mathematics

Article

Optimizing Traffic Light Green Duration under Stochastic Considerations

Krasimira Stoilova * and Todor Stoilov

Institute of Information and Communication Technologies-Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria
* Correspondence: krasimira.stoilova@iict.bas.bg

Abstract: An optimization model for traffic light control in an urban network of intersections is derived. The model is based on store-and-forward analytic relations, which account for the length of the queue of waiting vehicles in front of the traffic light intersection. The model is complicated with probabilistic relations that formalize the requirements for maintaining short queues of vehicles. Probabilistic inequalities apply to each intersection of the city network. Approximations of probability inequalities are given in the article. Quadratic deterministic inequalities, which are part of the set of the traffic flow control optimization problem, are derived. Numerical simulations are performed, applying mean estimated data for real traffic in an urban area of Sofia. The model predictive approach is applied to traffic light optimization and control. Empirical results give advantages of the obtained model compared to the classical store-and-forward optimization model for the total number of vehicles waiting in the considered urban network.

Keywords: traffic optimization; probabilistic inequality; approximations; model predictive approach

MSC: 90B20; 93A30; 90B22

check for updates

Citation: Stoilova, K.; Stoilov, T. Optimizing Traffic Light Green Duration under Stochastic Considerations. *Mathematics* **2023**, *11*, 496. https://doi.org/10.3390/math11030496

Academic Editor: Aleksandr Rakhmangulov

Received: 15 December 2022
Revised: 9 January 2023
Accepted: 15 January 2023
Published: 17 January 2023

1. Introduction

This study aims to derive an optimization model for traffic light control that takes into account the stochastic behavior of traffic flows. The importance of the problem of traffic control in an urban environment does not need a long argument. It is seen daily that traffic needs to be regulated by reducing the inconvenience to city dwellers from delays, traffic jams and heavy traffic that reduces traffic safety and the resulting inconveniences of increased fuel costs, environmental and noise pollution, etc. The main control effects that apply in traffic control are not so many: traffic light splitting, traffic light cycles and phase sequences. In general, green light duration is the main control tool that must be applied to control the intersection and network traffic. Quantitative traffic control models are continuously developed, verified and evaluated.

Our particular interest in this paper is how to consider the stochastic behavior of traffic flows in urban intersections in a straightforward manner. Our main goal is to incorporate stochastic constraints into an analytically defined optimization problem for green light duration estimation. Random constraints formalize the probabilistic requirements for maintaining small queues or vehicle counts.

2. Overview of the Traffic Models with Stochastic Parameters

In [1,2] the classification of traffic control strategies is presented in three categories: isolated intersection control; traffic light control through fixed time settings; coordinated and adaptive traffic-dependent signal control. The main development-oriented strategy is the latter, which must take into account the stochastic nature of vehicular traffic flows in an urban environment.

The stochastic behavior of traffic flows was the reason why traffic light parameters were estimated and evaluated by applying simulations in a software environment. An overview of the use of traffic simulation software can be found in [3,4]. In [5], the advantages of a new generation of traffic simulators, which are used for dynamic traffic analysis and the congestion prediction of interconnected road segments, are presented. The use of a traffic simulator in a Web environment is a widely accepted solution that favors its repeated access by a wider audience [6]. Twenty-nine simulation packages addressing microscopic, macroscopic, mesoscopic, homogeneous, heterogeneous, discrete and continuous flows and traffic models are evaluated in [7]. Traffic simulation is accepted as a universal tool for evaluating traffic behavior by providing stochastic parameters of traffic flows.

The stochastic components in traffic management are considered through various formal, quantitative and qualitative approaches: fuzzy logic and neural network formalizations [8–10]; the application of artificial intelligence solutions [11]; forecasting traffic flows by time series [12]; by applying learning technologies [13,14]; by supporting meta modeling [15].

In this work, we aim at the quantification of traffic management defined by analytical models that formalize in a clear way the random nature of the components of traffic behavior. These are mainly the flows that enter a city network and represent management objects. The turning and parking of vehicles inside the roads are also random in nature.

In [16], the probability of the occurrence of a suitable traffic flow scenario and the delay of a vehicle are calculated. For the set of scenarios, robust timing is estimated that gives the minimum average delay per vehicle for the set of all the traffic scenarios. The optimization problem has a nonlinear objective function that is minimized under a set of linear equalities. The stochastic nature of transport flows is formalized by probabilistic relations for modeling traffic dynamics. In [17], unknown demands and queuing uncertainty affect the control decisions. In [18], stochastic travel demands are considered by minimizing the total travel delay over successive time periods. In [19], a metaheuristic approach is applied to identify a robust plan for fixed-time signals. A robust solution is found among several scenarios of demands. In [20], a stochastic model predictive control is applied by defining and sequentially solving an optimization problem. It contains chance components from turning and parking vehicles. At each step of the predictive control, the values of the chance parameters are given by algorithmic applications of random generators. The stochastic predictive control model is presented in [21]. The uncertainty considered relates to the volumes of the traffic flows and the turning ratios of the vehicles in the flow. The uncertainty of traffic demands is formalized in a robust stochastic problem [22].

In this paper, we aim to quantify the traffic management performed in an optimization problem that contains analytical random connections that describe the stochastic nature of traffic processes in an urban network. The main feature of this research is that it applies probabilistic relations to consider stochastic events in traffic behavior that occur through turning, stopping and the parking of vehicles, events that cannot be formalized in deterministic relations. To define the traffic management optimization problem, we use the well-known store-and-forward model. We formalize the random nature of the traffic through probabilistic inequalities. These inequalities are then approximated as nonlinear algebraic relations. Numerical simulations and the evaluation of this model are performed by applying the model predictive approach. In Section 3, we present an analysis for analytically incorporating probabilistic relations into an optimization problem for green light duration estimation.

3. Basic Formalization of Traffic Management

Traffic control is aimed at reducing congestion [23]. Additional goals can be pursued by reducing vehicle emissions [24]. Mainly, traffic management is carried out through sections of traffic signals. Additions to the supporting traffic light control solutions are used, such as the use of waiting zones [25], variable speed limits and ramp metering [26]. The formal model called "store-and-forward" is intensively used to define and solve optimization problems. It originates from the works of [27]. The model is quite simple and formalizes the analogy with fluid dynamics. The idea of the model is that the traffic of a given section is a linear result of the inflows and outflows. Therefore, it was time for its intensive use to optimize the duration of green lights in urban networks [28–30]. The model is successively complicated and applied to green wave optimization [31]; to the coordination of two-way arterial routes [32]; to the application of model predictive control [33,34].

The theoretical basis of the store-and-forward model is the physical law of conservation: that the outflow in a network must equal the inflow [35,36]:

$$\boldsymbol{q}^{\boldsymbol{out}}(t+1) = \boldsymbol{q}^{\boldsymbol{in}}(t) \tag{1}$$

where $\boldsymbol{q}^{\boldsymbol{out}}(t+1)$ and $\boldsymbol{q}^{\boldsymbol{in}}(t)$ are the incoming and outgoing flows of vehicles at two consecutive times t with dimensions (vehicles/time), Figure 1.

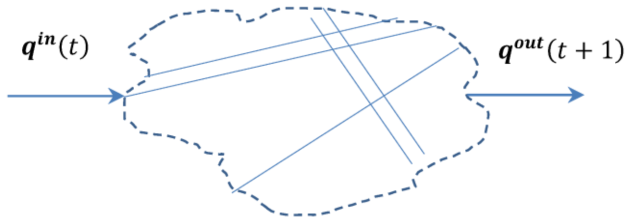

Figure 1. Conservation law as formal background of the store-and-forward model.

This model is derived from the traffic light optimization case of [27]. The formal derivation can be illustrated by applying the cellular transmission model [37]. The continuity of the flows from (1) formally insists on the fulfillment of the equality:

$$\frac{\partial q}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \tag{2}$$

where $\rho(t)$ is the traffic flow density at time t and quantified as vehicles per unit distance (vehicles/distance). Replacing the first derivatives by their approximation, it follows:

$$\frac{\rho(t+1) - \rho(t)}{T} \approx \frac{-q(t+1) + q(t)}{l} \tag{3}$$

Given that the density ρ is equal to the number of vehicles x at a distance l, $\boldsymbol{x} = \rho\,(.)l$ and the flow q in a time period T gives the number of vehicles, $\boldsymbol{x} = T\boldsymbol{q}()$, Relation (3) is modified in (4) and (6):

$$\boldsymbol{x}(t+1) = \boldsymbol{x}(t) + T[\boldsymbol{q}(t) - \boldsymbol{q}(t+1)], \tag{4}$$

where T is the duration of the control cycle. Given (1), Equation (4) becomes

$$\boldsymbol{x}(t+1) = \boldsymbol{x}(t) + T\left[\boldsymbol{q}^{in}(t) - \boldsymbol{q}^{\boldsymbol{out}}(t+1)\right]. \tag{5}$$

The product $T\boldsymbol{q}^{in}(t) = \boldsymbol{x}_{in}$ represents the vehicles entering the network. Respectively, $T\boldsymbol{q}^{out}(t+1) = \boldsymbol{x}_{out}$ gives the number of vehicles that leave the transport network for the end of the control period T. The store-and-forward model thus reduces to a simple algebraic equality:

$$\boldsymbol{x}(t+1) = \boldsymbol{x}(t) + \boldsymbol{x}_{in}(t) - \boldsymbol{x}_{out}(t), \tag{6}$$

saying that the resulting number of vehicles $\boldsymbol{x}(t+1)$ in the transport network is obtained as an algebraic summation of the current value $\boldsymbol{x}(t)$ and the difference between the incoming and outgoing vehicles. In particular, the output flow is proportional to the duration of the green light $\boldsymbol{u}$ (time) and depends on the geometric and structural configuration of the network, denoted as $\boldsymbol{s}$ (vehicles/per time). Downscaling the network to a single section of the road, the ratio (6) can be interpreted as the number of vehicles that are on a section of the street between two intersections and/or the waiting vehicles in front of a traffic light at a junction, and in discrete form this relation is:

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) + \boldsymbol{x}_{in}(k) - \boldsymbol{su}(k),\ k - \text{discrete time.} \tag{7}$$

Between two consecutive intersections with traffic lights, there may be additional uncontrolled areas where vehicles can enter/exit or stop/park. Accordingly, except $\boldsymbol{x}_{in}(k)$ additional vehicles can enter or leave the main controlled intersection, Figure 2.

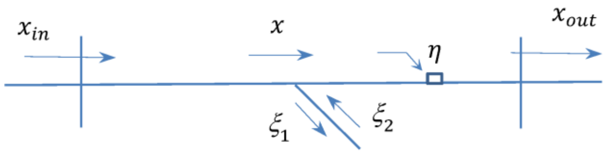

Figure 2. Unexpected fluctuations in the number of vehicles on the street.

Thus, an additional number of outgoing ζ_1 and incoming ζ_2 as well as those of parking and/or stopping η stochastically influence and affect the total value of vehicles x. The influence of these factors ζ_1, ζ_2, η is random and the relation (7) can be formally rewritten as:

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) + \boldsymbol{x}_{in}(k) - \boldsymbol{su}(k) + \boldsymbol{\varepsilon}(k), \quad \boldsymbol{\varepsilon} = (\zeta_1, \zeta_2, \eta) \tag{8}$$

but the difference with (7) is that the value of $\boldsymbol{x}$ becomes stochastic. Accordingly, the control problem that must estimate the duration of the green light $\boldsymbol{u}$ must take into account that Relation (8) is stochastic. The motivation of this work is based on the existence of stochastic events ζ_1, ζ_2, η, which cannot be evaluated and formalized with deterministic and formal relations. This is the case of turning, stopping and the parking of vehicles in road sections in an urban network, which are not taken into account for the control of traffic lights in the deterministic case.

In the deterministic case, the optimization problem is defined by the minimization of the objective function up to the green light duration $\boldsymbol{u}$ and/or x, given the deterministic Equality (7):

$$\min_{\boldsymbol{u},\boldsymbol{x}} F(\boldsymbol{x},\boldsymbol{u}), \tag{9}$$

$$\boldsymbol{x}(k+1) \leq \boldsymbol{x}(k) + \boldsymbol{x}_{in}(k) - \boldsymbol{su}(k),$$

$$u_{min} \leq \boldsymbol{u}(k) \leq u_{max}, 0 \leq \boldsymbol{x}(k) \leq \boldsymbol{x}_{max},$$

where $\boldsymbol{x}$ and $\boldsymbol{u}$ are vectors between the upper and lower bounds. The relation (7) transforms into an inequality. The scale of the transport network and its controlled intersections determine the size of the vectors $\boldsymbol{u}$ and $\boldsymbol{x}$. The objective function $F(\boldsymbol{x},\boldsymbol{u})$ can formalize the minimization of the sum of the number of vehicles or the queue lengths for the entire considered network, $\sum_{i=network} x_i$ [1], the Total Time Spent and/or the maximization of the output flows $\boldsymbol{q}^{out}$, which formalization is also related to the sum value, $\sum_{i=network} x_i$, [1,38,39].

In reality, there is uncertainty not only about the driver's decision about the formation of the volume value $\boldsymbol{x}$, but also from the traffic conditions that form the value of the incoming vehicles $\mathbf{x}_{in}$. The formal description of (8) allows the stochastic component $\boldsymbol{\varepsilon}(k)$ to take into account the stochastic and/or random nature of traffic demands.

As a consequence of (8), the values of $\boldsymbol{x}$ are also random variables. This insists that the analytic constraint $0 \leq \boldsymbol{x} \leq x_{max}$ takes a probabilistic form. This article applies a probabilistic inequality of the form:

$$P(\boldsymbol{x} \leq x_{max}) \geq \gamma \quad \text{or} \quad P(\boldsymbol{x}(k) \geq x_{max}) \leq 1 - \gamma, \tag{10}$$

which means that the probability that the number of vehicles $\boldsymbol{x}$ is less than the upper bound x_{max} on a road section in the network must be higher than γ, where γ is a confidence level.

The Relation (10) is applied in portfolio theory, saying that the probability of losses of an investment x should be lower than the value x_{max} and higher than the level of γ [40]. This relationship in portfolio theory is called Value at Risk (VaR), which is applied to assess the risk of investment decisions.

For the case of traffic control, the probabilistic Relation (10) can be applied as a constraint to the optimization problem or appropriately used as an objective function (this is the case in this study). For the constraint case and considering (10), the optimization problem has the following description:

$$\min_{\boldsymbol{u},\boldsymbol{x}} F(\boldsymbol{x},\boldsymbol{u}), \tag{11}$$

$$\boldsymbol{x}(k+1) \le \boldsymbol{x}(k) + \boldsymbol{x}_{in}(k) - \boldsymbol{su}(k),$$

$$u_{min} \le \boldsymbol{u}(k) \le u_{max},$$

$$P(\boldsymbol{x}(k) \ge x_{max}) \le 1 - \gamma.$$

Problem (11) contains probabilistic dependencies that make it difficult to solve. In [20], such a class of problems is solved by stochastically generating values of the random variable ε. Then, a predictive control model approach is applied to solve the discrete dynamic problem (11). However, for actual implementation, only the first element of the optimal solution is applied. Then, a new generation of ε and a solution of the new problem (11) are performed. An alternative solution to (11) is also demonstrated by a neural network approximation of the probabilistic Relation (10), which is given by training a neural network with historical data.

In [41], such an optimization problem with probabilistic inequalities was solved by a reduced gradient approach. The gradients are estimated as a combination of the probability of univariate and bivariate normal distributions. The estimators simultaneously estimate the probabilistic bounds and gradients of the objective function for each computational iteration. This computational approach is quite complex and time-consuming.

In [42], the probabilistic constraint (10) is transformed into a deterministic one. Unfortunately, the applied way of approximating the initial probabilistic constraint to a deterministic one contains many integral components. This makes the resulting nonlinear optimization problem difficult to solve.

The approach taken here is to approximate the probability inequality with a deterministic analytic relation. The approximation is made by using the VaR concept of risk assessment in portfolio investment theory. This approximation allows the resulting deterministic relation to have a quadratic form. This approach allows adding analytical relations that consider stochastic events in traffic behavior that cannot be formalized and taken into account in a deterministic way. Stochastic relations after approximations can be used as additional constraints and/or as an objective function. Our particular solution, which differs from the analysis provided above, is to define a stochastic parameter as an objective function in a traffic light control optimization problem. This allows traffic control to explicitly take into account stochastic changes in traffic behavior.

4. Deterministic Approximation of the Probabilistic Inequality

Without a lack of generalization, we assume that the stochastic values of the number of vehicles on road section i in the transport network are x_i, which values have a normal distribution. We assumed a normal distribution of x_i for physical and engineering reasons. In an urban network environment, the road sections are not that long (this is not the case with the highway), which gives reason to assume that the variance of x_i will not be higher than the mean value of the x_i of that section. The physical capacity of a given road section gives us reason for this assumption. This assumption leads to the shapes of the Probability Density Function (PDF) and the Cumulative Density Function (CDF) given in Figures 3 and 4.

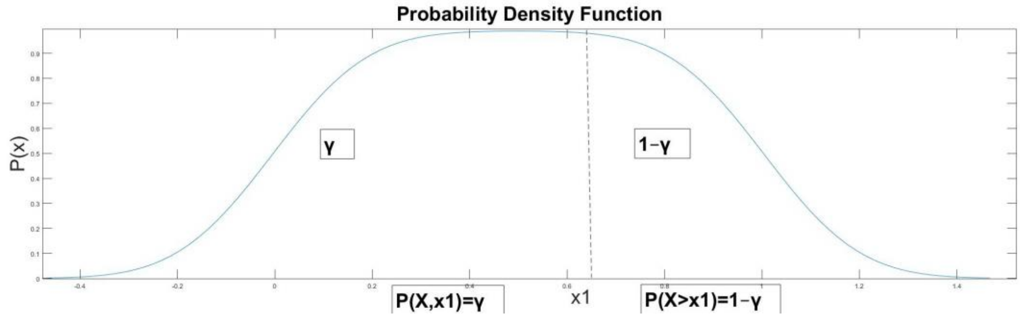

Figure 3. Probability Density Function of Normal distribution.

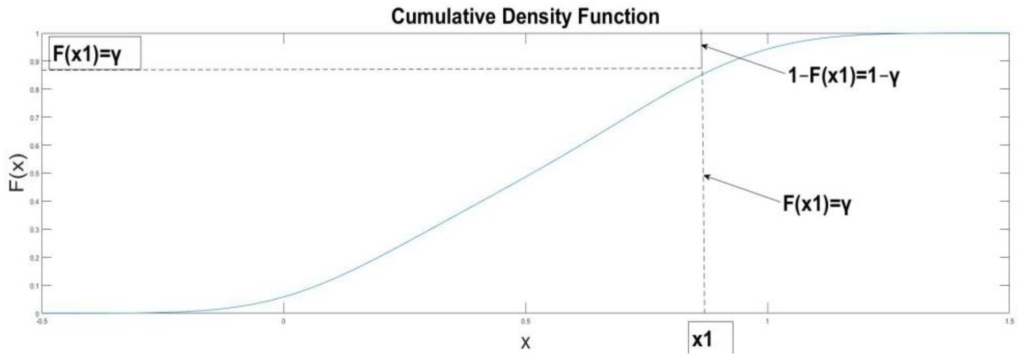

Figure 4. Cumulative Density Function of Normal distribution.

The following relations are included in the following conclusions derived from the relationship between the CDF and PDF functions.

$$F(x) = P(x < x_{\max}) > \gamma \quad \text{or} \quad 1 - F(x) = P(x > x_{\max}) \leq 1 - \gamma, \tag{12}$$

where $P()$ is the PDF and $F()$ is the CDF function. x is the set of all values that the stochastic variable can take and x_{max} is one of those values; γ is the confidence level.

The probabilistic constraint (10) is considered in the form:

$$P(x \leq x_{max}) \ \geq \ \gamma. \tag{13}$$

The inner inequality of the probability function $P()$ is multiplied by (-1) to change the direction of the inequality:

$$P(-x \geq -x_{max}) \ \geq \ \gamma. \tag{14}$$

The relation (14) is normalized for a stochastic variable with zero mean $E_x = 0$ and standard deviation equal to 1, $\sigma_x = 1$. The Relation (14) retains its form, but the values in the PDF are normalized:

$$P(\frac{-x - E_x}{\sigma_x^2} \geq \ \frac{-x_{max} - E_x}{\sigma_x^2}) \geq \gamma. \tag{15}$$

Using relation (12), (15) can be rewritten in the form:

$$P(\frac{-x - E_x}{\sigma_x^2} \geq \ \frac{-x_{max} - E_x}{\sigma_x^2}) = 1 - \mathrm{F}(\frac{-x_{max} - E_x}{\sigma_x^2}) \geq \gamma$$

or

$$F(\frac{-x_{max} - E_x}{\sigma_x^2}) \leq 1 - \gamma. \tag{16}$$

By multiplying both sides of inequality (16) by the inverse function F^{-1}, it follows:

$$-\ x_{max} \ -E_x \leq \sigma_X^2 F^{-1}(1 - \gamma). \tag{17}$$

The value $F^{-1}(1-\gamma)$ is the Z-score of a normalized stochastic function with a normal distribution. This value can be taken from tables [43]. Taking Relation (12), if the confidence level is $\gamma = 90\%$, it means that the probability that the number of vehicles x is lower than the value x_{max}, is supported with a probability of 90%. Accordingly, the probability that the number of vehicles is greater than x is the value $1-\gamma = 10\%$. From [43], the Z-score value is $F^{-1}(1-\gamma) = -1.282$. Using the notation $\delta = F^{-1}(1-\gamma)$, relation (17) takes the deterministic algebraic form

$$-E_x + \delta\sigma_x^2 \le x_{max} \quad (18)$$

where x_{max} is the predefined value of the number of vehicles that can be accepted on this road section and that participate in the upper bounds of the optimization problem (9), or $x = x_{max}$. We substitute the probabilistic inequality (18) into (11) and the problem becomes nonlinear deterministic. The graphical interpretation of the dependence (18) is given in Figure 5. The difference between the mean value E_x and the normalized volatility $\delta\sigma_x^2$ of the stochastic variable x must be greater than the value $-x_{max}$, or $E_x - \delta\sigma_x^2 \ge -x_{max}$. This means that the stochastic variable must be a value in the set $-x_{max} \le \boldsymbol{x} \le +x_{max}$ as shown in Figure 5.

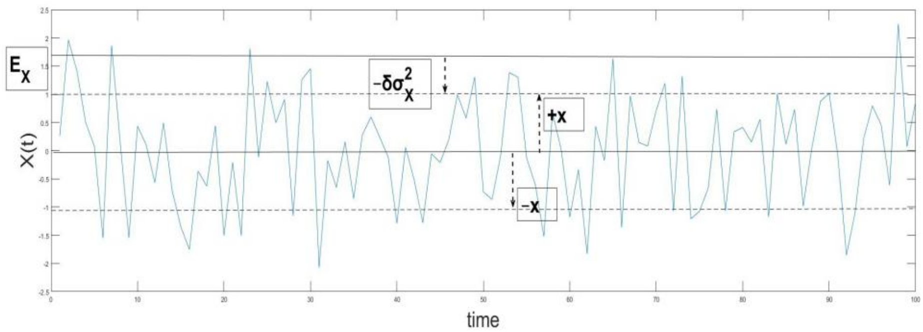

Figure 5. Graphical interpretation of inequality (18).

The probability inequality can be incorporated into the optimization problem in the constraint set as in (11) or in the objective function. The control problem that is defined here uses relation (18) as the optimization objective function. The problem will estimate the duration of green lights $\boldsymbol{u}$, which will minimize the number of vehicles x at a suitable intersection for the set of values $-x_{max} \le \boldsymbol{x} \le +x_{max}$. Thus, the objective function of the control problem will be the minimization to the arguments $\boldsymbol{u}$ of the sum of vehicles $\sum\limits_{i \in junctions} x_{maxi}(\boldsymbol{u})$, where the value $x_{maxi}(\boldsymbol{u})$ must be obtained for each junction according to the relation (18).

To incorporate (18) as the objective function of an optimization problem such as (9), the values E_X and σ_X^2 values must be described as functions of the green light duration $\boldsymbol{u}$. Therefore, the optimization problem (11) is transformed into (19) with an objective function according to the probability inequality (18). The optimization problem becomes:

$$\min_{\boldsymbol{u}(k)} F(u) = \sum_{k=1}^{K} [x_{max}(\boldsymbol{u}(k)) = -E_x(\boldsymbol{x}(k), \boldsymbol{u}(k)) + \delta\sigma_x^2(\boldsymbol{x}(k), \boldsymbol{u}(k))] \quad (19)$$

$$\boldsymbol{x}(k+1) \le \boldsymbol{x}(k) + \boldsymbol{x}_{in}(k) - \boldsymbol{su}(k),$$

$$u_{min} \le \boldsymbol{u}(k) \le u_{max},\ K - \text{ horizon for optimization.}$$

5. A model Predictive Framework for Traffic Light Control

The idea of the predictive control model is to predict the future state of the control system and evaluate the appropriate control influence. In dynamic systems, control is applied only to the first control step. The future state of the system is then estimated again, leading to a new control influence for the next control step. For the optimization problem (19), we will predict the vehicle values at intersections by considering the previous historical values of these vehicle queues. A graphical illustration of this sliding mode control policy is presented in Figure 6. For this study, the historical period is truncated and the predictions of the number of vehicles are based on their values, which are estimated at the last control step. For the first control step, the predicted control value ($\boldsymbol{u}^{opt}$) is for k = 3, Figure 6.

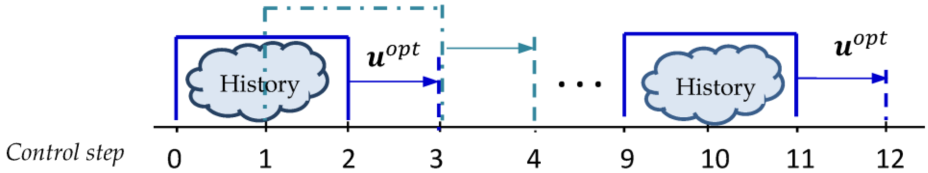

Figure 6. Graphical illustration of sliding mode control.

Taking into account the historical data, the average values of the vehicles at the intersection E_x and the corresponding volatility σ_x^2 are calculated. This makes problem (19) well-defined according to the values of its parameters. The $\boldsymbol{u}^{\text{opt}}$ solutions will be implemented. The resulting values of vehicles will be taken into consideration for the new estimations of $E_x(\boldsymbol{u})$ and $\sigma_x^2(\boldsymbol{u})$. By keeping the same amount of data for the historical period, a sliding policy is implemented to use the more recent historical data.

For this study, the historical period is truncated and the prediction of the number of vehicles is based on their values, which are estimated at the last control step. Formally, this control frame is analytically derived from the following relations:

- We assume that at the beginning of the control step, the value of vehicles at an intersection is x_0;
- For the start of the next control step (k = 1), the number of vehicles x_1 will change according to the store-and-forward relationship (7) or

$$\boldsymbol{x}(k=1) = \boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{x}_{in} - \boldsymbol{su}.$$

We take the value of x_{in} as deterministic, since the stochastic nature of x is considered by relation (18).

The average value of x for these two data types is:

$$E_X = \frac{1}{2}(\boldsymbol{X}_0 + \boldsymbol{X}_1) = \frac{1}{2}(\boldsymbol{X}_0 + \boldsymbol{X}_1 + \boldsymbol{X}_{in} - \boldsymbol{su}) = \boldsymbol{X}_0 + \frac{1}{2}\boldsymbol{X}_{in} - \frac{1}{2}\boldsymbol{su}. \quad (20)$$

The volatility of x is calculated as:

$$V_x = \sigma_x^2 = \frac{1}{2}[(\boldsymbol{x}_0 - E_x)^2 + (\boldsymbol{x}_1 - E_x)^2] = \frac{1}{2}[(\boldsymbol{x}_0 - E_x)^2 + (\boldsymbol{x}_0 + \boldsymbol{x}_{in} - \boldsymbol{su} - E_x)^2]. \quad (21)$$

After substituting E_x from (20) into (21), the analytical relation for the volatility is:

$$V_x = \sigma_x^2 = \frac{1}{4}(\boldsymbol{x}_{in} - \boldsymbol{su})^2 \quad (22)$$

Relations (20) and (22) analytically give the relations $E_x = E_x(u)$ and $\sigma_x^2 = \sigma_x^2(\boldsymbol{u})$. After substituting (20) and (22) into (18), which is the objective function of (19), the stochastic approximation (18) takes the form of a quadratic function to $\boldsymbol{u}$:

$$-\left(\boldsymbol{x}_0 + \frac{1}{2}\boldsymbol{x}_{in} - \frac{1}{2}\boldsymbol{su}\right) + \delta\frac{1}{4}(\boldsymbol{x}_{in} - \boldsymbol{su})^2 \leq x_{max} \quad (23)$$

The optimization problem (19) becomes deterministic with an objective function that formalizes the probabilistic requirement (18) or:

$$\min_{\boldsymbol{u}} F(\boldsymbol{u}) = \sum_{k=0}^{k=1} \boldsymbol{x_{max}}(\boldsymbol{u}) = -\left(\boldsymbol{x}_0 + \frac{1}{2}\boldsymbol{x}_{in} - \frac{1}{2}\boldsymbol{su}\right) + \delta\frac{1}{4}(\boldsymbol{x}_{in} - \boldsymbol{su})^2 \quad (24)$$

$$\boldsymbol{x}(k=1) = x_1 \leq \boldsymbol{x}_0 + \boldsymbol{x}_{in} - \boldsymbol{su},$$

$$u_{min} \leq \boldsymbol{u} \leq u_{max},$$

Relations (23) must be described analytically for each intersection of the transport network, which adds additional components to the objective function $F(\boldsymbol{u})$.

For this study, historical data for mean E_x and volatility $V_x = \sigma_x^2$ estimates are taken for only one previous control step. This short time horizon allows the shortest impacts and changes in traffic requirements to be taken into account. Increasing the history in the control step will change the analytical definition of E_x and $V_x = \sigma_x^2$ against the governing influences $\boldsymbol{u}$ and relations (20) and (21) will be modified accordingly.

Problem (23) can be modified by changing the store-and-forward constraints (7) in such a way that the current number of vehicles at an intersection plus inbounds must be less than the outbound. Thus, formally the control problem will preserve the presence of congestion. The analytical definition of this problem will be modified from (23) as:

$$\min_{\boldsymbol{u}}\{F(\boldsymbol{u}) = \sum_{k=0}^{k=1} x_{max}(\boldsymbol{u}) = -(\boldsymbol{x}_0 + \frac{1}{2}\boldsymbol{x}_{in} - \frac{1}{2}\boldsymbol{su}) + \delta\frac{1}{4}(\boldsymbol{x}_{in} - \boldsymbol{su})^2\}, \quad (25)$$

$$\boldsymbol{su} \geq \boldsymbol{x}_0 + \boldsymbol{x}_{in}$$

$$u_{min} \leq \boldsymbol{u} \leq u_{max}.$$

Problem (25) contains as an argument only the duration of green light $\boldsymbol{u}$. The application of the model predictive framework of the model allows the evaluation at each control step the value of the existing vehicles in intersection $\boldsymbol{x}_0$ and a new problem (25) will define the solutions of $\boldsymbol{u}$ for the next control step. The evaluation of $\boldsymbol{x}_0$ can be performed by using the estimated $\boldsymbol{u}^{opt}$ in the store-and-forward relation (7) in a recursive manner for each control step.

6. Stochastic Optimization of Traffic Management in an Urban Environment

The aim of the study is to develop a traffic management model reducing the number of vehicles in sections of the urban transport network, taking into account the stochastic nature of the traffic. The optimization model (25) obtained in the previous Section 5 is used. The objective function reflects the stochastic nature of the traffic flow and our objective is to minimize the sum of all vehicles in the network and/or waiting at the traffic light at the intersection as functions of the green traffic light duration $\boldsymbol{u}$. We consider a real network of intersections in an urban area in Sofia, which is a mixture of areas with residential, business and commercial activities and buildings. The network consists of four intersections and the number of vehicles in the main transport section is x_i, $i = 1, \ldots, 15$, Figure 7. Vehicles can go straight or turn left or right at the intersection. The main traffic flow passes through the horizontal street, while the traffic is not intense through the perpendicular streets. This consideration is formalized by multiplying turning vehicles from/to the perpendicular streets by coefficients $l < 1$. Right and left turning vehicles are multiplied by l_1 and l_2, respectively. As initial and known values for the model, we use statistics on the street capacity $\boldsymbol{s}$, the number of initial vehicles $\boldsymbol{x}_0$ and the intersection entries $\boldsymbol{x}_{in}$.

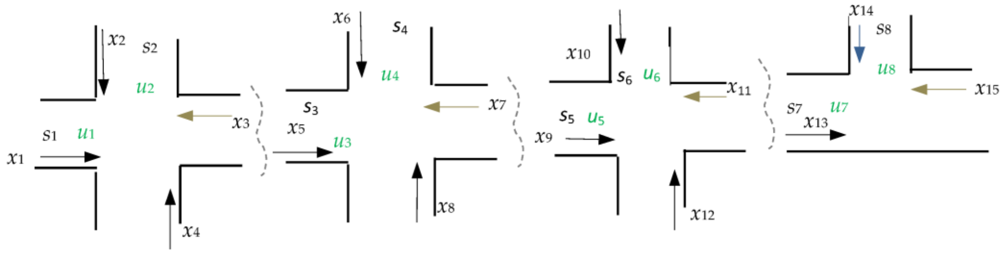

Figure 7. Arterial city network.

The objective function of (25) according to (18) can be briefly given as:

$$\min_{u} \left[x_{max}(u) = -E_X(\boldsymbol{u}) + \delta\sigma_X^2(\boldsymbol{u})\right] \tag{26}$$

and we need to determine the sum of all vehicles x_i, $i = 1, \ldots, 15$ as functions of $\boldsymbol{u}$ (u_j, $j = 1, \ldots, 8$).

6.1. Evaluation of the Number of Vehicles $x_1(u)$

To model the stochastic nature of the traffic, we consider two control cycles for $k = 0$ and $k = 1$ at the beginning, Figure 6. When $k = 0$, the first value of vehicles is:

$$x_1(k=0) = x_1(0) = x_{1_0}. \tag{27}$$

The value of x_{1_0} is known (pre-measured). For the next control step: $k = 1$, the value of the vehicles is determined according to the store-and-forward model and according to (7) is:

$$x_1(k=1) = x_1(k=0) + x_{1_{in}}(k=1) - s_1u_1(k=1)$$

or

$$x_1(1) = x_1(0) + x_{1_{in}}(1) - s_1u_1(1) \tag{28}$$

To formalize the outgoing flow, we divide it into straight-ahead direction by value s_1u_1, right-turning vehicles by value $l_1s_1u_1$ and the corresponding left-turning by value $l_2s_1u_1$. The total value of the outflow becomes $(1 + l_1 + l_2)s_1u_1$. Then, the number of vehicles for the second control cycle according to (28) becomes:

$$x_1(1) = x_1(0) + x_{1_{in}}(1) - (1 + l_1 + l_2)s_1u_1(1) \tag{29}$$

We replace the expression $(1 + l_1 + l_2)$ by L_1 and for simplicity (29) can be represented as:

$$x_1 = x_{1_0} + x_{1_{in}} - L_1s_1u_1. \tag{30}$$

We have to define both stochastic components of (26): E_x and σ_x^2 as functions of $\boldsymbol{u}$.

Analytical definition of the mean value as a function of u $E_x(u_1)$: the mean value E_x analytically for the case of one historical period is:

$$E_x = \frac{1}{2}[x_1(0) + x_1(1)]. \tag{31}$$

The value of $x_1(0)$ is estimated from (27) and $x_1(1)$—from (30). These two components of (31) are given from (27) and (30) and the mean $E_x(u_1)$ analytically is:

$$E_x(u_1) = \frac{1}{2}[x_1(0) + x_1(1)] = \frac{1}{2}(x_{1_0} + x_{1_0} + x_{1_{in}} - L_1s_1u_1) = x_{1_0} + \frac{x_{1_{in}}}{2} - \frac{L_1s_1u_1}{2} \tag{32}$$

Analytical definition of volatility $\sigma_x^2(u)$ as a function of u: the volatility σ_x^2 for the case of one historical period is:

$$\sigma_x^2(u_1) = \frac{1}{2}[(x_1(0) - E_x)^2 + (x_1(1) - E_x)^2]. \quad (33)$$

Next, substituting into (33) the values obtained from (27), (30) and (32) and applying a set of transformations, it follows:

$$\sigma_x^2(u_1) = \frac{1}{4}\left(x_{1_{in}} - L_1 s_1 u_1\right)^2 \quad (34)$$

Relations (33) and (34) have to be applied to any relation of the form (6) that will give the analytic components included in the objective function (16). For illustration purposes, these transformations are illustrated for the case of the number of vehicles $x_1(u_1)$ as a function of the estimated green light u_1.

Determination of the components of the objective function given by $x_1(u_1)$

We substitute Equation (32) for E_x and (34) for $\sigma_x^2(u)$ in (26):

$$x_{max1}(u_1) = x_{1_0} + \frac{x_{1_{in}}}{2} - \frac{L_1 s_1 u_1}{2} + \frac{1}{4}\delta\left(x_{1_{in}} - L_1 s_1 u_1\right)^2.$$

After processing the above equation, it follows:

$$x_{max1}(u_1) = \frac{\delta}{4}L_1^2 s_1^2 u_1^2 + \left(\frac{L_1 s_1}{2} - \frac{\delta}{2}x_{1in} s_1 L_1\right)u_1 \quad (35)$$

Thus, the objective function must have quadratic components $\frac{\delta}{4}L_1^2 s_1^2 u_1^2$ and linear component $\left(\frac{L_1 s_1}{2} - \frac{\delta}{2}x_{1in} s_1 L_1\right)u_1$.

6.2. Definition of the Objective Function Components for All Vehicles $x_i(u)$, i = 2, ... ,15

Considering the urban network topology of Figure 7, the relations for $x_2(u_2)$ following the case of (35) are:

$$x_{max2}(u_2) = \frac{\delta}{4}L_1^2 s_2^2 u_2^2 + \left(\frac{L_1 s_2}{2} - \frac{\delta}{2}x_{2in} s_2 L_1\right)u_2 \quad (36)$$

The value x_3 depends on the outflows from the second junction, Figure 7. This outflow from the second intersection depends on the duration of the green light u_3 and the corresponding street capacity s_3 or $s_3 u_3$, plus the turning flows from the perpendicular street from the second intersection: $(l_1 + l_2)s_4 u_4$. To simplify the notation, we replace the sum of the turning coefficients by $L_2 = (l_1 + l_2)$. This means that the number of vehicles x_3 depends not only on the duration of the green light u_1 at the first intersection, but also on u_3 and u_4 from the second traffic light. The initial value for the control cycle $k = 0$ is $x_3(u) = x_{3_0}$, Figure 6.

Following the store-and-forward model, the vehicles x_3 for the next control cycle $k = 1$ will have a value:

$$x_3(u) = x_{3_0} + s_3 u_3 + L_2 s_4 u_4 - L_1 s_1 u_1.$$

The mean value E_X of vehicles x_3 is given according to relation (32) or:

$$E_x(u) = \frac{1}{2}(x_{30} + x_{30} + s_3 u_3 + L_2 s_4 u_4 - L_1 s_1 u_1)$$

$$E_x(u) = x_{30} + \frac{s_3 u_3}{2} + \frac{L_2 s_4 u_4}{2} - \frac{L_1 s_1 u_1}{2} \quad (37)$$

The corresponding volatility $\sigma_x^2(u)$ of vehicles x_3 is given according to (33):

$$\sigma_x^2(u) = \frac{1}{2}\Big[\Big(x_3(0) - E_x\Big)^2 + \Big(x_3(1) - E_x\Big)^2\Big]$$

$$\sigma_x^2(u) = \tfrac{1}{2}[(x_{30} - x_{30} - \tfrac{s_3u_3}{2} - \tfrac{L_2s_4u_4}{2} + \tfrac{L_1s_1u_1}{2})^2 + \\ +(x_{3_0} + s_3u_3 + L_2s_4u_4 - L_1s_1u_1 - x_{30} - \tfrac{s_3u_3}{2} - \tfrac{L_2s_4u_4}{2} + \tfrac{L_1s_1u_1}{2})^2].$$

After a set of transformations, it follows:

$$\sigma_x^2(u) = \frac{1}{4}(s_3u_3 + L_2s_4u_4 - L_1s_1u_1)^2 \tag{38}$$

Using the relations for the mean $E_x(u)$ from (37) and the volatility $\sigma_x^2(u)$ from (38), the value of x_3 takes the analytical form:

$$x_{max3}(u) = -\tfrac{s_3u_3}{2} - \tfrac{L_2s_4u_4}{2} + \tfrac{L_1s_1u_1}{2} + \tfrac{\delta}{4}(s_3^2u_3^2 + L_2^2s_4^2u_4^2 + L_1^2s_1^2u_1^2 + \\ 2s_3u_3L_2s_4u_4 - -2s_3u_3L_1s_1u_1 - 2L_2s_4u_4L_1s_1u_1) \tag{39}$$

Relation (39) gives additional components for the objective function (26) of the optimization problem with quadratic components $\frac{\delta}{4}(s_3^2u_3^2 + L_2^2s_4^2u_4^2 + L_1^2s_1^2u_1^2)$ and linear components $-\frac{s_3u_3}{2} - \frac{L_2s_4u_4}{2} + \frac{L_1s_1u_1}{2} + \frac{\delta}{4}(2s_3u_3L_2s_4u_4 - 2s_3u_3L_1s_1u_1 - 2L_2s_4u_4L_1s_1)$. These components come from the x_{max3} vehicles.

Following the network topology in Figure 7, the expression for vehicles $x_4(u)$ is analogous to that of $x_1(u)$ and is of the form:

$$x_{max4}(u_2) = \frac{\delta}{4}L_1^2s_2^2u_2^2 + \left(\frac{L_1s_2}{2} - \frac{\delta}{2}x_{4in}s_2L_1\right)u_2 \tag{40}$$

Relation (40) gives the corresponding quadratic and linear components for the objective function (26) from x_{max4}.

The length of the queue $x_5(u)$ depends not only on the duration of the second green traffic light u_3, but also on the traffic coming from the first intersection. At the start of the control loop for $k = 0$, the value of x_5 is $x_5(0) = x_{5_0}$. For the next control loop, for $k = 1$ the value of $x_5(u)$ is evaluated using the store-and-forward relation:

$$x_5(u) = x_{5_0} + s_1u_1 + L_2s_2u_2 - L_1s_3u_3$$

For the case according to (31), the mean value E_X of x_5 is given by the ratio:

$$E_x(u) = x_{50} + \frac{s_1u_1}{2} + \frac{L_2s_2u_2}{2} - \frac{L_1s_3u_3}{2}. \tag{41}$$

According to (33), the corresponding volatility $\sigma_X^2(u)$ is:

$$\sigma_x^2(u) = \frac{1}{4}(s_1u_1 + L_2s_2u_2 - L_1s_3u_3)^2. \tag{42}$$

Using the relations for the mean $E_x(u)$ from (41) and the volatility $\sigma_X^2(u)$ from (42), the value of x_5 takes the analytical form

$$x_{max5}(u) = -\tfrac{s_1u_1}{2} - \tfrac{L_2s_2u_2}{2} + \tfrac{L_1s_3u_3}{2} + \tfrac{\delta}{4}(s_1^2u_1^2 + L_2^2s_2^2u_2^2 + L_1^2s_3^2u_3^2 \\ +2s_1u_1L_2s_2u_2 - 2s_1u_1L_1s_3u_3 - 2L_2s_2u_2L_1s_3u_3) \tag{43}$$

The relation (43) gives the corresponding additional quadratic and linear components for the objective function coming from x_5.

Through the same formal approach, the relations for the subsequent values of the vehicles are derived, which gives additional quadratic and linear components for the objective function (26) of the optimization problem. The general set of the analytical definition of the critical level of vehicles $x_{max}(u)$ for each road section of the urban network is given in the Appendix A.

6.3. Analytical Description of the Traffic Light Optimization Problem

The objective function of the task is given in the form (25). This refers to the minimization of the sum of all vehicles x_i, $i = 1, \ldots, 15$ in the urban network. The values of the number of vehicles are derived analytically, following the ratios (35), (36), (39), (40), (43) and the corresponding ones from the Appendix A. The sum of the number of vehicles is a non-linear quadratic function, which is represented in (45) in vector form. The constraints of the optimization problem mean that the outgoing traffic flows are required to be greater than the incoming traffic flows for each network junction. The constraints of the optimization problem according to the store-and-forward model are given by following the network topology of Figure 7 and this gives a set of inequalities:

$$\begin{gathered}
L_1 s_1 u_1 \geq x_{1_0} + x_{1_{in}} \\
L_1 s_2 u_2 \geq x_{2_0} + x_{2_{in}} \\
L_1 s_1 u_1 - s_3 u_3 - L_2 s_4 u_4 \geq x_{3_0} \\
L_1 s_2 u_2 \geq x_{4_0} + x_{4_{in}} \\
L_1 s_3 u_3 - s_1 u_1 - L_2 s_2 u_2 \geq x_{5_0} \\
L_1 s_4 u_4 \geq x_{6_0} + x_{6_{in}} \\
L_1 s_3 u_3 - s_5 u_5 - L_2 s_6 u_6 \geq x_{7_0} \\
L_1 s_4 u_4 \geq x_{8_0} + x_{8_{in}} \\
L_1 s_5 u_5 - s_3 u_3 - L_2 s_4 u_4 \geq x_{9_0} \\
L_1 s_6 u_6 \geq x_{10_0} + x_{10_{in}} \\
L_1 s_5 u_5 - s_7 u_7 - l_1 s_8 u_8 \geq x_{11_0} \\
L_1 s_6 u_6 \geq x_{12_0} + x_{12_{in}} \\
L_3 s_7 u_7 - s_5 u_5 - L_2 s_6 u_6 \geq x_{13_0} \\
L_2 s_8 u_8 \geq x_{14_0} + x_{14_{in}} \\
L_4 s_7 u_7 \geq x_{15_0} + x_{15_{in}} \\
u_1 + u_2 \leq 0.9 c_1 \\
u_3 + u_4 \leq 0.9 c_2 \\
u_5 + u_6 \leq 0.9 c_3 \\
u_7 + u_8 \leq 0.9 c_4
\end{gathered} \tag{44}$$

The last four constraints are relationships between the duration of the green light cycle and the traffic light, where $c_1, \ldots, c_4$ are the durations of the traffic light cycle on the control intersections. The yellow light is assumed to have a fixed length of 1/10th of a cycle.

The values of the duration of the green light u_i, $i = 1, \ldots, 8$ according to the stochastic nature of the traffic are estimated from the ratios (35), (36), (39), (40), (44).

The optimization problem can be represented in vector form:

$$\begin{gathered}
\min_u \frac{1}{2}\left(u^T Q u + R^T u\right) \\
Au \geq b
\end{gathered} \tag{45}$$

Both the objective function matrices of (45), Q and R, have descriptions, given in the Appendix A. The matrix Q has eight rows and eight columns, since the control variables of the considered network are u_j, $j = 1, \ldots, 8$, Figure 7:

$$Q = \begin{Vmatrix} q1_1 & q1_2 & q1_3 & q1_4 & q1_5 & q1_6 & q1_7 & q1_8 \\ q2_1 & q2_2 & q2_3 & q2_4 & q2_5 & q2_6 & q2_7 & q2_8 \\ q3_1 & q3_1 & q3_3 & q3_4 & q3_5 & q3_6 & q3_7 & q3_8 \\ q4_1 & q4_2 & q4_3 & q4_4 & q4_5 & q4_6 & q4_7 & q4_8 \\ q5_1 & q5_2 & q5_3 & q5_4 & q5_5 & q5_6 & q5_7 & q5_8 \\ q6_1 & q6_2 & q6_3 & q6_4 & q6_5 & q6_6 & q6_7 & q6_8 \\ q7_1 & q7_2 & q7_3 & q7_4 & q7_5 & q7_6 & q7_7 & q7_8 \\ q8_1 & q8_2 & q8_3 & q8_4 & q8_5 & q8_6 & q8_7 & q8_8 \end{Vmatrix} \quad (46)$$

The matrix elements of (46) are given in the Appendix A.

The matrix R from (45) consists of factors of the linear components of u:

$$R^T = \begin{Vmatrix} r_1 & r_2 & r_3 r_4 & r_5 & r_6 r_7 & r_8 \end{Vmatrix}.$$

The relevant elements of the R^T matrix are given in the Appendix A.

The elements of the matrices $\mathbf{A}$ and $\mathbf{b}$ for problem (45) are given in the Appendix A.

7. Numerical Simulations and Results

The optimization problem (45) is solved in a MATLAB environment with statistical data from a real urban area in Sofia. The solution of this linear-quadratic optimization problem shows a decrease in the values of the vehicles along the sections of the urban network. The dynamic behavior x_1, x_2, x_3 and x_4 for the first junction is given in Figure 8. The values of x_1, x_2 and x_4 after applying $\boldsymbol{u}^{opt}$ control solutions for the first control cycle decrease to zero values. The internal transport flow x_3, coming from the second intersection, decreases. Applying this control strategy for several control cycles, the value of x_3 tends to decrease to a zero value, Figure 8.

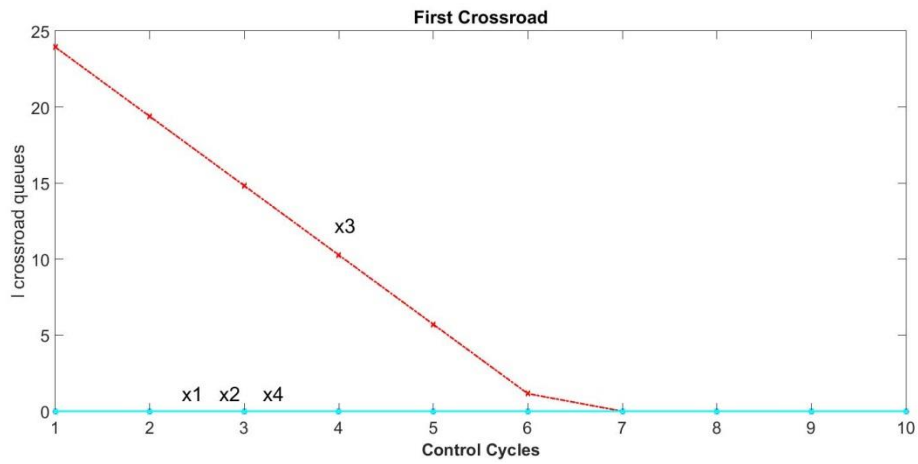

Figure 8. Traffic dynamics at the first intersection.

Applying the estimated optimal durations of the green lights $\boldsymbol{u}^{opt}$ for the first control cycle, the corresponding vehicle values for the second and third sections of the network give a further reduction to a zero level of the number of vehicles x_5, x_6, x_7 and x_8, Figure 9 and x_9, x_{10}, x_{11} and x_{12}, Figure 10.

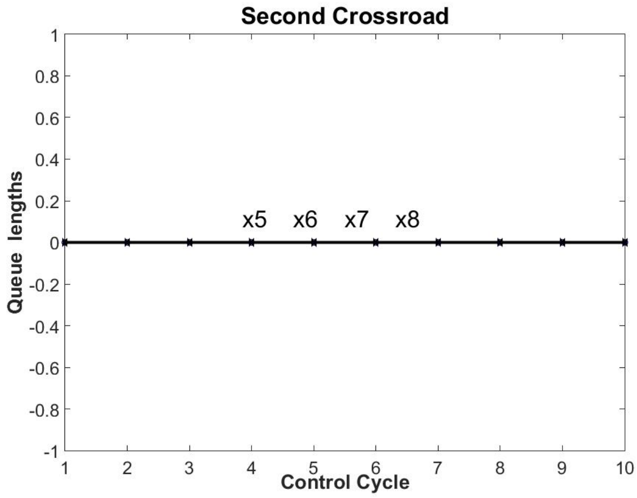

Figure 9. Traffic dynamics at the second intersection.

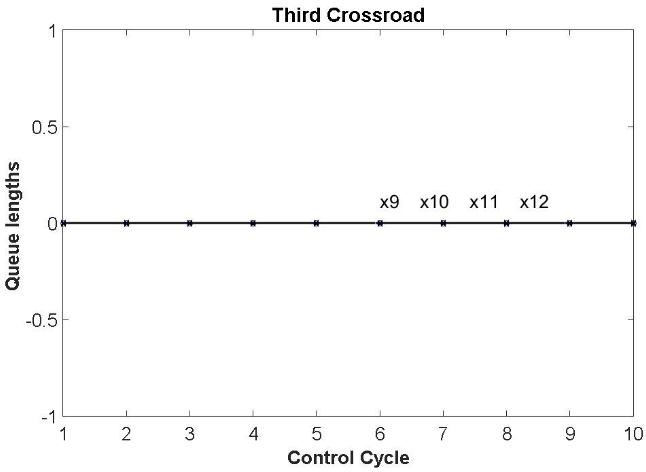

Figure 10. Traffic dynamics at the third intersection.

The only car flow that does not support decreasing dynamics is x_{13}, Figure 11. However, this control keeps its value at a constant level, which is good for minimizing congestion. The traffic authorities should take measures to increase road capacity s_7 to reduce x_{13}. The remaining vehicle flows x_{14} and x_{15} also maintain a decreasing character and decrease to zero after the first control cycle, Figure 11.

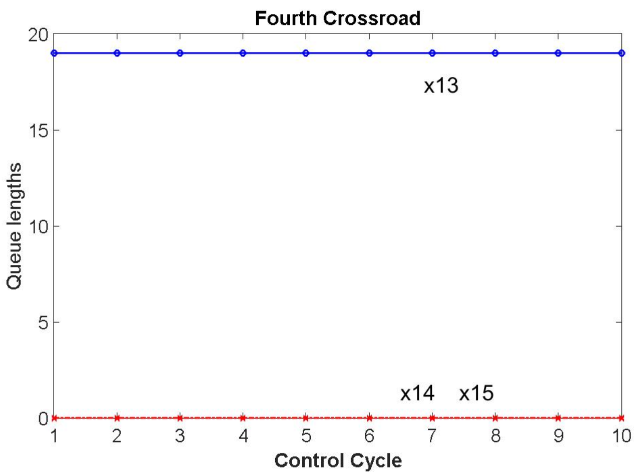

Figure 11. Traffic dynamics at the fourth intersection.

The decreasing character of the transport flows for the internal sections of the transport network x_3, x_9 and x_{13} are given in Figure 12, where the control cycles applying task (45) are successively performed.

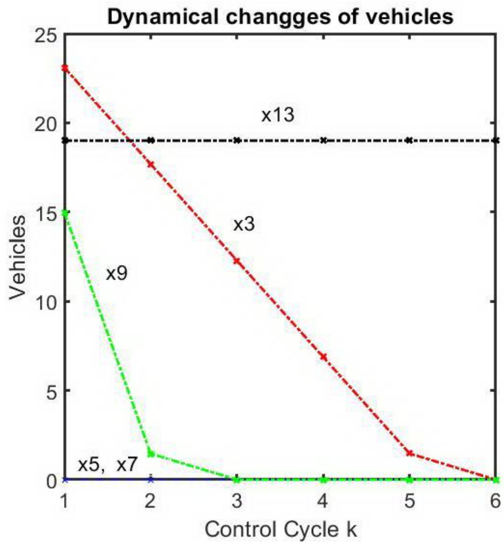

Figure 12. Dynamics of interconnected flows.

To evaluate the benefits of the control problem (45), the dynamics of the total number of vehicles in the transport network are also evaluated. The vehicle totals after each control cycle are illustrated in Figure 13.

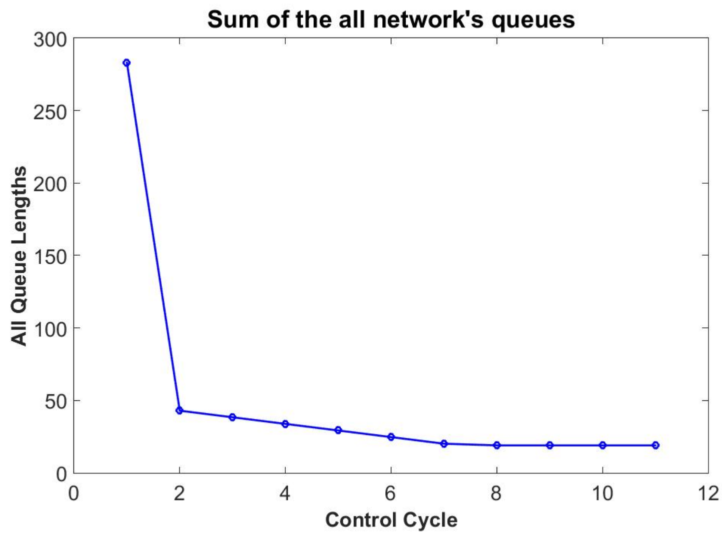

Figure 13. Change of the sum of all vehicles in the network.

It is obvious that this control policy of sequentially applying the solutions to problem (45) leads to a significant reduction of the sum of all queues. This is a useful result for the application of the developed optimization model, which considers probabilistic approximations in its objective function. To evaluate the results obtained from the resulting optimization model for traffic management with probabilistic demands, comparisons are made with the corresponding deterministic optimization problem. The numerical evaluations and comparisons are illustrated in the next section.

8. Comparisons between the Developed Probabilistic Control Model and the Corresponding Deterministic Model

The deterministic traffic management problem is applied to the same transport network presented in Figure 7. The deterministic optimization problem has the same set of constraints as (44). Since here the green light duration values do not take into account the random nature of the traffic, the objective function aims to minimize the sum of the number of vehicles x and green light duration u:

$$\begin{gathered} \min_{x,u}\left(x^2+u^2\right) \\ x_{out_i} \geq x_{0_i} + x_{in_i},\ i\ =\ 1,\ldots,\ 15 \\ u_1 + u_2 \leq 0.9c_1 \\ u_3 + u_4 \leq 0.9c_2 \\ u_5 + u_6 \leq 0.9c_3 \\ u_7 + u_8 \leq 0.9c_4. \end{gathered} \tag{47}$$

This optimization problem is solved in a MATLAB environment, and a comparison between the traffic flows of the respective urban sections is given in Figures 13–15. In these figures, the deterministic model gives the dynamic changes of the vehicles as a result of the solution of the problem (47) and the successive application of the duration of the green light $\boldsymbol{u}$. The probabilistic problem gives the dynamic changes of the vehicles by successively estimating the duration of the green light $\boldsymbol{u}$ through the solution of the resulting problem (45). This problem considers probabilistic requirements for the number of vehicles. The dynamic changes of vehicles x_3 for the two optimization problems are illustrated in Figure 14.

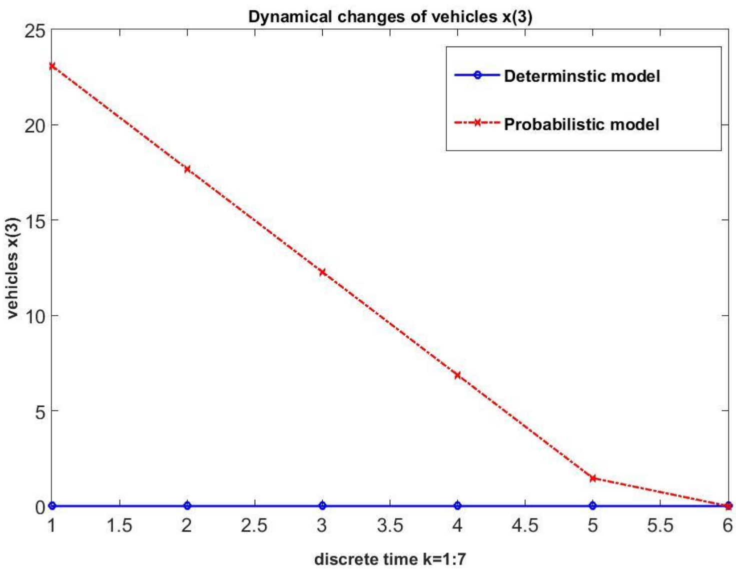

Figure 14. Comparison of the change in the number of vehicles x_3 in the network.

This result demonstrates a preference for the deterministic model only at the beginning of traffic management, as its influence causes x_3 to decrease to zero. Figures 15–17 show the changes in traffic flows x_5, x_9 and x_{13} for the two cases: deterministic and probabilistic. For all three vehicle flows, the deterministic values of x_5, x_9 and x_{13} have larger values than the probabilistic ones, which favors the developed probabilistic optimization model.

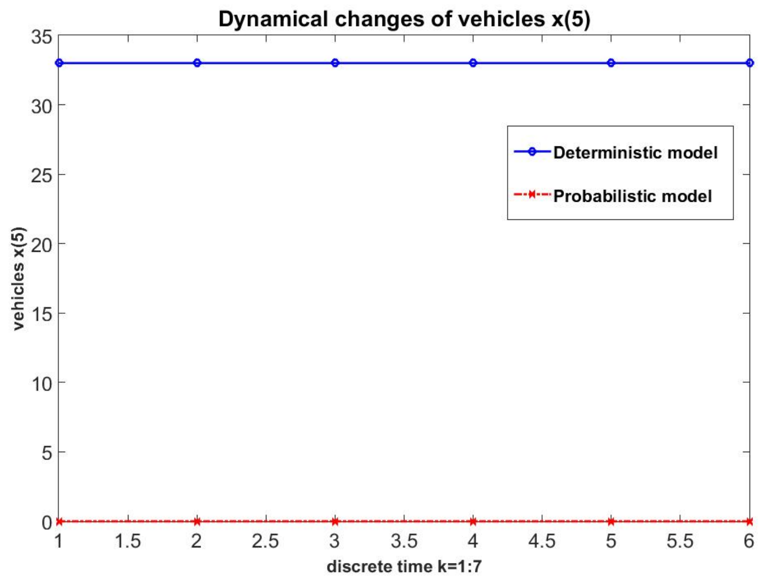

Figure 15. Comparison of the change in traffic flow x_5 from the network.

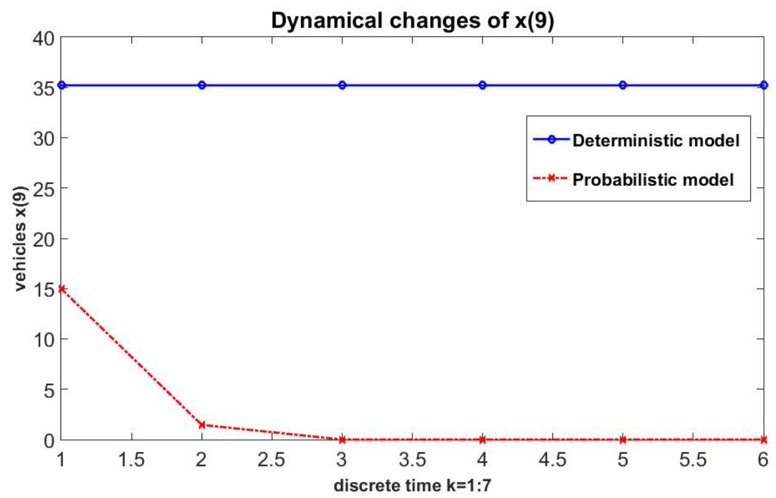

Figure 16. Comparison of the change in traffic flow x_9 from the network.

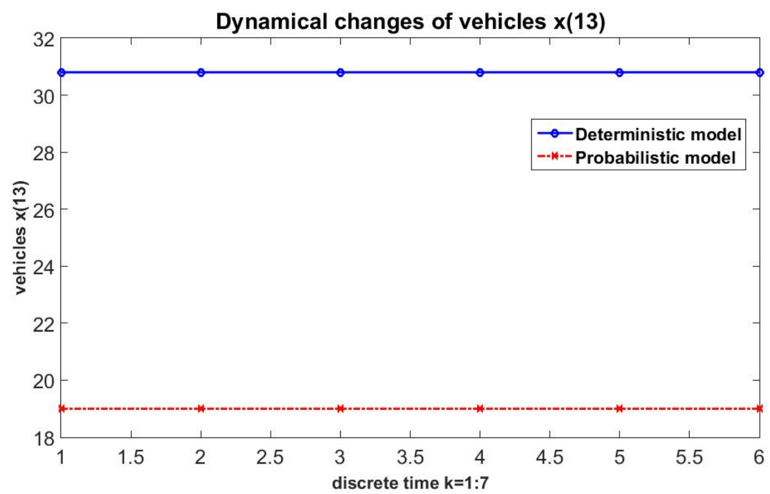

Figure 17. Comparison of change in traffic flow x_{13} from the network.

Therefore, this comparison provides advantages for the derived optimization problem that considers additional probabilistic requirements. The most representative comparison of the total number of vehicles in the urban network is given in Figure 18.

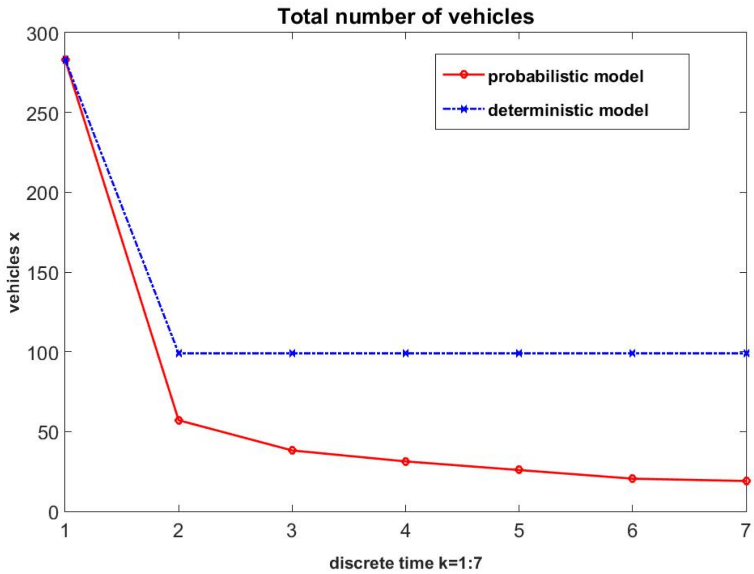

Figure 18. Comparison of the change in the sum of all traffic flows.

The graphical results in Figure 18 further prove that the derived optimization problem that considers probabilistic demands gives better results in reducing traffic flows in the transport network compared to the corresponding deterministic optimization problem. It is obvious that the developed model leads to a lower number of vehicles through the appropriately defined durations of the green traffic lights.

9. Discussion

Optimization shows better probabilistic results compared to deterministic ones. The comparison is illustrated for traffic flows between adjacent intersections of the main direction of travel. There is one exception regarding the dynamics of the queue length x_3. Its value also decreases to zero after several control iterations, but the deterministic approach reaches zero values of x_3 at the beginning, Figure 14. This means that both the deterministic and probabilistic approaches give a positive result with a slight preference to the deterministic one for x_3, since after the first control loop, the deterministic optimization gives zero queue length x_3. This small exception cannot detract from the results obtained, as it can be seen that the sum of all queue lengths is significantly smaller when applying the probabilistic optimization compared to the deterministic one.

The results, numerically obtained by using the developed model and defining and solving an appropriate optimization problem, provide useful and practically applicable results. Comparisons of the obtained results with the classical use of the store-and-forward model give advantages to the derived task. Our explanations for these advantages are due to the case that some of the stochastic events in traffic behavior are explicitly reflected in the control task. This is not the case for the classic store-and-forward model. Our experiments were conducted for a real part of the network in the city of Sofia. In practice, it is not easy to make comparisons if urban networks have different infrastructures, as the traffic behavior will be completely different. For comparison, we performed experiments with an optimization problem based on a classical deterministic store-and-forward model of the form (9). Our results, given the stochastic requirements, outperform the deterministic optimization case. Because a predefined infrastructure and previous traffic data collection were used, we were able to quantify the results of the derived stochastic model and the corresponding deterministic model.

10. Conclusions

The paper presents a model for optimizing traffic management considering the random nature of traffic flows. The objective function aims to minimize the green light duration of the considered urban area. The novelty of the model is that the green light duration of the model reflects the dynamic changes in traffic by formalizing through some statistical variables such as mean and standard deviation. Two historical intervals are considered for probabilistic approximations of traffic behavior. The derived optimization problem applies the well-known store-and-forward model but is complicated by probabilistic requirements for traffic behavior. The applied control strategy is based on the model predictive control approach, where a sliding procedure is applied for each control cycle. The completed task was empirically verified with real data from an urban part of Sofia. The obtained results are evaluated and compared with traffic optimization with a corresponding deterministic problem. Numerical results give an advantage and better potential to the probabilistic model as the traffic flows decrease significantly. Our assessment of the limitations of the developed model refers to the workload that must be performed in defining the optimization problem. The set of constraints defining the critical value for the throughput of a road section x_{max} must be obtained analytically for each road section of the urban network. Thus, if the network structure changes or the problem has to be applied to a different network, additional off-line workload must be allocated. From the point of view of control applications, this is not a critical issue since it is performed in the off-line stages of control design. However, the solution in the model predictive approach insists on the centralized collection of all data on the current traffic behavior: respectively, the number of vehicles for each road section. For a large-scale network, this raises technical issues for the engineering design and implementation of such a centralized technical system. However, this difficulty is common to all centralized control systems. From an algorithmic point of view, which relates to the inferred model and task, such constraints are not so restrictive. We see the potential and future development of the resulting model in the simultaneous optimization of both n of green light duration and traffic light cycle duration. This increases the control space of the problem, which is a prerequisite for satisfying additional traffic management objectives and/or constraints. A potential approach for this extension of the control domain is to apply bi-level hierarchical optimization to the control task. Our intention in future work is to use bi-level optimization to be able to optimize not only the green light but also the cycle time of the traffic light. Such a hierarchical formalization approach allows the extension of the optimization by including more variables, objective functions and constraints. The developed optimization task allows the automation of the necessary evaluations through suitable software for managing transport flows.

Author Contributions: The conceptualization of the study was conducted by T.S. The writing of the original draft and preparation of the paper were conducted by K.S. and T.S. All authors contributed to the writing of the review and final editing. The supervision was conducted by T.S. Visualizations were conducted by K.S. All authors have read and agreed to the published version of the manuscript.

Funding: The research leading to these results has received funding from project KP 06-H37/6 "Modeling and optimization of urban traffic in network of crossroads" with the Bulgarian Research Fund.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Notation Used

$\boldsymbol{x}$—Number of vehicles in the road sections of the urban network, (vehicles)
$\boldsymbol{x_{in}}$—Incoming vehicles for the start of the control cycle, (vehicles)
$\boldsymbol{x_{out}}$—Outgoing vehicles at the end of the control cycle, (vehicles)
$\boldsymbol{u}$¯The set of the green light durations, (time)
s¯Capacity of a road junction, (vehicle/time)

$\boldsymbol{\varepsilon} = (\zeta_1, \zeta_2, \eta)$—Random variable for describing stopping, turning, parking vehicles
x_{max}—Critical value about the capacity of a road section, (vehicle)
γ—Confidence level of probability, (%)
E_x—Mean value of vehicles on a corresponding road section
$V_x = \sigma_x^2$—Volatility and standard deviation of the vehicles on a corresponding road section
$\delta = F^{-1}(1-\gamma)$—Z-score of normalized stochastic function with normal distribution for the value of γ
$\boldsymbol{x}_0, \boldsymbol{x}_1$—Number of vehicles for the previous and current control cycle in the road sections, applied for the model predictive framework for traffic light control
l_1, l_2—Proportion of the turning right, left vehicles on a crossroad junction
c_i, i = 1, … ,4—durations of the traffic lights cycles for the fourth junctions
Analytical definition of the critical level of vehicles $x_{max}(u)$ for each road section of the urban network

$$x_{max1}(u_1) = \frac{\delta}{4}L_1^2 s_1^2 u_1^2 + \left(\frac{L_1 s_1}{2} - \frac{\delta}{2}x_{1in} s_1 L_1\right)u_1$$

$$x_{max2}(u_2) = \frac{\delta}{4}L_1^2 s_2^2 u_2^2 + \left(\frac{L_1 s_2}{2} - \frac{\delta}{2}x_{2in} s_2 L_1\right)u_2$$

$$x_{max3}(u) = -\tfrac{s_3 u_3}{2} - \tfrac{L_2 s_4 u_4}{2} + \tfrac{L_1 s_1 u_1}{2} + \tfrac{\delta}{4}(s_3^2 u_3^2 + L_2^2 s_4^2 u_4^2 + L_1^2 s_1^2 u_1^2 + 2 s_3 u_3 L_2 s_4 u_4 - \\ -2 s_3 u_3 L_1 s_1 u_1 - 2 L_2 s_4 u_4 L_1 s_1 u_1)$$

$$x_{max4}(u_2) = \frac{\delta}{4}L_1^2 s_2^2 u_2^2 + \left(\frac{L_1 s_2}{2} - \frac{\delta}{2}x_{4in} s_2 L_1\right)u_2$$

$$x_{5max}(u) = x_{5_0} + s_1 u_1 + L_2 s_2 u_2 - L_1 s_3 u_3$$

$$x_{max6}(u_4) = \frac{\delta}{4}L_1^2 s_4^2 u_4^2 + \left(\frac{L_1 s_4}{2} - \frac{\delta}{2}x_{6in} s_4 L_1\right)u_4$$

$$x_{max7}(u) = \tfrac{L_2 s_6 u_6}{2} + \tfrac{L_1 s_3 u_3}{2} + \tfrac{\delta}{4}(s_5^2 u_5^2 + L_2^2 s_6^2 u_6^2 + L_1^2 s_3^2 u_3^2 + 2 s_5 u_5 L_2 s_6 u_6 - 2 s_5 u_5 L_1 s_3 u_3 - \\ -2 L_2 s_6 u_6 L_1 s_3 u_3)$$

$$x_{max8}(u_4) = \frac{\delta}{4}L_1^2 s_4^2 u_4^2 + \left(\frac{L_1 s_4}{2} - \frac{\delta}{2}x_{8in} s_4 L_1\right)u_4$$

$$x_{max9}(u) = \tfrac{s_3 u_3}{2} - \tfrac{L_2 s_4 u_4}{2} + \tfrac{L_1 s_5 u_5}{2} + \tfrac{\delta}{4}(s_3^2 u_3^2 + L_2^2 s_4^2 u_4^2 + L_1^2 s_5^2 u_5^2 + 2 s_3 u_3 L_2 s_4 u_4 \\ - - 2 s_3 u_3 L_1 s_5 u_5 - 2 L_2 s_4 u_4 L_1 s_5 u_5)$$

$$x_{max10}(u) = \frac{\delta}{4}L_1^2 s_6^2 u_6^2 + \left(\frac{L_1 s_6}{2} - \frac{\delta}{2}x_{10in} s_6 L_1\right)u_6$$

$$x_{max11}(u) = \tfrac{s_7 u_7}{2} - \tfrac{l_1 s_8 u_8}{2} + \tfrac{L_1 s_5 u_5}{2} + \tfrac{\delta}{4}(s_7^2 u_7^2 + l_1^2 s_8^2 u_8^2 + L_1^2 s_5^2 u_5^2 + 2 s_7 u_7 l_1 s_8 u_8 - \\ -2 s_7 u_7 L_1 s_5 u_5 - 2 l_1 s_8 u_8 L_1 s_5 u_5)$$

$$x_{max12}(u_4) = \frac{\delta}{4}L_1^2 s_6^2 u_6^2 + \left(\frac{L_1 s_6}{2} - \frac{\delta}{2}x_{12in} s_6 L_1\right)u_6 \quad \text{(A1)}$$

$$x_{max13}(u) = -\tfrac{s_5 u_5}{2} - \tfrac{L_2 s_6 u_6}{2} + \tfrac{L_3 s_7 u_7}{2} + \tfrac{\delta}{4}(s_5^2 u_5^2 + L_2^2 s_6^2 u_6^2 + L_3^2 s_7^2 u_7^2 + 2 s_5 u_5 L_2 s_6 u_6 - \\ -2 s_5 u_5 L_3 s_7 u_7 - 2 L_2 s_6 u_6 L_3 s_7 u_7)$$

where $L_3 = 1 + l_2$.

$$x_{max14}(u) = \frac{\delta}{4}L_2^2 s_8^2 u_8^2 + \left(\frac{L_2 s_8}{2} - \frac{\delta}{2}x_{14in} s_8 L_2\right)u_8$$

$$x_{max15}(u) = \frac{\delta}{4}L_4^2 s_7^2 u_7^2 + \left(\frac{L_4 s_7}{2} - \frac{\delta}{2}x_{15in} s_7 L_4\right)u_7,$$

where $L_4 = 1 + l_1$.

The matrix elements of the optimization problem

Element of matrix Q of (45)

$$q_{1_1} = \frac{\delta}{2}L_1^2 s_1^2 + \frac{\delta}{4}s_1^2 = \left(\frac{\delta}{2}L_1^2 + \frac{\delta}{4}\right)s_1^2; \quad q_{1_2} = \frac{\delta}{4}L_2 s_1 s_2; \quad q_{1_3} = -\frac{\delta}{2}L_1 s_1 s_3; \quad q_{1_4} = -\frac{\delta}{4}L_1 L_2 s_1 s_4;$$

$$q_{1_5} = 0; \quad q_{1_6} = 0; \quad q_{1_7} = 0; \quad q_{1_8} = 0;$$

$$q_{2_1} = \tfrac{\delta}{4}L_2 s_1 s_2; \quad q_{2_2} = \delta\left(\tfrac{L_1^2}{2} + \tfrac{L_2^2}{4}\right)s_2^2; \quad q_{2_3} = -\tfrac{\delta}{4}L_1 L_2 s_2 s_3; \quad q_{2_4} = 0; \quad q_{2_5} = 0;$$

$$q_{2_6} = 0; \quad q_{2_7} = 0; \quad q_{2_8} = 0;$$

$$q_{3_1} = -\frac{\delta}{2}L_1 s_1 s_3; \quad q_{3_2} = -\frac{\delta}{4}L_1 L_2 s_2 s_3; \quad q_{3_3} = \frac{\delta}{2}\left(1 + L_1^2\right)s_3^2; \quad q_{3_4} = \frac{\delta}{2}L_2 s_3 s_4;$$

$$q_{3_5} = -\frac{\delta}{2}L_1 s_3 s_5; \quad q_{3_6} = -\frac{\delta}{4}L_1 L_2 s_3 s_6; \quad q_{3_7} = 0; \quad q_{3_8} = 0;$$

$$q_{4_1} = -\tfrac{\delta}{4}L_1 L_2 s_1 s_4; \quad q_{4_2} = 0; \quad q_{4_3} = \tfrac{\delta}{24}L_2 s_3 s_4; \quad q_{4_4} = \tfrac{\delta}{2}\left(L_1^2 + L_2^2\right)s_4^2;$$

$$q_{4_5} = -\tfrac{\delta}{4}L_1 L_2 s_4 s_5;$$

$$q_{4_6} = 0; \quad q_{4_7} = 0; \quad q_{4_8} = 0;$$

$$q_{5_1} = 0; \quad q_{5_2} = 0; \quad q_{5_3} = -\frac{\delta}{2}L_1 s_3 s_5; \quad q_{5_4} = -\frac{\delta}{4}L_1 L_2 s_4 s_5; \quad q_{5_5} = \frac{\delta}{4}\left(3 + L_1^2\right)s_5^2;$$

$$q_{5_6} = \frac{\delta}{2}L_2 s_5 s_6; \quad q_{5_7} = -\frac{\delta}{4}(L_1 + L_3)s_5 s_7; q_{5_8} = -\frac{\delta}{4}l_1 L_1 s_5 s_8;$$

$$q_{6_1} = 0; \quad q_{6_2} = 0; \quad q_{6_3} = -\frac{\delta}{4}L_1 L_2 s_3 s_6; \quad q_{6_4} = 0; \quad q_{6_5} = \frac{\delta}{2}L_2 s_5 s_6; \quad q_{6_6} = \frac{\delta}{2}\left(L_1^2 + L_2^2\right)s_6^2;$$

$$q_{6_7} = -\frac{\delta}{4}L_2 L_3 s_6 s_7; \quad q_{6_8} = 0;$$

$$q_{7_1} = 0; \quad q_{7_2} = 0; \quad q_{7_3} = 0; \quad q_{7_4} = 0; \quad q_{7_5} = -\frac{\delta}{4}(L_1 + L_3)s_5 s_7; q_{7_6} = -\frac{\delta}{4}L_2 L_3 s_6 s_7;$$

$$q_{7_7} = \frac{\delta}{4}\left(1 + L_3^2 + L_4^2\right)s_7^2; \quad q_{7_8} = \frac{\delta}{4}l_1 s_7 s_8;$$

$$q_{8_1} = 0; \quad q_{8_2} = 0; \quad q_{8_3} = 0; \quad q_{8_4} = 0; \quad q_{8_5} = -\frac{\delta}{4}l_1 L_1 s_5 s_8; \quad q_{8_6} = 0; \quad q_{8_7} = \frac{\delta}{4}l_1 s_7 s_8;$$

$$q_{8_8} = \frac{\delta}{4}\left(l_1^2 + L_2^2\right)s_8^2.$$

Elements of matrix R of (45)

$$r_1 = \left(L_1 - \frac{\delta}{2}x_{1_{in}}L_1 - \frac{1}{2}\right)s_1; \quad r_2 = L_1\left(1 - \frac{\delta}{2}x_{2_{in}} - \frac{\delta}{2}x_{4_{in}}\right)s_2 - \frac{L_2}{2}s_2; \quad r_3 = (L_1 - 1)s_3;$$

$$r_4 = \left(L_1 - \frac{\delta}{2}x_{6_{in}}L_1 - L_2 - \frac{\delta}{2}x_{8_{in}}L_1\right)s_4; \quad r_5 = (L_1 - 1)s_5;$$

$$r_6 = \left(L_1 - \frac{\delta}{2}x_{10_{in}}L_1 - \frac{\delta}{2}x_{12_{in}}L_1 - L_2\right)s_6; \quad r_7 = \left(\frac{L_3}{2} + \frac{L_4}{2} - \frac{\delta}{2}x_{15_{in}}L_4 - \frac{1}{2}\right)s_7;$$

$$r_8 = \frac{1}{2}\left(L_2 - l_1 - \delta x_{14_{in}}L_2 - \frac{1}{2}\right)s_8.$$

*Elements of matrices **A** and **b** of the constraints of (45)*

$$\boldsymbol{A} = \left\| \begin{matrix} L_1 s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_1 s_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_1 s_1 & 0 & -s_3 & -L_2 s_4 & 0 & 0 & 0 & 0 \\ 0 & L_1 s_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s_1 & -L_2 s_2 & L_1 s_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 s_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_1 s_3 & 0 & -s_5 & -L_2 s_6 & 0 & 0 \\ 0 & 0 & 0 & L_1 s_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -s_3 & -L_2 s_4 & L_1 s_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_1 s_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_1 s_5 & 0 & -s_7 & -l_1 s_8 \\ 0 & 0 & 0 & 0 & 0 & L_1 s_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s_5 & -L_2 s_6 & L_3 s_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_2 s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_4 s_7 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{matrix} \right\| ; \; \boldsymbol{b} = \left\| \begin{matrix} x_{1_0} + x_{1_{in}} \\ x_{2_0} + x_{2_{in}} \\ x_{3_0} \\ x_{4_0} + x_{4_{in}} \\ x_{5_0} \\ x_{6_0} + x_{6_{in}} \\ x_{7_0} \\ x_{8_0} + x_{8_{in}} \\ x_{9_0} \\ x_{10_0} + x_{10_{in}} \\ x_{11_0} \\ x_{12_0} + x_{12_{in}} \\ x_{13_0} \\ x_{14_0} + x_{14_{in}} \\ x_{15_0} + x_{15_{in}} \\ 0.9c_1 \\ 0.9c_2 \\ 0.9c_3 \\ 0.9c_4 \end{matrix} \right\|$$

References

1. Papageorgiou, M.; Diakaki, C.; Dinopoulou, V.; Kotsialos, A.; Wang, Y. Review of road traffic control strategies. *Proc. IEEE* **2003**, *91*, 2043–2067. [CrossRef]
2. Elbaum, Y.; Novoselsky, A.; Kagan, E. A Queueing Model for Traffic Flow Control in the Road Intersection. *Mathematics* **2022**, *10*, 3997. [CrossRef]
3. Kotusevski, G.; Hawick, K.A. A review of traffic simulation software. *Res. Lett. Inf. Math. Sci.* **2009**, *13*, 35–54. Available online: https://www.massey.ac.nz/massey/fms/Colleges/College%20of%20Sciences/IIMS/RLIMS/Volume13/TrafficSimulatorReview_arlims.pdf (accessed on 9 December 2022).
4. Mubasher, M.M.; Jaffry, S.W.Q. Systematic literature review of vehicular traffic flow simulators. In Proceedings of the 2015 International Conference on Open Source Software Computing OSSCOM 2015, Amman, Jordan, 10–13 September 2015; pp. 1–6. Available online: http://faculty.pucit.edu.pk/swjaffry/rpr/OSSCOM15trafficSimulators.pdf (accessed on 9 December 2022). [CrossRef]
5. Mostafi, S.; Alghamdi, T.; Elgazzar, K. RegTraffic: A Regression based Traffic Simulator for Spatiotemporal Traffic Modeling, Simulation and Visualization. In Proceedings of the International Joint Conference on Neural Networks (IJCNN), Padova, Italy, 18–23 July 2022; pp. 1–6. Available online: https://ieeexplore.ieee.org/document/9892185 (accessed on 9 December 2022). [CrossRef]
6. Ramadhan, H.; Nugraha, I.G.B.B. Web-based macroscopic road traffic simulator. In Proceedings of the 11th International Conference on Telecommunication Systems Services and Applications (TSSA), Lombok, Indonesia, 26–27 October 2017; pp. 1–6. Available online: https://ieeexplore.ieee.org/document/8272906 (accessed on 9 December 2022). [CrossRef]
7. Rehmat, U.; Khattak, K.; Zawar, H.K.; Khan, M.A.; Minallah, N.; Khan, A.N. Vehicular Traffic Simulation Software: A Systematic Comparative Analysis. *Pak. J. Eng. Technol.* **2021**, *4*, 66–78. Available online: https://www.researchgate.net/publication/350466666_Vehicular_Traffic_Simulation_Software_A_Systematic_Comparative_Analysis (accessed on 9 December 2022).
8. Jafari, S.; Shahbazi, Z.; Byun, Y.C. Improving the Road and Traffic Control Prediction Based on Fuzzy Logic Approach in Multiple Intersections. *Mathematics* **2022**, *10*, 2832. [CrossRef]
9. Amini, M.; Hatwagner, M.F.; Koczy, L.T. A Combined Approach of Fuzzy Cognitive Maps and Fuzzy Rule-Based Inference Supporting Freeway Traffic Control Strategies. *Mathematics* **2022**, *10*, 4139. [CrossRef]
10. Olayode, I.O.; Tartibu, L.K.; Okwu, M.O.; Ukaegbu, U.F. Development of a Hybrid Artificial Neural Network-Particle Swarm Optimization Model for the Modelling of Traffic Flow of Vehicles at Signalized Road Intersections. *J. Appl. Sci.* **2021**, *11*, 8387. [CrossRef]
11. Galvan-Correa, R.; Olguin-Carbajal, M.; Herrera-Lozada, J.C.; Sandoval-Gutierrez, J.; Serrano-Talamantes, J.F.; Cadena-Martinez, R.; Aquino-Ruíz, C. Micro Artificial Immune System for Traffic Light Control. *Appl. Sci.* **2020**, *10*, 7933. [CrossRef]
12. Shah, I.; Muhammad, I.; Ali, S.; Ahmed, S.; Almazah, M.M.A.; Al-Rezami, A.Y. Forecasting Day-Ahead Traffic Flow Using Functional Time Series Approach. *Mathematics* **2022**, *10*, 4279. [CrossRef]
13. Khan, M.U.; Hosseinzadeh, M.; Mosavi, A. An Intersection-Based Routing Scheme Using Q-Learning in Vehicular Ad Hoc Networks for Traffic Management in the Intelligent Transportation System. *Mathematics* **2022**, *10*, 3731. [CrossRef]

14. Kušić, K.; Ivanjko, E.; Vrbanić, F.; Gregurić, M.; Dusparic, I. Spatial-Temporal Traffic Flow Control on Motorways Using Distributed Multi-Agent Reinforcement Learning. *Mathematics* **2021**, *9*, 3081. [CrossRef]
15. Huang, W.; Hu, Y.; Zhang, X. Enhancing Model-Based Anticipatory Traffic Signal Control with Metamodeling and Adaptive Optimization. *Mathematics* **2022**, *10*, 2640. [CrossRef]
16. Yin, Y. Robust optimal traffic signal timing. *Transp. Res. B Methodol.* **2008**, *42*, 911–924. Available online: https://www.sciencedirect.com/science/article/pii/S0191261508000350 (accessed on 9 December 2022). [CrossRef]
17. Tettamanti, T.; Luspay, T.; Kulcsár, B.; Péni, T.; Varga, I. Robust Control for Urban Road Traffic Networks. *IEEE Trans. Intell. Transp. Syst.* **2014**, *15*, 385–398. Available online: https://ieeexplore.ieee.org/document/6615947 (accessed on 9 December 2022). [CrossRef]
18. Chiou, S.W. A two-stage model for period-dependent traffic signal control in a road networked system with stochastic travel demand. *Inf. Sci.* **2019**, *476*, 256–273. Available online: https://www.sciencedirect.com/science/article/pii/S0020025518308120 (accessed on 9 December 2022). [CrossRef]
19. Shirke, C.; Sabar, N.; Chung, E.; Bhaskar, A. Metaheuristic approach for designing robust traffic signal timings to effectively serve varying traffic demand. *J. Intell. Transp. Syst.* **2022**, *26*, 343–355. Available online: https://www.tandfonline.com/doi/10.1080/15472450.2021.1878892 (accessed on 9 December 2022). [CrossRef]
20. Ye, B.-L.; Wu, W.; Gao, H.; Lu, Y.; Cao, Q.; Zhu, L. Stochastic Model Predictive Control for Urban Traffic Networks. *J. Appl. Sci.* **2017**, *7*, 588. Available online: https://www.mdpi.com/2076-3417/7/6/588 (accessed on 9 December 2022). [CrossRef]
21. Pham, V.H.; Ahn, H.S. Distributed Stochastic Model Predictive Control for an Urban Traffic Network. *arXiv* **2022**, arXiv:2201.07949. Available online: https://www.semanticscholar.org/reader/0fedb0e9c2e500b6a12c2068e3fce800d4ebc4d9 (accessed on 9 December 2022).
22. Chiou, S.-W. Robust Stochastic Design of Signal-Controlled Road Network under Uncertain Travel Demands. *IEEE Trans. Autom. Control.* **2017**, *62*, 3152–3164. Available online: https://ieeexplore.ieee.org/document/7731219 (accessed on 9 December 2022). [CrossRef]
23. Wong, C.; Lee, Y. Lane-Based Traffic Signal Simulation and Optimization for Preventing Overflow. *Mathematics* **2020**, *8*, 1368. Available online: https://www.mdpi.com/2227-7390/8/8/1368 (accessed on 9 December 2022). [CrossRef]
24. Shepelev, V.; Glushkov, A.; Fadina, O.; Gritsenko, A. Comparative Evaluation of Road Vehicle Emissions at Urban Intersections with Detailed Traffic Dynamics. *Mathematics* **2022**, *10*, 1887. Available online: https://www.mdpi.com/2227-7390/10/11/1887 (accessed on 9 December 2022). [CrossRef]
25. Xia, X.; Ma, X.; Wang, J. Control Method for Signalized Intersection with Integrated Waiting Area. *J. Appl. Sci.* **2019**, *9*, 968. Available online: https://www.mdpi.com/2076-3417/9/5/968 (accessed on 9 December 2022). [CrossRef]
26. Vrbanić, F.; Ivanjko, E.; Kušić, K.; Čakija, D. Variable Speed Limit and Ramp Metering for Mixed Traffic Flows: A Review and Open Questions. *Appl. Sci.* **2021**, *11*, 2574. Available online: https://www.mdpi.com/2076-3417/11/6/2574 (accessed on 9 December 2022). [CrossRef]
27. Gazis, D.C.; Potts, R.B. The oversaturated intersection. In *International Symposia on Traffic Theory*, 2nd ed.; International Business Machines Corporation: London, UK, 1963; pp. 221–237.
28. Aboudolas, K.; Papageorgiou, M.; Kosmatopoulos, E. Store-and-forward based methods for the signal control problem in large-scale congested urban road networks. *Transp. Res. C Emerg. Technol.* **2009**, *17*, 163–174. [CrossRef]
29. Aboudolas, K.; Papageorgiou, M.; Kouvelas, A.; Kosmatopoulos, E. A rolling-horizon quadratic-programming approach to the signal control problem in large-scale congested urban road networks. *Transp. Res. Part C Emerg. Technol.* **2010**, *18*, 680–694. [CrossRef]
30. Tettamanti, T.; Varga, I. Distributed traffic control system based on model predictive control. *Civ. Eng.* **2010**, *54*, 3–9. [CrossRef]
31. Warberg, A.; Larsen, J.; Jørgensen, R.M. Green Wave Traffic Optimization—A Survey. Informatics and Mathematical Modelling. D T U Compute, 2008, Technical Report No. 2008-01. Available online: https://www.researchgate.net/profile/Jesper-Larsen/publication/267938298_Green_Wave_Traffic_Optimization_-_A_Survey/links/555f873308ae6f4dcc92705c/Green-Wave-Traffic-Optimization-A-Survey.pdf (accessed on 9 December 2022).
32. Xu, H.; Zhuo, Z.; Chen, J.; Fang, X. Traffic signal coordination control along oversaturated two-way arterials. *J. Comput. Sci.* **2020**, *6*, e319. Available online: https://peerj.com/articles/cs-319/ (accessed on 9 December 2022). [CrossRef]
33. Lin, S.; De Schutter, B.; Xi, Y.; Hellendoorn, J. A simplified macroscopic urban traffic network model for model-based predictive control. In Proceedings of the 12th IFAC Symposium on Transportation Systems, Redondo Beach, CA, USA, 2–4 September 2009; pp. 286–291. Available online: https://www.dcsc.tudelft.nl/~{}bdeschutter/pub/rep/09_028.pdf (accessed on 9 December 2022).
34. Wei, H.; Zheng, G.; Gayah, V.; Li, Z. A Survey on Traffic Signal Control Methods. 2020. Available online: https://arxiv.org/abs/1904.08117v3 (accessed on 9 December 2022).
35. Lighthill, M.J.; Whitham, G.B. On Kinematic Waves: I: Flow Movement in Long Rivers, II: A Theory of Traffic Flow on Long Crowded Roads. *Proc. R. Soc. Lond. A* **1995**, *229*, 317–345.
36. Richards, P.I. Shock Waves on the Highway. *Oper. Res.* **1956**, *4*, 42–57. [CrossRef]
37. Daganzo, C. The cell transmission model, part II: Network traffic. *Transp. Res. Part B Methodol.* **1995**, *29*, 79–93. Available online: https://ideas.repec.org/a/eee/transb/v29y1995i2p79-93.html (accessed on 9 December 2022). [CrossRef]

38. Sheikh, M.S.; Peng, Y. A Comprehensive Review on Traffic Control Modeling for Obtaining Sustainable Objectives in a Freeway Traffic Environment. *J. Adv. Transp.* **2022**, *2022*, 1012206. Available online: https://www.hindawi.com/journals/jat/2022/1012206/ (accessed on 9 December 2022). [CrossRef]
39. Papageorgiou, M.; Kotsialos, A. Freeway ramp metering: An overview. *IEEE Trans. Intell. Transp. Syst.* **2003**, *3*, 271–281. Available online: https://www.researchgate.net/publication/3427844_Freeway_Ramp_Metering_An_Overview (accessed on 9 December 2022). [CrossRef]
40. Chen, J.M. On Exactitude in Financial Regulation: Value-at-Risk, Expected Shortfall, and Expectiles. *J. Risks* **2018**, *6*, 61. Available online: https://www.mdpi.com/2227-9091/6/2/61 (accessed on 9 December 2022). [CrossRef]
41. Li, P.; Wendt, M.; Wozny, G. A probabilistically constrained model predictive controller. *Automatica* **2002**, *38*, 1171–1176. Available online: https://www.sciencedirect.com/science/article/pii/S000510980200002X (accessed on 9 December 2022). [CrossRef]
42. Zhou, X.; Ye, B.L.; Lu, Y.; Xiong, R. A Novel MPC with Chance Constraints for Signal Splits Control in Urban Traffic Network. *IFAC Proc. Vol.* **2014**, *47*, 11311–11317. Available online: https://www.sciencedirect.com/science/article/pii/S1474667016434142 (accessed on 9 December 2022). [CrossRef]
43. Appendix-Z-Score Percentile for Normal Distribution. Available online: https://www.pindling.org/Math/Learning/Statistics/z_scores_table.htm (accessed on 9 December 2022).