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Article

Two Velichko-like Theorems for $C(X)$ †

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Abstract: This paper provides two new Velichko-like theorems for the weak counterpart of the locally convex space $C_k(X)$ of all real-valued functions defined on a Tychonoff space X equipped with the compact-open topology τ_k.

Keywords: Velichko's theorem; K-analytic framed; angelic space; σ-compact space; (relatively) sequentially complete set

MSC: 54C35; 46A03; 54H05; 46A50

1. Preliminaries

Henceforth, unless otherwise stated, X will be a nonempty completely regular Hausdorff space. We represent by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions defined on X equipped with the *pointwise* topology τ_p. The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when provided with the weak* topology $\sigma(L(X), C(X))$, so that $\tau_p = \sigma(C(X), L(X))$. The linear space $C(X)$ equipped with the *compact-open* topology τ_k is represented by $C_k(X)$. In what follows, we shall denote by $C_w(X)$ the weak counterpart of the locally convex space $C_k(X)$, i.e., the space $C(X)$ equipped with the weak topology $\sigma(C(X), E)$, where E stands for the dual of $C_k(X)$.

The current work supplements the research carried out in [1,2] and is related to [3], Chapter 9, and [4–6]. It must be regarded as a part of the important growth of C_p-theory and C_k-theory that is taking place nowadays (see, for example, [7–10]).

2. Introduction

A set M of $C_p(X)$ is called (*relatively*) *sequentially complete* if each Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ of $C_p(X)$ contained in M converges in $C_p(X)$ to a function $f \in M$ (respectively, to some $f \in C(X)$). The classical theorem of N. V. Velichko ([11], 1.2.1 Theorem) together with two different generalizations reads as follows:

Theorem 1. *Each of the following statements implies that X is finite:*

1. *The space $C_p(X)$ is σ-compact* (Velichko).
2. *The space $C_p(X)$ is σ-countably compact* (Tkachuk and Shakhmatov, [12]).
3. *The space $C_p(X)$ is σ-bounded relatively sequentially complete* (Ferrando, Kąkol and Saxon, [2], Corollary 3.2).

Here, $C_p(X)$ is said to have a *σ-topological property* if there is a sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of $C(X)$, such that each set A_n enjoys this τ_p-topological property. Also, the term *bounded* is meant in the locally convex sense ([13], 1.4.5 Definition). The first and second

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statements of the previous theorem are equivalent by virtue of Theorem 2 below, which is a part of [14], Proposition 1 (see also [3], Proposition 9.6).

In the next example, $C_p^b(X)$ denotes the linear subspace of $C(X)$ consisting of bounded functions equipped with the relative pointwise topology. This example shows that the third statement of Theorem 1 does not work for $C_p^b(X)$ instead of $C_p(X)$.

Example 1. *If $C_p^b(X)$ is covered by a sequence of pointwise-bounded sequentially complete bounded sets, X need not be finite.*

Proof. If B stands for the closed unit ball of the Banach space $\left(C^b(\mathbb{N}), \|\cdot\|_\infty\right)$, then $C^b(\mathbb{N}) = \bigcup_{n=1}^{\infty} nB$. So, $\{nB : n \in \mathbb{N}\}$ is a sequence of pointwise-bounded sets. Since $C_p(\mathbb{N}) = \mathbb{R}^{\mathbb{N}}$ is sequentially complete, if $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C_p^b(\mathbb{N})$ contained in B, there exists $f \in \mathbb{R}^{\mathbb{N}}$, such that $f_n \to f$ in $\mathbb{R}^{\mathbb{N}}$. But, as $|f_n(y)| \leq 1$ for all $(n, y) \in \mathbb{N} \times Y$, we obtain $|f(y)| \leq 1$ for all $y \in Y$. Thus, $f \in B$, which shows that B is sequentially complete in $C_p^b(\mathbb{N})$. Consequently, each set nB is sequentially complete. But $X = \mathbb{N}$ is infinite. □

Let us recall that a *resolution* for a topological space X is a covering $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of X, such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ coordinate-wise, i.e., if $\alpha(i) \leq \beta(i)$ for every $i \in \mathbb{N}$. If E is a locally convex space, a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for E is called *bounded* if each A_α is a bounded set in E. An increasing covering $\{Q_n : n \in \mathbb{N}\}$ of a locally convex space E consisting of absolutely convex bounded sets is a trivial example of a bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for E, by setting $A_\alpha = Q_{\alpha(1)}$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$. The next result was obtained in [14], Proposition 1.

Theorem 2 (Ferrando–Kąkol)**.** *The space $C_p(X)$ has a bounded resolution if and only if $C_p(X)$ is both K-analytic-framed in $\mathbb{R}^X$ and angelic.*

The last statement of Theorem 1 is actually a consequence of [3], Lemma 9.5 together with the following characterization for X to be a P-space, in which the requirement of pointwise boundedness for the covering sequence, which is required in the third statement of Theorem 1, has been dropped ([2], Theorem 3.1).

Theorem 3 (Ferrando–Kąkol–Saxon)**.** *The space $C_p(X)$ is σ-relatively sequentially complete if and only if X is a P-space.*

In this paper, we are going to prove the following two Velichko-type theorems for the weak counterpart $C_w(X)$ of $C_k(X)$.

Theorem 4. *The space $C_w(X)$ is covered by a sequence of relatively sequentially complete sets if and only if X is a P-space.*

Theorem 5. *The space $C_w(X)$ is covered by a sequence of pointwise-bounded relatively sequentially complete sets if and only if X is finite.*

Theorems 4 and 5 are, respectively, the $C_w(X)$-version of Theorem 3 and of the third statement of Theorem 1.

3. An Auxiliary Result

Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ in $C(X)$ is called *pointwise eventually constant* (*cf.* [2]) if for each $x \in X$ there exists $f(x) \in \mathbb{R}$, such that $f_n(x) = f(x)$ for all but finitely many $n \in \mathbb{N}$. So, if $\{f_n\}_{n=1}^{\infty}$ is a pointwise eventually constant sequence in $C(X)$, there always exists some $f \in \mathbb{R}^X$, such that $f_n \to f$ pointwise on X. We shall require the following theorem, which is contained (but not explicitly stated) in [2], it being a consequence of [2], Theorem 1.1, and of the equivalence of statements (7) and (7′) of paper [2] (see [2], p. 910).

Theorem 6. *The following statements are equivalent.*

1. *Every uniformly bounded pointwise eventually constant sequence converges in $C_p(X)$.*
2. *X is a P-space.*

If X is a compact space and μ is a regular countably additive real-valued measure defined on the Borel algebra $\mathfrak{B}(X)$ of X, we shall denote by $L_0(\mu)$ the linear space of all (classes of) real-valued μ-measurable functions defined on X, and we shall denote by $rca(\mathfrak{B}(X))$ the linear space of regular countably additive Borel real measures on $\mathfrak{B}(X)$. As for the pointwise topology, a subset M of $C_w(X)$ is called *relatively sequentially complete* if each Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ of $C_w(X)$ contained in M converges in $C_w(X)$ to some $f \in C(X)$.

Theorem 7. *Let X be a compact set. The space $C_w(X)$ is σ-relatively sequentially complete if and only if X is finite.*

Proof. First, let us show that every uniformly bounded sequence in $C(X)$ that is pointwise convergent in $\mathbb{R}^X$ is a Cauchy sequence in $C_w(X)$. So, let $\{g_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence in $C(X)$ pointwise convergent in $\mathbb{R}^X$. If B denotes the closed unit ball of the Banach space $C_k(X)$, there is $\delta > 0$, such that $g_n \in \delta B$ for every $n \in \mathbb{N}$, so that $\sup_{x\in X}|g_n(x)| \leq \delta$ for all $n \in \mathbb{N}$. As $g_n \to g$ pointwise on X, then $g \in L_0(\mu)$ and

$$\langle g_n - g, \mu\rangle = \int_X (g_n - g)d\mu \to 0$$

for every $\mu \in rca(\mathfrak{B}(X))$. Thus, using the fact that $rca(\mathfrak{B}(X)) = C_k(X)^*$, we establish that $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C_w(X)$, as stated.

The following argument is based on (but not identical to) the proof of [2], Theorem 3.1. Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of weakly relatively sequentially complete subsets of $C_k(X)$. As $C_k(X) = (C(X), \|\cdot\|_\infty)$ is a Banach space, the Baire category theorem provides $m \in \mathbb{N}$, such that $\overline{A_m}$ has an interior point $h \in C(X)$ where the closure is in the norm-topology of $C_k(X)$. So, if B stands again for the closed unit ball of $C_k(X)$, there is $\epsilon > 0$, such that

$$h + \epsilon B \subseteq \overline{A_m}.$$

If $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded pointwise eventually constant sequence in $C(X)$, there is $f \in \mathbb{R}^X$, such that $f_n \to f$ in $\mathbb{R}^X$ and there exists $\delta > 0$, such that $\delta f_n \in \epsilon B$ for every $n \in \mathbb{N}$. So, $\{h + \delta f_n\}_{n=1}^{\infty}$ is a uniformly bounded pointwise eventually constant sequence in $\overline{A_m}$ that converges to $h + \delta f$ in $\mathbb{R}^X$. Clearly, for each $n \in \mathbb{N}$ there is $g_n \in A_m$, such that

$$\|h + \delta f_n - g_n\|_\infty < n^{-1},$$

where $\|\cdot\|_\infty$ denotes the norm of $C_k(X)$.

All this implies that the sequence $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded and converges pointwise to $h + \delta f$ in $\mathbb{R}^X$, so that, according to the first part of the proof, $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C_w(X)$. But, as $\{g_n\}_{n=1}^{\infty} \subseteq A_m$ and A_m is weakly relatively sequentially complete, it follows that $g_n \to h + \delta f$ in $C_w(X)$. Particularly, one has $h + \delta f \in C(X)$. As $h \in C(X)$, this means that $f \in C(X)$. Therefore, $\{f_n\}_{n=1}^{\infty}$ converges in $C_p(X)$.

Hence, according to Theorem 6, X must be a P-space. But every compact P-space is finite (see [15], Problem 4K). □

Remark 1. *Theorem 7 clearly implies the well-known fact that if X is a compact space the Banach space $C_k(X) = (C(X), \|\cdot\|_\infty)$ is weakly sequentially complete if and only if X is finite (note that if the compact set X is infinite, then $(C(X), \|\cdot\|_\infty)$ contains an isomorphic copy of c_0, which is not weakly sequentially complete).*

4. Proofs of Theorems 4 and 5

In the present section, we prove Theorems 4 and 5, which are concerned with the weak topology τ_w of $C_k(X)$ rather than the pointwise topology τ_p. We shall need the following result, which is an extension of a classic result of C_p-theory (see [16], Proposition 4.1), which we have borrowed from [17].

Lemma 1 ([17], Lemma 1)**.** *Let X be completely regular. If Q is a metrizable and compact subspace of X there exists a continuous-linear-extender map $\varphi : C_k(Q) \to C_k(X)$, i.e., such that $\varphi(f)|_Q = f$ for every $f \in C(Q)$.*

4.1. Proof of Theorem 4

Proof. Suppose that $C(X)$ is covered by a sequence $\{A_n : n \in \mathbb{N}\}$ consisting of weakly relatively sequentially complete sets. We claim that all compact sets of X are finite.

Assume for the sake of contradiction that there is an infinite compact set Q in X that is, hence, metrizable by the Urysohn metrizability theorem. By Lemma 1, there is a linear-continuous map $\varphi : C_k(Q) \to C_k(X)$, such that $\varphi(f)|_Q = f$, i.e., a continuous linear extender when $C(Q)$ is regarded as a Banach space. Consequently, the linear map φ is weak-to-weak continuous and is, hence, uniformly continuous for the weak topologies. Let us observe that $\{\varphi^{-1}(A_n) : n \in \mathbb{N}\}$ is a countable covering of $C_k(Q)$ consisting of weakly relatively sequentially complete sets. In fact, if $\{f_n\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in $\varphi^{-1}(A_m)$ then $\{\varphi(f_n)\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in A_m, due to φ being uniformly continuous, so that there is $h \in C(X)$, such that $\varphi(f_n) \to h$ in $C_w(X)$. Now the restriction map $T : C_k(X) \to C_k(Q)$ defined by $Tf = f|_Q$ is continuous, which implies that T is also weak-to-weak continuous. Therefore, $T\varphi(f_n) \to Th$ in $C_w(Q)$. Hence, setting $f := Th = h|_Q \in C(Q)$ and employing $T\varphi(f_n) = \varphi(f_n)|_Q = f_n$, it follows that $f_n \to f$ in $C_w(Q)$. Thus, the set $\varphi^{-1}(A_m)$ is relatively sequentially complete in $C_w(Q)$, which shows, as stated, that the sequence $\{\varphi^{-1}(A_n) : n \in \mathbb{N}\}$ is a covering of $C_k(Q)$ consisting of weakly relatively sequentially complete sets. Now the application of Theorem 7 guarantees that Q is finite, as desired.

As every compact set of X is finite, we obtain $C_p(X) = C_w(X) = C_k(X)$, which means that the space $C_p(X)$ is σ-relatively sequentially complete. Hence, Theorem 3 yields that X is a P-space.

Conversely, if X is a P-space, the compact sets of X are finite and, consequently, we obtain $C_p(X) = C_w(X)$. So, it follows from Theorem 3 that $C_w(X)$ is σ-relatively sequentially complete. □

4.2. Proof of Theorem 5

Proof. Finite X ensures a suitable sequence, with each $A_n = nB$, where B is the closed unit ball in the finite-dimensional Banach space $C_p(X) = C_w(X) = C_k(X)$. To prove the converse, recall the well-known property that a Tychonoff space X is pseudocompact if and only if $C_p(X)$ is σ-bounded (in the locally convex sense). Hence, if we suppose that space $C(X)$ is covered by a sequence $\{A_n : n \in \mathbb{N}\}$ of pointwise-bounded weakly relatively sequentially complete sets, then X is pseudocompact, and by Theorem 4 we obtain that X is a P-space. Therefore, X must be finite. □

We thank our anonymous reviewer for the simplification of this proof by employing the mentioned characterization of pseudocompactness of a Tychonoff space X.

Remark 2. *It is shown in [3], Proposition 9.18, that if $C_p(X)$ has a fundamental sequence consisting of bounded sets (i.e., a sequence of bounded sets that swallows the bounded sets) then X is finite. Of course, this property does not hold for $C_w(X)$, because if X is any compact set and B stands for the closed unit ball of $C_k(X)$ then $\{nB : n \in \mathbb{N}\}$ is a fundamental sequence of bounded sets for $C_w(X)$.*

5. Conclusions

In this paper, we have provided a complement to the research of [1,2] with Theorems 4 and 5, stating that *if X is a Tychonoff space, then $C_k(X)$ is covered by a sequence of*

1. *Weakly relatively sequentially complete sets if and only if X is a P-space.*
2. *Pointwise-bounded weakly relatively sequentially complete sets if and only if X is finite.*

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