mathematics

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Energy Decay Estimates of a Timoshenko System with Two Nonlinear Variable Exponent Damping Terms

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Abstract: This paper is concerned with the asymptotic behavior of the solution of a Timoshenko system with two nonlinear variable exponent damping terms. We prove that the system is stable under some specific conditions on the variable exponent and the equal wave speeds of propagation. We obtain exponential and polynomial decay results by using the multiplier method, and we prove that one variable damping is enough to have polynomial and exponential decay. We observe that the decay is not necessarily improved if the system has two variable damping terms. Our results built on, developed and generalized some earlier results in the literature.

Keywords: Timoshenko system; energy decay; variable exponents; nonlinear dampings

MSC: 35B37; 35B40; 74D99; 93D15; 93D20

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Citation: Al-Mahdi, A.M.; Al-Gharabli, M.M. Energy Decay Estimates of a Timoshenko System with Two Nonlinear Variable Exponent Damping Terms. *Mathematics* **2023**, *11*, 538. https://doi.org/10.3390/math11030538

Academic Editors: Xin-Guang Yang, Baowei Feng and Xingjie Yan

Received: 25 November 2022
Revised: 2 January 2023
Accepted: 5 January 2023
Published: 19 January 2023

1. Introduction

In 1921, Timoshenko [1] introduced the following system of hyperbolic partial differential equations as a model to describe the dynamics of a thick beam:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) = 0 & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{1}$$

where L, b, K, ρ_1, ρ_2 are positive physical constants, φ is the transverse displacement, and $(-\psi)$ is the rotational angle of the filament of the beam. For almost a century, a great number of researchers have devoted a considerable amount of time and effort studying this model. As a product, many results concerning the well-posedness and long-time behavior of the system have been established. For this matter of various types of dissipation, such as boundary and/or internal feedback, heat or thermoelasticity, finite and infinite memory, and Kelvin–Voigt damping, have been utilized. Various results regarding existence, polynomial, exponential and general decay have been proved. For example, the viscoelastic-type Timoshenko system had received a considerable attention since the work of Ammar-Khodja et al. [2] in which the authors studied the following system:

$$\begin{cases} \rho_1\phi_{tt} - K(\phi_x + \psi)_x = 0 & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(u_x + \psi) + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds = 0 & \text{in } (0,L)\times(0,+\infty), \\ \phi(0,t) = \phi(L,t) = \psi(0,t) = \psi(L,t) = 0 & \text{for } t \geq 0, \end{cases} \tag{2}$$

where g is a positive non-increasing differentiable L^1 function defined on $\mathbb{R}_+$, and it is called the relaxation function (kernel). They established the uniform stability of the system in the case of equal speeds of wave propagation. For the rate of decay, they obtained exponential and polynomial stability of the system for the relaxation functions g decaying exponentially and polynomially, respectively. Guesmia and Messaoudi [3] proved the same decay result of [2] by weakening some of the assumptions on the relaxation function g. Messaoudi and Mustafa [4] investigated the same system under more general assumptions on the relaxation function g and proved for the first time a general decay result from which the exponential and polynomial stability are only special cases.

For the Timoshenko system with frictional damping terms, a list of researchers have investigated the well-posedness and long-time behavior of the solutions of this system. For example, Kim and Renardy [5] investigated the uniform stabilization of the Timoshenko beam with two boundary control forces. They proved the exponential decay of the energy by using a multiplier method and provided numerical estimates of the eigenvalues of the operator associated with this system. Shi and Feng [6] considered a Timoshenko beam with two locally distributed pieces of feedback and established, using a frequency multiplier method, an exponential decay result for the energy. Muñoz and Racke [7] considered

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + d\psi_t = 0, \end{cases} \tag{3}$$

where d is a constant and established exponential and polynomial decay results. Mustafa and Messaoudi [8] considered the Timoshenko system (3) where $d\psi_t$ is replaced by $\alpha(t)h(\psi(t))$ and established some explicit and general decay results, depending on h and α, using some properties of convex functions. Similar results can be found in [9–17] and the references therein. In the above works, it is proved that the exponential stability of system (1) is achieved in the presence of linear damping mechanisms on both equations of (1) without imposing any condition on the speeds of wave propagation. However, if the damping effect is acting on only one equation, the system is exponentially stable if and only if it has equal speeds of wave propagation; that is,

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{4}$$

For the Timoshenko system with viscoelastic and nonlinear frictional dampings of the form

$$\begin{cases} \phi_{tt} - (\phi_x + \psi)_x = 0, \\ \psi_{tt} - \psi_{xx} + \phi_x + \psi + \int_0^t g(t-s)\psi_{xx}(s)ds + h(\psi_t) = 0, \end{cases} \tag{5}$$

Mustafa [18] obtained energy decay rates for (5) with general assumptions on the functions h and g. Al-Mahdi et al. [19] considered the system (5) with replacing the memory term $\int_0^t g(t-s)\psi_{xx}(s)ds$ with infinite memory $\int_0^{+\infty} g(s)\psi_{xx}(t-s)ds$ and obtained some new decay results in the case of equal speeds of wave propagation and just an upper bound estimate for the energy in the case of non-equal speeds of wave propagation. For more results on stability of Timoshenko systems with frictional and/or viscoelastic damping, we refer the reader to [20] and the references therein.

With the advancement of sciences and technology, many physical and engineering models require more sophisticated mathematical functional spaces to be studied and well understood. For instance, some models from physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, filtration processes in a porous media, nonlinear viscoelasticity, and image processing, give rise to such problems. The Lebesgue and Sobolev spaces with variable exponents proved to be efficient tools to study such problems, as well as other models such as filtration processes through a porous media and image processing. We cite [21,22] for further details on the electro-rheological fluids mathematical model. We briefly mention a few of the many references [23–29] that discuss the existence, blow-up, and stability of some problems with variable exponents. Messaoudi [30] focused in particular on the following equation:

$$u_{tt} - div(a(x)|\nabla u|^{r(.)-2}\nabla u) - \Delta u_t + |u_t|^{m(.)-2}u_t = 0, \tag{6}$$

and provided exponential and polynomial decay results with specified constraints on m and r. The following problem

$$u_{tt} - div\Big(|\nabla u|^{p(x)-2}\nabla u\Big) + |u_t|^{m(x)-2}u_t = |u|^{q(x)-2}u, \tag{7}$$

was recently studied by Li et al. [31], and a blow-up result has been produced for solutions with negative initial energy. The following problem

$$u_{tt} - \Delta u + u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m(\cdot)-2}u_t = u\ln|u|^k, \text{ on } \Omega \times (0, +\infty), \tag{8}$$

was studied by Al-Gharabli et al. [32], where g is a relaxation function, $m(.)$ is a variable exponent, and u_0 and u_1 are the given data. The authors produced explicit and general decay results for a large class of relaxation functions using the well-depth approach, as well as some specific requirements on the variable exponent function. Gao and Gao [33] and Park and Kang [34] who studied

$$u_{tt} - \Delta u + u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m(\cdot)-2}u_t = b|u|^{q(x)-2}u, \text{ on } \Omega \times (0, +\infty), \tag{9}$$

and proved the existence and blow-up results. Hassan et al. [35] treated Problem (9) when $b = 0$ and established an energy decay estimate. Mustafa et al. [36] considered the following wave equation with nonlinear damping having a variable exponent and a time-dependent coefficient

$$u_{tt} - \Delta u + \alpha(t)|u_t|^{m(x)-2}u_t = 0, \tag{10}$$

and established theoretical and numerical energy decay results depending on both α and m. Recently, Mustafa [37] studied the Timoshenko system (5) with replacing the frictional damping term $h(\psi_t)$ by the variable exponent damping term $|\psi_t|^{m(x)-2}\psi_t$ and established explicit energy decay rates where $m(x)$ is the variable-exponent function satisfying some spastic conditions.

In the present work, we consider the following nonlinear Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + \gamma|\varphi_t|^{p(\cdot)-2}\varphi_t = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta|\psi_t|^{q(\cdot)-2}\psi_t = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \end{cases} \tag{11}$$

where $(x,t) \in (0,L) \times (0,+\infty)$, γ, β are positive constants, $\varphi_0, \varphi_1, \psi_0, \psi_1$ are given data and $p(.)$ and $q(.)$ are the variable exponent functions satisfying some conditions to be specified in the next section. This system describes the transverse vibrations of a beam subject to the effect of the two nonlinear variable exponent damping terms $|\varphi_t|^{p(\cdot)-2}\varphi_t$ and $|\psi_t|^{q(\cdot)-2}\psi_t$ in the presence of a non-standard frictional damping due to the nature of the "smart" material. Our goal is to investigate System (11) and prove that the system is exponentially and polynomially stable and the stability results depend on the coefficient of the system and the variable exponents $p(.)$ and $q(.)$. We prove that one variable damping is enough to have polynomial and exponential decay. We observe that the decay is not necessarily improved if the system has two variable damping terms.

The paper is organized as follows: In Section 2, some preliminaries are given. We establish some technical lemmas in Section 3. In Section 4, we state and prove our main energy decay results. We present some conclusions in Section 5.

2. Preliminary and Assumptions

In this section, we present some preliminaries about the Lebesgue and Sobolev spaces with variable exponents (see [38–40]). Throughout this paper, c is used to denote a generic positive constant. Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. The Lebesgue space with a variable exponent $p(\cdot)$ is given by

$$L^{p(\cdot)}(\Omega) := \Big\{ v : \Omega \to \mathbb{R};\ \text{measurable in } \Omega :\ \varrho_{p(\cdot)}(\lambda v) < \infty,\ \text{for some } \lambda > 0 \Big\},$$

where

$$\varrho_{p(\cdot)}(v) = \int_\Omega |v(x)|^{p(x)} dx,$$

equipped with the following Luxembourg-type norm

$$\|v\|_{p(\cdot)} := \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{v(x)}{\lambda} \Big|^{p(x)} dx < \infty \Big\},$$

the space $L^{p(\cdot)}(\Omega)$ is a Banach space (see [39]), separable if $p(\cdot)$ is bounded and reflexive if $1 < p_1 \le p_2 < \infty$, where

$$p_1 := \operatorname{essinf}_{x\in\Omega} p(x), \quad p_2 := \operatorname{esssup}_{x\in\Omega} p(x).$$

The variable-exponent Sobolev space is defined as follows:

$$W^{1,p(\cdot)}(\Omega) = \Big\{ v \in L^{p(\cdot)}(\Omega) \ \text{such that } \nabla v \text{ exists and } |\nabla v| \in L^{p(\cdot)}(\Omega) \Big\},$$

which is a Banach space with respect to the norm $\|v\|_{W^{1,p(\cdot)}(\Omega)} = \|v\|_{p(\cdot)} + \|\nabla v\|_{p(\cdot)}$, and it is separable if $p(\cdot)$ is bounded and reflexive if $1 < p_1 \le p_2 < \infty$. Furthermore, we set $W_0^{1,p(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. We use the standard Lebesgue space $L^2(0, L)$ and Sobolev space $H_0^1(0, L)$ with their usual scalar products and norms and we assume the following hypotheses :

- **(A.1)** $p, q : \overline{\Omega} \to [1, \infty)$ are continuous functions such that

  $$p_1 := \operatorname{essinf}_{x\in\Omega} p(x), \quad p_2 := \operatorname{esssup}_{x\in\Omega} p(x).$$

  and $1 < p_1 \le p(x) \le p_2 < \infty$.

  $$q_1 := \operatorname{essinf}_{x\in\Omega} q(x), \quad q_2 := \operatorname{esssup}_{x\in\Omega} q(x).$$

  and $1 < q_1 \le q(x) \le q_2 < \infty$. Moreover, the variable functions p and q satisfy the log-Hölder continuity condition; that is, for any δ with $0 < \delta < 1$, there exists a constant $A > 0$ such that

  $$|p(x) - p(y)| \le -\frac{A}{\log|x-y|}, \ \text{for all } x, y \in \Omega, \ \text{with } |x - y| < \delta, \tag{12}$$

  and the same log-Hölder continuity condition for the variable function q.
- **(A.2)** The coefficients b, ρ_1, ρ_2, K satisfy $\frac{\rho_1}{\rho_2} = \frac{b}{K}$.

For completeness, we state, without proof, the global existence of System (11) which can be established by Faedo–Galerkin approximation, see [23,33].

Proposition 1. *Let* $(\varphi_0, \varphi_1), (\psi_0(.,0), \psi_1) \in H_0^1(0,L) \times L^2(0,L)$ *be given. Assume that* (**A.1**) *holds. Then, problem* (11) *has a unique global (weak) solution*

$$\varphi, \psi \in L^\infty\big([0,T); H_0^1(0,L)\big),$$

and

$$\varphi_t \in L^\infty\big((0,T); L^2(0,L)\big) \cap L^{p(\cdot)}\big((0,L) \times (0,T)\big),$$
$$\psi_t \in L^\infty\big((0,T); L^2(0,L)\big) \cap L^{q(\cdot)}\big((0,L) \times (0,T)\big).$$

We introduce the "modified" energy associated with problem (11)

$$E(t) := \frac{1}{2}\Big[\rho_1||\varphi_t^2||_2 + \rho_2||\psi_t^2||_2 + ||\psi_x^2||_2 + K||(\varphi_x + \psi)^2||_2\Big], \tag{13}$$

where $||\cdot||_2 = ||\cdot||_{L^2(0,L)}$. Direct differentiation, using (11), leads to

$$E'(t) = -\gamma \int_0^L |\varphi_t|^{p(\cdot)} dx - \beta \int_0^L |\psi_t|^{q(\cdot)} dx \le 0, \quad \forall t \ge 0. \tag{14}$$

3. Technical Lemmas

In this section, we state and establish several lemmas needed for the proof of our main result. We use $c > 0$ to denote a positive generic constant. The following lemmas will be of essential use in proving our decay results.

Lemma 1. *For any* $\eta > 0$ *and* $q_1 \ge 2$*, we have the following*

$$-\beta \int_0^L \psi|\psi_t|^{q(\cdot)-2}\psi_t dx \le c_1\eta\beta \int_0^L \psi_x^2 dx + \beta \int_0^L c_\eta(x)|\psi_t|^{q(x)} dx, \tag{15}$$

and, if $1 < q_1 < 2$*, we have*

$$-\beta \int_0^L \psi|\psi_t|^{q(\cdot)-2}\psi_t dx \le 2c\eta\beta \int_0^L \psi_x^2 dx + c_\eta\left[\beta \int_0^L |\psi_t|^{q(x)} dx + \left(\int_0^L \beta|\psi_t|^{q(x)} dx\right)^{q_1-1}\right], \tag{16}$$

where c_η *is a positive constant depends on* η *and* c_1 *and* $c_\eta(x)$ *are two positive constants defined in* (19) *and* (22)*, respectively.*

Lemma 2. *For any* $\lambda > 0$ *and* $p_1 \ge 2$*, we have the following*

$$-\gamma \int_0^L \psi|\varphi_t|^{p(\cdot)-2}\varphi_t dx \le c_2\lambda\gamma \int_0^L \varphi_x^2 dx + \gamma \int_0^L c_\lambda(x)|\varphi_t|^{p(x)} dx, \tag{17}$$

and, if $1 < p_1 < 2$*, we have*

$$-\gamma \int_0^L \varphi|\varphi_t|^{p(\cdot)-2}\varphi_t dx \le 2c\lambda\gamma \int_0^L \varphi_x^2 dx + c_\lambda\left[\gamma \int_0^L |\varphi_t|^{p(x)} dx + \left(\int_0^L \gamma|\varphi_t|^{p(x)} dx\right)^{p_1-1}\right], \tag{18}$$

where c_λ *is a positive constant that depends on* λ*,*

$$c_\lambda(x) = \lambda^{1-p(x)}(p(x))^{-p(x)}(p(x)-1)^{p(x)-1} > 0,$$

and

$$c_2 = \left(c_e^{p_1}\Big(2E(0)\Big)^{\frac{p_1-2}{}} + c_e^{p_2}\Big(2E(0)\Big)^{\frac{p_2-2}{}}\right) > 0.$$

Proof. We prove Lemma 1 and the proof of Lemma 2 will be in the same way. We start by applying Young's inequality with $\zeta(x) = \frac{q(x)}{q(x)-1}$ and $\zeta'(x) = q(x)$. Thus, for a.e $x \in (0, L)$ and any $\eta > 0$, we have

$$|\psi_t|^{q(x)-2}\psi_t\psi \leq \eta|\psi|^{q(x)} + c_\eta(x)|\psi_t|^{q(x)},$$

where

$$c_\eta(x) = \eta^{1-q(x)}(q(x))^{-q(x)}(q(x)-1)^{q(x)-1} > 0. \tag{19}$$

Hence,

$$-\beta\int_0^L \psi|\psi_t|^{q(x)-2}\psi_t dx \leq \eta\beta\int_0^L |\psi|^{q(x)}dx + \beta\int_0^L c_\eta(x)|\psi_t|^{q(x)}dx. \tag{20}$$

Next, using (13) and (14), Poincaré's inequality and the embedding property, we obtain

$$\begin{aligned}
\int_0^L |\psi|^{q(x)}dx &= \int_{\Omega_+} |\psi|^{q(x)}dx + \int_{\Omega_-} |\psi|^{q(x)}dx \\
&\leq \int_{\Omega_+} |\psi|^{q_2}dx + \int_{\Omega_-} |\psi|^{q_1}dx \\
&\leq \int_0^L |\psi|^{q_2}dx + \int_0^L |\psi|^{q_1}dx \\
&\leq c_e^{q_1}||\psi_x||_2^{q_1} + c_e^{q_2}||\psi_x||_2^{q_2} \\
&\leq \left(c_e^{q_1}||\psi_x||_2^{q_1-2} + c_e^{q_2}||\psi_x||_2^{q_2-2}\right)||\psi_x||_2^2 \\
&\leq \left(c_e^{q_1}\Big(2E(0)\Big)^{q_1-2} + c_e^{q_2}\Big(2E(0)\Big)^{q_2-2}\right)||\psi_x||_2^2 \\
&\leq c_1||\psi_x||_2^2,
\end{aligned} \tag{21}$$

where c_e is the embedding constant,

$$\Omega_+ = \{x \in (0, L) : |\psi(x, t)| \geq 1\}, \;\; \Omega_- = \{x \in (0, L) : |\psi(x, t)| < 1\}$$

and

$$c_1 = \left(c_e^{q_1}\Big(2E(0)\Big)^{q_1-2} + c_e^{q_2}\Big(2E(0)\Big)^{q_2-2}\right). \tag{22}$$

Then, from (20) and (21), we find that

$$-\beta\int_0^L \psi|\psi_t|^{q(x)-2}\psi_t dx \leq c_1\eta\beta\int_0^L \psi_x^2 dx + \beta\int_0^L c_\eta(x)|\psi_t|^{q(x)}dx. \tag{23}$$

Combining all the above estimations, estimate (15) is established. To prove (16), we set

$$\Omega_1 = \{x \in (0, L) : q(x) < 2\} \text{ and } \Omega_2 = \{x \in (0, L) : q(x) \geq 2\}.$$

Then, we have

$$-\beta\int_0^L \psi|\psi_t|^{q(x)-2}\psi_t dx = -\beta\int_{\Omega_1} \psi|\psi_t|^{q(x)-2}\psi_t dx - \beta\int_{\Omega_2} \psi|\psi_t|^{q(x)-2}\psi_t dx. \tag{24}$$

We notice that on Ω_1,

$$2q(x) - 2 < q(x), \;\text{ and }\; 2q(x) - 2 \geq 2q_1 - 2. \tag{25}$$

Therefore, by using Young's and Poincaré's inequalities and (25), we find that

$$\begin{aligned}
&-\beta\int_{\Omega_1}\psi|\psi_t|^{q(x)-2}\psi_t dx \leq \eta\beta\int_{\Omega_1}|\psi|^2 dx+\frac{\beta}{4\eta}\int_{\Omega_1}|\psi_t|^{2q(x)-2}dx\\
&\leq c\eta\beta||\psi_x||_2^2+c_\eta\beta\left[\int_{\Omega_1^+}|\psi_t|^{2q(x)-2}dx+\int_{\Omega_1^-}|\psi_t|^{2q(x)-2}dx\right]\\
&\leq c\eta\beta||\psi_x||_2^2+c_\eta\beta\left[\int_{\Omega_1^+}|\psi_t|^{q(x)}dx+\int_{\Omega_1^-}|\psi_t|^{2q_1-2}dx\right]\\
&\leq c\eta\beta||\psi_x||_2^2+c_\eta\beta\left[\int_{\Omega}|\psi_t|^{q(x)}dx+\left(\int_{\Omega_1^-}|\psi_t|^{2}dx\right)^{q_1-1}\right]\\
&\leq c\eta\beta||\psi_x||_2^2+c_\eta\beta\left[\int_0^L|\psi_t|^{q(x)}dx+\left(\int_{\Omega_1^-}|\psi_t|^{q(x)}dx\right)^{q_1-1}\right]\\
&\leq c\eta\beta||\psi_x||_2^2+c_\eta\left[\beta\int_0^L|\psi_t|^{q(x)}dx+\beta^{2-q_1}\left(\int_0^L\beta|\psi_t|^{q(x)}dx\right)^{q_1-1}\right],
\end{aligned}\tag{26}$$

where $c_\eta=\frac{1}{4\eta}$ and

$$\Omega_1^+=\{x\in\Omega_1:|\psi_t(x,t)|\geq 1\}\ \text{ and }\ \Omega_1^-=\{x\in\Omega_1:|\psi_t(x,t)|<1\}.\tag{27}$$

Next, we have, by the case of $q(x)\geq 2$,

$$-\beta\int_{\Omega_2}\psi|\psi_t|^{q(x)-2}\psi_t dx\leq c\eta\beta\int_0^L\psi_x^2 dx+\beta\int_\Omega c_\eta(x)|\psi_t|^{q(x)}dx.\tag{28}$$

Combining (26) and (28), the proof of (16) is completed. □

Lemma 3. *Assume that (**A.1**) holds. Then, the functional F defined by*

$$F(t):=-\rho_2\int_0^L\psi\psi_t dx$$

satisfies, for $q_1\geq 2$, *the following estimate*

$$\begin{aligned}
F'(t)\quad\leq\quad&-\rho_2\int_0^L\psi_t^2dx+\frac{K}{4}\int_0^L(\varphi_x+\psi)^2dx+(K+2b)\int_0^L\psi_x^2dx\\
&+c\beta\int_0^L|\psi_t|^{q(\cdot)}dx,
\end{aligned}\tag{29}$$

and for $1<q_1<2$, *the functional satisfies*

$$\begin{aligned}
F'(t)\quad\leq\quad&-\rho_2\int_0^L\psi_t^2dx+\frac{K}{4}\int_0^L(\varphi_x+\psi)^2dx+(K+2b)\int_0^L\psi_x^2dx\\
&+c\beta\int_0^L|\psi_t|^{q(\cdot)}dx+c\beta\left(\int_0^L|\psi_t|^{q(x)}dx\right)^{q_1-1}.
\end{aligned}\tag{30}$$

Proof. To prove (29), we start by differentiating F and using the equations in (11) to obtain

$$\begin{aligned}
F'(t)\quad=\quad&-\rho_2\int_0^L\psi_t^2dx-b\int_0^L\psi\psi_{xx}dx+K\int_0^L\psi(\varphi_x+\psi)dx\\
+\quad&\beta\int_0^L\psi|\psi_t|^{q(\cdot)-2}\psi_t dx.
\end{aligned}\tag{31}$$

Integrating by parts and estimating the last term in (31) using (15), applying Young and Poincaré's inequality, then for a positive constant ε, Equation (31) becomes

$$\begin{aligned} F'(t) \quad &\leq \quad -\rho_2 \int_0^L \psi_t^2 dx + b \int_0^L \psi_x^2 dx + K\varepsilon \int_0^L (\varphi_x + \psi)^2 dx + \frac{K}{4\varepsilon} \int_0^L \psi_x^2\, dx \\ &\quad + c_1 \eta \beta \int_0^L \psi_x^2 dx + \beta \int_0^L c_\eta(x) |\psi_t|^{q(\cdot)} dx. \end{aligned}$$

where c_1 is defined in (22). Selecting $\varepsilon = \frac{1}{4}$ and $\eta = \frac{b}{c_1\beta}$, the proof of (29) is completed. The proof of (30) is straightforward by imposing (16) for estimating the last term in (31). □

Lemma 4. *Under the condition (**A.1**), the functional I_1 defined by*

$$I_1(t) := -\int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) dx$$

satisfies for $p_1, q_1 \geq 2$ the following estimate

$$\begin{aligned} I_1'(t) \quad &\leq -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + 2K \int_0^L (\varphi_x + \psi)^2 dx + (2b + K) \int_0^L \psi_x^2 dx \\ &\quad + c\beta \int_0^L |\psi_t|^{q(\cdot)} dx + \gamma \int_0^L c_\lambda(x) |\varphi_t|^{p(\cdot)} dx, \end{aligned} \tag{32}$$

and for $1 < p_1, q_1 < 2$, the functional satisfies

$$\begin{aligned} I_1'(t) \quad &\leq -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + 2K \int_0^L (\varphi_x + \psi)^2 dx + (2b + K) \int_0^L \psi_x^2 dx \\ &\quad + c\beta \int_0^L |\psi_t|^{q(\cdot)} dx + \gamma \int_0^L c_\lambda(x) |\varphi_t|^{p(\cdot)} dx \\ &\quad + c\beta \left(\int_0^L |\psi_t|^{q(x)} dx \right)^{q_1 - 1} + c\gamma \left(\int_0^L |\varphi_t|^{p(x)} dx \right)^{p_1 - 1}. \end{aligned} \tag{33}$$

Proof. To prove (32), we use the equations of (11) to obtain

$$\begin{aligned} I_1'(t) \quad &= \quad -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + K \int_0^L (\varphi_x + \psi)^2 dx + b \int_0^L \psi_x^2 \\ &\quad + \gamma \int_0^L \varphi |\varphi_t|^{p(\cdot)-2} \varphi_t dx + \beta \int_0^L \psi |\psi_t|^{q(\cdot)-2} \varphi_t dx. \end{aligned}$$

Inserting (15) and (17) in the above equation, we find that

$$\begin{aligned} I_1'(t) \quad &\leq \quad -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx + K \int_0^L (\varphi_x + \psi)^2 dx + b \int_0^L \psi_x^2 dx + c_1 \eta \beta \int_0^L \psi_x^2 dx \\ &\quad + c_2 \lambda \gamma \int_0^L \varphi_x^2 dx + \gamma \int_0^L c_\lambda(x) |\varphi_t|^{p(\cdot)} dx + \beta \int_0^L c_\eta(x) |\psi_t|^{q(\cdot)} dx. \end{aligned} \tag{34}$$

Imposing the relation

$$\varphi_x^2 \leq 2(\varphi_x + \psi)^2 + 2\psi^2,$$

and using Poincaré's inequality on the term ψ^2, then (34) becomes

$$\begin{aligned} I_1'(t) \quad \leq \quad & -\int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2)dx + K\int_0^L (\varphi_x+\psi)^2 dx + b\int_0^L \psi_x^2 dx + c_1\eta\beta\int_0^L \psi_x^2 dx \\ & +(2c_2\gamma)\lambda\int_0^L (\varphi_x+\psi)^2 dx + 2c_2\lambda\gamma\int_0^L \psi_x^2 dx + \gamma\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx \\ + & \int_0^L c_\eta(x)|\psi_t|^{q(\cdot)}dx. \end{aligned}$$

Selecting $\eta = \frac{b}{c_1\beta}$ and $\lambda = \frac{K}{2c_2\gamma}$, so the proof of (32) is completed and the proof of (33) is straightforward by estimating the last two integrals in (34) using (16) and (18). □

Lemma 5. *Assume that (**A.1**) holds. Then, for any $0<\varepsilon<1$, the functional*

$$I_2(t) := \rho_2\int_0^L \psi_t(\varphi_x+\psi)dx + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_x dx$$

satisfies for $p_1, q_1 \geq 2$ the following estimate

$$\begin{aligned} I_2'(t) \quad \leq \quad & [b\psi_x\varphi_x]_{x=0}^{x=L} - \frac{K}{2}\int_0^L (\varphi_x+\psi)^2 dx \\ & +\rho_2\int_0^L \psi_t^2 dx + c\lambda\int_0^L \psi_x^2 dx + c\gamma\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx \\ & +\left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + c\beta\int_0^L |\psi_t|^{q(\cdot)}dx, \end{aligned} \tag{35}$$

and for $1<p_1,q_1<2$, the functional satisfies

$$\begin{aligned} I_2'(t) \quad \leq \quad & [b\psi_x\varphi_x]_{x=0}^{x=L} - \frac{K}{2}\int_0^L (\varphi_x+\psi)^2 dx \\ & +\rho_2\int_0^L \psi_t^2 dx + c\lambda\int_0^L \psi_x^2 dx + c\gamma\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx \\ & +\left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + c\beta\int_0^L |\psi_t|^{q(\cdot)}dx \\ & +c\beta\left(\int_0^L |\psi_t|^{q(x)}dx\right)^{q_1-1} + c\gamma\left(\int_0^L |\varphi_t|^{p(x)}dx\right)^{p_1-1}. \end{aligned} \tag{36}$$

Proof. To prove (35), we use the equations of (11), integrating by parts, recall (15) and (17) and apply Young's inequality to have the following:

$$\begin{aligned} I_2'(t) \quad = \quad & [b\psi_x\varphi_x]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx \\ & +\rho_2\int_0^L \psi_t^2 dx + \left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx \\ & -\beta\int_0^L (\varphi_x+\psi)|\psi_t|^{q(\cdot)-2}\psi_t dx - \frac{b\gamma}{K}\int_0^L \psi_x|\varphi_t|^{p(\cdot)-2}\varphi_t\, dx \\ \leq \quad & [b\psi_x\varphi_x]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx + \overline{c_1}\eta\beta\int_0^L (\varphi_x+\psi)^2 dx \\ & +\rho_2\int_0^L \psi_t^2 dx + \beta\int_0^L \overline{c_1}_\eta(x)|\psi_t|^{q(\cdot)}dx + \frac{b\overline{c_1}\lambda\gamma}{K}\int_0^L \psi_x^2\, dx \\ & +\left(\frac{b\rho_1}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + \frac{\gamma b}{K}\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx, \end{aligned}$$

where

$$\overline{c_1} = \left(2E(0)\right)^{q_1-2} + \left(2E(0)\right)^{q_2-2}. \tag{37}$$

Selection $\eta = \frac{K}{2\overline{c_1}\eta\beta}$, the proof of (35) is completed and the proof of (36) is straightforward by estimating the last two integrals in (35) using (16) and (18). □

Lemma 6. *Assume that (**A.1**) holds. Let $m(x) = 2 - \frac{4}{L}x$, for $x \in [0, L]$. Then, for any $0 < \varepsilon < 1$, the functional*

$$I_3(t) := \frac{b\rho_2}{4\varepsilon}\int_0^L m(x)\psi_t\psi_x dx + \varepsilon\frac{\rho_1}{K}\int_0^L m(x)\varphi_t\varphi_x dx$$

satisfies for $p_1, q_1 \geq 2$ the following estimate:

$$\begin{aligned} I_3'(t) \quad \leq \quad & -\frac{1}{4}\Big[(\psi_x(L,t))^2 + (b\psi_x(0,t))^2\Big] \\ & -\varepsilon\Big(\varphi_x^2(L,t) + \varphi_x^2(0,t)\Big) + \Big(\frac{1}{4} + c\varepsilon + c\varepsilon\lambda\Big)K\int_0^L (\varphi_x+\psi)^2 dx + c\varepsilon\rho_1\int_0^L \varphi_t^2 dx \\ & +\frac{c}{\varepsilon}\rho_2\int_0^L \psi_t^2 dx + \frac{c\eta}{\varepsilon^2}\int_0^L \psi_x^2 dx + \gamma\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx \\ + \quad & \beta\int_0^L c_\eta(x)|\psi_t|^{q(\cdot)}dx, \end{aligned} \tag{38}$$

and for $1 < p_1, q_1 < 2$, the functional satisfies

$$\begin{aligned} I_3'(t) \quad \leq \quad & -\frac{1}{4}\Big[(\psi_x(L,t))^2 + (b\psi_x(0,t))^2\Big] \\ & -\varepsilon\Big(\varphi_x^2(L,t) + \varphi_x^2(0,t)\Big) + \Big(\frac{1}{4} + c\varepsilon + c\varepsilon\lambda\Big)K\int_0^L (\varphi_x+\psi)^2 dx + c\varepsilon\rho_1\int_0^L \varphi_t^2 dx \\ & +\frac{c}{\varepsilon}\rho_2\int_0^L \psi_t^2 dx + \frac{c\eta}{\varepsilon^2}\int_0^L \psi_x^2 dx + \gamma\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx + \beta\int_0^L c_\eta(x)|\psi_t|^{q(\cdot)}dx \\ & +c\beta\left(\int_0^L |\psi_t|^{q(x)}dx\right)^{q_1-1} + c\gamma\left(\int_0^L |\varphi_t|^{p(x)}dx\right)^{p_1-1}. \end{aligned} \tag{39}$$

Proof. Exploiting the equations of (11), we have

$$\begin{aligned} I_3'(t) \quad = \quad & \frac{1}{4\varepsilon}\left[-(b\psi_x(L,t))^2 - (b\psi_x(0,t))^2 - \frac{1}{2}\int_0^L m'(x)(b\psi_x)^2 dx\right] \\ + \quad & \frac{1}{4\varepsilon}\left[-K\int_0^L m(x)(b\psi_x)(\varphi_x+\psi)dx - \frac{b\rho_2}{2}\int_0^L m'(x)\psi_t^2 dx\right] \\ - \quad & \frac{1}{4\varepsilon}\left[b\beta\int_0^L m(x)\psi_x|\psi_t|^{q(\cdot)-2}\psi_t dx\right] \\ & +\varepsilon\Big[-(\varphi_x^2(L,t) + \varphi_x^2(0,t)) + \int_0^L m(x)\varphi_x\psi_x dx \\ & -\frac{1}{2}\int_0^L m'(x)\varphi_x^2 dx - \frac{\rho_1}{2K}\int_0^L m'(x)\varphi_t^2 dx - \frac{\gamma}{K}\int_0^L m(x)\varphi_x|\varphi_t|^{p(\cdot)-2}\varphi_t dx\Big]. \end{aligned}$$

Using Young's and Poicaré's inequalities, (15) and (17), and the relation

$$\varphi_x^2 \leq 2(\varphi_x+\psi)^2 + 2\psi^2,$$

we find that

$$\begin{aligned} I_3'(t) \quad \leq \quad & \frac{1}{4\varepsilon}\Big[-(b\psi_x(L,t))^2-(b\psi_x(0,t))^2\Big] \\ & -\varepsilon\Big(\varphi_x^2(L,t)+\varphi_x^2(0,t)\Big)+\Big(\frac{1}{4}+c\varepsilon+c\varepsilon\lambda\Big)K\int_0^L(\varphi_x+\psi)^2dx \\ & +c\varepsilon\rho_1\int_0^L\varphi_t^2dx+\frac{c}{\varepsilon}\rho_2\int_0^L\psi_t^2dx+\frac{c\eta}{\varepsilon^2}\int_0^L\psi_x^2dx \\ & +\gamma\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx+\beta\int_0^L c_\eta(x)|\psi_t|^{q(\cdot)}dx. \end{aligned}$$

Hence, the proof of (38) is completed, and the proof of (39) is straightforward by using (16) and (18). □

Lemma 7. *Assume that (**A.1**) holds, then for any $\varepsilon>0$ small enough, the functional*

$$I(t):=3c\varepsilon I_1(t)+\mu I_2(t)+I_3(t)$$

satisfies for $p_1,q_1\geq 2$ the following estimate

$$\begin{aligned} I'(t) \quad \leq \quad & -\frac{K}{2}\int_0^L(\varphi_x+\psi)^2dx-2c_1\rho_1\int_0^L\varphi_t^2dx \\ & +c\rho_2\int_0^L\psi_t^2dx+c\int_0^L\psi_x^2dx+c\gamma\int_0^L|\varphi_t|^{p(\cdot)}dx \\ & +c\Big(\frac{b\rho_1}{K}-\rho_2\Big)\int_0^L\varphi_t\psi_{xt}dx+c\beta\int_0^L|\psi_t|^{q(\cdot)}dx. \end{aligned} \tag{40}$$

Proof. Using the estimates in Lemmas 4–6, we have

$$\begin{aligned} - \quad & \frac{K}{2}\Big(\mu-14c\varepsilon-2c\varepsilon\lambda-\frac{1}{2}\Big)\int_0^L(\varphi_x+\psi)^2dx \\ - \quad & 2c\varepsilon\rho_1\int_0^L\varphi_t^2dx+\mu\Big(\frac{b\rho_1}{K}-\rho_2\Big)\int_0^L\varphi_t\psi_{xt}dx \\ + \quad & \Big(-3c\varepsilon+\mu+\frac{c}{\varepsilon}\Big)\rho_2\int_0^L\psi_t^2dx \\ + \quad & \Big(3c\varepsilon(2b+K)+c\lambda\mu+\frac{c\eta}{\varepsilon^2}\Big)\int_0^L\psi_x^2dx \\ + \quad & c\varepsilon\gamma\int_0^L c_\lambda(x)|\varphi_t|^{p(\cdot)}dx+c\varepsilon\beta\int_0^L c_\eta(x)|\psi_t|^{q(\cdot)}dx. \end{aligned} \tag{41}$$

Then, choosing $\lambda=1$, $\mu=18c\varepsilon+\frac{1}{2}$ and ε so that $2c\varepsilon\geq 1$. Once ε is fixed, we set $c_1=c\varepsilon$ and obtain the required result. In the same way, using (16) and (18), we can prove that, for $1<p_1,q_1<2$,

$$\begin{aligned} I'(t) \quad \leq \quad & -\frac{K}{2}\int_0^L(\varphi_x+\psi)^2dx-2c_1\rho_1\int_0^L\varphi_t^2dx \\ & +c\rho_2\int_0^L\psi_t^2dx+c\int_0^L\psi_x^2dx+c\gamma\int_0^L|\varphi_t|^{p(\cdot)}dx \\ & +c\Big(\frac{b\rho_1}{K}-\rho_2\Big)\int_0^L\varphi_t\psi_{xt}dx+c\beta\int_0^L|\psi_t|^{q(\cdot)}dx \\ & +c\beta\Big(\int_0^L|\psi_t|^{q(x)}dx\Big)^{q_1-1}+c\gamma\Big(\int_0^L|\varphi_t|^{p(x)}dx\Big)^{p_1-1}. \end{aligned} \tag{42}$$

□

In the following lemma, we use the multiplier

$$w(x,t) = \frac{1}{L}\left(\int_0^L \psi(y,t)dy\right)x - \int_0^x \psi(y,t)dy$$

which satisfies, for some $c_2 > 0$,

$$\int_0^L w_x^2 dx \leq \int_0^L \psi^2 dx \qquad \text{and} \qquad \int_0^L w_t^2 dx \leq c_2 \int_0^L \psi_t^2 dx. \tag{43}$$

Lemma 8. *Assume that (**A.1**) holds. Then, the functional*

$$J(t) := \int_0^L (\rho_1 w \varphi_t + \rho_2 \psi \psi_t) dx, \tag{44}$$

satisfies for $\varepsilon_0 > 0$ and $p_1, q_1 \geq 2$, the estimate

$$\begin{aligned} J'(t) \quad &\leq \rho_1 \varepsilon_0 \int_0^L \varphi_t^2 dx + \frac{c\rho_2}{\varepsilon_0} \int_0^L \psi_t^2 dx - \frac{b}{2} \int_0^L \psi_x^2 dx + c\beta \int_0^L |\psi_t|^q dx \\ &+ c\gamma \int_0^L |\varphi_t|^p dx. \end{aligned} \tag{45}$$

Proof. Exploiting the equations of (11), (44), and integrating by parts, we obtain

$$\begin{aligned} J'(t) \quad &= \quad \rho_1 \int_0^L w_t \varphi_t dx + K \int_0^L w(\varphi_x + \psi)_x dx + \rho_2 \int_0^L \psi_t^2 dx - \gamma \int_0^L w|\varphi_t|^{p(\cdot)-2} \varphi_t dx \\ &+ \int_0^L \psi\Big(b\psi_{xx} - K(\varphi_x + \psi) - \beta|\psi_t|^{q(\cdot)-2}\psi_t\Big) dx \\ &= \quad \rho_1 \int_0^L w_t \varphi_t dx \underbrace{- K \int_0^L (w_x + \psi)(\varphi_x + \psi) dx}_{J_1} + \rho_2 \int_0^L \psi_t^2 dx \\ &- \quad b \int_0^L \psi_x^2 dx - \beta \int_0^L \psi|\psi_t|^{q(\cdot)-2}\psi_t dx - \gamma \int_0^L w|\varphi_t|^{p(\cdot)-2}\varphi_t dx. \end{aligned} \tag{46}$$

Using the fact that

$$w_x = \frac{1}{L}\left(\int_0^L \psi(y,t)dy\right) - \psi(x,t),$$

we can prove that $J_1 < 0$. Using this result, then (46) becomes

$$\begin{aligned} J'(t) \quad &\leq \quad \rho_1 \int_0^L w_t \varphi_t dx + \rho_2 \int_0^L \psi_t^2 dx \\ &- \quad b \int_0^L \psi_x^2 dx - \beta \int_0^L \psi|\psi_t|^{q(\cdot)-2}\psi_t dx - \gamma \int_0^L w|\varphi_t|^{p(\cdot)-2}\varphi_t dx. \end{aligned} \tag{47}$$

Using (15), (17) and (43) and Young's inequality, then (47) becomes

$$\begin{aligned} J'(t) \quad &\leq \quad \varepsilon_0 \rho_1 \int_0^L \varphi_t^2 dx + \frac{c_2}{\varepsilon_0}\rho_1 \int_0^L w_t^2 dx + \rho_2 \int_0^L \psi_t^2 dx - b \int_0^L \psi_x^2 dx \\ &+ c_1 \eta \beta \int_0^L \psi_x^2 dx + c_2 \lambda \gamma \int_0^L \psi_x^2 dx + \beta \int_0^L c_\lambda(x)|\psi_t|^q dx \\ &+ \gamma \int_0^L c_\lambda(x)|\varphi_t|^p dx. \end{aligned}$$

Using (43) again, taking $\lambda = \eta$ and $c_0 = \min\{c_1\beta, c_2\gamma\}$, the above estimate becomes

$$\begin{aligned} J'(t) \quad \leq \quad & \varepsilon_0\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon_0}\rho_2 \int_0^L \psi_t^2 dx - b\int_0^L \psi_x^2 dx \\ & + c_0\eta \int_0^L \psi_x^2 dx + \beta \int_0^L c_\lambda(x)|\psi_t|^q dx + \gamma \int_0^L c_\lambda(x)|\varphi_t|^p dx. \end{aligned}$$

Selecting $\eta = \frac{b}{2c_0}$, the proof of (45) is completed. Similarly, using (16) and (18), one can prove that, for $1 < p_1, q_1 < 2$,

$$\begin{aligned} J'(t) \quad & \leq \rho_1\varepsilon_0 \int_0^L \varphi_t^2 dx + \frac{c\rho_2}{\varepsilon_0} \int_0^L \psi_t^2 dx - \frac{b}{2}\int_0^L \psi_x^2 dx + c\beta \int_0^L |\psi_t|^q dx \\ & + c\gamma \int_0^L |\varphi_t|^p dx \\ & + c\beta \left(\int_0^L |\psi_t|^{q(x)} dx \right)^{q_1 - 1} + c\gamma \left(\int_0^L |\varphi_t|^{p(x)} dx \right)^{p_1 - 1}. \end{aligned} \tag{48}$$

□

Lemma 9. *Assume that (**A.1**) and (**A.2**) hold. Then, the functional $\mathcal{L}$ defined by*

$$\mathcal{L}(t) := NE(t) + N_2F(t) + I(t) + N_1J(t)$$

satisfies for some positive constant ϑ, $t \geq 0$ and suitable choices of $N, N_1, N_2 > 0$,

$$\mathcal{L}(t) \sim E(t) \tag{49}$$

and the estimates

$$\mathcal{L}'(t) \leq \begin{cases} -\vartheta E(t) + c\int_0^L \varphi_t^2 dx + c\int_0^L \psi_t^2 dx, & p_1, q_1 \geq 2; \\ -\vartheta E(t) + c\int_0^L \varphi_t^2 dx + c\int_0^L \psi_t^2 dx - cE^{-\alpha_1}(t)E'(t), & \gamma = 0, \beta \neq 0, 1 < p_1, q_1 < 2. \\ -\vartheta E(t) + c\int_0^L \varphi_t^2 dx + c\int_0^L \psi_t^2 dx - cE^{-\alpha_2}(t)E'(t), & \beta = 0, \gamma \neq 0, 1 < p_1, q_1 < 2. \\ -\vartheta E(t) + c\int_0^L \varphi_t^2 dx + c\int_0^L \psi_t^2 dx - cE^{-\alpha_3}(t)E'(t), & \gamma \neq 0, \beta \neq 0, 1 < p_1, q_1 < 2. \end{cases} \tag{50}$$

where $\alpha_1 = \frac{2-q_1}{q_1-1} > 0, \alpha_2 = \frac{2-p_1}{p_1-1} > 0, \alpha_3 = \frac{2-m_1}{m_1-1} > 0$ and $m_1 = \min\{p_1, q_1\}$.

Proof. It is a routine computation to establish that $\mathcal{L}(t) \sim E(t)$. To prove $(50)_2$, combining (29), (40) and (45), we obtain

$$\begin{aligned} \mathcal{L}'(t) \quad \leq \quad & -(N - cN_1 - c)\beta \int_0^L |\psi_t|^{q(\cdot)} dx \\ - \quad & (N - cN_1 - c)\gamma \int_0^L |\varphi_t|^{p(\cdot)} dx \\ & -\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - 2c_1\rho_1 \int_0^L \varphi_t^2 dx + N_1\varepsilon_0\rho_1 \int_0^L \varphi_t^2 dx \\ & -N_2\rho_2 \int_0^L \psi_t^2 dx + \left(\frac{cN_1}{\varepsilon_0} + c \right)\rho_2 \int_0^L \psi_t^2 dx \\ & -\left(\frac{bN_1}{2} - (K + 2b)N_2\eta - c \right) \int_0^L \psi_x^2 dx \\ & + c\gamma \left(\int_0^L |\varphi_t|^{p(\cdot)} dx \right)^{p_1 - 1} + c\beta \left(\int_0^L |\psi_t|^{q(\cdot)} dx \right)^{q_1 - 1}. \end{aligned}$$

We start by choosing $N_2 = \frac{c}{(K+2b)\eta}$ and $\varepsilon_0 = N_1$, then we choose N_1 large enough such that $\frac{bN_1}{2} - 2c > 1$. Finally, we select N large enough such that $N - cN_1 - c > 0$. Then, the above estimate becomes

$$\begin{aligned} \mathcal{L}'(t) &\leq -cE(t) + c\int_0^L \psi_t^2 dx + c\int_0^L \varphi_t^2 dx + c\gamma\left(\int_0^L |\varphi_t|^{p(\cdot)} dx\right)^{p_1-1} \\ &+ c\beta\left(\int_0^L |\psi_t|^{q(\cdot)} dx\right)^{q_1-1}. \end{aligned}$$

Recalling (14), the above estimate becomes

$$\mathcal{L}'(t) \leq -cE(t) + c\int_0^L \psi_t^2 dx + c\int_0^L \varphi_t^2 dx + c\gamma\big(-E'(t)\big)^{p_1-1} + c\beta\big(-E'(t)\big)^{q_1-1}.$$

We consider the case when $\gamma = 0$ and $\beta \neq 0$. In this case, we have

$$\mathcal{L}'(t) \leq -cE(t) + c\int_0^L \psi_t^2 dx + c\int_0^L \varphi_t^2 dx + c\beta\big(-E'(t)\big)^{q_1-1}. \tag{51}$$

Using Young's inequality with $\zeta = \frac{1}{q_1-1}$ and $\zeta^* = \frac{1}{2-q_1}$, then for any $\varepsilon > 0$, we estimate the last term as follows:

$$E^\alpha(t)\big(-E'(t)\big)^{q_1-1} \leq \varepsilon E^{\frac{\alpha}{2-q_1}}(t) + c_\varepsilon\big(-E'(t)\big).$$

Multiplying both sides by $E^{-\alpha}$ where $\alpha = \frac{2-q_1}{q_1-1}$ gives us

$$\big(-E'(t)\big)^{q_1-1} \leq \varepsilon E(t) + c_\varepsilon E^{-\alpha}(t)\big(-E'(t)\big).$$

Inserting this estimate in (52), we find that

$$\mathcal{L}'(t) \leq -(c-\varepsilon)E(t) + c\int_0^L \psi_t^2 dx + c\int_0^L \varphi_t^2 dx + c_\varepsilon E^{-\alpha}(t)\big(-E'(t)\big). \tag{52}$$

By taking ε small enough and using the non increasing property of E. The proof of $(50)_2$ is completed, and the proofs of the remaining cases are similar. □

Lemma 10. *Assume that* $(\boldsymbol{A.1})$ *holds. If* $p_1, q_1 \geq 2$, *then*

$$\begin{aligned} \int_0^1 \varphi_t^2 dx &\leq -cE'(t), \quad \text{if } p_2 = 2, \\ \int_0^1 \psi_t^2 dx &\leq -cE'(t), \quad \text{if } q_2 = 2, \end{aligned} \tag{53}$$

and

$$\begin{aligned} \int_0^1 \varphi_t^2 dx &\leq -cE'(t) + c\big(-E'(t)\big)^{\frac{2}{p_2}}, \quad \text{if } p_2 > 2 \\ \int_0^1 \psi_t^2 dx &\leq -cE'(t) + c\big(-E'(t)\big)^{\frac{2}{q_2}}, \quad \text{if } q_2 > 2. \end{aligned} \tag{54}$$

Proof. By recalling (14), it is easy to establish (53). To prove the first estimate in (54), we set the following partitions

$$\Omega_1 = \{x \in \Omega : |\varphi_t| \geq 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |\varphi_t| < 1\}. \tag{55}$$

Use of Hölder and Young inequalities and (13), we obtain for Ω_1,

$$\int_{\Omega_1} \varphi_t^2 dx \leq \int_\Omega |\varphi_t|^{p(x)} dx \leq -cE'(t), \tag{56}$$

and for Ω_2, we obtain

$$\begin{aligned}\int_{\Omega_2}\varphi_t^2\mathrm{d}x &\le c\left(\int_{\Omega_2}|\varphi_t|^{p_2}\mathrm{d}x\right)^{\frac{2}{p_2}}\\ &\le c\left(\int_{\Omega_2}|\varphi_t|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_2}} \le c\left(\int_{\Omega}|\varphi_t|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_2}} \le c\big(-E'(t)\big)^{\frac{2}{p_2}}.\end{aligned} \tag{57}$$

Combining (56) and (57), the first estimate in (54) is established and repeating the same steps to establish the second estimate in (54). □

4. Decay Estimates

In this section, we state and prove our decay results.

Theorem 1. *Assume that* $(\boldsymbol{A.1})$ *and* $(\boldsymbol{A.2})$ *hold and* $p_1, q_1 \ge 2$*. Then, the energy functional* (13) *satisfies, for some positive constants* $\lambda_i, \sigma_i, \mu_i > 0,\ i = 1,2,3$ *and for any* $t \ge 0$*,*

$$\begin{cases} E(t) < \mu_1 e^{-\lambda_1 t}, & \text{if } \gamma = 0,\ \beta \ne 0 \text{ and } q_2 = 2;\\ E(t) < \mu_2 e^{-\lambda_2 t}, & \text{if } \gamma \ne 0,\ \beta = 0 \text{ and } p_2 = 2;\\ E(t) < \mu_3 e^{-\lambda_3 t}, & \text{if } \gamma \ne 0,\ \beta \ne 0 \text{ and } p_2 = q_2 = 2, \end{cases} \tag{58}$$

and

$$\begin{cases} E(t) < \dfrac{\sigma_1}{(t+1)^{\left(\frac{q_2-2}{2}\right)}}, & \text{if } \gamma = 0,\ \beta \ne 0 \text{ and } q_2 > 2;\\ E(t) < \dfrac{\sigma_2}{(t+1)^{\left(\frac{p_2-2}{2}\right)}}, & \text{if } \gamma \ne 0,\ \beta = 0 \text{ and } p_2 > 2;\\ E(t) < \dfrac{\sigma_3}{(t+1)^{\left(\frac{m_2-2}{2}\right)}}, & \text{if } \gamma \ne 0,\ \beta \ne 0 \text{ and } p_2, q_2 > 2, \end{cases} \tag{59}$$

where $m_2 = \min\{p_2, q_2\}$.

Proof. To prove $(58)_1$, we impose Lemma 10 in $(50)_1$ to obtain

$$\mathcal{L}'(t) \le -c\mathcal{L}(t) + c(-E'(t)).$$

This gives

$$\mathcal{L}_1'(t) \le -c\mathcal{L}(t).$$

where $\mathcal{L}_1 = \mathcal{L} + cE \sim E$. Integrating the last estimate over the interval $(0,t)$ and using the equivalence properties $\mathcal{L}_1, \mathcal{L} \sim E$, the proof of $(58)_1$ is completed, and the proofs of $(58)_2$ and $(58)_3$ are similar. Now, we prove the estimate in $(59)_3$ and the remaining will be similar. In this case, we also impose Lemma 10 in $(50)_1$ to obtain

$$\mathcal{L}'(t) \le -c\mathcal{L}(t) + (-E'(t))^{\frac{2}{p_2}} + (-E'(t))^{\frac{2}{q_2}}.$$

Multiplying the last equation by E^α where $\alpha = \frac{p_2-2}{2} > 0$, then we obtain

$$E^q\mathcal{L}'(t) \le -cE^\alpha\mathcal{L}(t) + E^\alpha(-E'(t))^{\frac{2}{p_2}} + E^\alpha(-E'(t))^{\frac{2}{q_2}}.$$

Use of Young's inequality twice, we obtain for $\varepsilon > 0$

$$E^{\alpha}\mathcal{L}'(t) \le -cE^{\alpha+1}\mathcal{L}(t) + \varepsilon E^{\frac{\alpha p_2}{p_2-2}} + \varepsilon E^{\frac{\alpha q_2}{q_2-2}} + C_{\varepsilon}(-E'(t)).$$

Here, we will discuss ***two cases***:

Case A: If $p_2 < q_2$, we have

$$E^{\alpha}\mathcal{L}'(t) \le -cE^{\alpha+1}\mathcal{L}(t) + \varepsilon E^{\frac{\alpha p_2}{p_2-2}} + \varepsilon E^{\frac{\alpha p_2}{p_2-2}} E^{\frac{2\alpha(p_2-q_2)}{(p_2-2)(q_2-2)}} + C_{\varepsilon}(-E'(t)).$$

Using the non-increasing property of E, then we obtain

$$E^{\alpha}\mathcal{L}'(t) \le -(c - \varepsilon - c\varepsilon)E^{\alpha+1}\mathcal{L}(t) + C_{\varepsilon}(-E'(t)).$$

Taking ε small enough, the above estimate becomes:

$$\mathcal{L}_2(t) \le -cE^{\alpha+1}(t), \qquad \forall t \ge 0, \tag{60}$$

where $\mathcal{L}_2 = E^{\alpha}\mathcal{L} + cE \sim E$.

Integration over $(0,t)$, using $E \sim \mathcal{L}_2$, gives

$$E(t) < \frac{c_{p_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{61}$$

where $\alpha = \frac{p_2-2}{2}$.

Case B: If $q_2 < p_2$, in this case, we will obtain

$$E(t) < \frac{c_{q_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{62}$$

where $\alpha = \frac{q_2-2}{2}$. Thus, by taking $m_2 = \min\{p_2, q_2\}$, the proof of (59)$_3$ is completed. □

Theorem 2. *Assume that* $(\boldsymbol{A.1})$ *and* $(\boldsymbol{A.2})$ *hold,* $1 < p_1, q_1 < 2$ *and* $p_2 = q_2 = 2$*. Then, the energy functional* (13) *satisfies, for a positive constants* C_i*,* $i = 1,2,3$*, and for any* $t > 0$*,*

$$\begin{cases} E(t) < \dfrac{C_1}{(t+1)^{\left(\frac{q_1-1}{2-q_1}\right)}}, & \text{if } \gamma = 0 \text{ and } \beta \neq 0; \\[2ex] E(t) < \dfrac{C_2}{(t+1)^{\left(\frac{p_1-1}{2-p_1}\right)}}, & \text{if } \gamma \neq 0 \text{ and } \beta = 0; \\[2ex] E(t) < \dfrac{C_3}{(t+1)^{\left(\frac{m_1-1}{2-p_1}\right)}}, & \text{if } \gamma \neq 0 \text{ and } \beta \neq 0. \end{cases} \tag{63}$$

where $m_1 = \min\{p_1, q_1\}$.

Proof. To prove (63)$_1$, we apply Lemma 10 in (50)$_2$ to have

$$\mathcal{L}'(t) \le -cE(t) + \left(-E'(t)\right) + \left(-E'(t)\right) - cE^{-\alpha_1}(t)E'(t), \tag{64}$$

where $\alpha_1 = \frac{2-q_1}{q_1-1} > 0$. (64) becomes

$$\mathcal{L}_1'(t) \le -cE(t) - cE^{-\alpha_1}(t)E'(t), \tag{65}$$

where $\mathcal{L}_1 = \mathcal{L} + cE \sim E$. Multiplying (65) by E^{α_1} gives

$$E^{\alpha_1}(t)\mathcal{L}_1'(t) \le -cE^{\alpha_1+1}(t) - cE'(t).$$

which implies that

$$\mathcal{L}_2'(t) \leq -cE^{\alpha_1+1}(t)$$

where $\mathcal{L}_2 = E^{\alpha}\mathcal{L}_1 + cE \sim E$

Then, we have the following decay estimate

$$E(t) < \frac{c_{q_1}}{(t+1)^{1/\alpha_1}}, \ \forall\, t > 0, \tag{66}$$

where $\alpha_1 = \frac{2-q_1}{q_1-1}$. This completes the proof of $(63)_1$. The proof of $(63)_2$ and $(63)_3$ will be in the same way. □

Theorem 3. *Assume that* $(\boldsymbol{A.1})$ *and* $(\boldsymbol{A.2})$ *hold,* $1 < p_1, q_1 < 2$ *and* $p_2, q_2 > 2$*. Then, the energy functional* (13) *satisfies, for a positive constants* $C_i,\ i = 1,2,3$*, and for any* $t > 0$*,*

$$\begin{cases} E(t) < \dfrac{C_1}{(t+1)^{\left(\frac{2}{q_2-2}\right)}}, & \text{if } \gamma = 0 \text{ and } \beta \neq 0; \\[2ex] E(t) < \dfrac{C_2}{(t+1)^{\left(\frac{2}{p_2-2}\right)}}, & \text{if } \gamma \neq 0 \text{ and } \beta = 0; \\[2ex] E(t) < \dfrac{C_3}{(t+1)^{\left(\frac{2}{m_2-2}\right)}}, & \text{if } \gamma \neq 0 \text{ and } \beta \neq 0, \end{cases} \tag{67}$$

where $m_2 = \min\{p_2, q_2\}$

Proof. To prove $(67)_1$, we apply Lemma 10 in $(50)_1$ to obtain

$$\mathcal{L}'(t) \leq -cE(t) + (-E'(t))^{\frac{2}{p_2}} + (-E'(t))^{\frac{2}{q_2}} - cE^{-\alpha_1}(t)E'(t),$$

where $\alpha_1 = \frac{2-q_1}{q_1-1} > 0$. Multiplying by E^{α} where $\alpha = \frac{q_2-2}{2} > 0$, using $\alpha - \alpha_1 > 0$ and Young's inequality twice, we obtain for $\varepsilon > 0$

$$E^{\alpha}\mathcal{L}'(t) \leq -cE^{\alpha+1}(t) + \varepsilon E^{\frac{\alpha p_2}{p_2-2}} + \varepsilon E^{\frac{\alpha q_2}{q_2-2}} + C_{\varepsilon}(-E'(t)).$$

Assuming that $q_2 > p_2$, then

$$E^{\alpha}\mathcal{L}'(t) \leq -cE^{\alpha+1}\mathcal{L}(t) + \varepsilon E^{\frac{\alpha q_2}{q_2-2}} + \varepsilon E^{\frac{\alpha q_2}{q_2-2}} E^{\frac{2\alpha(q_2-p_2)}{(p_2-2)(q_2-2)}} + C_{\varepsilon}(-E'(t)).$$

Using the non-increasing property of E, then we obtain

$$E^{\alpha}\mathcal{L}'(t) \leq -(c - \varepsilon - c\varepsilon)E^{\alpha+1}\mathcal{L}(t) + C_{\varepsilon}(-E'(t)).$$

Taking a small enough ε, the above estimate becomes:

$$\mathcal{L}_2(t) \leq -cE^{\alpha+1}(t), \qquad \forall t \geq 0, \tag{68}$$

where $\mathcal{L}_2 = E^{\alpha}\mathcal{L} + cE \sim E$.

Integration over $(0,t)$, using $E \sim \mathcal{L}_2$, gives

$$E(t) < \frac{c_{q_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{69}$$

where $\alpha = \frac{q_2-2}{2}$. Thus, the proof of $(67)_1$ is completed, and the proofs of $(67)_2$ and $(67)_3$ will be the same. □

5. Conclusions

In this study, we considered the Timoshenko system with two nonzero dampings of variable exponent types. We discussed different cases, and we proved that the system is exponentially and polynomially stable, and the stability results depend on the values of p_1, p_2, q_1, q_2. In addition, we concluded that the decay estimate is not necessarily improved if the system has two dampings.

Author Contributions: Writing—original draft, A.M.A.-M.; Writing—review & editing, M.M.A.-G. All authors have read and agreed to the published version of the manuscript.

Funding: This work is funded by KFUPM through Project No. Grant No. INCB2215.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors would like to acknowledge the support provided by King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia. The support provided by the Interdisciplinary Research Center for Construction & Building Materials (IRC-CBM) at King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia, for funding this work through Project No. INCB2215, is also greatly acknowledged.

Conflicts of Interest: The authors declare that there is no conflict of interest.

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