^* + \tilde{\tilde{\eta}}_1 k^2\,A). \tag{17}$$

We also introduce the slow variable $z = \varepsilon_d \tau$ and assume that $\Phi = \Phi(\chi, z)$, $\varphi = \varphi(\zeta, \chi, z)$, where

$$\chi_\xi = 1, \quad \chi_\tau = -w(z), \quad \zeta_\xi = 0, \quad \zeta_\tau = s_3(z).$$

Then, the governing equations are obtained by separating the real and imaginary parts in Equation (17),

$$w\Phi\varphi_\chi - k\,(3 + 10\gamma\,k^2)(\Phi_{\chi\chi} - \Phi\,\varphi_\chi^2) - s_3\,\Phi\varphi_\zeta + p_r\Phi^3 - \varepsilon_d\Phi\varphi_z = 0, \tag{18}$$

$$k\,(3 + 10\gamma\,k^2)(\Phi\,\varphi_{\chi\chi} + 2\Phi_\chi\,\varphi_\chi) + w\Phi_\chi - \varepsilon_d(\tilde{p}_i\Phi^3 - \tilde{\tilde{\eta}}_1 k^2\,\Phi + \Phi_z) = 0. \tag{19}$$

The asymptotic solution to Equations (18) and (19) is sought as

$$\Phi = \Phi_0 + \varepsilon_d\Phi_1 + \ldots, \qquad \varphi = \varphi_0 + \varepsilon_d\varphi_1 + \ldots.$$

The leading-order problem is

$$w\Phi_0\varphi_{0,\chi} - k\,(3 + 10\gamma\,k^2)(\Phi_{0,\chi\chi} - \Phi_0\varphi_{0,\chi}^2) - s_3\,\Phi_0\varphi_{0,\theta} + p_r\Phi_0^3 = 0, \tag{20}$$

$$k\,(3 + 10\gamma\,k^2)(\Phi_0\,\varphi_{0,\chi\chi} + 2\Phi_{0,\chi}\,\varphi_{0,\chi}) + w\Phi_{0,\chi} = 0. \tag{21}$$

The solution to Equation (21) is

$$\varphi_0 \ = \zeta + s(z)\chi, \qquad w = -2k\,(3 + 10\gamma\,k^2)\,s. \tag{22}$$

The bell-shaped solitary wave solution is

$$\Phi_0 = \sqrt{\frac{-2k\,(3 + 10\gamma\,k^2)}{p_r}}\,m\,\mathrm{sech}(m\,\chi), \tag{23}$$

where

$$s_3 = -k\,(3 + 10\gamma\,k^2)(m^2 + s^2). \tag{24}$$

The next-order (ε_d) problem is

$$\begin{gathered} w(\Phi_1\varphi_{0,\chi} + \Phi_0\varphi_{1,\chi}) - k\,(3 + 10\gamma\,k^2)(\Phi_{1,\chi\chi} - 2\Phi_0\varphi_{0,\chi}\varphi_{1,\chi}) - \\ - s_3\,(\Phi_1\varphi_{0,\theta} + \Phi_0\varphi_{1,\theta}) + 3p_r\Phi_0^2\Phi_1 - \Phi_0\ \varphi_{0,z}\ = 0, \end{gathered} \tag{25}$$

$$\begin{gathered} k\,(3 + 10\gamma\,k^2)(\Phi_1\varphi_{0,\chi\chi} + \Phi_0\varphi_{1,\chi\chi} + 2\Phi_{1,\chi}\varphi_{0,\chi} + 2\Phi_{0,\chi}\varphi_{1,\chi}) + w\Phi_{1,\chi} - \\ - \tilde{p}_i\Phi_0^3 + \tilde{\tilde{\eta}}_1k^2\,\Phi_0 - \Phi_{0,z} = 0. \end{gathered} \tag{26}$$

In view of the leading-order solution, Equation (26) is rewritten as

$$2k\,(3 + 10\gamma\,k^2)(\Phi_0^2\varphi_{1,\chi})_\chi - \Phi_0\Phi_{0,z} - \tilde{p}_i\Phi_0^4 - \tilde{\tilde{\eta}}_1\Phi_0^2 = 0, \tag{27}$$

where

$$\Phi_{0,z} = \frac{m_Z}{m}\,\Phi_0 + \frac{m_Z}{m}\,\chi\,\Phi_{0,\chi}.$$

The solution to Equation (27) with Φ_0 defined by Equation (23) is obtained by a standard procedure for finding solutions to ordinary differential equations,

$$\begin{gathered} \varphi_1 = \frac{1}{k\,(3{+}10\gamma\,k^2)}\left(\frac{1}{2m}\left(\frac{m_Z}{4m^2} + \frac{\tilde{\tilde{\eta}}_1k^2}{2m} - \frac{2m\,\tilde{p}_i\,k\,(3{+}10\gamma\,k^2)}{3p_r}\right)\cosh(2m\chi) - \right. \\ \left. - \frac{2k\,\tilde{p}_is}{3p_r\,m}\log\Big(\cosh(m\chi)\Big) - \frac{m_z}{2m}\chi^2\right). \end{gathered} \tag{28}$$

Preventing exponential growth in the solution results in the following equation for m, and we assume

$$m_z + 2\tilde{\tilde{\eta}}_1k^2m - \frac{8m^3}{3}\,\frac{\tilde{p}_i}{p_r}\,k\,(3 + 10\gamma\,k^2) = 0. \tag{29}$$

The general solution of Equation (29) is

$$m = \frac{a_1}{\sqrt{\frac{4\tilde{p}_i(3+10\gamma\,k^2)\,a_1^2}{3p_r\tilde{\tilde{\eta}}_1\,k} + \exp(4\tilde{\tilde{\eta}}_1\,k^2z)}}, \tag{30}$$

where a_1 is a free parameter. Solution (30) accounts for a decrease in m caused by dissipation, and it is bounded and real for $\tilde{p}_i\ <\ 0$. When the contribution of the Focht model is negligibly small, one can assume $\tilde{\tilde{\eta}}_1 = 0$. Then, the general solution to Equation (29) has another form:

$$m = \frac{1}{\sqrt{m_0^{-2} - 16\tilde{p}_i/\,p_r\,(3 + 10\gamma k^2)\,k\,z}},$$

where m_0 is a free parameter. Again, the solution is decreasing, bounded, and real for $\tilde{p}_i\ <\ 0$.

The solution to Equation (25) does not contain exponentially growing parts when $\varphi_{0,z} = 0$ or when we assume $s_z = 0$.

The obtained solution accounts for an attenuation of the modulated wave (23) due to the variation of m described by Equation (30).

4. Discussion

Attenuation of the wave due to dissipation is caused by the processes undertaking on fast and slow variables. An influence of the Maxwell model contribution is described by the terms with coefficients κ_i. The linear term with coefficient κ_1 in Equation (3) gives rise to nonzero ω_I in Equation (6). It is responsible for a decrease in the fast variable. A nonlinear term with coefficient κ_2 provides the existence of the coefficient p_I at the nonlinear term in Equation (11). It affects the solution for variations of m, Equation (30); however, the decrease in m is mainly described by the coefficient $\tilde{\eta}_1$ following from the contribution of the linear part of the Focht model in Equation (3). There is no decrease in the fast variable when the coefficients κ_i are small. In this case, nonlinear dissipative terms are negligibly small, while the linear ones describe together a decrease in m, hence in the amplitude and velocity of the nonlinear modulated wave.

One has to note two kinds of attenuation. The first of these is described by only a decrease in the amplitude of solution (5) as $\exp(-\omega_i\, t)$; see Figure 1. Another attenuation is provided by the simultaneous variation in the amplitude and wave number due to Equations (23) and (30). A comparison of two kinds of attenuated waves is shown in Figure 2. They can be considered as linear and nonlinear attenuation, respectively.

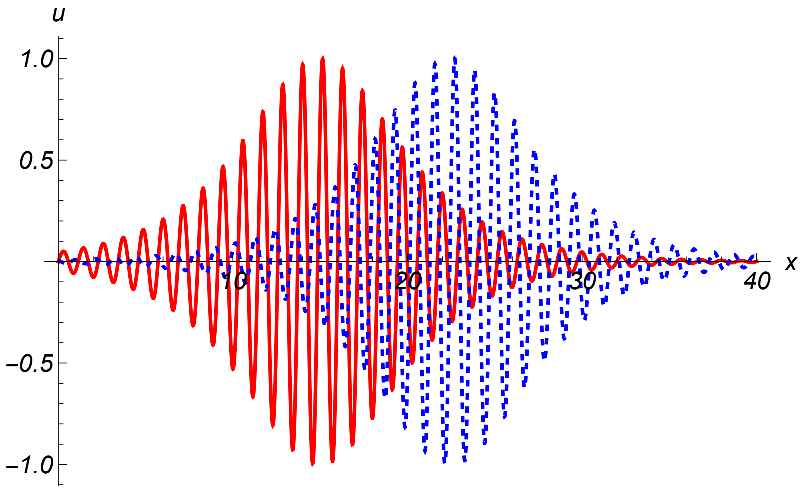

Figure 1. Dynamics of the attenuation of a modulated wave when only the envelope wave amplitude decreases. The initial profile is shown by the solid line. The dashed line accounts for the wave at some time.

The obtained modulated nonlinear wave solution contains free parameters that makes it possible to model the generation of such a wave from a rather arbitrary input. The higher-order nonlinear and dispersion terms in the original model Equation (3) affect the value and the sign of the coefficients in modulation Equation (11). It provides the existence of a bell-shaped modulated wave solution (15) and the values of its amplitude and velocity. This is important when a partly ice-covered sea is considered. The waves propagating in open sea and in the ice-covered area are governed by the equation with different coefficients. When the wave enters the covered area, it may suffer attenuation even in the absence of dissipation in the model. This process can be described numerically and will be the subject of future work.

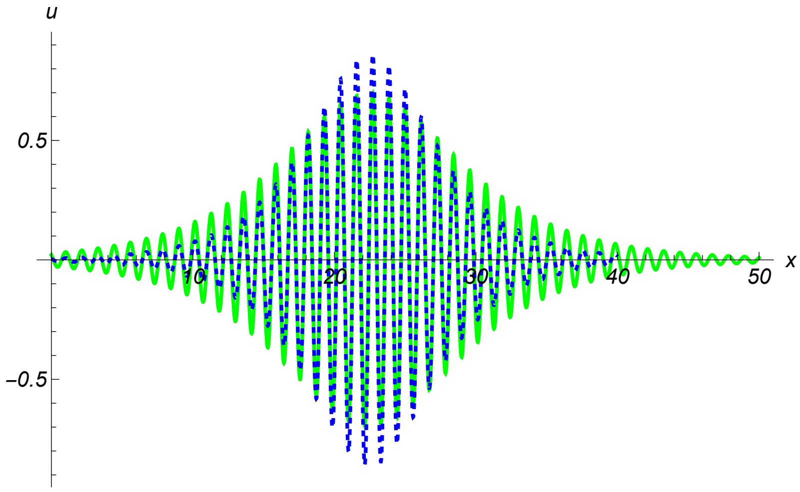

Figure 2. Comparison of the attenuation of modulated waves. The attenuation due to variation of both the envelope amplitude and wave number is shown by a solid line. The dashed line accounts for the wave when only the amplitude decreases.

Author Contributions: A.V.P. designed the research. A.V.P. and A.M.K. equally conducted the analysis and contributed to the writing of the article. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Government of the Russian Federation (state assignment 0784-2020-0021).

Data Availability Statement: Data is contained within the present article.

Conflicts of Interest: The authors declare no conflict of interest.

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